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The Siebeck–Marden–Northshield Theorem and the Real Roots of the Symbolic Cubic Equation

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Abstract

The isolation intervals of the real roots of the symbolic monic cubic polynomial $x^3 + ax^2 + bx + c$ are determined, in terms of the coefficients of the polynomial, by solving the Siebeck–Marden–Northshield triangle — the equilateral triangle that projects onto the three real roots of the cubic polynomial and whose inscribed circle projects onto an interval with endpoints equal to stationary points of the polynomial.

Mathematics Subject Classification Codes (2020): 26C10, 12D10, 11D25.

Keywords: Polynomials; Cubic equation; Siebeck–Marden–Northshield theorem; Roots; Isolation intervals; Root bounds.

1 Introduction

The elegant theorem of Siebeck and Marden (often referred to as Marden's theorem) [1]–[5] relates geometrically the complex non-collinear roots of a cubic polynomial with complex coefficients to a triangle whose vertices project onto them, on one hand, and, on the other, the critical points of the polynomial to the projections of the foci of the inellipse of this triangle. This ellipse is unique and is called Steiner inellipse [6]. It is inscribed in the triangle in such way that it is tangent to the sides of the triangle at their midpoints.

The real version of the Siebeck–Marden Theorem, as given by Northshield [7], states that the three real roots (not all of which are equal) of a cubic polynomial are projections of the vertices of some equilateral triangle in the plane. However, it is the inscribed circle of the equilateral triangle that projects onto an interval the endpoints of which are the stationary points of the polynomial.

The goal of this work is to consider a cubic equation with real coefficients and, using the Siebeck–Marden–Northshield theorem [7], solve the equilateral triangle and find the isolation intervals of the real roots of the symbolic monic cubic polynomial $x^3 + ax^2 + bx + c$.

2 Analysis

The vertices of the equilateral triangle that projects onto the three real roots of the cubic polynomial $x^3 + ax^2 + bx + c$ are points P , Q , and R with coordinates $(x_1, (x_2 - x_3)/\sqrt{3})$, $(x_2, (x_3 - x_1)/\sqrt{3})$, and $(x_3, (x_1 - x_2)/\sqrt{3})$, respectively [7].

Lemma 1. *The monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ with $b > a^2/3$ has only one real root.*

Proof. The discriminant of the monic cubic polynomial $x^3 + ax^2 + bx + c$ is

$$\Delta_3 = -27c^2 + (18ab - 4a^3)c + a^2b^2 - 4b^3. \quad (1)$$

It is quadratic in c and the discriminant of this quadratic is

$$\Delta_2 = 16(a^2 - 3b)^3 \quad (2)$$

As $b > a^2/3$, one has $\Delta_2 < 0$ for all a and thus $\Delta_3 < 0$ for all a and c . Hence, the cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ with $b > a^2/3$ has only one real root (and a pair of complex conjugate roots). \square

Lemma 2. *The monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ with $b \leq a^2/3$ has three real roots, provided that $c \in [c_2, c_1]$, where $c_{1,2}$ are the roots of the quadratic equation*

$$x^2 + \left(\frac{4}{27}a^3 - \frac{2}{3}ab\right)x - \frac{1}{27}a^2b^2 + \frac{4}{27}b^3 = 0, \quad (3)$$

namely:

$$c_{1,2}(a, b) = c_0 \pm \frac{2}{27} \sqrt{(a^2 - 3b)^3}, \quad (4)$$

where

$$c_0(a, b) = -\frac{2}{27} a^3 + \frac{1}{3} ab. \quad (5)$$

Proof. The discriminant $\Delta_3 = -27c^2 + (18ab - 4a^3)c + a^2b^2 - 4b^3$ of the monic cubic polynomial $x^3 + ax^2 + bx + c$ is positive between the roots of the equation $\Delta_3 = 0$, which is quadratic in c . This is exactly equation (3) and its roots are the ones given in (4) and (5). \square

Lemma 3 (Solving the Siebeck–Marden–Northshield Triangle). *The centre of the inscribed circle of the equilateral triangle that projects onto the three real roots of the monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ is point $(-a/3, 0)$, the projection onto the abscissa of the inflection point of $p(x)$, and the radius of the inscribed circle is $r = (1/3)\sqrt{a^2 - 3b}$. The radius of the circumscribed circle is $2r = (2/3)\sqrt{a^2 - 3b}$.*

Proof. The inflection point of the graph of the monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ occurs at the root of $p''(x) = 6x + 2a$, namely at $x = \phi = -a/3$.

