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1-D Schrödinger Operators with Local Point Interactions on a Discrete Set

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1-D Schrödinger operators with local point interactions on a discrete set

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ABSTRACT

Spectral properties of 1-D Schrödinger operators $H_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{x_n \in X} \alpha_n \delta(x - x_n)$ with local point interactions on a discrete set $X = \{x_n\}_{n=1}^\infty$ are well studied when $d_* := \inf_{n,k \in \mathbb{N}} |x_n - x_k| > 0$. Our paper is devoted to the case $d_* = 0$. We consider $H_{X,\alpha}$ in the framework of extension theory of symmetric operators by applying the technique of boundary triplets and the corresponding Weyl functions.

We show that the spectral properties of $H_{X,\alpha}$ like self-adjointness, discreteness, and lower semiboundedness correlate with the corresponding spectral properties of certain classes of Jacobi matrices. Based on this connection, we obtain necessary and sufficient conditions for the operators $H_{X,\alpha}$ to be self-adjoint, lower semibounded, and discrete in the case $d_* = 0$.

The operators with δ' -type interactions are investigated too. The obtained results demonstrate that in the case $d_* = 0$, as distinguished from the case $d_* > 0$, the spectral properties of the operators with δ - and δ' -type interactions are substantially different.

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1. Introduction

Differential operators with point interactions arise in various physical applications as exactly solvable models that describe complicated physical phenomena (numerous results as well as a comprehensive list of references may be found in [3,4,17,36]). An important class of such operators is formed by the differential operators with the coefficients having singular support on a disjoint set of points. The most known examples are the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ associated with the formal differential expressions

$$\ell_{X,\alpha,q} := -\frac{d^2}{dx^2} + q(x) + \sum_{x_n \in X} \alpha_n \delta_n, \quad \ell_{X,\beta,q} := -\frac{d^2}{dx^2} + q(x) + \sum_{x_n \in X} \beta_n (\cdot, \delta'_n) \delta'_n, \quad (1)$$

where $\delta_n := \delta(x - x_n)$ and δ is a Dirac delta-function. These operators describe δ - and δ' -interactions, respectively, on a discrete set $X = \{x_n\}_{n \in I} \subset \mathbb{R}$, and the coefficients $\alpha_n, \beta_n \in \mathbb{R}$ are called the strengths of the interaction at the point $x = x_n$. Investigation of these models was originated by Kronig and Penney [41] and Grossmann et al. [24] (see also [18]), respectively. In particular, the “Kronig–Penney model” ($\ell_{X,\alpha,q}$ with $X = \mathbb{Z}$, $\alpha_n \equiv \alpha$, and $q \equiv 0$) provides a simple model for a nonrelativistic electron moving in a fixed crystal lattice.

There are several ways to associate the operators with $\ell_{X,\alpha,q}$ and $\ell_{X,\beta,q}$. For example, a δ -interaction at a point $x = x_0$ may be defined using the *form method*, that is the operator $-\frac{d^2}{dx^2} + \alpha_0 \delta(x - x_0)$ is defined as an operator associated in $L^2(\mathbb{R})$ with the quadratic form

$$t[f] = \int_{\mathbb{R}} |f'(t)|^2 dt + \alpha_0 |f(x_0)|^2, \quad f \in W_2^1(\mathbb{R}).$$

Another way to introduce a local interaction at x_0 is to consider a symmetric operator $H_{\min} := H_{\min}^- \oplus H_{\min}^+$, where H_{\min}^- and H_{\min}^+ are the minimal operators generated by $-\frac{d^2}{dx^2}$ in $L^2(-\infty, x_0)$ and $L^2(x_0, +\infty)$, respectively, and to impose *boundary conditions* connecting x_{0+} and x_{0-} .

Both these methods have disadvantages if the set X is infinite. The form method works only for the case of lower semibounded operators. If we apply the method of boundary conditions, then the corresponding minimal operator H_{\min} has infinite deficiency indices and the description of self-adjoint extensions of H_{\min} is rather complicated problem in this case.

An alternative approach was proposed recently in [9] (see also [48] for the case of δ -type interactions). Namely, the operators with general local interactions on a discrete set X were defined as self-adjoint extensions such that the Lagrange brackets $[f, g] := \overline{f(x)g'(x)} - \overline{f'(x)g(x)}$ are continuous on \mathbb{R} for arbitrary elements f, g from the domain. It was shown in [9,48] that classical Sturm–Liouville theory with all its fundamental objects can be generalized to include local point interactions. In particular, Weyl’s alternative has been established in this case.

Nevertheless, to the best of our knowledge there are only a few results that describe the spectral properties of operators with local interactions in the case $d_* = 0$, where

$$d_* := \inf_{i,j \in I} |x_i - x_j| = 0. \tag{2}$$

Let us present a brief historical overview. Note that we are interested in the case when the set X is infinite (the case $|X| < \infty$ is considered in great detail in [3]). First we need some notation. Let \mathcal{I} be the semi-axis, $\mathcal{I} = [0, +\infty)$, and let $X = \{x_n\}_{n=1}^\infty \subset \mathcal{I}$ be a strictly increasing sequence, $x_{n+1} > x_n$, $n \in \mathbb{N}$, such that $x_n \rightarrow +\infty$. We denote $d_n := x_n - x_{n-1}$, $x_0 := 0$, and assume $q \in L^2_{\text{loc}}[0, +\infty)$.

