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QUOTIENT-TRANSITIVITY AND CYCLIC SUBGROUP-TRANSITIVITY

BRENDAN GOLDSMITH AND KETAO GONG*

ABSTRACT. We introduce two new notions of transitivity for Abelian p -groups based on isomorphism of quotients rather than the classical use of equality of height sequences associated with Abelian p -group theory. Unlike the classical theory where ‘most’ groups are transitive, these new notions lead to much smaller classes but even these classes are sufficiently large to be interesting.

1. INTRODUCTION

The notion of transitivity in the context of Abelian group and module theory was introduced by Kaplansky in [10] and developed further in [11, Section 18]; in this work we shall focus exclusively on groups. The basic idea is simple: given an Abelian group G and elements $x, y \in G$, G is said to be *transitive* if, when a certain necessary condition for the existence of an automorphism of G mapping $x \mapsto y$ is satisfied, then there really is an automorphism of G mapping $x \mapsto y$. In Kaplansky’s initial work a key role was played by the Ulm or height sequence of elements. Recall that if G is an Abelian p -group which is reduced (i.e., does not contain a divisible subgroup), then the height in G , $h_G(x)$ of an element x is the ordinal α if $x \in p^\alpha G \setminus p^{\alpha+1}G$, with the usual convention that $h_G(0) = \infty$. The Ulm or height sequence of x with respect to G is the sequence of ordinals or symbols ∞ given by $U_G(x) = (h_G(x), h_G(px), h_G(p^2x), \dots)$; the collection of such sequences is partially ordered pointwise. Thus in Kaplansky’s original work a reduced p -group G was said to be *transitive* if, for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$, there is an automorphism ϕ of G with $\phi(x) = y$.

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In terms of p -groups, Kaplansky's original theorem - see [11, Theorem 24] - was the following:

Theorem (Kaplansky): If M is a reduced Abelian group with the property that any two elements of M can be embedded in a countable direct summand of M , then M is transitive.

Kaplansky observed that both countable p -groups and groups with no elements of infinite height - the latter are usually called separable p -groups - are transitive. Interestingly, his proof was simply to cite an Exercise given after his celebrated proof of Ulm's Theorem [11, Theorem 14].

Since Kaplansky's early work there has been significant attention paid to the notion of transitivity, see for example, the early contributions of Corner [1], Griffith [7] and Hill [8]. The culmination of the work in this area of the last two authors, was a proof that the so-called totally projective groups, a large class of Abelian p -groups which contains all direct sums of countable groups, are transitive.

The necessary condition used by Kaplansky is rather particular to local situations such as Abelian p -groups, but another more general approach, which can be used in the category of all groups, has been considered by Strüngmann and the first author in [5, 6]. There, a group G is said to be *weakly transitive* if there exists an automorphism of G mapping the element x to the element y when there exist endomorphisms θ, ϕ of G with $\theta(x) = y$ and $\phi(y) = x$.

Both classical and weak transitivity are examples of phenomena which are widespread: in some not too precisely defined way, most groups have these properties. Furthermore, thanks to an observation of Corner [1], we know that these 'generic' types of transitivity are determined by the action of the endomorphism ring on the first Ulm subgroup, $p^\omega G$, of the group G in question; in particular, all separable p -groups possess both properties.

In this work we wish to consider another type of transitivity which is in some sense the polar opposite of the classical notion: not 'too many' groups will have this property but sufficiently many do possess it to make the class interesting.

Definition 1.1. A p -group G is said to be **transitive with respect to cyclic subgroups** if when X, Y are cyclic subgroups of G with (i) $X \cong Y$ and (ii) $G/X \cong G/Y$, there exists an automorphism ϕ of G with $\phi(X) = Y$.

We use the abbreviation G is CS-transitive for the full statement ‘ G is transitive with respect to cyclic subgroups’.

Remark 1.2. The notion of transitivity with respect to cyclic subgroups also makes sense in the category of all groups: the condition (ii) in the definition above is replaced by

(ii’) X, Y are subgroups of G of the same finite index.

Thus the notion of a CS-transitive non-Abelian group leads to a larger class than the so-called *order-transitive p -groups*, i.e., the p -groups where for each positive integer r , the elements of order p^r are transitively permuted by the automorphism group. We shall show later that all finite Abelian p -groups are CS-transitive but for non-Abelian groups the concept is very strong; for example, a simple calculation shows that the dihedral group of order 8, $D_4 = \langle s, t | s^4 = t^2 = 1, tst = s^3 \rangle$ is not transitive with respect to cyclic subgroups since the elements s^2 and t generate isomorphic subgroups of order 2 and hence both are subgroups of finite index 4, but no automorphism of D_4 can map $t \mapsto s^2$. On the other hand, the quaternion group $Q_8 = \langle a, b | a^4 = 1, a^2 = b^2, bab = a \rangle$ has a single cyclic subgroup of order 2 and three subgroups of order 4 (and hence of index 2.) However, it is well known that the automorphism group of Q_8 acts transitively on the elements of order 4 and hence Q_8 is CS-transitive.