Given that the vertices P , Q , and R of the triangle are points of coordinates $(x_1, (x_2 - x_3)/\sqrt{3})$, $(x_2, (x_3 - x_1)/\sqrt{3})$, and $(x_3, (x_1 - x_2)/\sqrt{3})$, respectively, the centroid of the triangle is point of coordinates $(-a/3, 0)$ — the first coordinate projection of the inflection point.

Each side of the triangle is equal to $\alpha = (\sqrt{12}/3)\sqrt{a^2 - 3b}$. The radius of a circle inscribed in equilateral triangle with side α is $r = \alpha/\sqrt{12} = (1/3)\sqrt{a^2 - 3b}$. The radius of the circumscribed circle of an equilateral triangle with side α is $2r = (2/3)\sqrt{a^2 - 3b}$ — see Figure 1. \square

Lemma 4. *The maximum distance between the three real roots of the monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ is $\sqrt{12}r = (\sqrt{12}/3)\sqrt{a^2 - 3b}$. In this case, one side of the equilateral triangle that projects onto the roots of the monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ is parallel to the abscissa.*

Proof. Given that the root $x_2 = \nu_2 = \phi = -a/3$ of the “balanced” cubic equation $x^3 + ax^2 + bx + c_0 = 0$, where $c_0 = -2a^3/27 + ab/3$, is the midpoint between its other two roots $x_{1,3} = \nu_{1,3} = -a/3 \pm \sqrt{a^2/3 - b}$, one has $x_1 - x_2$ (the second coordinate of point R) being equal to $x_2 - x_3$ (the second coordinate of point R). Hence P and R are both above the abscissa and are equidistant from it. Thus PR is parallel to the abscissa. Hence, the distance between x_3 and x_1 is exactly equal to the length $\alpha = (\sqrt{12}/3)\sqrt{a^2 - 3b}$ of the side PR . In any other case of three real roots ($c \in [c_2, c_1]$ and $c \neq c_0$), the side PR will not be parallel to the abscissa and hence the projection of PR onto the abscissa will be shorter than the length of PR , that is, the three real roots of the cubic polynomial will lie in an interval of length smaller than $\alpha = (\sqrt{12}/3)\sqrt{a^2 - 3b}$. \square

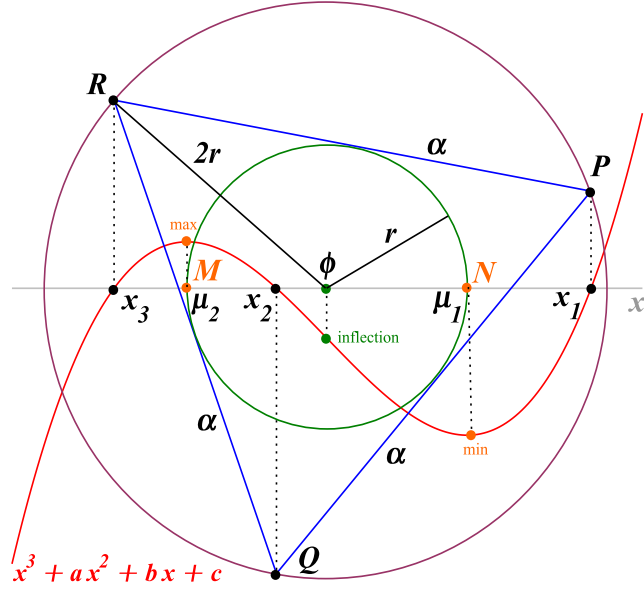


Figure 1

Siebeck–Marden–Northshield Theorem: Any three real numbers, not all equal, are the projections of the vertices of some equilateral triangle in the plane. For a cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ with three real roots (not all equal), the inscribed circle of the equilateral triangle that projects onto those roots itself projects to an interval with endpoints equal to critical points of $p(x)$ [7]. The vertices of the triangle are points P , Q , and R with coordinates $(x_1, (x_2 - x_3)/\sqrt{3})$, $(x_2, (x_3 - x_1)/\sqrt{3})$, and $(x_3, (x_1 - x_2)/\sqrt{3})$, respectively [7]. The centroid of the triangle is at the projection of the inflection point of $p(x)$. The radius of the inscribed circle is $r = (1/3)\sqrt{a^2 - 3b}$. The radius of the circumscribed circle is $2r = (2/3)\sqrt{a^2 - 3b}$. The maximum distance between the three real roots of the monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ is equal to the side of the triangle: $\alpha = \sqrt{12}r = (\sqrt{12}/3)\sqrt{a^2 - 3b}$. This occurs in the case of the “balanced” cubic (with $c = c_0 = -2a^3/27 + ab/3$), when one of the sides of the triangle is parallel to the abscissa. For any other c such that $c_2 \leq c \leq c_1$, the three real roots of the cubic lie within a shorter interval.