In $L^2(\mathcal{I})$, the minimal symmetric operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ are naturally associated with (1). Namely, define the operators $H_{X,\alpha,q}^0$ and $H_{X,\beta,q}^0$ by the differential expression

$$\tau_q := -\frac{d^2}{dx^2} + q(x) \tag{3}$$

on the domains, respectively,

$$\text{dom}(H_{X,\alpha,q}^0) = \left\{ f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) : f'(0) = 0, \begin{matrix} f(x_{n+}) = f(x_{n-}) \\ f'(x_{n+}) - f'(x_{n-}) = \alpha_n f(x_n) \end{matrix}, n \in \mathbb{N} \right\}, \tag{4}$$

$$\text{dom}(H_{X,\beta,q}^0) = \left\{ f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) : f'(0) = 0, \begin{matrix} f'(x_{n+}) = f'(x_{n-}) \\ f(x_{n+}) - f(x_{n-}) = \beta_n f'(x_n) \end{matrix}, n \in \mathbb{N} \right\}. \tag{5}$$

Let $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ be the closures of $H_{X,\alpha,q}^0$ and $H_{X,\beta,q}^0$, respectively. In general, the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ are symmetric but not automatically self-adjoint, even in the case $q \equiv 0$.

Spectral analysis of $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ consists (at least partially) of the following problems:

- (a) Finding self-adjointness criteria for $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ and description of self-adjoint extensions if the deficiency indices $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ are nontrivial.
- (b) Lower semiboundedness of the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$.
- (c) Discreteness of the spectra of the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$.
- (d) Characterization of continuous, absolutely continuous, and singular parts of the spectra of the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$.
- (e) Resolvent comparability of the operators $H_{X,\alpha^{(1)},q}$ and $H_{X,\alpha^{(2)},q}$ with $\alpha^{(1)} \neq \alpha^{(2)}$.

In the present paper, we confine ourselves to the case of bounded potentials $q \in L^\infty(\mathcal{I})$. Let us note that the case of unbounded q was studied in [7,19,20,53,9] and the case of q being a $W_{\text{loc}}^{2,-1}(\mathcal{I})$

distribution was studied in [25,26,51,52] (see also references therein). More precisely, it is shown in [9] (see also [48]) that $n_{\pm}(H_{X,\alpha,q}) \leq 1$ and the deficiency indices may be characterized in terms of the limit point and the limit circle classification for the endpoint $x = +\infty$. Brasche [7, Theorem 1] proved that $H_{X,\alpha,q}$ is self-adjoint and lower semibounded if the potential q is lower semibounded and the strengths $\alpha_n, n \in \mathbb{N}$, are nonnegative. Assuming the condition $d_* > 0$, Gesztesy and Kirsch [19], Shubin Christ and Stolz [53] (see also [9]) established self-adjointness of $H_{X,\alpha,q}$ for several classes of unbounded potentials q . In particular, Gesztesy and Kirsch [19, Theorem 3.1] proved that $H_{X,\alpha,q} = H_{X,\alpha,q}^*$ if $q \in L^\infty(\mathcal{I})$ and $d_* > 0$ (other proofs are given in [34] and [53]). Moreover, Shubin Christ and Stolz [53, pp. 495–496] showed that the condition $d_* > 0$ cannot be dropped there even if $q \equiv 0$. More precisely, they proved that $n_{\pm}(H_{X,\alpha,0}) = 1$ if $d_n = \frac{1}{n}$ and $\alpha_n = -2n - 1, n \in \mathbb{N}$. Note also that self-adjointness of $H_{X,\alpha,0}$ with arbitrary $X = \{x_n\}_{n=1}^\infty \subset \mathcal{I}$ was erroneously stated without proof in [44].

Finally, we emphasize that in contrast to δ -type interactions the operator $H_{X,\beta,0}$ is self-adjoint for arbitrary $\{\beta_n\}_{n=1}^\infty \subset \mathbb{R}$ (see [9, Theorem 4.7]). Let us also mention the recent papers [5,49] dealing with spectral properties of Hamiltonians with δ' -interactions on compact subsets of \mathbb{R} having Lebesgue measure zero.

In the present paper, we investigate problems (a)–(c) and (e) in the case $d_* = 0$ and $q \in L^\infty(\mathbb{R})$ (we postpone the study of the case of unbounded q as well as the problem (d) to our forthcoming paper). We consider the operators with point interactions in the framework of extension theory of symmetric operators. This approach allows one to treat the operators $H_{X,\alpha,q}$ and $H_{X,\beta,q}$ as self-adjoint (or symmetric) extensions of the minimal operator

$$H_{\min} := \bigoplus_{n \in \mathbb{N}} H_n, \quad H_n = -\frac{d^2}{dx^2} + q(x), \quad \text{dom}(H_n) = W_{0,2}^{2,2}[x_{n-1}, x_n], \quad (6)$$

being a direct sum of symmetric operators H_n with deficiency indices $n_{\pm}(H_n) = 2$. Let us mention the papers [16,42,55,56] dealing with direct sums of self-adjoint operators generated by quasi-differential expressions and describing other solvable models.