In this paper we shall not investigate CS-transitivity any further for non-Abelian groups. Consequently, in the sequel the word ‘group’ shall mean an additively written Abelian p -group and we shall further assume that all such groups are reduced, i.e., the Prüfer quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not a subgroup.

It is apparent that the notion of being transitive with respect to cyclic submodules is connected in some sense to the notion of the submodules being equivalent, however, we emphasize that are not taking either concept of isomorphism in the above

definition as being understood in the context of valuated groups. For an excellent discussion of that approach see the paper [9] by Hill and West. There is another reason why the notion of CS-transitivity is interesting. In his celebrated paper [8], Hill introduced the notion of a *potentially transitive group*: a reduced p -group G is said to be potentially transitive if, given any pair of elements x, y of G with $U_G(x) = U_G(y)$, there is an isomorphism $G/\langle x \rangle \cong G/\langle y \rangle$. Clearly a transitive group is potentially transitive and it is straightforward to show that if G is potentially transitive and CS-transitive, then it must in fact be transitive.

There are situations in which the condition (i) that $X \cong Y$ of Definition 1.1 may be dropped: for example, if G is finite then condition (i) is an immediate consequence of condition (ii). More generally, as we shall establish later, the same is true if G is an arbitrary homocyclic p -group. But condition (i) does not, in general, follow from condition (ii) and we find it convenient to make a formal distinction by defining:

Definition 1.3. A p -group M is said to be **quotient-transitive** if, given any pair of non-zero elements $x, y \in M$, with $M/\langle x \rangle \cong M/\langle y \rangle$, there is an automorphism ϕ of M with $\phi(x) = y$.

It is, however, immediate that for finite p -groups, the notions of quotient-transitivity and CS-transitivity coincide. Furthermore, a quotient-transitive group is always transitive with respect to cyclic subgroups. However, the concepts are different as we shall show shortly.

Remark 1.4. Definitions 1.1, 1.3 can also be extended to modules in a natural way but in this paper we will restrict our attention to abelian p -groups; in a subsequent work we will develop the more general notion, particularly in the context of p -adic modules.

We finish this introduction by giving a short summary of our principal results and some comments about notation and terminology.

A first objective of the paper is to establish (in Theorem 3.2) that finite p -groups are quotient-transitive and hence CS-transitive. Rather surprisingly we could find

no reference to such a result in the existing literature. We also establish that there are, however, many situations where infinite p -groups are also quotient-transitive and in Theorem 4.8 we obtain a classification of separable quotient-transitive p -groups: such a group is either semi-standard - see the paragraph below - or of the form $F \oplus \bigoplus_{\lambda} \mathbb{Z}(p^n)$ where λ is arbitrary and F is a finite p -group of exponent $< n$. In section 4 we illustrate by means of examples, the difficulties which arise in relation to the classification of CS-transitive p -groups, even when one restricts attention to direct sums of cyclic groups.

Our notation and terminology are largely standard and agree with those used in the texts [2, 3, 4] of Fuchs and of Kaplansky [11]. Given a positive integer n , we say that G is *homocyclic of exponent n* if G has the form $\bigoplus_{\kappa} \mathbb{Z}(p^n)$ for some (possibly infinite) cardinal κ ; we use the notation $e(G)$ to denote the exponent of a bounded p -group G i.e., the least integer n such that $p^n G = 0$. Of particular importance for this work is a notion named *semi-standard* by Corner: a p -group G is said to be semi-standard if, for a basic subgroup B of G , there is a decomposition $B = B_1 \oplus \cdots \oplus B_n \oplus \cdots$ where each B_n is homocyclic of exponent n and of finite rank i.e., B_n is of the form $\bigoplus_{r_n} \mathbb{Z}(p^n)$ with r_n finite or zero. This is, of course, equivalent to the finiteness of the Ulm invariants $f_n(G)$ where $0 \leq n < \omega$.

2. ELEMENTARY RESULTS

In this short section we present some simple results that are useful for later discussions.

Example 2.1. Suppose that $G = A \oplus B$ where A is an elementary p -group of infinite rank and $B = \langle b \rangle$ is cyclic of order p^2 . Then G is CS-transitive but not quotient-transitive.

Proof. Let x be an arbitrary element of G so that x can be expressed in the form $x = ra + sb$ for some integers r, s and $a \in A$. By absorbing suitable units and utilising standard properties of elementary groups, we may assume that $\langle a \rangle$ is a

summand of A and that $r \in \{0, 1\}$ and $s \in \{0, 1, p\}$. We determine the structure of $G/\langle x \rangle$ for the various cases.

Consider the situation where x has order p . Then x has the form $x = a, a + pb$ or pb . Clearly $G/\langle a \rangle \cong G$ and $G/\langle pb \rangle \cong A$. In the remaining case $G/\langle a + pb \rangle$ is easily shown to be isomorphic to G since we have

$$G/\langle a + pb \rangle = (\langle a + pb \rangle + \langle b \rangle / \langle a + pb \rangle) \oplus A_1 \cong \langle b \rangle / (\langle a + pb \rangle \cap \langle b \rangle) \oplus A_1,$$

where $A = \langle a \rangle \oplus A_1$, and the result follows since $\langle a + pb \rangle \cap \langle b \rangle = 0$.