Theorem 1. *The monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$, for which $b < a^2/3$ and $c \in [c_2, c_1]$, has three real roots $x_3 \leq x_2 \leq x_1$, at least two of which are different and any two of which are not farther apart than $(\sqrt{12}/3)\sqrt{a^2 - 3b}$, with the following isolation intervals:*

(I) For $c_2 \leq c \leq c_0$: $x_3 \in [\nu_3, \mu_2]$, $x_2 \in [\mu_2, \phi]$, and $x_1 \in [\nu_1, \xi_2]$.

(II) For $c_0 \leq c \leq c_1$: $x_3 \in [\xi_1, \nu_3]$, $x_2 \in [\phi, \mu_1]$, and $x_1 \in [\mu_1, \nu_1]$,

where:

(i) $\mu_{1,2}$ is the double root and $\xi_{1,2}$ is the simple root of $p_{1,2}(x) = x^3 + ax^2 + bx + c_{1,2}$, that is, $\mu_{1,2}$ are the roots of $3x^2 + 2ax + b = 0$, namely: $\mu_{1,2} = -a/3 \pm r = -a/3 \pm (1/3)\sqrt{a^2 - 3b}$ and $\xi_{1,2} = -a - 2\mu_{1,2} = -a/3 \mp 2r = -a/3 \mp (2/3)\sqrt{a^2 - 3b}$.

(ii) $\nu_{1,2,3}$ are the roots of the “balanced” cubic equation $p_0(x) = x^3 + ax^2 + bx + c_0$, namely: $\nu_{1,3} = -a/3 \pm \alpha/2 = -a/3 \pm (\sqrt{3}/3)\sqrt{a^2 - 3b}$ and $\nu_2 = \phi = -a/3$.

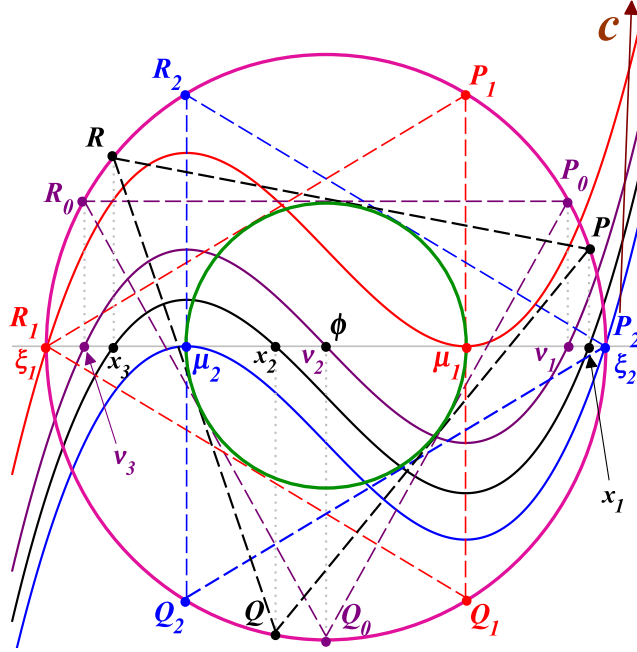


Figure 2

Presented here are four cubics: the “balanced” cubic $x^3 + ax^2 + bx + c_0$ (second from top), whose Siebeck–Marden–Northshield triangle is $P_0Q_0R_0$ with P_0R_0 parallel to the abscissa and whose roots are the symmetric $\nu_{1,3}$ and $\nu_2 = \phi = -a/3$; the two “extreme” cubics $x^3 + ax^2 + bx + c_{1,2}$ (top and bottom) which exhibit double real roots $\mu_{1,2}$ and a simple root $\xi_{1,2}$ and whose Siebeck–Marden–Northshield triangles $P_{1,2}Q_{1,2}R_{1,2}$ have a side perpendicular to the abscissa and a vertex on the abscissa; and a general cubic (second from bottom) $x^3 + ax^2 + bx + c$ with distinct real roots $x_3 < x_2 < x_1$ and whose Siebeck–Marden–Northshield triangle is PQR . Increasing c rotates the Siebeck–Marden–Northshield triangle counterclockwise about its centroid. Decreasing c results in its clockwise rotation. The isolation intervals of the roots of the latter can be immediately determined from the graph:

