2021-04-30

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Recommended Citation

Chekhlov, A., Danchev, P. & Goldsmith, B. (2021). On the socles of characteristically inert subgroups of Abelian p-groups. Forum Mathematicum, 33(4), 889-898. DOI: 10.1515/forum-2020-0348

ON THE SOCLES OF CHARACTERISTICALLY INERT SUBGROUPS OF ABELIAN p-GROUPS

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ABSTRACT. We define the notion of a *characteristically inert socle-regular* Abelian p-group and explore such groups by focussing on their socles, thereby relating them to previously studied notions of socle-regularity. We show that large classes of p-groups, including all divisible, totally projective and torsion-complete p-groups, share this property when the prime p is odd.

The present work generalises notions of full inertia studied recently by several authors and is a development of a recent work of the authors in the Mediterranean J. Math. (2021).

1. Introduction and Conventions

Recall that subgroups H, K of a group G are said to be *commensurable* if the intersection $H \cap K$ has finite index simultaneously in H and in K; this is conventionally denoted as $H \sim K$ and this operation is known to be an equivalence relation. Recently, motivated by questions arising within the context of the notion of intrinsic entropy of Abelian groups, the concept of a fully inert subgroup has been studied by various authors, see, for example, [1, 2, 3, 15, 16]: a subgroup H of a group G is said to be *fully inert* in G if the factor-group $(H + \phi(H))/H$ is finite for any endomorphism $\phi : G \to G$, that is, H is commensurable with $H + \phi(H)$ for any $\phi \in \mathrm{End}(G)$, the endomorphism group of G. In this current work we wish to study a natural generalisation of this concept.

Before giving the details, we comment that for this paper, all groups (other than automorphism groups) will be additively written Abelian p-groups; we note that

2010 *Mathematics Subject Classification.* 20K10.

Key words and phrases. socle-regular groups, characteristically inert subgroups, characteristically inert socle-regular groups, weakly characteristically inert socle-regular groups.

many of the concepts investigated may be considered in a much wider context, including some situations where the groups are not necessarily commutative. Our notation is largely standard and follows that of Fuchs [11, 12]. In particular, we shall say that a subgroup H of a group G is *fully invariant in G [characteristic in G]* if $\phi(H) \leq H$ for all endomorphisms [automorphisms] ϕ of G. We shall have occasion to use the notions of transitivity and full transitivity; here we mean the classic notions introduced by Kaplansky as found in [21]. Two further notions that are, perhaps, not so commonly used are 'starred groups' and 'semi-standard groups'. By the former we mean a reduced p-group having the property that the cardinality of the group is equal to the cardinality of a basic subgroup; the latter means a p-group G such that for each $n < \omega$, all the Ulm invariants $f_n(G)$ are finite.

We say that a subgroup H of a group G is *characteristically inert* in G if the factor-group $(H + \phi(H))/H$ is finite for any automorphism $\phi : G \to G$; notice that all finite and cofinite subgroups of G are both characteristically inert and fully inert in G and if H is characteristic [fully invariant] in G, then H is characteristically [fully] inert in G.

One 'feels' that fully inert [characteristically inert] subgroups should, in some sense, be not 'too far' from fully invariant [characteristic] subgroups. In the case of fully inert subgroups this expectation has been realised in certain situations, including those in which the group G is either torsion-complete or totally projective [15, 16, 22]. Working from a different perspective, the second- and third-named authors investigated fully invariant and characteristic subgroups of a p-group by focussing on their socles [6, 7]. This led to two interesting classes of p-groups: a p-group G is said to be *socle-regular [strongly socle-regular]* if for every fully invariant [characteristic] subgroup H of G, there is an ordinal α such that $H[p] = p^\alpha G[p]$. It seems totally natural to try to combine these various notions of socle-regularity and inertia; the present authors have tackled this problem in [3] merging socle-regularity and full inertia. As explained in [3], there are several possible ways to combine these notions. Here we shall focus primarily on one of these approaches and make the necessary modifications to replace fully inert subgroups with characteristically inert subgroups. For the benefit of the reader we give both concepts in a uniform way.

Definition 1.1. An Abelian p-group G is said to be *characteristically inert socle-regular [fully inert socle-regular]* if, for all infinite characteristically inert subgroups [fully inert subgroups] H of G, there exists an ordinal α, depending on H, such that $H[p] \sim p^{\alpha}G[p]$.

Remark 1.2. We remark at the outset that our real interest in this definition is the situation where H is infinite but not cofinite in G; if H is any cofinite subgroup of G then $H[p] \sim p^0G[p]$. It is also convenient to remove finite subgroups from our definition. Such subgroups are, as observed above, both fully and characteristically inert and are commensurable with the zero subgroup $\{0\}$. In the case of reduced groups, they cause no difficulty since the choice of α as the length of the group G shows they satisfy the requirements for both characteristically and fully inert socle-regularity. When we deal with groups which are not reduced, as here, it is more convenient to impose the requirement of infinite order as above.

