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On Newton's Rule of Signs

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Abstract

Analysing the *cubic sectors* of a real polynomial of degree n, a minor modification of Newton's Rule of signs is proposed with which stricter upper limits on the number of real roots can be found. A new necessary condition for reality of the roots of a polynomial is also proposed. Relationship between the quadratic elements of the polynomial is established through its roots and those of its derivatives. Some aspects of polynomial discriminants are also discussed.

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1 Introduction

Newton's rule, introduced without proof by Newton [1] and proven 182 years later by Sylvester [2], allows one to find upper limits on the number of positive roots and the number of negative roots of a real polynomial of degree n by considering the double sequence of simple elements (the coefficients of the polynomial) and the quadratic elements of the polynomial (obtained by taking the square of each of the coefficients of the polynomial and subtracting from it the product of its immediate neighbours).
For a polynomial with coefficients in binomial form, $p(x) = \sum_{k=n}^{0} \binom{n}{k} a_k x^k$, the quadratic elements are the discriminants of all possible quadratic polynomials that can be extracted from the given polynomial by taking, also in binomial form, three consecutive coefficients from it. Permanencies in sign in the sequence of quadratic elements are associated with real roots of the polynomial: if there is a permanence in sign in the corresponding simple elements too, then this indicates a possible negative root, while, if there is a variation in the sign in the corresponding simple elements, then there may be a positive root. This is a generalisation of Descartes' rule of signs [3] from 1637 (the Descartes rule was made more precise by Gauss [4] in 1876).
Newton states [1] that *if all roots of a polynomial are real, then all quadratic elements of the polynomial are positive.* Expressing the quadratic elements as inequalities for their positivity (the so called *Newton inequalities*), allows the reformulation of the necessary condition for reality of all roots of the polynomial: *if the roots of the polynomial are all real, then all Newton inequalities hold.*
Positive quadratic elements however are not always associated with real roots. For example, for the quartic polynomial $x^4 - 2x^3 - 2x^2 + 5x + 10$, all quadratic elements are positive (hence their sequence exhibits only permanencies of sign). But this quartic has two pairs of complex conjugate roots.
Performing the analysis at the level of cubic polynomials (instead of quadratic), that is, considering the *cubic sectors of a polynomial*, allows one to get a clearer picture of what type of roots are associated with the two quadratic elements of the cubic polynomial. As with the *quadratic sectors* of the polynomial, these cubic polynomials are all possible cubics that can be obtained by taking in binomial form four consecutive coefficients of the original polynomial. Negativity of either of the quadratic elements of a cubic polynomial warrants a pair of complex roots. However, the case of positive quadratic elements needs further investigation. Each quadratic element (except the two at either end of the polynomial, namely, a_n^2 and a_0^2) has two *adjacent coefficients* in the polynomial (simple elements of the polynomial belonging to different cubic sectors) — the two coefficients sitting on either side of those forming the quadratic element. Using a new criterion on the coefficients of the cubic polynomial for the reality of its roots, instead of considering the sign of its discriminant, two types of positive quadratic elements are identified in this work. Real roots of a cubic are associated with a positive quadratic element for which the adjacent coefficient in the cubic lies in a particular finite interval, the endpoints of which are determined by the coefficients of the cubic, forming the quadratic element. However, a cubic with a positive quadratic element with adjacent coefficient not in its

relevant interval has a pair of complex conjugate roots. If a positive quadratic element happens to have either of its adjacent coefficients outside of their prescribed relevant intervals (hence, associating with complex roots), such quadratic element, referred to in this work as *falsely positive quadratic element*, should, depending on its neighbours in the sequence of quadratic elements, have its sign changed to negative. The Newton Rule then gets a minor modification as it will then exhibit a maximum number of real roots reduced by exactly two (with the minimum number of complex roots increased, of course, by a pair).

The analysis of the *cubic sectors* of a real polynomial also allows the formulation of a new sufficient condition for the reality of its roots: *if the roots of a real polynomial are all real numbers, then each quadratic element of the polynomial is positive and each adjacent coefficient lies in its prescribed relevant interval.* An equivalent form of this new necessary condition is the following: *if all roots are real, then the polynomial cannot have negative, zero, or falsely positive quadratic elements.*

Relationship is also found between the different quadratic elements of a polynomial. It involves the roots of the polynomial and those of its derivatives and allows one to express the ratio of two quadratic elements as ratio of the sums of squares of the differences of these roots or their reciprocals.

An interesting connection between the discriminants of polynomials and the quadratic elements is also established in this paper.

This work starts with the presentation of a geometrical viewpoint with which the influence of the variation of the free term of a polynomial on the sign of the discriminant is established and, hence, on the possible number of real roots.

2 On the Changes of Sign of the Discriminant of a Real Polynomial

Consider a polynomial of degree n with real coefficients in binomial form

$$p(x) = \sum_{k=n}^{0} \binom{n}{k} a_k x^k. \tag{1}$$

(Without loss of generality, any polynomial can be written in binomial form.)

The polynomial (1) can be viewed as an element of a congruence of polynomials

$$\mathcal{P} = \left\{ \sum_{k=n}^{1} \binom{n}{k} a_k x^k + \alpha \mid \alpha \in \mathbb{R} \right\}, \tag{2}$$

the graphs of which foliate the xy-plane by the continuous variation of the foliation parameter α. The elements in this congruence differ from each other by the value of their free term and they all have the same set of stationary points $\{\mu_i\}_{i=1}^{N}$ — the roots of the equation $p'(x) = n \sum_{k=n}^{1} \binom{n-1}{k} a_k x^{k-1} = 0$. Clearly, $N = n - 1 - 2k$, where $k \leq (n-1)/2$ is a non-negative integer.

Within the congruence, there are exactly N (not necessarily all different) *"privileged"* polynomials

$$p_i(x) = p(x) - p(\mu_i), \tag{3}$$

for which the abscissa is tangent to their graph at the stationary point μ_i. Each *privileged* polynomial $p_i(x)$ has a zero discriminant as each one of them has (at least one) at least double real root.

The variation of the foliation parameter α produces continuous sets of polynomials such that all discriminants within a set have the same sign. Each time the parameter α "*traverses*" the value $a_0 - p(\mu_i)$, a new continuous set of polynomials is obtained and the discriminants in this new set have sign opposite to that of the previous set (unless α "traverses" an even number of $a_0 - p(\mu_i)$ at the same time; in which case the discriminants of the polynomials in the new set will bounce back from 0).

In this manner, different *"polynomial bands"* are formed in the xy-plane and within each of these polynomial bands, the discriminants of all polynomials have the same sign. These bands are bounded by the *privileged* polynomials — see Figure 1a and Figure 1b.