We investigate these operators by applying the technique of boundary triplets and the corresponding Weyl functions (see Section 2 for precise definitions). This new approach to extension theory of symmetric operators has been appeared and elaborated during the last three decades (see [22,13,14, 8] and references therein). The main ingredient is an abstract version of the Green formula for the adjoint A^* of a symmetric operator A (see formula (12)). A boundary triplet for A^* always exists whenever $n_+(A) = n_-(A)$, though it is not unique. Its role in extension theory is similar to that of a coordinate system in analytic geometry. It enables one to describe self-adjoint extensions in terms of (abstract) boundary conditions in place of the second J. von Neumann formula, though this description is simple and adequate only for a suitable choice of a boundary triplet. Note that construction of a suitable boundary triplet is a rather difficult problem if $n_{\pm}(A) = \infty$.

This approach was first applied to the spectral analysis of $H_{X,\alpha,q}$ by Kochubei in [35]. More precisely, he proved that in the case $d_* > 0$ (and $q \in L^\infty(\mathcal{I})$) a boundary triplet Π for H_{\min}^* can be chosen as a direct sum of triplets Π_n defined by (78), that is $\Pi := \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \bigoplus_{n=1}^\infty \Pi_n$, where

$$\mathcal{H} := \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n, \quad \Gamma_0 := \bigoplus_{n \in \mathbb{N}} \Gamma_0^{(n)}, \quad \Gamma_1 := \bigoplus_{n \in \mathbb{N}} \Gamma_1^{(n)}. \quad (7)$$

Based on this construction, he gave an alternative proof of the self-adjointness of $H_{X,\alpha,0}$ (see [19, Theorem 3.1]) and investigated the problem (e) as well.

The main difficulty in extending this approach to the case $d_* = 0$ (or unbounded q) is the construction of a suitable boundary triplet for the operator H_{\min}^* (see [34,35]). It looks natural that the triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ defined by (7) and (78) forms a boundary triplet for H_{\min}^* in this case too. Indeed, Green's identity holds for $f, g \in \text{dom}(H_{\min}^*)$ with compact supports in \mathcal{I} . However, $\text{dom}(\Gamma_0) \cap \text{dom}(\Gamma_1)$ is only a proper part of $\text{dom}(H_{\min}^*)$ and the boundary mapping $\Gamma := \{\Gamma_0, \Gamma_1\}$ cannot be extended

onto $\text{dom}(H_{\min}^*)$ if $d_* = 0$. In this case, erroneous construction of a boundary triplet for H_{\min}^* was announced in [44,45] (see Remark 4.2). Note also that the first example of the operator (6) with $q \notin L^\infty$ and such that Π is not a boundary triplet for H_{\min}^* was given in [34].

Recently Neidhardt and one of the authors proved that the triplet of the form (7) becomes a boundary triplet after appropriate regularization of the mappings $\Gamma_0^{(n)}$ and $\Gamma_1^{(n)}$, $n \in \mathbb{N}$ (see [43, Theorem 5.3]). Starting with this result, we investigate the problem in full generality. More precisely, we show that in general $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of the form (7) is only a boundary relation in the sense of [12] and we find a criterion for Π to form a boundary triplet for H_{\min}^* . Moreover, we present a general regularization procedure that enables us to construct a suitable boundary triplet Π for H_{\min}^* in the form $\Pi = \bigoplus_{n=1}^\infty \Pi_n$. Namely, in this boundary triplet the sets of Hamiltonians $H_{X,\alpha,0}$ and $H_{X,\beta,0}$ are parameterized by means of certain classes of Jacobi (tri-diagonal) matrices (the construction from [43] leads to multi-diagonal matrices). In turn, the latter leads to a correlation between spectral properties of the Hamiltonians (1) and the corresponding Jacobi matrices. Note that another technique for analyzing spectral properties of $H_{X,\alpha,0}$ and $H_{X,\beta,0}$ by means of second order difference operators was proposed by Phariseau [50] (see also [3, Chapter III.2.1]).

More precisely, in the case of δ -interactions, we show that the spectral properties of the operator $H_{X,\alpha,0}$ are closely connected with the corresponding spectral properties of the Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} r_1^{-2}(\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}) & (r_1 r_2 d_2)^{-1} & 0 & \dots \\ (r_1 r_2 d_2)^{-1} & r_2^{-2}(\alpha_2 + \frac{1}{d_2} + \frac{1}{d_3}) & (r_2 r_3 d_3)^{-1} & \dots \\ 0 & (r_2 r_3 d_3)^{-1} & r_3^{-2}(\alpha_3 + \frac{1}{d_3} + \frac{1}{d_4}) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{8}$$

where $r_n = \sqrt{d_n + d_{n+1}}$, $n \in \mathbb{N}$. We first show that $n_\pm(H_{X,\alpha,0}) = n_\pm(B_{X,\alpha})$ (Theorem 5.4) and hence that $n_\pm(H_{X,\alpha,0}) \leq 1$ (cf. [48,9]). Combining this with the Carleman criterion, we arrive at the following result (see Proposition 5.7):

the operator $H_{X,\alpha,q}$ with δ -interactions is self-adjoint for any $\alpha = \{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ provided that

$$\sum_{n \in \mathbb{N}} d_n^2 = \infty \quad \text{and} \quad q \in L^\infty(\mathcal{I}).$$

This result is sharp. Namely (see Proposition 5.9):

if $\sum_{n \in \mathbb{N}} d_n^2 < \infty$ and $X = \{x_n\}_{n \in \mathbb{N}}$ satisfies also some concave assumptions, then there exists $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ such that the operator $H_{X,\alpha,0}$ is symmetric with $n_\pm(H_{X,\alpha,0}) = 1$.