We finish this introduction by summarising the contents of the paper. There are four further sections, the first of which lays down preliminary results that help to simplify and clarify some of our more complicated arguments in later sections. In Section 3.5 we investigate characteristically inert subgroups of groups that are not reduced, showing in particular (Theorem 3.4) that an infinite characteristically inert subgroup C of a divisible p-group D is commensurable with $D[p^n]$ for some $n \geq 0$ and hence divisible p-groups are characteristically inert socle-regular.

In the main section, Section 4, we observe that if the automorphism group of G generates (additively) the full endomorphism group of G, then the notions of characteristically inert and fully inert coincide; thus many questions on characteristically inert subgroups reduce to questions on the more familiar, and more intensively studied, fully inert subgroups. An immediate consequence of this (Theorem 4.4) is that for p-groups with $p \neq 2$, the classes of totally projective and torsion-complete groups are characteristically inert socle-regular. The final theorem in that section (Theorem

4.6) shows an interesting 'family' connection with well-known results on transitivity/full transitivity [9] and socle-regularity/strong socle-regularity [7]: a group G is fully inert socle-regular if and only if $G \oplus G$ is characteristically socle-regular. The section finishes with an example showing that the class of characteristically inert socle-regular groups is not closed under direct sums.
The paper concludes with some remarks about a weaker notion of characteristically inert socle-regularity analogous to a notion introduced by the authors in [3].

2. Fundamental and Preliminary Results

We begin with a simple lemma that will enable us to produce examples of groups which are not characteristically socle-regular.

Lemma 2.1. *Suppose that G is a reduced unbounded separable p-group which is not starred. Then any characteristic subgroup of G is uncountable.*

Proof. Observe that as G is unbounded and not starred, it must be uncountable. Now if C is a characteristic subgroup of the separable group G, then it follows from [7, Corollary 2.2] that $C[p] = p^nG[p]$ for some $n < \omega$. Since G is unbounded, it follows that $p^nG[p]$ is infinite and then as G is reduced, $|p^nG[p]| = |p^nG|$. However, as G is not starred, $|G/p^nG| = |B/p^nB| < |G|$, where B is a basic subgroup of G. Thus $|G| = |p^nG|$ and it follows that $|C[p]| = |G|$ is uncountable. □

Our next result is based on an idea in [17, Theorem 2.4]; it provides an abundant supply of groups in which characteristically inert subgroups are not commensurable with any characteristic subgroup; in particular the groups are not characteristically inert socle-regular.

Example 2.2. Let G be a separable semi-standard p-group such that, in the Pierce decomposition $\mathrm{End}(G) = A \oplus E_s(G)$, A is the completion in the p-adic topology of a J_p-subalgebra F which is a free J_p-module of at most countable rank. Then there are characteristically inert subgroups of G which are not commensurable with any characteristic subgroup of G and G is not characteristically socle-regular.

Proof. Let H be any countable infinite subgroup of G and set $H^F = \sum_{\alpha \in F} \alpha(H)$; note that H^F is F-invariant, because F is closed under multiplication. We shall assume that F is actually of countably infinite rank, leaving the easier finite rank case to the reader. Let $F = \oplus_{n \in \mathbb{N}} J_p \alpha_n$. Clearly $\sum_{n \in \mathbb{N}} \alpha_n(H) \leq H^F$. Conversely let $\alpha \in F$. Then $\alpha = \sum_{1 \leq i \leq n} \pi_i \alpha_i$ for suitable $n \in \mathbb{N}$ and $\pi_i \in J_p$. It follows that $\alpha(H) \leq \sum_{1 \leq i \leq n} \alpha_i(H)$, so $H^F = \sum_{i \in \mathbb{N}} \alpha_i(H)$. This equality implies that H^F is countable, since each subgroup $\alpha_i(H)$ is countable, being an image of H; thus $H^F[p]$ is also countable. It follows exactly as in [17, Theorem 2.4], that $H^F[p]$ is characteristically inert in G; in fact, it is even fully inert in G. Furthermore, the group G is necessarily uncountable, see, for example, [17, Lemma 2.3], and as G is semi-standard, it is then not starred. Applying Lemma 2.1 above, we see that $H^F[p]$ cannot be commensurable with the socle of any characteristic subgroup of G. □

In [7, Theorem 2.4] it was shown that a transitive p-group was strongly socle-regular. Our next example shows that if we omit the requirement of transitivity, there are p-groups (for all primes p) which are not characteristically inert socle-regular.

Example 2.3. There is a fully transitive non-transitive p-group G which is not characteristically inert socle-regular.