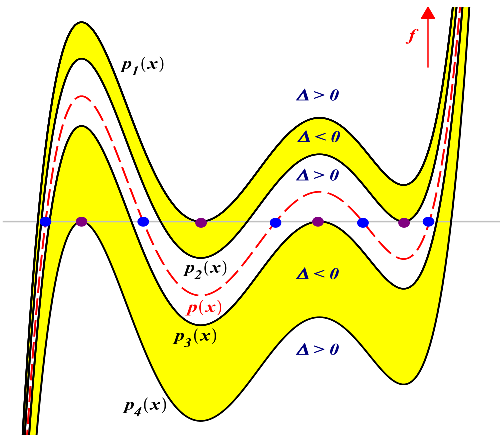

Figure 1a

The quintic polynomial $x^5 - 3x^4 - x^3 + 7x^2 - (3/2)x + f$ with varying free term f. The four *privileged* polynomials $p_i(x)$ have a double root each. When f is sufficiently large, the discriminant is positive and the polynomial has one real roots and two pairs of complex conjugate roots. In the polynomial band bounded by the privileged polynomials $p_1(x)$ and $p_2(x)$, all polynomials have a negative discriminant and, hence, 3 real roots and a pair of complex conjugate roots. In the next polynomial band, bounded by $p_2(x)$ and $p_3(x)$, all polynomials have positive discriminant and five real roots. With the decrease of the foliation parameter f, the situation starts to "unwind": the polynomial band between $p_3(x)$ and $p_4(x)$ is characterised by negative discriminants and, thus, 3 real roots and one pair of complex conjugate roots. Finally, "below" $p_4(x)$ all polynomials have positive discriminants and 1 real root and two pairs of complex conjugate roots.

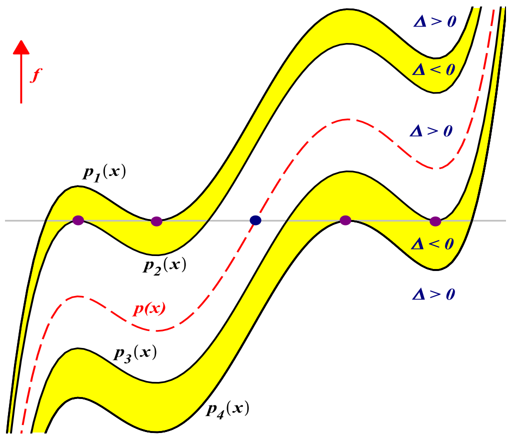

Figure 1b

Another quintic polynomial, $5x^5 + (1/10)x^4 - 8x^3 - (1/4)x^2 + 4x + f$, with varying free term f and, again, with four *privileged* polynomials $p_i(x)$ having a double root each. As the free term f is varied from $-\infty$ to $+\infty$, it can be seen that the number of real roots of the resulting polynomials does not necessarily increase monotonically from one polynomial band to another until only real roots are encountered (as in Figure 1a). In fact, the number of real roots may increase or decrease as one "shifts through" the different polynomial bands. This polynomial cannot have five real roots.

If the discriminant of a polynomial is positive, then the number of complex roots of the polynomial is a multiple of 4. If the discriminant is negative, then there are $2m+1$ pairs of complex-conjugate roots, where $m \leq (n-2)/4$.
If a polynomial has only real roots, then all of its derivatives have only real roots [1]. The converse is not true: if all derivatives of a polynomial have only real roots, it does not follow that the roots of the polynomial are all real [1]. For the example with the quartic polynomial $x^4 - 2x^3 - 2x^2 + 5x + 10$, which has four complex roots (referred to in the Introduction), all of the derivatives of the polynomial have real roots only. Hence, all of the derivatives of a polynomial having only real roots is a necessary condition for the polynomial to have only real roots. If any of the derivatives of a polynomial has at least one pair of complex roots, then the polynomial cannot have only real roots. This means that the number of stationary points of the polynomial will be strictly less than $n-1$. Hence, the number of *privileged* polynomials $p_i(x)$ will be strictly less than $n-1$. Therefore, the number of different polynomial bands and, hence, the number of continuous sets of polynomials with discriminants of the same sign will be strictly less than $n-1$. In result, under the variation of the foliation parameter α, there will not be enough polynomial bands in the congruence so that the polynomials could alternate the signs of their discriminants from one polynomial band into the other until a positive discriminant and possibility of real roots only is revealed: an even number of polynomial bands is eliminated by the reduction by an even number of *privileged* polynomials, which stems, in turn, from the reduction by an even number of the stationary points, due to some derivative(s) having an even number of complex roots. On the contrary, if a polynomial has only real roots, this could only happen in the inner-most polynomial band of n such bands — for which the discriminant is positive.
Note that, as the free term is varied from $-\infty$ to $+\infty$, the number of real roots of the resulting polynomials does not necessarily increase monotonically from one polynomial band to another (until a situation with real roots only is reached — as in Figure 1a). In fact, the number of real roots may increase or decrease as one "shifts through" the different polynomial bands (Figure 1b).
In order for the number of polynomial bands to be reduced (always by an even number), it would be sufficient if $p'(x)$ has at least one pair of complex conjugate roots. Again, note that this is not a necessary and sufficient condition: if $p'(x)$ has $n-1$ real roots, all polynomial bands will be present, but the polynomial $p(x)$ should not necessarily have n real roots (as in Figure 1b). In order that $p'(x)$ does not to have $n-1$ real roots, it would be sufficient if $p''(x)$ does not have $n-2$ real roots. Continuing recursively, one arrives at the bottom rung: $(d^{n-2}/dx^{n-2})p(x) = (n!/2!)(a_{n-2}\,x^2 + 2a_{n-1}\,x + a_n)$ having no real roots is a sufficient condition for the polynomial $p(x)$ to have at least a pair of complex-conjugate roots. Thus, if the discriminant of the bottom-rung quadratic polynomial, or the so called *quadratic element*,

$$A_{n-1} = a_{n-1}^2 - a_n\,a_{n-2} \tag{4}$$

is negative, it will guarantee the elimination of at least two of the polynomial bands and, hence, guarantee the impossibility for the polynomial to have real roots only.

3 The Incomplete and the Complete Newton's Rules

As one will be interested in the number of real roots of the polynomial, if not all powers of the variable are present in $p(x)$, the missing terms can be "restored" in a very simple manner — through a coordinate transformation of the type $x \to x + \beta$ (which may need to be repeated). This works in direction, opposite to that of depressing a polynomial. The new polynomial $p(x + \beta)$ differs from the original polynomial $p(x)$ only in that its graph is translated β units to the left (if β is positive) or β units to the right (if β is negative): both polynomials have the same number of real roots.