Moreover, we show that the equality $n_\pm(H_{X,\alpha,0}) = 1$ yields that the strengths α_n cannot tend to ∞ very fast (Proposition 5.11). This situation is illustrated by Example 5.12 (see also Proposition 5.13). More precisely, let $H_{X,\alpha,0}$ be the minimal closed symmetric operator associated with the differential expression $\ell_{X,\alpha,0}$, where $\mathcal{I} = \mathbb{R}_+$ and $X = \{x_n\}_{n \in \mathbb{N}}$ is defined by $d_n = x_n - x_{n-1} := \frac{1}{n}$, $n \in \mathbb{N}$. Then

- (i) $n_\pm(H_{X,\alpha,0}) = 0$ if either $\alpha_n \leq -(4n + 2) + O(n^{-1})$ or $\alpha_n \geq -Cn^{-1}$ with some $C > 0$,
- (ii) $n_\pm(H_{X,\alpha,0}) = 1$ if $\alpha_n = -a(4n + 2) + O(n^{-1})$ with $a \in (0, 1)$.

The latter enables us to construct a positive potential $q > 0$ (see Section 7) such that the operator $H_{X,\alpha,q}$ with $\alpha_n = -4n - 2$ and $d_n = x_n - x_{n-1} = 1/n$ is symmetric with $n_\pm(H_{X,\alpha,q}) = 1$. This shows that self-adjointness of $H_{X,\alpha,0}$ is not stable under positive perturbations in the case $d_* = 0$ (in the case $d_* > 0$, it was shown in [19, Theorem 3.1] that self-adjointness of $H_{X,\alpha,0}$ is stable under perturbations by positive potentials q).

Further, in the case $d_* = 0$ we solve the problems (b) and (c) in terms of the Jacobi operators (8). Namely, we show that the operator $H_{X,\alpha,0}$ is lower semibounded if and only if the operator $B_{X,\alpha}$ is also lower semibounded. As for discreteness of the spectrum of $H_{X,\alpha,0}$, we first note that any self-adjoint extension of $H_{X,\alpha,0}$ has discrete spectrum whenever $n_\pm(H_{X,\alpha,0}) = 1$. In the case $H_{X,\alpha,0} = H_{X,\alpha,0}^*$, the operator $H_{X,\alpha,0}$ has discrete spectrum if and only if $d_n \rightarrow 0$ and $B_{X,\alpha}$ is discrete (Theorem 5.17).

Using recent advances in the spectral theory of unbounded Jacobi operators (see [29,30,11]), we obtain necessary and sufficient conditions for discreteness and lower semiboundedness of the operator $H_{X,\alpha,0}$ in the case $d_* = 0$. We show that condition

$$\frac{\alpha_n}{d_n + d_{n+1}} \geq C, \quad n \in \mathbb{N}, \text{ for some } C \in \mathbb{R}, \tag{9}$$

is sufficient for semiboundedness. If $d_* > 0$, then (9) reads $\inf_{n \in \mathbb{N}} \alpha_n > -\infty$ and it is also necessary (see [7] and also Corollary 5.25). If $d_* = 0$, then the situation becomes more complicated. In Proposition 5.28, we show that the operator $H_{X,\alpha,0}$ might be non-semibounded even if $\alpha_n \rightarrow 0$.

Further (see Proposition 5.18), *the operator $H_{X,\alpha,0} = H_{X,\alpha,0}^*$ is discrete provided that*

$$\lim_{n \rightarrow \infty} d_n = 0, \quad \lim_{n \rightarrow \infty} \frac{|\alpha_n|}{d_n} = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{d_n \alpha_n} > -\frac{1}{4}. \tag{10}$$

The third condition in (10) is sharp (cf. Remark 5.27). Besides, (10) implies that $H_{X,\alpha,0}$ may be discrete if $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ is bounded. Also (10) enables us to construct operators, which are discrete but not lower semibounded. For instance, *the operator $H_C = -\frac{d^2}{dx^2} - \sum_{n \in \mathbb{N}} C \sqrt{n} \delta(x - \sqrt{n})$ with $C > 8$ has discrete spectrum though it is not lower semibounded* (other approaches for constructing non-lower semibounded Schrödinger operators with discrete spectra can be found in [28,23]).

Let us stress that the spectral properties of the operators $H_{X,\alpha,0}$ and $H_{X,\beta,0}$ are completely different in the case $d_* = 0$. This becomes clear because of the structure of the boundary operators $B_{X,\alpha}$ and $B_{X,\beta}$ that parameterize the Hamiltonians $H_{X,\alpha,0}$ and $H_{X,\beta,0}$, respectively. Namely, we show that the spectral properties of the operator with δ' -interactions are closely connected with the Jacobi matrix

$$B_{X,\beta} := R_X^{-1/2}(I + U)B_\beta^{-1}(I + U^*)R_X^{-1/2}, \quad B_\beta = \text{diag}(-\beta_n - d_n), \quad R_X = \text{diag}(d_n), \tag{11}$$

and U is unilateral shift on $l_2(\mathbb{N})$. On the other hand, the operator (11) is closely connected with the Krein string spectral theory (see Section 2.2). Namely, in the case when $\beta_n + d_n > 0, n \in \mathbb{N}$, the difference expression associated with (11) describes the motion of the nonhomogeneous string with the mass distribution

$$M_\beta(x) = \sum_{x_{n-1} < x} d_n, \quad x \geq 0; \quad x_n - x_{n-1} = \beta_n + d_n, \quad x_0 = 0.$$