Proof. Let G be Corner's [5] non-transitive fully transitive p-group with $p^\omega G$ elementary of infinite rank and R the $\mathbb{Z}(p)$-algebra freely generated by indeterminates $a_0, a_1, \ldots$ where $\mathrm{End}G \restriction p^\omega G = R$ and $\mathrm{Aut}G \restriction p^\omega G = U(R) = \{1, 2, \ldots (p-1)\}$. Let $X = X[p]$ be an infinite, co-infinite subgroup of $p^\omega G$. Then the action of $\mathrm{Aut}G$ on $p^\omega G$ gives us that X is characteristic in G; note that $X[p]$ is not commensurable with $p^\omega G[p]$. Since $X[p] \leq p^\omega G[p] \leq p^n G[p]$ for every integer $n < \omega$, we cannot have that $X[p]$ is commensurable with any $p^n G[p]$ for $n < \omega$. Finally, as $p^{\omega+1} G = 0$, we cannot have that X is commensurable with any $p^\beta G[p]$ and so G is not characteristically inert socle-regular. □

Examples 2.2 and 2.3 show that there is no simple connection between transitivity and failure of characteristically inert socle-regularity.

There is, however, a simple way of producing characteristically inert socle-regular groups. We shall make use of this result in Section 4.

Proposition 2.4. *If G is a reduced p-group with the property that every characteristically inert subgroup is commensurable with a characteristic subgroup of G, then if G is strongly socle-regular, it is also characteristically inert socle-regular.*

Proof. Suppose that H is characteristically inert in G. Then, by hypothesis, there is a characteristic subgroup C of G with $H \sim C$. Since G is strongly socle-regular, there is an ordinal β with $C[p] = p^\beta G[p]$, whence $H[p] \sim C[p] = p^\beta G[p]$. □

Our next result gathers together some simple properties that we will use in various places. The results are probably well known and we give just outlines of the simple proofs.

Proposition 2.5. *(i) If C is characteristically inert in G and $H \sim C$, then H is characteristically inert in G; in particular if C is characteristic in G and $H \sim C$, then H is characteristically inert in G;*
(ii) if $G = A \oplus B$, $\pi : G \to A$ is the projection and C is characteristic in G, then $f(\pi(C)) \leq C \cap B$ for any $f \in \mathrm{Hom}\,(A, B)$;
(iii) if two subgroups A, B of G are commensurable then A is reduced if and only if B is also reduced.

Proof. (i) Since $H \sim C$, if α is an arbitrary automorphism of G then $\alpha(C) \sim \alpha(H)$ and hence we get $\alpha(H) \cap H \cap C \sim \alpha(C) \cap C$ and this latter is commensurable with $\alpha(C)$ since C is characteristically inert in G. Thus $\alpha(H) \cap H \cap C \sim \alpha(C) \sim \alpha(H)$ and the quotient $\alpha(H)/(\alpha(H) \cap H \cap C)$ is finite. The result follows since $\alpha(H)/(\alpha(H) \cap H)$ is a homomorphic image of $\alpha(H)/(\alpha(H) \cap H \cap C)$.
(ii) The matrix $\Delta = \left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)$ represents an automorphism of G, and if $c = a + b \in C$, $a \in A$, $b \in B$ then $\Delta(c) = a + (b + f(a)) \in C$ and hence $f(a) = f(\pi(c)) \in C$.
(iii) Since the relation $\sim$ is symmetric it suffices to show that if A is reduced then so too is B. Suppose, for a contradiction, that B is not reduced, then $A \sim B = D \oplus K$, where $0 \neq D$ is divisible. Consequently, $(D \oplus K)/(A \cap B) \geq (D + (A \cap B))/(A \cap B) \cong$

$D/(D \cap A \cap B)$. Since the latter term is both divisible and finite, it must be 0. Thus $D = (D \cap A \cap B) \leq A$ – contradiction to A being reduced. □

Our next result is analogous to a well-known result on fully inert subgroups, see, for example, [3, Lemma 2.1] or [16, Lemma 3.3].

Lemma 2.6. *If every automorphism of a subgroup A of G extends to an automorphism of G and if C is characteristically inert in G, then $C \cap A$ is characteristically inert in A.*

Proof. Suppose that α is an arbitrary automorphism of A and let ψ be an automorphism of G such that $\psi \restriction A = \alpha$. Then $(C \cap A) + \alpha(C \cap A) \leq [C + \psi(C)] \cap A$ and so we have $[(C \cap A) + \alpha(C \cap A)]/(C \cap A) \leq ([C + \psi(C)] \cap A)/(C \cap A)$ and the latter term is a homomorphic image of $[C + \psi(C)]/C$ which is finite since C is characteristically inert in G and ψ is an automorphism of G. Since α was arbitrary, $C \cap A$ is characteristically inert in A. □

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Notice that an immediate consequence of Lemma 2.6 is that if C is characteristically inert in $G = A \oplus B$, then $C \cap A, C \cap B$ are characteristically inert in A, B respectively. Furthermore, for any $n < \omega$ and any G, $C \cap p^n G$ is characteristically inert in $p^n G$. A consequence of Zippin's theorem - see [11, Corollary 77.4, Theorem 83.4] - is that n can be replaced by any ordinal provided that the group G is countable, or more generally totally projective.