- (I) For $c_2 \leq c \leq c_0$ (shown on this Figure): $x_3 \in [\nu_3, \mu_2]$, $x_2 \in [\mu_2, \phi]$, and $x_1 \in [\nu_1, \xi_2]$.
- (II) For $c_0 \leq c \leq c_1$: $x_3 \in [\xi_1, \nu_3]$, $x_2 \in [\phi, \mu_1]$, and $x_1 \in [\mu_1, \nu_1]$.

Proof. Due to Lemma 2, the discriminant $\Delta_3 = -27c^2 + (18ab - 4a^3)c + a^2b^2 - 4b^3$ of the monic cubic polynomial $x^3 + ax^2 + bx + c$ is non-negative for all a and $b \leq a^2/3$, if c is between the roots $c_{1,2} = c_0 \pm (2/27)\sqrt{(a^2 - 3b)^3}$ (with $c_0 = -2a^3/27 + ab/3$) of the quadratic equation $x^2 + (4a^3/27 - 2ab/3)x - a^2b^2/27 + 4b^3/27 = 0$. Then $x^3 + ax^2 + bx + c$ will have three real roots. The two “extreme” cases, the cubics $x^3 + ax^2 + bx + c_1$ and $x^3 + ax^2 + bx + c_2$, will each have a double root (as Δ_3 vanishes for $c = c_{1,2}$) and a simple root. Otherwise, for $c_2 < c < c_1$, the cubic polynomial will have three distinct roots. Let $\mu_{1,2}$ denote the double root of the “extreme” cubic $x^3 + ax^2 + bx + c_{1,2}$ and $\xi_{1,2}$ — the corresponding simple root. When $c = c_{1,2}$, one has, due to Viète formulæ: $2\mu_i + \xi_i = -a$, $\mu_i^2 + 2\mu_i\xi_i = b$, and $\mu_i^2\xi_i = -c$ (for $i = 1, 2$). Expressing from the first $\xi_i = -a - 2\mu_i$ and substituting into the second yields $-3\mu_i^2 - 2a\mu_i - b = 0$, that is, the double roots $\mu_{1,2}$ of each of the “extreme” cubics $x^3 + ax^2 + bx + c_{1,2}$ are the roots of the quadratic equation $3x^2 + 2ax + b = 0$, that is $\mu_{1,2} = -a/3 \pm r = -a/3 \pm (1/3)\sqrt{a^2 - 3b}$.

Hence one finds: $\xi_{1,2} = -a - 2\mu_{1,2} = -a/3 \mp 2r = -a/3 \mp (2/3)\sqrt{a^2 - 3b}$.

Due to Lemma 4, the biggest distance between the roots of the cubic will be $\alpha = (\sqrt{12}/3)\sqrt{a^2 - 3b}$.

The roots of the “balanced” cubic equation $x^3 + ax^2 + bx - 2a^3/27 + ab/3 = 0$ (see the proof of Lemma 4) are symmetric with respect to the centre of the inscribed circle: $\nu_3 = -a/3 - \sqrt{a^2/3 - b}$, $\nu_2 = \phi = -a/3$, and $\nu_1 = -a/3 + \sqrt{a^2/3 - b}$. The “balanced” equation has triangle $P_0Q_0R_0$ and the side P_0R_0 is parallel to the abscissa (Figure 2).

When $c = c_1 > c_0$, the Siebeck–Marden–Northshield triangle is $P_1Q_1R_1$ and its side P_1Q_1 is perpendicular to the abscissa. Hence the roots x_2 and x_1 coalesce into the double root μ_1 . The vertex R_1 is on the abscissa at the smallest root ξ_1 (Figure 2).

When $c = c_2 < c_0$, the Siebeck–Marden–Northshield triangle is $P_2Q_2R_2$ and its side R_2Q_2 is perpendicular to the abscissa. The roots x_3 and x_2 coalesce into the double root μ_2 , while the biggest root x_1 is equal to ξ_2 , as the vertex P_2 is on the abscissa at ξ_2 (Figure 2).