Suppose now that x has order p^2 , so that $x = b$ or $a + b$. Clearly $G/\langle b \rangle \cong A$, while if $x = a + b$ we have as before

$$G/\langle a + b \rangle = A_1 \oplus (\langle a + b \rangle + \langle b \rangle) / \langle a + b \rangle \cong A_1 \oplus \langle b \rangle / (\langle a + b \rangle \cap \langle b \rangle) \cong A,$$

the last isomorphism coming from the fact that $\langle a + b \rangle \cap \langle b \rangle = \langle pb \rangle$.

Now suppose that $x, y \in G$ satisfy $o(x) = o(y)$ and $G/\langle x \rangle \cong G/\langle y \rangle$. If x (and hence y) has order p , then we must have that $\{x, y\} = \{a, a + pb\}$ and as both $a, a + pb$ have the same Ulm sequence and G is a direct sum of cyclic groups, it follows from Kaplansky's theorem, or by direct construction that there is an automorphism ϕ of G interchanging x and y .

On the other hand if $o(x) = p^2$, then $\{x, y\} = \{b, a + b\}$ and again the existence of the required automorphism comes from the fact that b and $a + b$ have the same Ulm sequence.

So G is certainly CS-transitive; it is not quotient-transitive since $G/\langle b \rangle \cong A \cong G/\langle pb \rangle$ but clearly no automorphism of G can map $b \mapsto pb$. $\square$

There are, however, situations other finiteness in which quotient-transitivity and CS-transitivity coincide.

Proposition 2.2. *If G is a finite p -group or a homocyclic p -group of arbitrary rank, then G is CS-transitive if, and only if, G is quotient-transitive.*

Proof. We have already observed the equivalence of the **two** transitivity notions when G is finite.

If G is quotient-transitive then certainly it is CS-transitive.

Suppose that G is homocyclic of exponent n and G is CS-transitive. Then if X, Y are cyclic subgroups of G with $G/X \cong G/Y$, then we can find elements $x, y \in G$ with $X = \langle p^r x \rangle, Y = \langle p^s y \rangle$ and x, y are both elements of order p^n . Consequently $\langle x \rangle$ and $\langle y \rangle$ are summands of G : say $G = \langle x \rangle \oplus G_1 = \langle y \rangle \oplus G_2$ and note that $G_1 \cong G_2$. Now we have

$$\mathbb{Z}(p^r) \oplus G_1 \cong G/X \cong G/Y \cong \mathbb{Z}(p^s) \oplus G_2.$$

It follows immediately that $r = s$ and so $X \cong Y$. Thus there is an automorphism ϕ of G with $\phi(x) = y$ and G is quotient-transitive, as required. $\square$

A simple adaptation of the argument above yields:

Corollary 2.3. *If G is an arbitrary homocyclic p -group then G is transitive with respect to cyclic subgroups. In particular, arbitrary homocyclic p -groups are also quotient-transitive*

3. TORSION GROUPS

Our objective in this section is to show that finite p -groups are quotient-transitive. The corresponding result for transitivity in the original sense of Kaplansky is reasonably straightforward although, as mentioned in the Introduction, it is based on quite a deep result, surprisingly, the argument to establish quotient-transitivity is quite complex. Clearly a first step in this process is an understanding of the structural significance of a quotient of the type $G/\langle x \rangle$. Fortunately, the quotients that we shall need to handle are of a very specific type and this eases the calculations required to understand them.

Suppose that G is a p -group and $x \in G$ has Ulm sequence $U_G(x) = \{r_0, \dots, r_n = \infty\}$ where the $r_i (i < n)$ are integers. If $r_{n_1}, r_{n_2} \dots r_{n_t} = r_n$ denote the entries following a gap in the Ulm sequence, then by a well-known result [3, Lemma 65.4] or [4, 10, Lemma 1.4] (attributed by Fuchs to Baer), there is a finite direct summand $C = \langle c_1 \rangle \oplus \langle c_2 \rangle \oplus \dots \oplus \langle c_t \rangle$ of G where $o(c_i) = p^{n_i + k_i}$, $x = p^{k_1} c_1 + p^{k_2} c_2 + \dots + p^{k_t} c_t$ and the following relationships hold:

- (i) $k_1 = r_0, k_2 = r_{n_1} - n_1, \dots, k_t = r_{n_{t-1}} - n_{t-1}$;
- (ii) $0 < n_1 < \dots < n_t$ and $0 \leq k_1 < \dots < k_t$.

We remark that in the following proposition we interpret the group $\mathbb{Z}(p^0)$ as the trivial group.