To illustrate this, consider the polynomial $p(x)$ and, starting from a_n, let a_m (with $m \leq n$) be the last non-zero coefficient. One has $p(x) = \ldots + \binom{n}{m} a_m x^m + \binom{n}{m-2} a_{m-2} x^{m-2} + \ldots$ The missing term $a_{m-1}x^{m-1}$ can be introduced, with arbitrary coefficient a_{m-1}, through the coordinate transformation: $x \to x + (a_{m-1}/a_m)/(n - m + 1)$. Hence, the number of real roots of the original polynomial $\ldots + \binom{n}{m} a_m x^m + \binom{n}{m-2} a_{m-2} x^{m-2} + \ldots$ will be the same as the number of real roots of the new polynomial $\ldots + \binom{n}{m} a_m x^m + \binom{n}{m-1} a_{m-1} x^{m-1} + \binom{n}{m-2} (1/a_m)\{(n - m - 2)/[2(n - m + 1)]a_{m-1}^2 + a_{m-2}a_m\}x^{m-2} + \ldots$ Although the polynomial obtained in this way is different from the original, their graphs exhibit translational invariance and the two polynomials have the same number of real roots.

Without loss of generality, for the purpose of real root counting, one can assume that none of the coefficients of the polynomial $p(x)$ is zero.

Having already encountered the quadratic element $A_{n-1} = a_{n-1}^2 - a_n\, a_{n-2}$, consider the polynomial, reciprocal to $p(x)$:

$$p^{\dagger}(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}. \tag{5}$$

As the roots of two reciprocal polynomials are reciprocal, the polynomials $p(x)$ and $p^{\dagger}(x)$ have the same number of real roots. Applying to $p^{\dagger}(x)$ the arguments applied earlier to $p(x)$, one arrives at the *quadratic element*

$$A_1 = a_1^2 - a_2\, a_0. \tag{6}$$

If it is negative, it is then sufficient for $p^{\dagger}(x)$, and hence for $p(x)$, to have at least a pair of complex conjugate roots.

Applying these arguments to the reciprocal polynomial of the first derivative of $p^{\dagger}(x)$, that is to $n \sum_{k=n-1}^{0} \binom{n-1}{k} a_k x^k$, yields the *quadratic element*

$$A_{n-2} = a_{n-2}^2 - a_{n-1}\, a_{n-3}. \tag{7}$$

If it is negative, the polynomial $p(x)$ cannot have n real roots.

From the polynomial, reciprocal to $n \sum_{k=n-1}^{0} \binom{n-1}{k} a_k x^k$, one gets the *quadratic element*

$$A_2 = a_2^2 - a_3\, a_1. \tag{8}$$

If it is negative, the polynomial $p(x)$ cannot have n real roots.
Continuing in this vein, one can "*resolve*" the polynomial $p(x)$ into a collection of *quadratic elements*

$$A_{m-1} = a_{m-1}^2 - a_m\, a_{m-2}, \tag{9}$$

where $2 \leq m \leq n$ (see also [5, 6, 7]).
These *quadratic elements* are the discriminants of all possible quadratic polynomials which can be extracted from the original polynomial by taking in binomial form three consecutive coefficients from it. These quadratic polynomials will be referred to as *quadratic sectors* of the polynomial $p(x)$.
Newton states the following on page 365 of [1]: "*Although it is a certain Criterion, that there are two impossible Roots as often as the Square of any Term (...) is deficient of the product of the Terms adjacent, yet it is no Proof that the Roots are real if the Square of any Term (...) exceeds the product of the adjacent Terms;...*" and, ibidem, "*Lastly, every Rule, depending upon the Comparison of the Square of a Term with the Product of the adjacent Terms on either Side, must sometimes fail to discover the impossible Roots, because the Number of such Comparisons being always less by Unity than the Number of Quantities in the Equation,...*"
Hence, a *sufficient condition for the existence of complex roots is the occurrence of a non-positive quadratic element* (see also Theorem 4 in [5]).
Newton did not prove his Rule and it remained unproven for 182 years — until 1865 when Sylvester [2] proved it.
Suppose that the quadratic elements are all non-zero (if some are, then a coordinate translation of the type considered earlier, would cure this; vanishing quadratic elements were not considered by Newton, but such trick was alluded to by him [1], pages 373 and further, and used by Sylvester in his proof [2] of Newton's Rule). Introduce next the quadratic elements $A_n = a_n^2$ and $A_0 = a_0^2$, put into a sequence all *quadratic elements* and consider their signs. When presenting his proof of Newton's rule, Sylvester writes [2]: "*By a group of negative signs, or a negative group, if we understand a sequence of negative signs, with no positive sign intervening, this incomplete rule may be stated otherwise, as follows: The number of imaginary roots of an algebraic function cannot be less than the number of negative groups in the complete series of its quadratic elements*".
This is referred to as *Newton's Incomplete Rule.* As the two end quadratic elements are both positive, this lower bound is always an even number.
The coefficients a_m $(0 \leq m \leq n)$ of the polynomial are termed *simple elements.* Consider their natural sequence, together with the natural sequence of quadratic elements, written underneath them in such way that a_m is directly above A_m. Sylvester [2] terms $\begin{smallmatrix} a_m \\ A_m \end{smallmatrix}$ an *adjacent coefficient couple of elements* and $\begin{smallmatrix} a_m & a_{m+1} \\ A_m & A_{m+1} \end{smallmatrix}$ — an *adjacent coefficient couple of successions.* Within an adjacent coefficient couple of successions, there are four possibilities: pP or a *double permanence* — when the simple elements are of the same sign and the quadratic elements are also of the same sign; vV or a *double variation* — when the simple elements have opposite sign and the quadratic elements have opposite sign; pV or a *permanence variation* — when the simple elements have the same sign and

the quadratic elements have opposite sign; and, finally, vP or a *variation permanence* — when the simple elements have opposite sign, but the quadratic elements have the same sign [2].
The *Complete Newton's Rule*, stated by Newton [1] and proved by Sylvester [2], is:

(1) *The number of negative roots is less than or equal to the number of double permanencies (pP).*

(2) *The number of positive roots is less than or equal to the number of variation permanencies (vP).*

Two immediate corollaries follow [2]:

(1) *The number of real roots is less than or equal to the number of double permanencies plus the number of variation permanencies, or, simply, the number of permanencies in the sequence of quadratic elements.*

(2) *The number of complex roots is greater than or equal to the number of variations in the sequence of quadratic elements.*

Newton's claim that "*it is no Proof that the Roots are real if the Square of any Term (...) exceeds the product of the adjacent Terms*" [1], presents a necessary condition for the reality of all roots of a polynomial. Namely, *if all roots of a polynomial are real, then all Newton inequalities hold, that is:*

$$A_{m-1} = a_{m-1}^2 - a_m\, a_{m-2} > 0 \tag{10}$$

for all $m = 0, 1, \ldots, n$. The converse, as already discussed, is not true. See also Theorem 3 in [5].