Increasing c rotates the Siebeck–Marden–Northshield triangle counterclockwise about its centroid. Decreasing c results in its clockwise rotation. The isolation intervals of the roots of the cubic polynomial are then easily read geometrically — see Figure 2. \square

Theorem 2. *The monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$, for which $b < a^2/3$ and*

(I) $c < c_2$, *has only one real root: $x_1 > \xi_2 = -a - 2\mu_2 = -a/3 + 2r = -a/3 + (2/3)\sqrt{a^2 - 3b}$ (it can be bounded from above by a polynomial root bound).*

(II) $c > c_1$, *has only one real root: $x_1 < \xi_1 = -a - 2\mu_1 = -a/3 - 2r = -a/3 - (2/3)\sqrt{a^2 - 3b}$ (it can be bounded from below by a polynomial root bound).*

Proof. See the caption of Figure 3. \square

As polynomial upper root bound, one can take one of the many existing root bounds. For example, it could be the bigger of 1 and the sum of the absolute values of all negative coefficients [8]. Or one can consider the bound [9]: $1 + \sqrt[k]{H}$, where $k = 1$ if $a < 0$, $k = 2$ if $a > 0$ and $b < 0$, and $k = 3$ if $a > 0$ and $b > 0$, and $c < 0$ (if a , b , and c are all positive, the upper root bound is zero). H is the biggest absolute value of all negative coefficients in $x^3 + ax^2 + bx + c$.

The lower root bound is the negative of the upper root bound of $-x^3 + ax^2 - bx + c$.

Theorem 3. *The monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$, for which $b = a^2/3$ and*

(I) $c < (1/27)a^3$, *has only one real root: $x_1 = -a/3 + \sqrt[3]{a^3/27 - c} > -a/3$.*

(II) $c = (1/27)a^3$, *has a triple real root: $x_1 = x_2 = x_3 = -a/3$.*

(III) $c > (1/27)a^3$, *has only one real root: $x_1 = -a/3 + \sqrt[3]{a^3/27 - c} < -a/3$.*

Proof. See the caption of Figure 4. \square

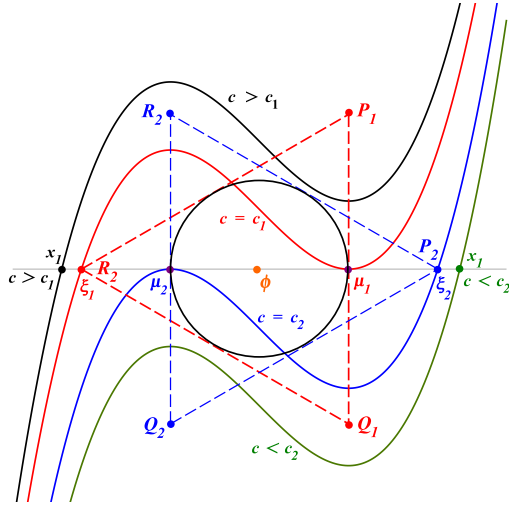


Figure 3

Proof of Theorem 2

The two “extreme” cubics — with $c = c_1$ (second from top) and with $c = c_2$ (second from bottom) with their triangles $P_1Q_1R_1$ and $P_2Q_2R_2$, respectively. Each of these cubics has a double root $\mu_{1,2}$ and a simple root $\xi_{1,2}$, respectively. Cubics with c such that $c_2 < c < c_1$ are between those two and they are the only ones with three distinct real roots. When $c > c_1$ (uppermost cubic), there is a pair of complex conjugate roots and a single real root $x_1 < \xi_1$. When $c < c_2$ (lowermost cubic), there is a pair of complex conjugate roots and a single real root $x_1 > \xi_2$. The isolation intervals of the single real root for either of the two latter cubics can be found by the determination of the lower (respectively, upper) root bound of the cubic.