Proposition 3.1. *Suppose that $C = \langle c_1 \rangle \oplus \langle c_2 \rangle \oplus \dots \oplus \langle c_t \rangle$ is a p -group and $o(c_i) = p^{n_i+k_i}$ for each $1 \leq i \leq t$. If $x = p^{k_1}c_1 + p^{k_2}c_2 + \dots + p^{k_t}c_t$, with $0 < n_1 < \dots < n_t$ and $0 \leq k_1 < \dots < k_t$, then the quotient group*

$$C/\langle x \rangle \cong \mathbb{Z}(p^{r_0}) \oplus \mathbb{Z}(p^{r_{n_1}}) \oplus \dots \oplus \mathbb{Z}(p^{r_{n_{t-1}}}),$$

where $r_0 = k_1, r_{n_1} = k_2 + n_1, \dots, r_{n_{t-1}} = k_t + n_{t-1}$. In particular, C and $C/\langle x \rangle$ do not have any cyclic summands of the same order.

Proof. Notice firstly that the properties of the n_i, k_i and the form of the element x ensure that $U_C(x) = \{r_0, r_1, \dots, r_n = \infty\}$ and that the $r_{n_1}, r_{n_2}, \dots, r_{n_t} = r_n$ are precisely the entries following a gap in the Ulm sequence so that the relationships in (i) above are satisfied; equivalently the relationships in the final part of the statement of the Proposition hold.

The proof is by induction on the number t of summands in the decomposition of C . If $t = 1$, the result holds since $C/\langle x \rangle$ is then just cyclic of order p^{k_1} and $k_1 = r_0$. So suppose C has $t > 1$ summands and the result is true for groups of the same form with less than t summands.

For convenience we write $X = \langle x \rangle$ and $A = \langle c_t \rangle + X$. If $c = v_1c_1 + v_2c_2 + \dots + v_tc_t$ (for suitable integers v_i) is an arbitrary element of C , observe that $p^{n_{t-1}+k_t}c = p^{n_{t-1}+k_t}v_tc_t$ since the conditions on the n_i, k_i ensure that $p^{n_{t-1}+k_t} \geq o(c_i)$ for all $1 \leq i \leq t-1$. Furthermore, $p^{n_{t-1}}x = p^{n_{t-1}+k_t}c_t$ so $p^{n_{t-1}+k_t}c \in X$ for all $c \in C$. Hence C/X has exponent at most $n_{t-1} + k_t$.

Notice however, that $p^{n_{t-1}+k_t-1}c_t \notin X$ for otherwise it would follow that

$$p^{n_{t-1}+k_t-1}c_{t-1} = p^{n_{t-1}-1}x - p^{n_{t-1}+k_t-1}c_t \in X$$

and a simple linear independence argument shows that this cannot be true. Thus C/X has exponent exactly $n_{t-1} + k_t = r_{n_{t-1}}$. It also follows from this argument that

$A/X = \langle c_t + X \rangle$ is a cyclic subgroup of C/X with exponent equal to the exponent of C/X . Now it is known [11, Lemma 4], that if G is a p -group with $p^r G = 0$ and x is an element of order p^r , then the cyclic summand generated by x is a direct summand of G . Thus A/X is a direct summand of C/X , say $C/X = A/X \oplus B/X$, of order $p^{r_{n_t-1}}$. Let $C' = \langle c_1 \rangle \oplus \cdots \oplus \langle c_{t-1} \rangle$ so that

$$B/X \cong C/A = (\langle c_t \rangle + X + C')/A \cong C'/(C' \cap A).$$

Consider the intersection $C' \cap A$. Clearly $\langle p^{k_1} c_1 + \cdots + p^{k_{t-1}} c_{t-1} \rangle = \langle x - p^{k_t} c_t \rangle \in C' \cap A$; however, if $z \in C' \cap A$, then $z = uc_t + vx = w_1 c_1 + \cdots + w_{t-1} c_{t-1}$ and linear independence gives $(w_1 - vp_1^k) c_1 = 0, \dots, (w_{t-1} - vp^{k_{t-1}}) c_{t-1} = 0$ and so $z = w_1 c_1 + \cdots + w_{t-1} c_{t-1} = vd$, where $d = p^{k_1} c_1 + \cdots + p^{k_{t-1}} c_{t-1} = x - p^{k_t} c_t$. Thus $z \in \langle d \rangle$ and so $C' \cap A = \langle d \rangle$. It follows that $B/X \cong C'/\langle d \rangle$ and since the element d of C' satisfies the requirements in the statement of the proposition, a simple induction gives that B/X is a direct sum of cyclic groups with exponents equal to the terms following the jumps in the Ulm sequence in C' of d . However, it is immediate that these cyclic groups are then of exponents $r_0, r_1, \dots, r_{n_t-2}$ and $C/\langle x \rangle$ has the desired form.