The reader familiar with [3, Lemma 2.1] or [16, Lemma 3.3] will observe immediately that Lemma 2.6 above is a great deal weaker than the results one can obtain for fully inert subgroups. This weakness derives from the fact that characteristic subgroups of a group G do not behave as nicely as fully invariant ones when G has the form $G = A \oplus B$. We can, however, regain some of the strength of the fully inert situation by imposing an extra condition.

Proposition 2.7. *Let $G = A \oplus B$ and suppose that C is characteristically inert in G. Then* *(i) $C \cap A, C \cap B$ are characteristically inert in A, B respectively;*

(ii) if the identity 1_A of A is the sum of two automorphisms (in particular if 2.1_A is a unit of End$A$*), then $C \sim (C \cap A) \oplus (C \cap B)$;*

(iii) if $\pi : G \to A$ is the canonical projection of G onto A along B, and f is any homomorphism $f : A \to B$, then $C \sim C + f\pi(C) \sim C + f(C \cap A)$.

Proof. Part (i) follows from Proposition 2.6 above.

For part (ii) write $1_A = \alpha_1 + \alpha_2$ where α_1, α_2 are automorphisms of A. Define matrices $\Delta = \Delta_1 + \Delta_2$ by setting $\Delta_1 = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & 1_B \end{smallmatrix}\right)$ and $\Delta_2 = \left(\begin{smallmatrix} \alpha_2 & 0 \\ 0 & 1_B \end{smallmatrix}\right)$; clearly then $\Delta = \left(\begin{smallmatrix} 1_A & 0 \\ 0 & 2.1_B \end{smallmatrix}\right)$. Then $C + \Delta(C) = [C + \Delta_1(C)] + [C + \Delta_2(C)]$ giving the quotient

$$(C + \Delta(C))/C = ([C + \Delta_1(C)] + [C + \Delta_2(C)])/(C + C);$$

since the displayed term is an homomorphic image of $[C+\Delta_1(C)]/C \oplus [C+\Delta_2(C)]/C$, it is finite as Δ_1, Δ_2 are automorphisms of G and C is characteristically inert in G. We claim that $C + \Delta(C) = C + \pi_A(C) = C + \pi_B(C)$, where π_A, π_B are the canonic projections of G onto A, B respectively. Suppose for the moment that we have established this claim, then $(C + \Delta(C))/C = (C + \pi_A(C))/C \cong \pi_A(C)/(C \cap \pi_A(C))$ is finite; similarly $\pi_B(C)/(C \cap \pi_B(C))$ is finite. Now $C \leq \pi_A(C) \oplus \pi_B(C)$, $C \cap A = C \cap \pi_A(C)$ and $C \cap B = C \cap \pi_b(C)$, so we have

$$C/((C \cap A) \oplus (C \cap B)) \leq \pi_A(C)/(C \cap \pi_A(C)) \oplus \pi_B(C)/(C \cap \pi_B(C))$$

is finite, giving $C \sim (C \cap A) \oplus (C \cap B)$. Thus to complete the proof it suffices to establish the claim above.

To establish the claim, observe that if $x \in \pi_A(C)$, then $x = (a, 0)$ for some $(a, b) \in C$ and thus $(a, 0) = 2(a, b) - (a, 2b) \in C + \Delta(C)$ as $\Delta((a, b)) = (a, 2b)$. On the other hand, if $y \in \Delta(C)$ then $y = (a, 2b)$ for some $(a, b) \in C$. Hence, $y = 2(a, b) - (a, 0) \in C + \pi_A(C)$. So we have $C + \pi_A(C) \leq C + \Delta(C) \leq C + \pi_A(C)$ giving equality. A similar argument establishes that $C + \Delta(C) = C + \pi_B(C)$.

For part(iii), set $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right) \in \mathrm{Aut}(G)$, so that $C \sim C + \alpha(C) = C + f\pi(C)$. Since $C \cap A \leq \pi(C)$, we have that $f(C \cap A) \leq f\pi(C)$ and hence $C + f(C \cap A) \leq C + f\pi(C)$, giving us the desired $C \sim C + f(C \cap A)$. $\square$

3. Groups which are not reduced

In this short section we consider groups G which are not reduced i.e., G can be expressed as $G = D \oplus A$ where $D \neq 0$ is divisible.

We begin with two simple lemmas which will be useful when we look at divisible groups.