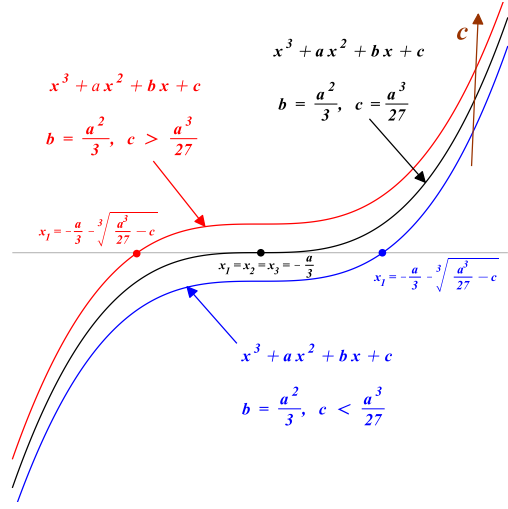


Figure 4

Proof of Theorem 3

The special case of $b = a^2/3$. One immediately gets that $c_1 = c_2 = a^3/27$ in this case. The only cubic with three real roots is the one with $c = a^3/27$. This is the cubic $x^3 + ax^2 + (a^2/3)x + a^3/27 = (x + a/3)^3$ (middle curve). Clearly, this cubic has a triple real root $x_1 = x_2 = x_3 = -a/3$. If one increases c above $a^3/27$ (top cubic), there is a pair of complex conjugate roots and a single root $x_1 < -a/3$. If one increases c above $a^3/27$ (bottom cubic), there is a pair of complex conjugate roots and a single root $x_1 > -a/3$. The single real root for either of the two latter cubics can be immediately found completing the cube: $x_1 = -a/3 + \sqrt[3]{a^3/27 - c}$.

Theorem 4. The only real root x_1 of monic cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ with $b > a^2/3$ (due to Lemma 1) has the following isolation interval:

- (I) If $a \geq 0$ and $c \leq 0$: $0 \leq x_1 \leq -c/b$.
- (II) If $a \geq 0$ and $c > 0$: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$.
- (III) If $a < 0$ and $c < 0$: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$.
- (IV) If $a < 0$ and $c \geq 0$: $-c/b \leq x_1 \leq 0$.

Proof. Re-write the cubic equation $x^3 + ax^2 + bx + c = 0$ as $x^3 + ax^2 = -bx - c$. Such “split” of polynomial equations of different degrees has been proposed and studied in [10, 11, 12]

The rest of the proof is graphic — see the captions of Figures 5–8 for the four cases (I)–(IV) respectively. \square

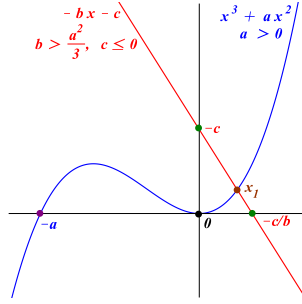


Figure 5

Proof of Theorem 4 (I)

When $a \geq 0$ and $c \leq 0$, the isolation interval of the single root x_1 is: $0 \leq x_1 \leq -c/b$.

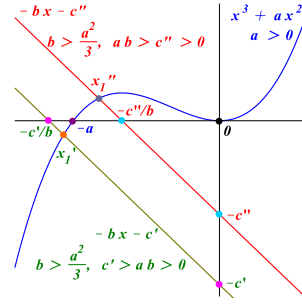


Figure 6

Proof of Theorem 4 (II)

When $a \geq 0$ and $c > 0$, the isolation interval of the single root x_1 is: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$.

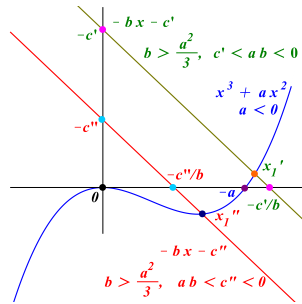


Figure 7

Proof of Theorem 4 (III)

When $a < 0$ and $c < 0$, the isolation interval of the single root x_1 is: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$.

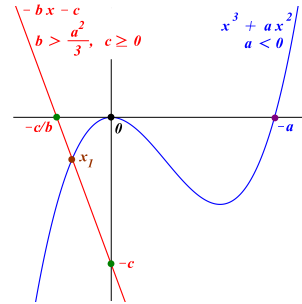


Figure 8

Proof of Theorem 4 (IV)

When $a < 0$ and $c \geq 0$, the isolation interval of the single root x_1 is: $0 \leq x_1 \leq -c/b$.

3 Roles of the Coefficients of the Symbolic Cubic Equation $x^3 + ax^2 + bx + c$ and Isolation Intervals of its Real Roots — Summary and Application of the Analysis

- (a) The coefficient a of the quadratic term of $x^3 + ax^2 + bx + c$ selects the centre $\phi = -a/3$ of the inscribed circle of the equilateral triangle that projects onto the roots of $x^3 + ax^2 + bx + c$, in the case of three real roots. The centre of this circle is also the projection of the inflection point of the graph of $x^3 + ax^2 + bx + c$ onto the abscissa. The inscribed circle projects to an interval on the abscissa with endpoints equal to the projections of the stationary points of $x^3 + ax^2 + bx + c$ (Figure 1).