Finally, the conditions on the n_i, k_j ensure that $r_{n_i} = n_i + k_{i+1}$ can never be equal to $n_j + k_j$ for any j . $\square$

Summarizing the last proposition 3.1 we see that $C/\langle x \rangle$ determines the Ulm sequence of x in C : the terms following the jumps in $U_C(x)$ can be read off from the quotient and the remaining terms in the Ulm sequence just increase by 1 at each step, so the full sequence can be reconstructed uniquely from knowledge of the quotient $C/\langle x \rangle$. We can now use this to obtain the main result of this section.

Theorem 3.2. *If G is a finite p -group and $G/\langle x \rangle \cong G/\langle y \rangle$ for elements $x, y \in G$, then $U_G(x) = U_G(y)$ and there is an automorphism ϕ of G with $\phi(x) = y$.*

Proof. Notice immediately that from the isomorphism between the quotients, it must follow that x, y have the same order. Suppose that $U_G(x) = (r_0, \dots, \uparrow r_{n_1}, \dots, \uparrow$

$r_{n_t} = \infty$) and $U_G(y) = (s_0, \dots, \uparrow s_{m_1}, \dots, \uparrow s_{m_u} = \infty)$, where $\uparrow$ indicates a gap in the Ulm sequence; note then that $n_t = m_u$.

Embed x, y in summands C, D as in Proposition 3.1 and let $G = C \oplus H = D \oplus K$; thus C is a direct sum of cyclic groups of the form $C = \langle c_1 \rangle \oplus \dots \oplus \langle c_t \rangle$ with $o(c_i) = p^{n_i+k_i}$ where $0 < n_1 < \dots < n_t$ and $0 \leq k_1 < k_2 < \dots < k_t$; similarly D is a direct sum of cyclic groups $\langle d_1 \rangle \oplus \dots \oplus \langle d_u \rangle$ with $o(d_j) = p^{m_j+\ell_j}$. Note that the summands of C, D each consist of a single copy of the given cyclic group, so that the Ulm invariants of C, D will be either 1 or 0.

Now $G/\langle x \rangle = C/\langle x \rangle \oplus H \cong G/\langle y \rangle = D/\langle y \rangle \oplus K$ and we also have, for all α , the relations

$$f_\alpha(G) = f_\alpha(C) + f_\alpha(H) = f_\alpha(D) + f_\alpha(K) \quad (1)$$

where the f_α are Ulm invariants. Furthermore, $f_\alpha(C/\langle x \rangle) + f_\alpha(H) = f_\alpha(D/\langle y \rangle) + f_\alpha(K)$. Since all the Ulm invariants are integers we can write $f_\alpha(H) = f_\alpha(G) - f_\alpha(C)$ and a similar expression for $f_\alpha(K)$ from which we deduce that

$$f_\alpha(C/\langle x \rangle) + f_\alpha(D) = f_\alpha(D/\langle y \rangle) + f_\alpha(C) \quad (2).$$

Let $\alpha = n_1 + k_1 - 1$, so that $f_\alpha(C) = 1$ and hence it follows from Proposition 3.1 that $f_\alpha(C/\langle x \rangle) = 0$. Substituting in (2), we get that $f_\alpha(D) = f_\alpha(D/\langle y \rangle) + 1$ and we conclude that $f_\alpha(D) \neq 0$. In particular, D has a summand isomorphic to $\mathbb{Z}(p^{n_1+k_1})$. Repeating this argument for the various values of $\alpha = n_i + k_i - 1$ we deduce that D has a subgroup isomorphic to C .

Now choose α to take the various values $m_j + \ell_j$ and repeat the argument above. From this we can conclude that C has a subgroup isomorphic to D , whence $D \cong C$. Since the group G is finite we must have that $H \cong K$ and $C/\langle x \rangle \cong D/\langle y \rangle$. An isomorphism ϕ between D and C maps $D/\langle y \rangle$ isomorphically onto $C/\langle \phi(y) \rangle$ and so we conclude from Proposition 3.1 that $U_C(x) = U_C(\phi(y))$. Since C is a summand of G and ϕ is an isomorphism, we then have $U_G(x) = U_G(y)$. The existence of an automorphism ϕ of G mapping $x \mapsto y$ follows from Kaplansky's theorem. $\square$

3.1. Infinite Torsion Groups. Our first result in this subsection guarantees the existence of an adequate supply of automorphisms in certain situations; our arguments are based on those of Pierce in [12, Lemma 2.4].

Lemma 3.3. *Let G be an arbitrary unbounded reduced p -group which is semi-standard and $x, y \in G$ are such that (i) $\langle x \rangle \cap p^\omega G = 0 = \langle y \rangle \cap p^\omega G$ and (ii) $G/\langle x \rangle \cong G/\langle y \rangle$. Then there is an automorphism of G mapping $y \mapsto x$.*