Lemma 3.1. *Let $G = D \oplus A$ where D, A are non-zero with D divisible and let π denote the canonical projection of G onto A along D. Then if X is characteristically inert in G and $\pi(X)$ is unbounded, $D \leq X$.*

Proof. Let $D = \bigoplus_{i \in I} D_i$ where each D_i is a rank one divisible p-group. Choose a fixed but arbitrary D_i. Since $\pi(X)$ is unbounded, there is a homomorphism $\phi : \pi(X) \to D_i$ which is onto D_i. Since D_i is injective, ϕ extends to a map $\psi : A \to D_i$ with $\psi(\pi(X)) = D_i$. Applying Proposition 2.7(iii), we get that $X \sim X + \psi(\pi(X)) = X + D_i$ so that $(X + D_i)/X \cong D_i/(X \cap D_i)$ is finite and divisible, hence zero. So $D_i = X \cap D_i \leq X$. Since D_i was arbitrary and each element of D is just a finite sum of elements from the various D_i, we have $D \leq X$, as required. □

Corollary 3.2. *If G is divisible and X is a reduced characteristically inert subgroup of G, then X is bounded.*

Proof. If G is of rank 1, then X reduced implies that X is even finite. So suppose that G has rank at least 2 and assume for a contradiction that X is unbounded. Let $G = D \oplus D_1$, where D_1 is of rank 1 and let π_1 is the canonical projection of G onto D_1. We claim that $\pi_1(X)$ is unbounded: if not, $p^n\pi_1(X) = 0$ for some $n < \omega$. Hence $p^nX \leq \mathrm{Ker}\pi_1 = D$. Since X is unbounded, so too is p^nX and then $p^nX = D$, since the latter is divisible of rank 1. Thus $\pi_1(X)$ is unbounded which on applying Lemma 3.1 above, leads to the contradiction that the reduced group X contains D. □

Lemma 3.3. *Let $C = C_0 \oplus C_1$, where C_0, C_1 are infinite rank homocyclic groups of exponents n_0, n_1 respectively. If $D = D_0 \oplus D_1$ is a divisible hull of C with*

$C_0 \leq D_0, C_1 \leq D_1$, *then if* $n_0 < n_1$, *the subgroup* C *is not characteristically inert in* D.

Proof. By working with suitable summands if necessary, we may assume that C_0 and C_1 are both of rank $\aleph_0$. Suppose, for a contradiction, that C is characteristically inert in D.

Let $\pi : D \to D_1$ be the canonical projection of D onto D_1. Now embed, via a map f say, C_1 into D_0 in such a way that $f(C_1) \geq C_0$ – this is possible since $C_1 \cong D[p^{n_1}]$ and $C_0 \cong D_0[p^{n_0}]$. Since D_0 is injective, we can extend f to a map $D_1 \to D_0$ which we will continue to call f. By Proposition 2.7(iii), we have that $C \sim C + f\pi(C) \sim C + f(C \cap D_1) = C + f(C_1)$. But then the finite quotient $f(C_1)/(C \cap f(C_1)) = D_0[p^{n_1}]/D_0[p^{n_0}]$, contrary to D_0 being of infinite rank. Thus C cannot be characteristically inert in D, as required. □

We first consider the situation in which we have a divisible group.

Theorem 3.4. *Let* D *be a divisible group and* C *an infinite characteristically inert subgroup of* D. *Then* C *is commensurable with* $D[p^n]$ *for some* $n \geq 0$ *and* D *is characteristically inert socle-regular.*

Proof. Firstly note that we may assume D has rank at least 2: the only infinite subgroup of $\mathbb{Z}(p^\infty)$ is the whole group itself, so in this situation we have that $C = D = D[p^0]$. We split the argument into two cases : (a) C is not reduced (b) C is reduced.

Case(a). Assume $C = D_0 \oplus K$ and $D = D_0 \oplus D_1$ where D_0 is the maximal divisible subgroup of C. We claim that D_1 must then be zero, giving us the desired result that again $C = D$.

Suppose then that $D_1 \neq 0$, so there exists a homomorphism $f : D_0 \to D_1$ with $f(D_0)$ a non-zero divisible subgroup of D_1. If π denotes the canonical projection of D onto D_0, then $\phi = 1_D + f\pi$ is an automorphism of D and so $C \sim C + \phi(C) = C + f\pi(D)$. It then follows that $f\pi(D)/(C \cap f\pi(D)) = f(D_0)/(f(D_0) \cap C)$ is both finite and divisible, whence it is zero, i.e., $C \cap f(D_0) = f(D_0)$ giving the contradiction that $f(D_0)$ is a divisible subgroup of C which is not contained in D_0.

Case(b). It follows from Corollary 3.2 that in this situation C must be reduced and bounded. So C is a direct sum of a finite number of homocyclic groups. Moreover, as a homocyclic group of finite rank is finite, we may replace C by another characteristically inert subgroup which is a finite direct sum of homocylic groups where each component is of infinite rank. Thus there is no loss to assume that C itself has this form.

Now embed C as an essential subgroup of some summand D_1 of D; say $D = D_1 \oplus D_2$. Observe firstly that D_2 cannot be of infinite rank. For if so, by working with a suitable summand of D_2, we may without loss assume that D_2 itself is of countable rank. Now choose a summand C_0 of C of countably infinite rank and let $D_0 \leq D_1$ be a divisible hull of C_0. Embed C_0 in D_2 via a map, ϕ say, and extend this to a (still monic) map from $D_0 \to D_2$ which we will continue to call ϕ. Now ϕ extends trivially to an endomorphism $\psi : D_1 \to D_2$ and this gives rise to an automorphism of D, say Δ, where $\Delta = \begin{pmatrix} 1_{D_1} & 0 \\ \psi & 1_{D_2} \end{pmatrix}$. Since C is characteristically inert in D, the quotient $C + \Delta(C)/C$ is finite, yielding that $\psi(C)/(C \cap \psi(C))$ is finite; since $C \cap \psi(C) = 0$, this would make $\psi(C)$ finite, impossible as $\psi(C) \geq \phi(C)$ and ϕ is monic on C. Thus D_2 is of finite rank, as claimed.