Based on this connection, we obtain the following criteria for the operator $H_{X,\beta,0}$ to be self-adjoint, lower semibounded, and discrete² (Theorem 6.3 and Propositions 6.9, 6.11 and 6.16):

(a) $H_{X,\beta,0}$ is self-adjoint if and only if either $\mathcal{I} = \mathbb{R}_+$ or

$$\sum_{n \in \mathbb{N}} \left[d_{n+1} \sum_{i=1}^n (\beta_i + d_i)^2 \right] = \infty.$$

(b) For the operator $H_{X,\beta,0}$ to be lower semibounded it is necessary that

$$\frac{1}{\beta_n} \geq -C_1 d_n - \frac{1}{d_n} \quad \text{and} \quad \frac{1}{\beta_n} \geq -C_1 d_{n+1} - \frac{1}{d_{n+1}}, \quad n \in \mathbb{N},$$

² Here we can consider the case when \mathcal{I} is a bounded interval.

and it is sufficient that

$$\frac{1}{\beta_n} \geq -C_2 \min\{d_n, d_{n+1}\}, \quad n \in \mathbb{N},$$

with some positive constants $C_1, C_2 > 0$ independent of $n \in \mathbb{N}$.

(c1) Let $\mathcal{I} = \mathbb{R}_+$. The spectrum of $H_{X,\beta,0}$ is not discrete if one of the following conditions hold

- $\lim_{n \rightarrow \infty} x_n \sum_{j=n}^{\infty} d_j^3 > 0$,
- $\beta_n \geq -Cd_n^3, n \in \mathbb{N}, C > 0$,
- $\beta_n^- \leq -C(d_n^{-1} + d_{n+1}^{-1}), n \in \mathbb{N}$ ($\beta_n^- := \beta_n$ if $\beta_n < 0$ and $\beta_n^- := -\infty$ if $\beta_n > 0$).

(c2) If $d_n + \beta_n \geq 0$ for all $n \in \mathbb{N}$, then the spectrum of $H_{X,\beta,0}$ is discrete if and only if

$$\lim_{n \rightarrow \infty} x_n \sum_{j=n}^{\infty} d_j^3 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n \sum_{j=n}^{\infty} (\beta_j + d_j) = 0.$$

Note that (a) and (c2) follow, respectively, from Hamburger’s theorem and the Kac–Krein discreteness criterion for the operator (11). The results are demonstrated by Example 6.14.

In conclusion let us briefly describe the content of the paper.

Section 2 is preparatory. It contains necessary definitions and statements on theory of boundary triplets of symmetric operators and the Krein string spectral theory.

In Section 3, for arbitrary family of symmetric operators $\{S_n\}_{n \in \mathbb{N}}$, we investigate a direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets Π_n for $S_n^*, n \in \mathbb{N}$. We obtain two criteria for Π to form a boundary triplet for the operator $A^* = \bigoplus_1^{\infty} S_n^*$ and regularization procedures for Π_n are given.

Sections 4–6 are devoted to the spectral analysis of operators with δ - and δ' -interactions on a discrete set X . We confine ourselves to the case $q \in L^\infty$. In Section 4, we construct boundary triplets for the operator H_{\min}^* . Spectral analysis of the Hamiltonians $H_{X,\alpha,0}$ and $H_{X,\beta,0}$ are provided in Sections 5 and 6, respectively. More precisely, we study self-adjointness of the minimal operators $H_{X,\alpha,0}$ and $H_{X,\beta,0}$, discreteness of their spectra, and their lower semiboundedness.

In Section 7, we show that self-adjointness of the operator $H_{X,\alpha,q}$ with δ -interactions is not stable under perturbation by positive unbounded potentials if $d_* = 0$.

Preliminary version of this paper was published as a preprint [37].

Notation. $\mathfrak{H}, \mathcal{H}$ stand for the separable Hilbert spaces. $[\mathfrak{H}, \mathcal{H}]$ denotes the set of bounded operators from \mathfrak{H} to \mathcal{H} ; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$ and $\mathfrak{S}_p(\mathfrak{H}), p \in (0, \infty)$, is the Neumann–Schatten ideal in $[\mathfrak{H}]$. $\mathcal{C}(\mathfrak{H})$ and $\tilde{\mathcal{C}}(\mathfrak{H})$ are the sets of closed operators and linear relations in \mathfrak{H} , respectively. Let T be a linear operator in a Hilbert space \mathfrak{H} . In what follows, $\text{dom}(T), \ker(T), \text{ran}(T)$ are the domain, the kernel, the range of T , respectively; $\sigma(T), \rho(T)$, and $\hat{\rho}(T)$ denote the spectrum, the resolvent set, and the set of regular type points of T , respectively; $R_T(\lambda) := (T - \lambda I)^{-1}, \lambda \in \rho(T)$, is the resolvent of T .