Proof. To see this note that since (i) holds, we can find finite subgroups C_1, C_2 with $x \in C_1 \leq B_1$ where B_1 is a basic subgroup of G ; similarly for $y \in C_2 \leq B_2$. Furthermore C_1, C_2 are summands of G , say $G = C_1 \oplus E_1 = C_2 \oplus E_2$ for suitable E_1, E_2 . In fact since $B_1 \cong B_2$ are both direct sums of cyclic groups we can arrange that $C_1 \cong C_2$, via ψ say. As the groups $C_1 \cong C_2$ are finite, it follows from standard cancellation theory that we also have $E_1 \cong E_2$; let σ be an isomorphism $E_1 \cong E_2$. Then we have

$$C_1/\langle x \rangle \oplus E_1 = G/\langle x \rangle \cong G/\langle y \rangle = C_2/\langle y \rangle \oplus E_2.$$

Since G is semi-standard, each Ulm invariant $f_n(G)$ is finite and hence each invariant $f_n(E_1) = f_n(E_2)$ is also finite. The finiteness of the quotients $C_1/\langle x \rangle$ and $C_2/\langle y \rangle$ then yields that $C_1/\langle x \rangle \cong C_2/\langle y \rangle$.

It follows that $C_2/\langle \psi(x) \rangle \cong C_1/\langle x \rangle \cong C_2/\langle y \rangle$, and so by Theorem 3.2 above there is an automorphism θ of C_2 with $\theta(y) = \psi(x)$. Now define $\chi : G \rightarrow G$ by $\chi = \psi^{-1}\theta \oplus \sigma$ and note that χ is then an automorphism of G with $\chi(y) = \psi^{-1}\theta(y) = x$. $\square$

In particular, we have established

Corollary 3.4. *Let G be a reduced semi-standard separable p -group, then G is quotient-transitive.*

Proof. Since G is separable, condition (i) of Lemma 3.3 holds for all $x \in G$ and so the result follows if G is unbounded. If G is bounded and semi-standard then it is finite and the result follows from Theorem 3.2. $\square$

The necessity of the condition that C be semi-standard follows from Example 2.1 in Section 2.

4. QUOTIENT-TRANSITIVE p -GROUPS

In this section we obtain a complete classification of separable torsion quotient-transitive p -groups. First we establish a general result.

Proposition 4.1. *Suppose that G is quotient-transitive [CS-transitive] and H is a summand of G which is weakly transitive, then H is also quotient-transitive [CS-transitive].*

Proof. We give a proof for quotient-transitivity, the simple modification to obtain the corresponding result for CS-transitive groups is left to the reader. So suppose that G is quotient-transitive and H is a weakly transitive summand of G , say $G = H \oplus A$. Now if x, y are nonzero elements of H with $H/\langle x \rangle \cong H/\langle y \rangle$, then $z = (x, 0), w = (y, 0)$ are nonzero elements of G with $G/\langle z \rangle \cong G/\langle w \rangle$ and so there exists an automorphism ϕ of G mapping $z \mapsto w$; the inverse of ϕ then maps $w \mapsto z$. So there is a matrix $\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ sending $z \mapsto w$. It follows that $\alpha(x) = y$ for the endomorphism $\alpha \in \text{End}(H)$. Similarly, using the inverse of ϕ , we get an endomorphism α_1 of H with $\alpha_1(y) = x$. Since H is weakly transitive there is an automorphism of H mapping $x \mapsto y$ and H is quotient-transitive, as required. $\square$

Our next example is elementary but useful; it is a simple extension of part of Example 2.1.

Example 4.2. If $G = A \oplus B$ where $A = \bigoplus_{\lambda} \mathbb{Z}(p^n)$, with λ infinite and $B = \langle b \rangle$ is cyclic of order p^m with $m > n$, then G is not quotient-transitive.

Proof. As $m > n \geq 1$, $p^n b \neq 0$ and so

$$G/\langle p^n b \rangle \cong A \oplus \mathbb{Z}(p^n) \cong A \cong G/\langle b \rangle$$

since λ is infinite. Clearly no automorphism of G can map $b \mapsto p^n b$, and G is not quotient-transitive. $\square$

An immediate consequence of Example 4.2 is the following:

Proposition 4.3. *If G is a quotient-transitive group and the Ulm invariant $f_G(n-1)$ is infinite for some integer $n \geq 1$, then $f_G(m) = 0$ for all integers $m \geq n$.*

Proof. Since $f_G(n-1)$ is infinite G has a summand of the form $\bigoplus_{\lambda} \mathbb{Z}(p^n)$ with λ infinite. If now $f_G(m) \neq 0$ for any $m \geq n$, then G also has a summand H of the form $\bigoplus_{\lambda} \mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$. Since the group H is clearly weakly transitive - it is even transitive - it follows from Proposition 4.1 that H is quotient-transitive contrary to Example 4.2. Thus $f_G(m) = 0$ for all $m \geq n$. $\square$

It is now easy to deduce a useful finiteness type result.

Proposition 4.4. *If G is a quotient-transitive p -group and $f_G(n-1)$ is infinite for some integer $n \geq 1$, then G is p^n -bounded.*

Proof. From Proposition 4.3 we have that $f_G(m) = 0$ for all integers $m \geq n$ which implies that a basic subgroup of G is p^n -bounded, whence G itself is p^n -bounded. $\square$

Corollary 4.5. *If G is a quotient-transitive p -group then either G is semi-standard or G is of the form $G = F \oplus \bigoplus_{\lambda} \mathbb{Z}(p^n)$, where λ is infinite and F is finite of exponent $< n$.*

Before proceeding to the classification of separable quotient-transitive p -groups we need a further result.