4 Relationship between the Quadratic Elements through the Roots of the Real Polynomial and Its Derivatives

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the zeros of the polynomial (1); $\mu_1, \mu_2, \ldots, \mu_{n-1}$ be the zeros of its first derivative; $\nu_1, \nu_2, \ldots, \nu_{n-2}$ — those of its second derivative and so forth until θ_1, θ_2 — the zeros of its derivative of order $n-2$.
Using the relationship between the zeros of a polynomial and those of its derivative, see corollary (2.2) in Whitely [9] and, also, [10], one gets:

$$\begin{aligned}
\sum_{i>j}(\lambda_i - \lambda_j)^2 &= \frac{n^2}{(n-1)(n-2)}\sum_{i>j}(\mu_i - \mu_j)^2 \\
&= \frac{n^2}{(n-1)(n-2)}\,\frac{(n-1)^2}{(n-2)(n-3)}\sum_{i>j}(\nu_i - \nu_j)^2 \\
&= \ldots = \frac{1}{4}n^2(n-1)(\theta_2 - \theta_1)^2 \\
&= \frac{1}{4}n^2(n-1)(a_{n-1}^2 - a_n\, a_{n-2}) = \frac{1}{4}n^2(n-1)A_{n-1},
\end{aligned} \tag{11}$$

given that the derivative of order $(n-2)$ of the polynomial (1) is $(n!/2!)(a_n x^2 + 2a_{n-1}x + a_{n-2})$ and, hence, $(\theta_1 - \theta_2)^2$ is its discriminant, that is $(\theta_1 - \theta_2)^2 = a_{n-1}^2 - a_n\, a_{n-2} = A_{n-1}$. Therefore, the quadratic element $A_{n-1} = a_{n-1}^2 - a_n\, a_{n-2}$ is equal to (modulo positive multiplicative factors) the sums of the squares of the differences of the roots of all derivatives of the polynomial from order 0 (the polynomial itself) to order $n-2$ — the quadratic polynomial $(n!/2!)(a_n x^2 + 2a_{n-1}x + a_{n-2})$; all these sums have the same sign as the quadratic element A_{n-1}:

$$
\begin{aligned}
A_{n-1} &= \frac{4}{n^2(n-1)} \sum_{i>j} (\lambda_i - \lambda_j)^2 = \frac{4}{(n-1)^2(n-2)} \sum_{i>j} (\mu_i - \mu_j)^2 \\
&= \frac{4}{(n-2)^2(n-3)} \sum_{i>j} (\nu_i - \nu_j)^2 = \ \ldots \ = (\theta_2 - \theta_1)^2. \qquad (12)
\end{aligned}
$$

If all of the roots of the polynomial are real, then all these sums will be positive, including A_{n-1}. On the other hand, positivity of A_{n-1} and, hence, positivity of all these sums, does not necessarily mean that the roots of the polynomial are real: should one or more pair of complex roots is present, then the sum of the squares of the differences of the roots could be negative, zero, or positive. However, if in the sequence (12), one of the sums is negative, then all others are negative too, together with A_{n-1}. And vice versa. For example, if A_{n-1} is positive, then the quadratic polynomial $(d^{n-2}/dx^{n-2})p(x)$ has two real roots $\theta_{1,2}$. Hence, the roots of the cubic polynomial $(d^{n-3}/dx^{n-3})p(x)$ may be all real or there may be a complex pair, however, the sum of the squares of their differences will also be positive. This applies to all earlier derivatives of the polynomial. If one considers again the reciprocal polynomial (5), following the same arguments, a similar relationship can be established between A_m and the sums of the squares of the differences of the roots of all derivatives of the reciprocal polynomial of order $m-2$ to order 0. The roots of the reciprocal polynomial are $1/\lambda_i$ $(i = 1, 2, \ldots n)$. Let q_i $(i = 1, 2, \ldots, n-1)$, r_i $(i = 1, 2, \ldots, n-2)$, $\ldots, s_{1,2}$ be the roots of its first, second, $\ldots$, $(n-2)^{\text{th}}$ derivative, respectively. In a similar manner, one gets:

$$
\begin{aligned}
A_1 &= \frac{4}{n^2(n-1)} \sum_{i>j} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right)^2 = \frac{4}{(n-1)^2(n-2)} \sum_{i>j} (q_i - q_j)^2 \\
&= \frac{4}{(n-2)^2(n-3)} \sum_{i>j} (r_i - r_j)^2 = \ \ldots \ = (s_2 - s_1)^2. \qquad (13)
\end{aligned}
$$

Using (12) and (13), one can relate the two quadratic elements through the roots of the polynomial or the roots of its derivatives:

$$
\frac{A_1}{A_{n-1}} = \frac{\sum_{i>j} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right)^2}{\sum_{i>j} (\lambda_i - \lambda_j)^2}. \qquad (14)
$$

Likewise, considering $p'_n(x)$ and its reciprocal polynomial (whose roots are $1/\mu_i$ $i = 1, 2, \ldots, n-1$), one can relate A_2 with A_{n-1}:

$$\frac{A_2}{A_{n-1}} = \frac{\sum\limits_{i>j} \left(\frac{1}{\mu_i} - \frac{1}{\mu_j} \right)^2}{\sum\limits_{i>j} (\mu_i - \mu_j)^2}. \tag{15}$$

From $p''_n(x)$ and its reciprocal polynomial (whose roots are $1/\nu_i$ $i = 1, 2, \ldots, n-2$), it is straightforward to obtain another relation:

$$\frac{A_3}{A_{n-1}} = \frac{\sum\limits_{i>j} \left(\frac{1}{\nu_i} - \frac{1}{\nu_j} \right)^2}{\sum\limits_{i>j} (\nu_i - \nu_j)^2} \tag{16}$$

and so forth. Hence:

$$\frac{A_1}{A_2} = \frac{\sum\limits_{i>j} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right)^2}{\sum\limits_{i>j} \left(\frac{1}{\mu_i} - \frac{1}{\mu_j} \right)^2}, \qquad \frac{A_1}{A_3} = \frac{\sum\limits_{i>j} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right)^2}{\sum\limits_{i>j} \left(\frac{1}{\nu_i} - \frac{1}{\nu_j} \right)^2}, \qquad \frac{A_2}{A_3} = \frac{\sum\limits_{i>j} \left(\frac{1}{\mu_i} - \frac{1}{\mu_j} \right)^2}{\sum\limits_{i>j} \left(\frac{1}{\nu_i} - \frac{1}{\nu_j} \right)^2}, \quad \text{etc.} \tag{17}$$