Let X be a discrete subset of $\mathcal{I} \subseteq \mathbb{R}$. By $W^{2,2}(\mathcal{I} \setminus X), W_0^{2,2}(\mathcal{I} \setminus X)$, and $W_{\text{loc}}^{2,2}(\mathcal{I} \setminus X)$ we denote the Sobolev spaces

$$\begin{aligned} W^{2,2}(\mathcal{I} \setminus X) &:= \{f \in L^2(\mathcal{I}): f, f' \in AC_{\text{loc}}(\mathcal{I} \setminus X), f'' \in L^2(\mathcal{I})\}, \\ W_0^{2,2}(\mathcal{I} \setminus X) &:= \{f \in W^{2,2}(\mathcal{I}): f(x_k) = f'(x_k) = 0, \text{ for all } x_k \in X\}, \\ W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) &:= \{f \in W^{2,2}(\mathcal{I} \setminus X): \text{supp } f \text{ is compact in } \mathcal{I}\}. \end{aligned}$$

Let I be a subset of \mathbb{Z} , $I \subseteq \mathbb{Z}$. We denote by $l_2(I, \mathcal{H})$ the Hilbert space of \mathcal{H} -valued sequences such that $\|f\|^2 = \sum_{n \in I} \|f_n\|_{\mathcal{H}}^2 < \infty$; $l_{2,0}(I, \mathcal{H})$ is the set of sequences with only finitely many values being nonzero; we also abbreviate $l_2 := l_2(\mathbb{N}, \mathbb{C})$, $l_{2,0} := l_{2,0}(\mathbb{N}, \mathbb{C})$.

2. Preliminaries

2.1. Boundary triplets and Weyl functions

In this section we briefly review the notion of abstract boundary triplets and associated Weyl functions in the extension theory of symmetric operators (we refer to [13,14,22] for a detailed study of boundary triplets).

2.1.1. Linear relations, boundary triplets, and self-adjoint extensions

1. The set $\tilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in \mathcal{H} is the set of closed linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. Recall that $\text{dom}(\Theta) = \{f: \{f, f'\} \in \Theta\}$, $\text{ran}(\Theta) = \{f': \{f, f'\} \in \Theta\}$, and $\text{mul}(\Theta) = \{f': \{0, f'\} \in \Theta\}$ are the domain, the range, and the multi-valued part of Θ . A closed linear operator A in \mathcal{H} is identified with its graph $\text{gr}(A)$, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in \mathcal{H} is viewed as a subset of $\tilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator if and only if $\text{mul}(\Theta)$ is trivial. For the definition of the inverse, the resolvent set and the spectrum of linear relations we refer to [15]. We recall that the adjoint relation $\Theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$ of $\Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \{\{h, h'\}: (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \text{ for all } \{f, f'\} \in \Theta\}.$$

A linear relation Θ is said to be *symmetric* if $\Theta \subset \Theta^*$ and *self-adjoint* if $\Theta = \Theta^*$.

For a symmetric linear relation $\Theta \subset \Theta^*$ in \mathcal{H} the multi-valued part $\text{mul}(\Theta)$ is the orthogonal complement of $\text{dom}(\Theta)$ in \mathcal{H} . Setting $\mathcal{H}_{\text{op}} := \overline{\text{dom}(\Theta)}$ and $\mathcal{H}_{\infty} = \text{mul}(\Theta)$, one arrives at the orthogonal decomposition $\Theta = \Theta^{\text{op}} \oplus \Theta^{\infty}$, where Θ^{op} is a symmetric operator in \mathcal{H}_{op} and is called *the operator part* of Θ , and $\Theta^{\infty} = \{\{0, f'\}: f' \in \text{mul}(\Theta)\}$ is a “pure” linear relation in \mathcal{H}_{∞} .

2. Let A be a densely defined closed symmetric operator in the separable Hilbert space \mathfrak{H} with equal deficiency indices $n_{\pm}(A) = \dim \mathfrak{N}_{\pm i} \leq \infty$, $\mathfrak{N}_z := \ker(A^* - z)$.

Definition 2.1. (See [22].) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *boundary triplet* for the adjoint operator A^* if \mathcal{H} is a Hilbert space and $\Gamma_0, \Gamma_1: \text{dom}(A^*) \rightarrow \mathcal{H}$ are bounded linear mappings such that the abstract Green identity

$$(A^* f, g)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A^*), \tag{12}$$

holds and the mapping $\Gamma := \{\Gamma_0, \Gamma_1\}: \text{dom}(A^*) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

First note that a boundary triplet for A^* exists since the deficiency indices of A are assumed to be equal. Moreover, $n_{\pm}(A) = \dim(\mathcal{H})$ and $A = A^* \upharpoonright (\ker(\Gamma_0) \cap \ker(\Gamma_1))$ hold. Note also that a boundary triplet for A^* is not unique.

A closed extension \tilde{A} of A is called *proper* if $A \subseteq \tilde{A} \subseteq A^*$. Two proper extensions \tilde{A}_1 and \tilde{A}_2 of A are called *disjoint* if $\text{dom}(\tilde{A}_1) \cap \text{dom}(\tilde{A}_2) = \text{dom}(A)$ and *transversal* if in addition $\text{dom}(\tilde{A}_1) \dot{+} \text{dom}(\tilde{A}_2) = \text{dom}(A^*)$. The set of proper extensions of A is denoted by $\text{Ext } A$. Fixing a boundary triplet Π one can parameterize the set $\text{Ext } A$ in the following way.