Proposition 4.6. *Suppose that $G = F \oplus B$ where F is finite, B is homocyclic of exponent n and of arbitrary rank, and $e(F) < n$, then G is quotient-transitive.*

Proof. Let $0 \neq x, y \in G$ with $G/\langle x \rangle \cong G/\langle y \rangle$. The $x = f + p^k b, y = f_1 + p^u b_1$ for some integers $k, u \geq 0$ and elements b, b_1 of height zero in B . Since b, b_1 generate summands of B , we may without loss in generality assume $b = b_1$. So $x = p^k b, y = p^u b$; let $B = \langle b \rangle \oplus B_1$ and set $X = \langle x \rangle, Y = \langle y \rangle$.

Now $G/\langle x \rangle = ((F \oplus \langle b \rangle)/\langle x \rangle) \oplus B_1 \cong G/\langle y \rangle = ((F \oplus \langle b \rangle)/\langle y \rangle) \oplus B_1$.

If $b+X$ and $b+Y$ both have order $< n$, it follows that $(F \oplus \langle b \rangle / \langle x \rangle)$ and $(F \oplus \langle b \rangle / \langle y \rangle)$ are both of exponent $< n$, then by comparing Ulm invariants one sees immediately that $(F \oplus \langle b \rangle / \langle x \rangle) \cong (F \oplus \langle b \rangle / \langle y \rangle)$. Since $F \oplus \langle b \rangle$ is finite, it follows from Theorem 3.2 that there is an automorphism of $F \oplus \langle b \rangle$, and hence of G , mapping $x \mapsto y$.

The next possibility is that both $b+X, b+Y$ have order $= p^n$. It follows from Lemma 4.7 below that $F \oplus \langle b \rangle / X = \mathbb{Z}(p^n) \oplus W$ while $F \oplus \langle b \rangle / Y = \mathbb{Z}(p^n) \oplus V$; here $W \cong F / \langle f \rangle, V \cong F / \langle f_1 \rangle$ and hence both are finite of order $\leq |F|$. Since W, V have exponents $\leq e(F) < n$, it again follows from $G/X \cong G/Y$ that $W \cong V$. Applying Theorem 3.2 again, we get an automorphism of G mapping $x \mapsto y$.

The remaining possibility is that one of $b+X, b+Y$ has order p^n while the other has order $< p^n$; say $o(b+X) = p^n, o(b+Y) < p^n$; we show that this case cannot arise. It follows from Lemma 4.7 below that $F \oplus \langle b \rangle / X = \mathbb{Z}(p^n) \oplus W$, with $W \cong F / \langle f \rangle$ while $F \oplus \langle b \rangle / Y$ is finite of exponent $< n$. So $G/X \cong \mathbb{Z}(p^n) \oplus W \oplus B_1 \cong (F \oplus \langle b \rangle / Y) \oplus B_1$. Comparison of Ulm invariants again gives

$$F / \langle f \rangle \cong W \cong (F \oplus \langle b \rangle / Y).$$

If $o(Y) < p^n$ then in the displayed equation above, the left-hand side has order $\leq |F|$ while the right-hand side has order $> |F|$ – contradiction.

So we conclude that $|Y| = o(y) = p^n$. But then we must have $|F / \langle f \rangle| = |F|$ which forces $f = 0$. Hence $p^k b = x$ and since, by assumption, $o(b+X) = p^n$ we are forced to conclude that $k \geq n$, whence $x = 0$ – contradiction. So this case does not arise. Since we have established the existence of an automorphism of G mapping $x \mapsto y$ in the other two cases, the proof will be complete once we have established Lemma 4.7. $\square$

Lemma 4.7. *Let $H = F \oplus \langle b \rangle$, where b is of order p^n and F is finite of exponent strictly less than n . If $a = f + p^k b \in H$ and $A = \langle a \rangle$, then either H/A is finite of exponent $< n$ or $H/A \cong \mathbb{Z}(p^n) \oplus (F / \langle f \rangle)$.*

Proof. The first possibility arises if $o(b+A) < p^n$, since in this situation all the generators of H/A have order $< p^n$. If, however, $o(b+A) = p^n$, then the group

generated by $b + A$ has order p^n and hence is a summand of H by [11, Lemma 4]. Thus H/A splits as $(\langle b \rangle + A/A) \oplus W$ and clearly $W \cong (F \oplus \langle b \rangle)/(\langle b \rangle + A) = (F \oplus \langle b \rangle)/\langle f \rangle \oplus \langle b \rangle \cong F/\langle f \rangle$. $\square$

The classification we sought is then:

Theorem 4.8. *A separable p -group G is quotient-transitive if, and only if, it is either semi-standard or of the form $G = F \oplus \bigoplus_{\lambda} \mathbb{Z}(p^n)$, where λ is arbitrary and F is finite of exponent $< n$.*

Proof. The necessity has already been established in Corollary 4.5. For the sufficiency we simply quote Proposition 3.4 and Proposition 4.6. $\square$

We finish this section with a short discussion of the problem of classifying reduced separable CS-transitive p -groups. In fact, such a classification seems rather difficult even in the simplified situation where we consider only direct sums of cyclic groups. We begin with a simple general result.