The final step in our argument is to show that C must, in fact, be homocyclic. This, however, follows easily from Lemma 3.3

Then C is homocyclic of infinite rank and exponent N, say. Now we have $D = D_1 \oplus D_2$ with D_2 of finite rank and C essential in D_1. Taking p^N-socles we get, $D[p^N] = D_1[p^N] \oplus D_2[p^N] \sim D_1[p^N]$, as D_2 is of finite rank. However, $D_1[p^N] = C[p^N] = C$, so $C \sim D[p^N]$, as required.

Finally the claim that D is characteristically socle-regular is then immediate.

□

We now consider the characteristically inert subgroups of groups which are not divisible but are also not reduced.

Proposition 3.5. *If $G = D \oplus A$ where D, A are non-zero with D divisible and A reduced, then for any unbounded characteristically inert subgroup C of G, we have $D \leq C$.*

Proof. Since C is characteristically inert in G, we have by Proposition 2.6 that $C \cap D$ is characteristically inert in D. If $C \cap D$ is not reduced, it follows from the proof of case(a) of Proposition 3.4 that $C \cap D = D$, giving $D \leq C$, as required. So it suffices to consider the case in which C is reduced. We also know from Corollary 3.2 that $C \cap D$ is bounded.

We split our argument into two cases: (a) D is of infinite rank and (b) D is of finite rank. Denote the canonical projection of G onto A by π.

Case(a). Since D is of infinite rank, we can write $D = D_1 \oplus D_2$ with $D \cong D_1 \cong D_2$. Then, as the endomorphism ring of D is isomorphic to the full 2×2 matrix ring over $\mathrm{End}(D)$ itself, the identity 1_D can be expressed as a sum of two automorphisms of D and it follows from Proposition 2.7 (ii) that $C \sim (C \cap D) \oplus (C \cap A)$. Thus $(C \cap D) \oplus (C \cap A)$ is, by Proposition 2.5(i), characteristically inert in G; it is also unbounded since it is a finite-index subgroup of the unbounded group C. As $C \cap D$ is bounded we must have that $C \cap A$ is unbounded. Setting $X = (C \cap D) \oplus (C \cap A)$, we have that $C \cap A) \leq \pi(X)$ so that $\pi(X)$ is unbounded. It follows immediately from Corollary 3.2 that $D \leq X \leq C$, as required.

Case(b). If $\pi(C)$ is bounded, then for some $n < \omega$, we have $\pi(p^n C) = 0$ so that $p^n C \leq \mathrm{Ker}\pi = D$. Now, $p^n C$ is reduced since we are in the situation where C is reduced. Moreover, D is of finite rank so that $p^n C$ is actually finite – contrary to C being unbounded. Thus we may assume that $\pi(C)$ is unbounded. This, however, is impossible: arguing exactly as in the proof of Lemma 3.1, we get an onto map $f : \pi(C) \to D_1$, where D_1 is a rank 1 summand of D and this map extends by injectivity to a map $\psi : A \to D_1$ with $\psi\pi(C) = f\pi(C) = D_1$. It follows from Proposition 2.7(iii) that $C \sim C + \psi\pi(C) = C + D_1$, so that we must have that $D_1/(C \cap D_1)$ is finite and also divisible, whence it is zero and thus $D_1 = C \cap D_1 \leq C$ - contradiction, as C is reduced. Thus case(b) does not occur and the proof is complete. $\square$

4. Main Results

In this section we shall work exclusively with reduced p-groups.

Our first result gives an interesting connection between the notions of characteristically inert and fully inert subgroups in certain situations; it is clear that a fully inert subgroup is always characteristically inert; the converse is not true as shown in our next example.

Example 4.1. There is a group G with a characteristically inert subgroup which is not a fully inert subgroup.

Proof. We will use a realisation theorem of Corner [4, Theorem 6.1] to establish the result. Let $H = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \ldots$ be an elementary group of countably infinite rank and let Φ be the subring of $\mathrm{End}(H)$ generated by $1_H, \phi$ where ϕ is the Bernoulli shift map $\phi : e_i \mapsto e_{i+1}$ for each $i \geq 1$. Clearly Φ can be construed as the polynomial ring $F[\phi]$ where F is the Galois field of p elements; note that the group of units of Φ is then just the units of F. Now apply Corner's result to get a p-group G with $p^\omega G = H$ and $\mathrm{End}(G)$ acts on $p^\omega G$ as Φ, while $\mathrm{Aut}(G)$ acts on $p^\omega G$ as the units of $\Phi = F \setminus \{0\}$.