Proposition 2.2. (See [14].) Let A be as above and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . Then the mapping

$$(\text{Ext } A \ni) \tilde{A} \rightarrow \Gamma \text{ dom}(\tilde{A}) = \{\{\Gamma_0 f, \Gamma_1 f\}: f \in \text{dom}(\tilde{A})\} =: \Theta \in \tilde{\mathcal{C}}(\mathcal{H}) \tag{13}$$

establishes a bijective correspondence between the sets $\text{Ext } A$ and $\tilde{\mathcal{C}}(\mathcal{H})$. We put $A_{\Theta} := \tilde{A}$ where Θ is defined by (13), i.e. $A_{\Theta} := A^* \upharpoonright \Gamma^{-1} \Theta = A^* \upharpoonright \{\{f \in \text{dom}(A^*): \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}$. Then:

- (i) A_Θ is symmetric (self-adjoint) if and only if Θ is symmetric (self-adjoint), and $n_\pm(A_\Theta) = n_\pm(\Theta)$ holds.
- (ii) The extensions A_Θ and A_0 are disjoint (transversal) if and only if $\Theta \in \mathcal{C}(\mathcal{H})$ ($\Theta \in [\mathcal{H}]$). In this case A_Θ admits a representation $A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta \Gamma_0)$.

It follows immediately from Proposition 2.2 that the extensions $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ are self-adjoint. Clearly, $A_j = A_{\Theta_j}$ ($j = 0, 1$), where the subspaces $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$ are self-adjoint relations in \mathcal{H} . Note that Θ_0 is a “pure” linear relation.

2.1.2. Weyl functions, γ -fields, and Krein type formula for resolvents

1. In [13,14] the concept of the classical Weyl–Titchmarsh m -function from the theory of Sturm–Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh m -function in the spectral theory of singular Sturm–Liouville operators.

Definition 2.3. (See [13].) Let A be a densely defined closed symmetric operator in \mathfrak{H} with equal deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . The operator-valued functions $\gamma : \rho(A_0) \rightarrow [\mathcal{H}, \mathfrak{H}]$ and $M : \rho(A_0) \rightarrow [\mathcal{H}]$ defined by

$$\gamma(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0), \tag{14}$$

are called the γ -field and the Weyl function, respectively, corresponding to the boundary triplet Π .

The γ -field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (14) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations hold (see [13])

$$\gamma(z) = (I + (z - \zeta)(A_0 - z)^{-1})\gamma(\zeta), \tag{15}$$

$$M(z) - M(\zeta)^* = (z - \bar{\zeta})\gamma(\zeta)^*\gamma(z), \tag{16}$$

$$\gamma^*(\bar{z}) = \Gamma_1(A_0 - z)^{-1}, \quad z, \zeta \in \rho(A_0). \tag{17}$$

Identity (16) yields that $M(\cdot)$ is an $R_{\mathcal{H}}$ -function (or Nevanlinna function), that is, $M(\cdot)$ is an ($[\mathcal{H}]$ -valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ and

$$\text{Im } z \cdot \text{Im } M(z) \geq 0, \quad M(z^*) = M(\bar{z}), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{18}$$

Besides, it follows from (16) that $M(\cdot)$ satisfies $0 \in \rho(\text{Im } M(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Since A is densely defined, $M(\cdot)$ admits an integral representation (see, for instance, [14])

$$M(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma_M(t), \quad z \in \rho(A_0), \tag{19}$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on \mathbb{R} satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (19) is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators the measure $\Sigma_M(\cdot)$ is not necessarily orthogonal. However, the measure Σ_M is uniquely determined by the Nevanlinna function $M(\cdot)$. The operator-valued measure Σ_M is called the spectral measure of $M(\cdot)$. If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence (see [14, 40]). Due to this fact, spectral properties of A_0 can be expressed in terms of $M(\cdot)$.

2. The following result provides a description of resolvents and spectra of proper extensions of the operator A in terms of the Weyl function $M(\cdot)$ and the corresponding boundary parameters.

Proposition 2.4. (See [13].) For any $\Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ the following Krein type formula holds

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\bar{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \quad (20)$$

Moreover, if $z \in \rho(A_0)$, then

$$z \in \sigma_i(A_\Theta) \iff 0 \in \sigma_i(\Theta - M(z)), \quad i \in \{p, c, r\}.$$

Formula (20) is a generalization of the well-known Krein formula for canonical resolvents (cf. [2]). We note also that all objects in (20) are expressed in terms of the boundary triplet Π .

The following result is deduced from (20).

Proposition 2.5. (See [13].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $\Theta_1, \Theta_2 \in \tilde{\mathcal{C}}(\mathcal{H})$. Then:

(i) For any $z \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})$, $\zeta \in \rho(\Theta_1) \cap \rho(\Theta_2)$ the following equivalence holds

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (\Theta_1 - \zeta)^{-1} - (\Theta_2 - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \quad (21)$$

(ii) If, in addition, $\Theta_1, \Theta_2 \in \mathcal{C}(\mathcal{H})$ and $\text{dom}(\Theta_1) = \text{dom}(\Theta_2)$, then

$$\overline{\Theta_1 - \Theta_2} \in \mathfrak{S}_p(\mathcal{H}) \implies (A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}). \quad (22)$$

(iii) Moreover, if $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then implication (22) becomes equivalence.