5 The Cubic Sectors of a Real Polynomial and the Falsely Positive Quadratic Elements

Instead of considering Newton's *quadratic inequalities*, Rosset [8] introduces a set of *cubic inequalities* in the following manner. If $x_1, x_2, \ldots, x_n$, with $n \geq 3$, are real numbers; $e_1 = \sum_{i=1}^n x_i$, $e_2 = \sum_{i<j} x_i x_j$, $\ldots, e_n = \prod_{i=1}^n x_i$ are the elementary symmetric functions; and $E_m = \binom{n}{m}^{-1} e_m$ are the normalized elementary symmetric functions, the *cubic inequalities* proposed by Rosset are [8]:

$$6E_m E_{m+1} E_{m+2} E_{m+3} - 4E_m E_{m+2}^3 - E_m^2 E_{m+3}^2 - 4E_{m+1}^3 E_{m+3} + 3E_{m+1}^2 E_{m+2}^2 \geq 0, \tag{18}$$

for $m = 0, 1, \ldots, n-3$.
Rosset points out [8] that if $E_0 = 1, E_1, \ldots, E_n$ are real numbers satisfying the cubic inequalities (18) for $m = 0, 1, \ldots, n-3$, then they also satisfy Newton's quadratic inequalities $E_m^2 > E_{m-1} E_{m+1}$, with $m = 1, 2, \ldots, n-1$. The converse is not true and, hence, Rosset finds [8] that the cubic inequalities are stronger than Newton's quadratic inequalities.
Indeed, performing the analysis at the level of cubic polynomials (instead of quadratic), allows one to get a clearer picture of what type of roots are associated with the two quadratic elements of the cubic polynomial.
In this work, approach that is different from Rosset's is proposed.

Consider the general real cubic polynomial with coefficients in binomial form (see also [11, 12]):

$$c(x) = a_3x^3 + 3a_2x^2 + 3a_1x + a_0. \tag{19}$$

The discriminant of this cubic polynomial is

$$\Delta_3 = 27[-a_3^2a_0^2 + 2a_2(3a_1a_3 - 2a_2^2)a_0 + 3a_1^2a_2^2 - 4a_1^3a_3]. \tag{20}$$

It is quadratic in the free term a_0 and the discriminant of this quadratic is

$$\Delta_2 = 16(a_2^2 - a_1a_3)^3 = 16A_2^3. \tag{21}$$

If the quadratic element A_2 is negative and, hence, $\Delta_2 < 0$, then $\Delta_3 < 0$ for all a_0. Thus, the cubic polynomial $c(x)$ with $A_2 < 0$ has only one real root (and a pair of complex conjugate roots).
If the quadratic element A_2 is non-negative, then the cubic polynomial $c(x)$ has three real roots, provided that $a_0 \in [a_2^{(0)}, a_1^{(0)}]$, where $a_{1,2}^{(0)}$ are the roots of the quadratic in a_0 equation $\Delta_3 = 0$ — see [11, 12]:

$$a_3^2a_0^2 - 2a_2(3a_1a_3 - 2a_2^2)a_0 - 3a_1^2a_2^2 + 4a_1^3a_3 = 0, \tag{22}$$

namely:

$$a_{1,2}^{(0)} = \frac{-3a_2A_2 + a_2^3 \pm 2A_2^{3/2}}{a_3^2}. \tag{23}$$

If $a_0 \notin [a_2^{(0)}, a_1^{(0)}]$, then the cubic polynomial $c(x)$ will have one real root and a pair of complex conjugate roots.
Hence, this is a *new criterion for the reality of the roots of the cubic polynomial.* It is a condition on the coefficients of the polynomial, rather than the sign of the discriminant: *the cubic polynomial* $c(x) = a_3x^3 + 3a_2x^2 + 3a_1x + a_0$ *has three real roots if, and only if, the quadratic element* $A_2 = a_2^2 - a_1a_3$ *is non-negative and the adjacent coefficient* a_0 *lies in the interval* $[a_2^{(0)}, a_1^{(0)}]$*, with* $a_{1,2}^{(0)}$ *given by* (23).
If one considers instead the reciprocal cubic polynomial $C^\dagger = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ and applies the same arguments to it, three real roots will exist if, and only if, the quadratic element $A_1 = a_1^2 - a_0a_2$ is non-negative and the adjacent coefficient a_3 lies in the interval $[a_2^{(3)}, a_1^{(3)}]$, with $a_{1,2}^{(3)}$ given by

$$a_{1,2}^{(3)} = \frac{-3a_1A_1 + a_1^3 \pm 2A_1^{3/2}}{a_0^2}. \tag{24}$$

If $c(x)$ has three real roots, $c^\dagger(x)$ will also have three real roots (and vice versa). Hence, $A_1 \geq 0$ and $a_3 \in [a_2^{(3)}, a_1^{(3)}]$ is equivalent to $A_2 \geq 0$ and $a_0 \in [a_2^{(0)}, a_1^{(0)}]$: if either of these holds, the other one holds too and the cubic (and, of course, its reciprocal) will have

three real roots.
If both quadratic elements are negative, the cubic will have one real root, as already shown.
However, negativity of either of the two quadratic elements of the cubic does not mean that the other quadratic element is also negative. Clearly, the negative quadratic element leads straight away to a situation with one real root and a pair of complex conjugate roots. Hence, the reciprocal cubic should be with the same root structure and, if the associated quadratic element is non-negative, then the adjacent coefficient must necessarily lie outside of its prescribed interval. Otherwise, the reciprocal cubic will have three real roots and this is, obviously, impossible.

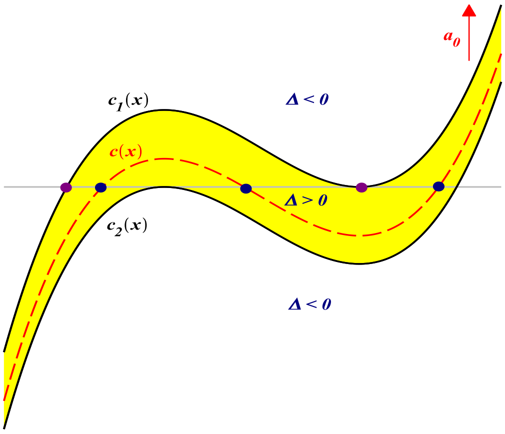

Figure 2a
The quadratic element A_2 of the cubic $a_3x^3+3a_2x^2+3a_1x+a_0$ is positive and the variation of adjacent coefficient (the free term) a_0 results in variation of the sign of the cubic discriminant. When $a_0 = a_{1,2}^{(0)}$, the cubic $c_{1,2}(x)$ is obtained and the cubic discriminant is zero in this case. All cubics in the highlighted band between $c_1(x)$ and $c_2(x)$ (as the dashed curve), that is, all cubics with positive quadratic element A_2 and $a_0 \in [a_2^{(0)}, a_1^{(0)}]$, have positive discriminants and three real roots. All other cubics (outside the highlighted band) have negative discriminant and one real root together with a pair of complex conjugate roots.