Now choose $X = \langle e_1 \rangle \oplus \langle e_3 \rangle \oplus \ldots \langle e_{2n-1} \rangle \oplus \ldots$; clearly X (like every subgroup of $p^\omega G$) is characteristic in G and hence certainly characteristically inert in G. However, X is not fully inert in G: the quotient $(X + \phi(X))/X = H/X$ is infinite. □

Recall that a group G is said to have unit sum number $\leq \omega$ if the automorphism group of G generates (additively) the full endomorphism group of G.

Proposition 4.2. *If $usn(G) \leq \omega$ then every characteristically inert subgroup C of G is fully inert in G.*

Proof. Let ϕ be an arbitrary endomorphism of G, then there are finitely many automorphisms $\alpha_1, \ldots, \alpha_n$ of G with $\phi = \alpha_1 + \alpha_2 + \ldots \alpha_n$. But then, $(C + \phi(C))/C \leq (\alpha_1(C) + C)/C + \ldots (\alpha_n(C) + C)/C$ and as C is characteristically inert in G, each term on the RHS is finite, whence $(C + \phi(C))/C$ is finite as required. □

A consequence of this simple observation is that the study of characteristically inert subgroups reduces, in the case where the automorphism group generates the full endomorphism group, to the study of the more familiar fully inert subgroups. We pick out just two consequences of this below.

Proposition 4.3. *Let H be an infinite characteristically inert subgroup of the group $G = \bigoplus_{i \in I} G_i$, where $G_i \cong G_j$ for all $i, j \in I$ and the index set I is infinite. If π_i denotes the canonic projection of G onto G_i, then H is commensurable with $\bigoplus_{i \in I} \pi_i(H)$, the $\pi_i(H)$ are fully inert in the G_i and almost all $\pi_i(H)$ are fully invariant in the G_i. Furthermore, there is a finite subset $S \subseteq I$ such that $\bigoplus_{i \in I \setminus S} \pi_i(H)$ is fully invariant in $\bigoplus_{i \in I \setminus S} G_i$.*

Proof. In this situation $usn(G) \leq 3$ and so it follows from Proposition 4.2 above that H is actually fully inert in G. The desired result now follows directly from [3, Lemma 2.2]. □

Our second application of Proposition 4.2 yields the interesting:

Theorem 4.4. *If G is a reduced p-group with $p \neq 2$, which is either totally projective or torsion-complete, then G is characteristically inert socle-regular.*

Proof. When $p \neq 2$, both of these classes have unit sum number 2, see [18, Theorem 4.1], [14, Theorem 3.5]. Furthermore, a fully inert subgroup of such a group is commensurable with a fully invariant subgroup by recent results of Keef, Goldsmith and Salce - see [22, 16]. Thus a characteristically inert subgroup H of G is commensurable with a fully invariant subgroup of G. However, such a fully invariant subgroup has a socle of the form $p^\beta G[p]$ since torsion-complete and totally projective groups are socle-regular as shown in [6], so $H[p]$ is commensurable with a subgroup of the form $p^\beta G[p]$, as require. □

The restriction to $p \neq 2$ in the case of torsion-complete groups can be removed by adding in a restriction relating to Ulm invariants.

Corollary 4.5. *If G is a torsion-complete group in which almost all non-zero Ulm invariants $f_G(n)$, $n < \omega$ are at least 2, then G is characterisically inert socle-regular.*

Proof. Let B_0 be a basic subgroup of G, then we can write $B_0 = B \oplus B'$ where all the non-zero Ulm invariants of B are at least 2 and B' is a finite group. Then $G = B' \oplus \bar{B}$. The condition on the Ulm invariants ensures, by [14, Theorem 3.6], that every endomorphism of $\bar{B}$ is a sum of two automorphisms and it follows, as in the proof of the previous theorem, that $\bar{B}$ is characteristically inert socle-regular: there is no restriction to $p \neq 2$ in [16]. Now if C is a characteristically inert subgroup of G, then $G[p] = B'[p] \oplus \bar{B}[p] \sim \bar{B}[p]$ and hence $C[p] \sim (C[p] \cap \bar{B}[p])$. However, we know from Proposition 2.7(i) that $C \cap \bar{B}$ is characteristically inert in $\bar{B}$, and so there is an $n < \omega$ such that $(C \cap \bar{B})[p] \sim p^n \bar{B}[p] \sim p^n G[p]$. Hence $C[p] \sim (C \cap \bar{B})[p] \sim p^n G[p]$ and G is characteristically inert socle-regular. □

We are not, however, able to decide what happens in general for these classes when the prime $p = 2$.

In [7, Theorem 3.6] an interesting connection between the notions of socle-regularity and strong socle-regularity was established: a reduced p-group G is socle-regular if and only if $G \oplus G$ is strongly socle-regular. Thus these notions have a connection similar to that established in [9] for the notions of full transitivity and transitivity. Interestingly, a similar connection exists between fully inert socle-regularity and characteristically inert socle-regularity even though, as noted in Section 2, there is no obvious connection between transitivity and characteristically inert socle-regularity.