- (b) For any given a , the coefficients b of the linear term of $x^3 + ax^2 + bx + c$ determines the radius $r = (1/3)\sqrt{a^2 - 3b}$ of the inscribed circle. The circumscribed circle of the equilateral triangle has radius $2r = (2/3)\sqrt{a^2 - 3b}$.

If a cubic polynomial has two stationary points, the distance between them is always $2r = (2/3)\sqrt{a^2 - 3b}$.

The inflection point of the graph of $x^3 + ax^2 + bx + c$ is always the midpoint $(-a/3)$ between the stationary points of the cubic polynomial.

Hence, the analysis of the cubic polynomial $x^3 + ax^2 + bx + c$ should start with what the value of b , relative to $a^2/3$, is.

(I) If $b < a^2/3$ and if:

(i) $c_2 \leq c \leq c_0$, then the polynomial $x^3 + ax^2 + bx + c$ has three real roots with the following isolation intervals: $x_3 \in [\nu_3, \mu_2]$, $x_2 \in [\mu_2, \phi]$, and $x_1 \in [\nu_1, \xi_2]$ (Figure 2).

(ii) $c_0 \leq c \leq c_1$, then the polynomial $x^3 + ax^2 + bx + c$ has three real roots with the following isolation intervals: $x_3 \in [\xi_1, \nu_3]$, $x_2 \in [\phi, \mu_1]$, and $x_1 \in [\mu_1, \nu_1]$ (Figure 2).

In the above, $c_{1,2} = c_0 \pm (2/27)\sqrt{(a^2 - 3b)^3}$, with $c_0 = -2a^3/27 + ab/3$, are the values of c for which, for any a and $b < a^2/3$, the discriminant Δ_3 of the cubic polynomial $x^3 + ax^2 + bx + c$ is zero (Δ_3 positive for c between c_2 and c_1). Namely, these are the roots of the quadratic equation (3): $x^2 + (4a^3/27 - 2ab/3)x - a^2b^2/27 + 4b^3/27 = 0$.

Also in the above, $\nu_3 = -a/3 - \sqrt{a^2/3 - b}$, $\nu_2 = \phi = -a/3$, and $\nu_1 = -a/3 + \sqrt{a^2/3 - b}$ are three real roots of the “balanced” cubic polynomial $x^3 + ax^2 + bx + c_0$ (Figure 2).

The roots of the “extreme” cubic $x^3 + ax^2 + bx + c_1$ are the double root $\mu_1 = -a/3 + (\sqrt{3}/3)\sqrt{a^2/3 - b}$ and the simple root $\xi_1 = -a - 2\mu_1 = -a/3 - 2r = -a/3 - (2/3)\sqrt{a^2 - 3b}$. Likewise, the roots of the “extreme” cubic $x^3 + ax^2 + bx + c_2$ are the double root $\mu_2 = -a/3 - (\sqrt{3}/3)\sqrt{a^2/3 - b}$ and the simple root $\xi_2 = -a - 2\mu_2 = -a/3 + 2r = -a/3 + (2/3)\sqrt{a^2 - 3b}$ (Figure 2 and Figure 3).

The biggest distance between any two of the three real roots of the cubic equation $x^3 + ax^2 + bx + c = 0$ is $\alpha = \sqrt{12}r = (\sqrt{12}/3)\sqrt{a^2 - 3b}$ — achieved for the roots of the “balanced” cubic equation $x^3 + ax^2 + bx + c_0$ (Figure 2). For any other cubic equation with $c_2 \leq c \leq c_1$, the three real roots are within an interval of length $3r = \sqrt{a^2 - 3b} < \alpha$ (Figure 2).

(iii) $c < c_2$, then the polynomial $x^3 + ax^2 + bx + c$ has only one real root: $x_1 > \xi_2 = -a - 2\mu_2 = -a/3 + 2r = -a/3 + (2/3)\sqrt{a^2 - 3b}$ (Figure 3).

The root x_1 can be bounded from above by a polynomial root bound.

(iv) $c > c_1$, then the polynomial $x^3 + ax^2 + bx + c$ has only one real root: $x_1 < \xi_1 = -a - 2\mu_1 = -a/3 - 2r = -a/3 - (2/3)\sqrt{a^2 - 3b}$ (Figure 3).