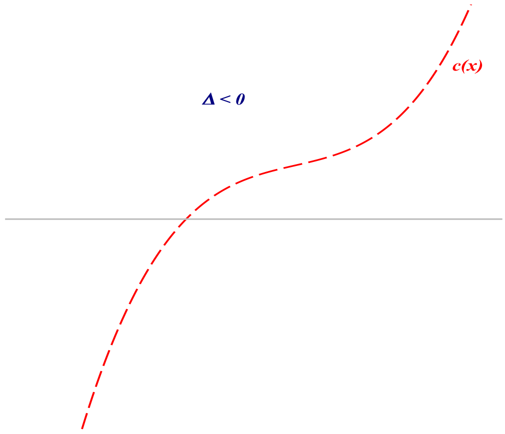

Figure 2b
This is a cubic with a negative quadratic element A_2. The discriminant of the cubic is also negative. The derivative of the cubic, the quadratic polynomial $a_3x^2+2a_2x+a_1$ has a pair of complex conjugate roots. The cubic has no stationary points and, hence, there are no *privileged* polynomials associated with it. In result, two of the polynomial bands are not present. There is one real root and a pair of complex conjugate roots.

Hence, the possible situations for the quadratic elements of the cubic and their adjacent coefficients can be summarised as follows (see also Figure 2a and Figure 2b): **(a)** both quadratic elements positive and both adjacent coefficients lying in their respective prescribed intervals, resulting in the cubic having three real roots; **(b)** both quadratic elements negative (the respective intervals for the adjacent coefficients are not over the reals in such case) and the cubic having one real root and a pair of complex conjugate roots; and **(c)** one quadratic element negative and the other quadratic element positive with the adjacent coefficient of the latter not in its relevant prescribed interval, yielding a cubic also with one real root and a pair of complex conjugate roots.

The question of what happens in the case of a vanishing quadratic element should also be addressed. If, for the cubic polynomial $c(x)$, it happens so that $A_2 = 0$, then, if also $a_0 = a_2^3/a_3^2$, see (23), the cubic polynomial will have a triple real root equal to $-a_2/a_3$; for any other value of a_0, the cubic polynomial will have one real root and a pair of complex conjugate roots, see also [12]. Hence, if the cubic polynomial is an element of the cubic sectors of a polynomial of degree 4 or higher, then the higher-degree polynomial will have a reduced number of stationary points (not counted with their individual multiplicities) and, thus, a vanishing quadratic element of the cubic associates with complex roots up in the hierarchy of polynomials of higher degrees.

Returning to the positive quadratic elements, one can clearly see that a positive quadratic element, considered in its own right, does not necessarily associates with real roots. One has to also consider the adjacent coefficient. For a cubic polynomial, each of its two quadratic elements is an "end" one, that is, it involves the end 3-tuples of coefficients of the cubic. Each quadratic element has only one adjacent coefficient — the remaining coefficient of the cubic. However, if one considers a polynomial of degree 4 or higher, all quadratic elements, except the ones at either "end" of the polynomial will have two adjacent coefficients — one on either "side" of the quadratic element. Then two cubic sectors of a polynomial of degree 4 or more will share a quadratic element, but the shared quadratic element will associate with different adjacent coefficients — from the different cubic sectors of the polynomial. And this is exactly what will be considered next: the "resolution" of a polynomial of degree n into its cubic sectors.

The cubic sectors of a polynomial are obtained by taking four consecutive coefficients and using them to write cubic polynomials with coefficients also in binomial form. This is achieved by a suitable sequence of differentiations and passages to reciprocal polynomials. Differentiating $p(x)$ $n-3$ times yields the cubic sector $c_{n-3}(x) = (n!/3!)(a_n x^3 + 3a_{n-1}x + 3a_{n-2}x + a_{n-3})$. Differentiating the reciprocal polynomial $p^\dagger(x) = \binom{n}{n} a_0 x^n + \binom{n}{n-1} a_1 x^{n-1} + \ldots + \binom{n}{0} a_n$ gives $\binom{n}{n} n a_0 x^{n-1} + \binom{n}{n-1}(n-1) a_1 x^{n-2} + \binom{n}{n-2}(n-2) a_2 x^{n-3} + \ldots + \binom{n}{1} a_{n-1} = n[\binom{n-1}{n-1} a_0 x^{n-1} + \binom{n-1}{n-2} a_1 x^{n-2} + \binom{n-1}{n-3} a_2 x^{n-3} + \ldots + \binom{n-1}{1} a_{n-1}]$. Taking the reciprocal polynomial of this derivative, gives the polynomial of degree $n-1$, $n[\binom{n-1}{1} a_1 x^{n-1} + \binom{n-1}{2} a_2 x^{n-2} + \ldots + \binom{n-1}{n-1} a_0]$, and differentiating it $n-4$ times yields the cubic sector $c_{n-4}(x) = (n!/3!)(a_{n-1}x^3 + 3a_{n-2}x + 3a_{n-3}x + a_{n-4})$. Descending down the ladder to a polynomial of degree $n-2$, gives the next cubic sector: $c_{n-5}(x) = (n!/3!)(a_{n-2}x^3 + 3a_{n-3}x^2 + 3a_{n-4}x + a_{n-5})$. The two "bottom rungs" are clearly $(n!/4!)[a_4 x^4 + 4a_3 x^3 + 6a_2 x^2 + 4a_1 x + a_0]$, from which, after differentiation, one gets the second last cubic sector $c_1(x) = (n!/3!)(a_4 x^3 + 3a_3 x^2 + 3a_2 x + a_1)$, and $(n!/3!)[a^3 x^3 + 3a_2 x^2 + 3a_1 x + a_0]$ which, itself, is the last cubic sector $c_0(x)$.

Note that a polynomial with non-zero coefficients has $n-2$ cubic sectors. A cubic polynomial ($n = 3$) coincides with its only cubic sector.

Consider the chain of cubic sectors and their quadratic elements. There are two quadratic elements for each cubic sector. Two neighbouring cubic sectors, on the other hand, share a quadratic element (the cubic sectors overlap), but the shared quadratic element associates with a different adjacent coefficient in each of the neighbouring cubic sectors.

If one of the two quadratic elements of a cubic sector is positive and the adjacent coefficient from this cubic sector lies in its relevant prescribed interval, the cubic sector will have three real roots. Hence, the other quadratic element of the same cubic sector must also be positive and its adjacent coefficient from this cubic sector must also lie in its relevant prescribed interval. On the other hand, if one of the two quadratic elements of a cubic sector is positive, but the adjacent coefficient from this cubic sector is not in its relevant prescribed interval, the cubic sector will have only one real root and, hence, the other quadratic element of this cubic sector will be either negative or will be positive and its adjacent coefficient from this cubic sector will also be outside its relevant prescribed interval.