Theorem 4.6. *A group G is fully inert socle-regular if and only if $G \oplus G$ is characteristically inert socle-regular.*

Proof. Suppose that G is fully inert socle-regular and C is characteristically inert in $K = G_1 \oplus G_2$, where $G_1 \cong G \cong G_2$. A well-known result of Kaplansky then tells us that every endomorphism of K is the sum of 3 automorphisms of K and so it follows from Proposition 4.2 above that C is fully inert in K. Writing $C_1 = C \cap G_1, C_2 = C \cap G_2$, it then follows from standard results on full inertia, see, for example, [**?**, Lemma 2.1], that $C \sim C_1 \oplus C_2$ and C_1, C_2 are fully inert in G_1, G_2 respectively. Furthermore, if $\phi : G_1 \to G_2, \psi : G_2 \to G_1$ are arbitrary homomorphisms then $\phi(C_1) + C_2 \sim C_2$ and $\psi(C_2) + C_1 \sim C_1$

Since the G_i are fully inert socle-regular, there are ordinals α, β such that $(C_1)[p] \sim p^\alpha G_1[p]$ and $(C_2)[p] \sim p^\beta G_2[p]$. Now choosing ϕ, ψ to be the isomorphisms identifying G_1 and G_2, we see that $C_1 \sim C_1 + C + 2 \sim C_2$; in other words $p^\alpha G_1[p] \sim p^\beta G[p] \sim p^\alpha G_2[p]$ and thus $C[p] \sim p^\gamma G_1[p] \oplus p^\gamma G_2[p]$ where $\gamma = \max\{\alpha, \beta\}$. Thus $C[p] \sim p^\gamma K[p]$ and K is then characteristically socle-regular.

Conversely, suppose that $G \oplus G$ is characteristically socle-regular and let F be an arbitrary fully inert subgroup of G. Then, as observed in [3, Section 2], $F \oplus F$ is fully inert in $G \oplus G$, and hence $F \oplus F$ is certainly characteristically inert in $G \oplus G$. Thus there is an ordinal β such that $F[p] \oplus F[p] \sim p^\beta G[p] \oplus p^\beta G[p]$. It follows easily from standard properties of commensurability that $F[p] \sim p^\beta G[p]$. Since F was arbitrary, G is characteristically inert socle-regular. □

We finish this section with an example showing that the direct sum of two characteristically inert socle-regular groups need not be characteristically inert socle-regular.

Example 4.7. The class of characteristically inert socle-regular groups is not closed under the taking of direct sums.

Proof. Our example is based on [17, Example 3.6]. Let $G = B_1 \oplus \bar{B}_2$ where $B_1 \cong B_2$ are standard basic p-groups with $p \neq 2$. Then G is a direct sum of two characteristically inert socle-regular groups by Theorem 4.4 above; set $C = \{0\} \oplus \bar{B}_2[p]$.

Now an automorphism of G has the form $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where $\gamma : \bar{B}_2 \to B_1$. By a well-known result of Megibben (see [12, Exercise 14, p317]) that γ is small, so there is an $N \in \mathbb{N}$ with $\gamma(p^N \bar{B}_2[p]) = 0$. Since $\bar{B}_2$ is standard, it follows that $\gamma(\bar{B}_2[p])$ is finite and a straightforward calculation shows that $C + \Delta(C) = \gamma(\bar{B}_2[p]) \oplus \bar{B}_2[p] \sim C$. Thus C is certainly characteristically inert in G. However, $C[p] = \{0\} \oplus \bar{B}_2[p]$ while for all $n \in \mathbb{N}$, we have that $p^n G[p] = p^n B_1[p] \oplus p^n \bar{B}_2[p]$ has an infinite first component. Hence $C[p] \nsim p^n G[p]$ for any $n \in \mathbb{N}$ and G is not characteristically inert socle-regular. □

5. Concluding Remarks

There is an analogous concept to the notion weakly socle-regular which was investigated in [3]:

Definition 5.1. A reduced p-group G is said to be *weakly characteristically inert socle-regular* if, for all infinite fully inert subgroups H of G, there exists an ordinal α, depending on H, such that $p^\alpha G \neq 0$ and $H[p] \cap p^\alpha G[p]$ is of finite index in $p^\alpha G[p]$.

In this work we haven't carried out a detailed investigation of this concept; we hope this will be undertaken in a future work. An obvious question to be considered in this context is whether an analogue of Theorem 4.6 holds for the weak versions of the concepts.

We have seen in our discussions in Section 2 that transitivity and full transitivity are not fundamentally connected to the notions of fully and characteristically inert socle-regular groups, nonetheless there seem to be some parallels between the theories, so we close with an obvious question.

Problem 5.2 Are the classes of fully inert socle-regular and characteristically inert socle-regular groups independent of each other?

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