Proposition 4.9. *If G is a p -group and $m \geq 0$ is an integer such that the Ulm invariants $f_G(m)$ and $f_G(2m+1)$ are both infinite, then G is not CS-transitive. In particular, the group $\bigoplus_{\lambda} \mathbb{Z}(p) \oplus \bigoplus_{\mu} \mathbb{Z}(p^2)$ is not CS-transitive if λ, μ are both infinite.*

Proof. If both Ulm invariants are infinite then G has a decomposition of the form $G = A \oplus B \oplus H$, where $A = \bigoplus_{\lambda} \mathbb{Z}(p^{m+1})$, $B = \bigoplus_{\mu} \mathbb{Z}(p^{2m+2})$ and λ, μ are infinite. Suppose that x, z are canonical summands of A, B respectively and let $y = p^{m+1}z$. Then $o(x) = o(y)$ and we have $G/\langle x \rangle \cong G$ since λ is infinite. Furthermore, if $B = \langle z \rangle \oplus B_1$, then $G/\langle y \rangle \cong A \oplus H \oplus B_1 \oplus (\langle z \rangle/\langle p^{m+1}z \rangle) \cong G$. Since the heights of x and y differ, no automorphism of G can map $x \mapsto y$ and thus G is not CS-transitive. $\square$

Thus it is not true that arbitrary direct sums of cyclic p -groups are transitive on cyclic subgroups; this is in marked contrast to the situation pertaining in classical transitivity, where Kaplansky's theorem tells us that all such groups are transitive.

It also indicates that no direct analogue of Corner's use of endomorphism rings discussed in the Introduction, can hold.

Our next set of examples illustrates how CS-transitivity of direct sums of cyclic groups depends heavily on arithmetical properties of the non-zero Ulm invariants.

Example 4.10. Let $G = A \oplus B$, where $A = \bigoplus_{\omega} \langle e_i \rangle$, $B = \bigoplus_{\omega} \langle f_j \rangle$ and for all i, j $o(e_i) = p$, $o(f_j) = p^3$. Then G is CS-transitive.

Proof. Let x, y be arbitrary elements of G with $G/\langle x \rangle \cong G/\langle y \rangle$. Since G has exponent 3, there are just 3 classes of elements to consider:

(i) Elements of order p . There are essentially only three types here: those of the form e , those of the form $e + p^2 f$ and those of the form $p^2 f$, where $e \in A$, $f \in B$. Since all elements of the form $e, e + p^2 f$ have Ulm sequence equal to $(0, \infty, \dots)$, the classical theory of transitivity ensures that there is an automorphism of G interchanging such elements. Clearly then $G/\langle e \rangle \cong G$ and it will suffice to show that for the remaining case $G/\langle p^2 f \rangle$ is not isomorphic to G . This is immediate since $G/\langle p^2 f \rangle$ has a direct summand isomorphic to $\mathbb{Z}(p^2)$.

(ii) Elements of order p^2 . Here the situation again arises where there are possibilities of the form $e + pf, pf$, with f of order p^3 . However, if f is an arbitrary element of B of order p^3 , then $G/\langle pf \rangle \cong G$. On the other hand an easy calculation gives that $G/\langle e + pf \rangle \cong G \oplus \mathbb{Z}(p^2)$. So in this situation we never have isomorphic quotients to deal with.

(iii) Elements of order p^3 . In this case the only possibilities are that the elements are of the form $e + f, f$. However both types of element will have Ulm sequence $(0, 1, 2, \infty, \dots)$ and so there is, as above, an automorphism interchanging them.

Thus in all situations where $G/\langle x \rangle \cong G/\langle y \rangle$, we have an automorphism of G mapping $x \mapsto y$ and so G is CS-transitive. $\square$

Example 4.11. Let $G = \bigoplus_{\omega} \langle e_i \rangle \oplus \bigoplus_{\omega} \langle f_i \rangle \oplus \bigoplus_{\omega} \langle g_i \rangle$, where for all $i < \omega$, $o(e_i) = p^r$, $o(f_i) = p^s$ and $o(g_i) = p^t$ with $r < s < t$ and $r + t = 2s$. Then G is not CS-transitive.

Proof. Take $x = p^r f_0, y = p^s g_0$ so that $o(x) = p^{s-r} = p^{t-s} = o(y)$. Note that as x, y have different heights in G , there cannot be an automorphism of G mapping $x \mapsto y$. However, a straightforward calculation shows that $G/\langle x \rangle \cong G \cong G/\langle y \rangle$ and so G is not CS-transitive. $\square$

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