The root x_1 can be bounded from below by a polynomial root bound.

(II) If $b = a^2/3$ and if:

- (i) $c < (1/27)a^3$, then the polynomial $x^3 + ax^2 + bx + c$ has only one real root: $x_1 = -a/3 + \sqrt[3]{a^3/27 - c} > -a/3$ (Figure 4).
- (ii) $c = (1/27)a^3$, then the polynomial $x^3 + ax^2 + bx + c$ has a triple real root: $x_1 = x_2 = x_3 = -a/3$ (Figure 4).
- (iii) $c > (1/27)a^3$, then the polynomial $x^3 + ax^2 + bx + c$ has only one real root: $x_1 = -a/3 + \sqrt[3]{a^3/27 - c} < -a/3$ (Figure 4).
- (III) If $b > a^2/3$, the discriminant of the cubic polynomial is negative and thus $x^3 + ax^2 + bx + c$ has one real root x_1 and a pair of complex conjugate roots. The isolation interval of x_1 depends on the signs of a and c and is as follows:
- (i) If $a \geq 0$ and $c \leq 0$: $0 \leq x_1 \leq -c/b$ (Figure 5).
- (ii) If $a \geq 0$ and $c > 0$: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$ (Figure 6).
- (iii) If $a < 0$ and $c < 0$: $\min\{-a, -c/b\} \leq x_1 \leq \max\{-a, -c/b\}$ (Figure 7).
- (iv) If $a < 0$ and $c \geq 0$: $-c/b \leq x_1 \leq 0$ (Figure 8).
- (c) The coefficient c of $x^3 + ax^2 + bx + c$ rotates the equilateral triangle (which exists if $b < a^2/3$) that projects onto the roots $x_3 \leq x_2 \leq x_1$ (at least two of which are different) of the cubic polynomial. The vertices P , Q , and R of the triangle are points of coordinates $(x_1, (x_2 - x_3)/\sqrt{3})$, $(x_2, (x_3 - x_1)/\sqrt{3})$, and $(x_3, (x_1 - x_2)/\sqrt{3})$, respectively. Point Q is always below the abscissa and points P and R — always above it.
- When $c = c_0 = -2a^3/27 + ab/3$, the side PR is parallel to the abscissa. This corresponds to the “balanced” cubic equation $x^3 + ax^2 + bx - 2a^3/27 + ab/3 = 0$, the roots of which are symmetric with respect to the centre of the inscribed circle: $\nu_3 = -a/3 - \sqrt{a^2/3 - b}$, $\nu_2 = \phi = -a/3$, and $\nu_1 = -a/3 + \sqrt{a^2/3 - b}$. The “balanced” equation has triangle $P_0Q_0R_0$ (Figure 2).
- When c increases from c_0 towards $c_1 > c_0$, the equilateral triangle PQR rotates counterclockwise around its centre from the position of triangle $P_0Q_0R_0$ of the “balanced” equation. When $c = c_1$, the roots x_2 and x_1 coalesce into the double root μ_1 , while the smallest root x_3 becomes equal to $\xi_1 = -a - 2\mu_1 = -a/3 - 2r = -a/3 - (2/3)\sqrt{a^2 - 3b}$. The triangle in this case is $P_1Q_1R_1$ and its side P_1Q_1 is perpendicular to the abscissa. The vertex R_1 is on the abscissa. The triangle cannot be rotated further counterclockwise as, when $c > c_1$, the polynomial $x^3 + ax^2 + bx + c$ has only one real root (Figure 2).
- When c decreases from c_0 towards $c_2 < c_0$, the equilateral triangle PQR rotates clockwise around its centre from the position of triangle $P_0Q_0R_0$ of the “balanced” equation. When $c = c_2$, the roots x_3 and x_2 coalesce into the double root μ_2 , while the biggest root x_1 becomes equal to $\xi_2 = -a - 2\mu_2 = -a/3 + 2r = -a/3 + (2/3)\sqrt{a^2 - 3b}$. The triangle in this case is $P_2Q_2R_2$ and its side R_2Q_2 is perpendicular to the abscissa. The vertex P_2 is on the abscissa. The triangle cannot be rotated further clockwise as, when $c < c_2$, the polynomial $x^3 + ax^2 + bx + c$ has only one real root (Figure 2).

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