Due to the overlap of the cubic sectors, negativity of a quadratic element in one cubic sector "spills over" into the neighbouring cubic sector by the shared quadratic element in the following manner: a negative quadratic element in a cubic sector leads to either a negative quadratic element in the cubic sector adjacent to the terms of this quadratic element, or to a positive one for which the associated adjacent coefficient does not lie in its prescribed relevant interval. If the "shared" quadratic element is also negative, then the neighbouring cubic sector will also have one real root only. However, if the "shared" quadratic element is positive, then, as it associates with a different adjacent coefficient in the neighbouring cubic sector, this different adjacent coefficient in the neighbouring cubic sector may lie or may not lie in its relevant prescribed interval.

Hence, if there is a negative quadratic element, then its neighbouring quadratic element will either be also negative or will be non-negative and its adjacent coefficient will not lie in its relevant prescribed interval. In other words, a positive quadratic element with its adjacent coefficient in its relevant interval can only be a neighbour to a positive quadratic element and can not be a neighbour to a negative quadratic element.

A positive quadratic element, for which one or both of its adjacent coefficients does not lie in its relevant prescribed intervals, is clearly associated with complex roots. For the purposes of this work, such positive quadratic elements will be referred to as *falsely positive quadratic elements* and the change of the sign of some of them, under certain conditions, results in a minor modification of Newton's Rules.

6 Proposed Minor Modification of Newton's Rules

The arguments put forward so far allow one to propose a minor modification of Newton's Incomplete and Complete Rules. This involves changing the signs of some of the *falsely positive quadratic elements*. This should be done under strict conditions and as follows.

One or more falsely positive quadratic element can be neighboured by negative quadratic elements on both sides. In this case, the sign of the falsely positive quadratic element(s) should not be changed as, by doing so, the neighbouring negative groups would be merged and, in result, more permanencies in the sequence of quadratic elements would ensue and thus, the minimum number of complex roots would decrease, rather than increase, while the number of real roots would increase. The Newton rules in this case are not modified.

When one or more falsely positive quadratic elements are neigboured by a positive quadratic element on one side and a negative quadratic element on the other, changing the sign of the falsely positive quadratic element(s) would form a negative group with the neighbouring negative quadratic element or enlarge the existing neighbouring negative group. There would be no change in the minimum number of complex roots, under Newton's incomplete rule, as advocated by Sylvester [2]. Hence, the sign of the falsely positive quadratic element(s) should also not be changed. Newton's rules will not be modified in this case either.

The proposed minor modification of Newton's rule is to be applied only when one or more falsely positive quadratic elements are neighboured by positive quadratic elements or by groups of positive quadratic elements. Only in such cases, the signs of these falsely positive quadratic elements should be changed. In result, an existing positive group (of at least three) will be broken by the introduction of a negative quadratic element or a group of negative quadratic elements within it. Hence, the number of permanencies in the sequence of quadratic elements will decrease by two and, with it, the maximum number of real roots will be reduced by two and the minimum number of complex roots will increase by two. This is just another way of saying that falsely positive quadratic elements associate with complex roots, not with real ones.

As example, consider the polynomial $x^8 - 16x^7 + 28x^6 + 112x^5 - 70x^4 + (28/5)x^3 + 28x^2 + 16x + 1$. There are two positive roots, two negative roots and two pairs of complex conjugate roots.

All quadratic elements of this polynomial are positive. However, $A_3 = a_3^2 - a_2 a_4 = 101/100 > 0$, but none of its two adjacent coefficients, $a_1 = 5$ and $a_5 = 2$, are in their relevant prescribed intervals. Also, $A_2 = a_2^2 - a_1 a_3 = 4/5 > 0$, but its associated adjacent coefficients are such that $a_0 = 1$ belongs to its prescribed interval, while $a_4 = -1$ does not. Likewise, $A_4 = a_4^2 - a_3 a_5 = 4/5 > 0$, but its associated adjacent coefficients are such that $a_2 = 1$ does not belong to its prescribed interval, while $a_6 = 1$ does. Hence, A_2, A_3 and A_4 are all *falsely positive quadratic elements*. As these form a group and this group is neighboured by positive quadratic elements ($A_1 = 3$ and $A_5 = 5$), the signs of A_2, A_3 and A_4 should be all changed from plus to minus.

If the original Newton's Rule is applied, as all quadratic elements are positive, the number of sign changes in the sequence of coefficients (simple elements) gives an upper limit on the number of positive roots. There are four sign changes, hence, there are maximum 4 positive roots. The number of permanencies in the sequence of coefficients gives an upper limit on the number of negative roots. There are 4 permanencies, hence there are maximum 4 negative roots.

With the proposed modification, the signs of A_2, A_3 and A_4 are now negative. Applying the Complete Newton's Rule to the modified sequence of quadratic elements yields maximum 3 positive roots and maximum 3 negative roots. These are stricter limits on the positive and negative roots than the ones provided by the unmodified sequence of quadratic elements, as in Newton's original Rule.

7 New Necessary Condition for the Reality of All Roots of a Real Polynomial

The analysis of the cubic sectors of a real polynomial also allows the formulation of a new sufficient condition for the reality of its roots: *if the roots of a real polynomial are all real numbers, then each quadratic element of the polynomial is positive and each of its adjacent coefficients lies in its relevant prescribed interval.*
An equivalent form of this new necessary condition is the following: *if all of the roots are real, then the polynomial cannot have negative, vanishing, or falsely positive quadratic elements.*
It should be noted that a polynomial cannot have a single falsely positive quadratic element. This can be seen in the following manner. If there is one cubic sector with a falsely positive quadratic element, then the reciprocal cubic polynomial, that is, a neighbouring cubic sector, will have either another falsely positive quadratic element (meaning that the polynomial has more than one falsely positive quadratic elements) or a negative quadratic element. As the passage from a negative quadratic element to a positive quadratic element necessarily happens through a falsely positive quadratic element (with either one or both of its associated adjacent coefficients outside of their relevant prescribed intervals), there cannot be a polynomial with a single falsely positive quadratic element.

8 Relationship between the Discriminants of Real Polynomials, the Discriminants of their Derivatives, and the Quadratic Elements

As seen earlier (in Section 5), the discriminant $\mathit{\Delta}_3$ of the cubic polynomial $a_3x^3+3a_2x^2+3a_1x+a_0$ is quadratic in the free term a_0 and the discriminant $\mathit{\Delta}_2$ of the discriminant $\mathit{\Delta}_3$ is proportional to the third power of quadratic element $A_2 : \ \mathit{\Delta}_2 = 16A_2^3$. Hence, the sign of the discriminant of the quadratic, which is the discriminant in a_0 of the cubic is determined entirely by the sign of the quadratic element A_2. As discussed, if $A_2 \geq 0$, it is the adjacent coefficient, a_0, which controls the number of real roots of the cubic.
The discriminant of the derivative of the cubic, that is, the discriminant of $3(a_3x^2 + 2a_2x + 3a_1)$ is equal to $-36A_2$.
For the quartic polynomial $a_4x^4 + 4a_3x^3 + 6a_2x^2 + 4a_1x + a_0$, the discriminant is $\mathit{\Delta}_4 = 256[a_4^3a_0^3+(-12a_1a_3a_4^2-18a_2^2a_4^2+54a_2a_3^2a_4-27a_3^4)a_0^2+(54a_1^2a_2a_4^2-6a_1^2a_3^2a_4-180a_1a_2^2a_3a_4+108a_1a_2a_3^3+81a_2^4a_4-54a_2^3a_3^2)a_0-27a_1^4a_4^2+108a_1^3a_2a_3a_4-64a_1^3a_3^3-54a_1^2a_2^3a_4+36a_1^2a_2^2a_3^2]$ — a cubic in the free term a_0.
The discriminant of this cubic in a_0 is $19683\,(a_1a_4^2-3a_2a_3a_4+2a_3^3)^2\,(-a_1^2a_4^2+6a_1a_2a_3a_4-4a_1a_3^3-4a_2^3a_4+3a_2^2a_3^2)^3$. The sign of this discriminant is determined by the sign of the expression that is cubed. Its discriminant in a_1 is given by $16A_3^3$.
The discriminant of the derivative of the quartic is $6912[-a_1^2a_4^2+(6a_2a_3a_4-4a_3^3)a_1-4a_2^3a_4+3a_3^2a_2^2]$. This is a quadratic in the free term a_1 of the derivative of the quartic

and the discriminant of this quadratic is $764411904A_3^3$.
The discriminant of the second derivative of the quartic is $-576A_3$.
Such type of relationships exist between the discriminants of polynomials of higher degrees, their derivatives, and their quadratic elements.
It is enticing to consider the *quartic sectors* (or higher) of a polynomial, instead of its *cubic sectors*.
Returning to the quartic, one immediately sees that if A_3 is negative, then the discriminant of the cubic in a_0, $19683\,(a_1a_4^2 - 3a_2a_3a_4 + 2a_3^3)^2\,(-a_1^2a_4^2 + 6a_1a_2a_3a_4 - 4a_1a_3^3 - 4a_2^3a_4 + 3a_2^2a_3^2)^3$ is negative for all a_1 (except the value of a_1 for which the squared term is zero). Hence, the discriminant Δ_4 of the quartic, viewed as a cubic in the free term a_0, will have one real root only and will change its sign only once with the variation of the free term a_0. The quartic will have only one stationary point, as the derivative of the quartic is quadratic in its free term a_1 and the discriminant of this quadratic is proportional to A_3^3 and hence negative. Therefore, the quartic will either have no real roots (positive discriminant) or it will have 2 real roots and a pair of complex conjugate roots (negative discriminant).
If, on the other hand, A_3 is positive and a_1 is such that the squared term is not zero, then the above discriminant of the cubic in a_0 is positive. There will be three real roots of the cubic in a_0 and the discriminant Δ_4 of the quartic will change sign three times. Thus, with the variation of the free term a_0, the quartic with $a_4 > 0$ will "shift through" the following bands with the increase of a_0: negative discriminant with 2 real roots and a pair of complex roots; then a positive discriminant with four real roots; then a negative discriminant with two real roots and a pair of complex roots again; and finally — a positive discriminant with no real roots.
The values of the free term a_0 at which the discriminant of the quartic changes sign (once or three times) are roots of cubic polynomials and, for the general cubic polynomial, these cannot be given in a form that would allow analysis.
Thus, unfortunately, for a polynomial of degree 4 or more, it is not possible to determine the number of real roots just by knowing which derivative(s) do not posses real roots only or by knowing how many real roots each of its derivatives has. Only for polynomials of degree 3 or less this can be done. Moreover, with the increase of the degree of the polynomial, the Abel–Ruffini theorem prevents from explicitly knowing the points at which the discriminant of the polynomial changes sign (by variation of the free term of the polynomial).

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References

[1] *Arithmeticae Universalis Liber Primus*, written by or belonging to I. Newton, notes deposited to the Cambridge University library late in 1683 or early in 1684;
I. Newton, *Arithmetica Universalis; Sive de Compsoitione et Resolutione Arithmetica Liber*, published by W. Whiston, Cambridge (1707);
I. Newton, *Universal Arithmetick or, A Treatise of Arithmetical Composition and Resolution*, translated from Latin by J. Ralphson and J. Cunn, W. Johnston London (1769).

[2] J.J. Sylvester, *On an Elementary Proof and Generalization of Sir Isaac Newton's Hitherto Undemonstrated Rule for Discovery of Imaginary Roots*, Proceedings of the London Mathematical Society **I**, 1–16 (1865–1866).

[3] R. Descartes, *La Géométrie*, French Academy of Sciences, Paris (1637).

[4] C.F. Gauss, *Werke, Dritter Band*, Society of Industrial and Applied Mathematics (SIAM), Göttingen (1876), p. 67.

[5] C.G. Wagner, *Newton's Inequality and a Test for Imaginary Roots*, The Two-Year College Mathematics Journal **8(3)**, 145–147 (1977), doi: 10.2307/3027264.

[6] C.P. Niculescu, *A New Look at Newton's Inequalities*, Journal of Inequalities in Pure and Applied Mathematics, **1(2)**, article 17 (2000).

[7] D.J. Acosta, *Newton's Rule of Signs for Imaginary Roots*, The American Mathematical Monthly **110(8)**, 694–706 (2003), doi: 10.2307/3647852

[8] S. Rosset, *Normalized Symmetric Functions, Newton's Inequalities and a New Set of Stronger Inequalities*, The American Mathematical Monthly **96(9)**, 815–819 (1989), doi: 10.2307/2324844.

[9] J.N. Whiteley, *On Newton's Inequalities for Real Polynomials*, The American Mathematical Monthly **76(8)**, 905–909 (1969), doi: 10.1080/00029890.1969.12000362.

[10] V.V. Monov, *Newton's Inequalities for Families of Complex Numbers*, Journal of Inequalities in Pure and Applied Mathematics **6(3)**, Article 78 (2005).

[11] E.M. Prodanov, *Isolation Intervals of the Real Roots of the Parametric Cubic Equation*, arXiv: 2108.02009.

[12] E.M. Prodanov, *The Siebeck–Marden–Northshield Theorem and the Real Roots of the Symbolic Cubic Equation*, arXiv: 2107.01847.