2.1.3. Extensions of a nonnegative operator

Assume that a symmetric operator $A \in \mathcal{C}(\mathfrak{H})$ is nonnegative. Then the set $\text{Ext}_A(0, \infty)$ of its nonnegative self-adjoint extensions is non-empty (see [2,33]). Moreover, there is a maximal nonnegative extension A_F (also called Friedrichs' or hard extension) and there is a minimal nonnegative extension A_K (Krein's or soft extension) satisfying

$$(A_F + x)^{-1} \leq (\tilde{A} + x)^{-1} \leq (A_K + x)^{-1}, \quad x \in (0, \infty), \quad \tilde{A} \in \text{Ext}_A(0, \infty)$$

(for details we refer the reader to [2,22]).

Proposition 2.6. (See [13].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 = A_0^* \geq 0$. Let $M(\cdot)$ be the corresponding Weyl function. Then $A_0 = A_F$ ($A_0 = A_K$) if and only if

$$\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty \quad \left(\lim_{x \uparrow 0} (M(x)f, f) = +\infty \right), \quad f \in \mathcal{H} \setminus \{0\}. \quad (23)$$

It is said that $M(\cdot)$ uniformly tends to $-\infty$ for $x \rightarrow -\infty$ if for any $a > 0$ there exists $x_a < 0$ such that $M(x_a) < -a \cdot I_{\mathcal{H}}$. In this case we will write $M(x) \rightrightarrows -\infty, x \rightarrow -\infty$.

Proposition 2.7. (See [13].) Let A be a nonnegative symmetric operator in \mathfrak{H} . Assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* such that $A_0 = A_F$, and let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions

- (i) a linear relation $\Theta \in \tilde{\mathcal{C}}_{\text{self}}(\mathcal{H})$ is semibounded below,
- (ii) a self-adjoint extension A_Θ is semibounded below,

are equivalent if and only if $M(x) \rightrightarrows -\infty$ for $x \rightarrow -\infty$.

2.1.4. Generalized boundary triplets and boundary relations

In many applications the notion of a boundary triplet is too strong. Therefore it makes sense to relax its definition. To do this we follow [14, Section 6].

Definition 2.8. (See [14].) Let A be a closed densely defined symmetric operator in \mathfrak{H} with equal deficiency indices. Let $A_* \supseteq A$ be a not necessarily closed extension of A such that $(A_*)^* = A$. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triplet for A^** if \mathcal{H} is a Hilbert space and $\Gamma_j: \text{dom}(A_*) \rightarrow \mathcal{H}, j = 0, 1$, are linear mappings such that

- (G1) Γ_0 is surjective,
- (G2) $A_{*0} := A_* \upharpoonright \ker(\Gamma_0)$ is a self-adjoint operator,
- (G3) Green's formula holds

$$(A_* f, g)_{\mathfrak{H}} - (f, A_* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A_*) = \text{dom}(\Gamma). \quad (24)$$

Note that one always has $A \subseteq A_* \subseteq A^* = \overline{A_*}$. The following properties of a generalized boundary triplet have been established in [14].

Lemma 2.9. (See [14].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^* . Then:

- (i) $\mathfrak{N}_z^* := \text{dom}(A_*) \cap \mathfrak{N}_z$ is dense in \mathfrak{N}_z and $\text{dom}(A_*) = \text{dom}(A_0) + \mathfrak{N}_z^*$.
- (ii) $\overline{\Gamma_1 \text{dom}(A_0)} = \mathcal{H}$.
- (iii) $\ker(\Gamma) = \text{dom}(A)$ and $\overline{\text{ran}(\Gamma)} = \mathcal{H} \oplus \mathcal{H}$, where $\Gamma := \{\Gamma_0, \Gamma_1\}$.

For any generalized boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ we set $A_{*j} := A^* \upharpoonright \ker(\Gamma_j), j = 0, 1$. Note that the extensions A_{*0} and A_{*1} are always disjoint but not necessarily transversal.

Starting with Definition 2.8, one can introduce concepts of the (generalized) γ -field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ corresponding to a generalized boundary triplet Π in just the same way as it was done for (ordinary) boundary triplets (for details see [14]). Let us mention only the following proposition (cf. [14, Proposition 6.2]).

Proposition 2.10. (See [14].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^* , $A_* = A^* \upharpoonright \text{dom}(\Gamma)$, and let $M(\cdot)$ be the corresponding Weyl function. Then:

- (i) $M(\cdot)$ is an $[\mathcal{H}]$ -valued Nevanlinna function satisfying $\ker(\text{Im } M(z)) = \{0\}, z \in \mathbb{C}_+$.
- (ii) Π is an ordinary boundary triplet if and only if $0 \in \rho(\text{Im } M(i))$.

We also need the following definition.

Definition 2.11. (See [12].) Let A be as in Definition 2.8 and let \mathcal{H} be an auxiliary Hilbert space. A linear relation (multi-valued mapping) $\Gamma: \mathfrak{H} \rightarrow \mathcal{H}^2$ is called a boundary relation for A^* if:

- (i) $\text{dom}(\Gamma)$ is dense in $\text{dom}(A^*)$, and identity

$$(A_* f, g)_{\mathfrak{H}} - (f, A_* g)_{\mathfrak{H}} = (l', h)_{\mathcal{H}} - (l, h')_{\mathcal{H}}, \quad (25)$$

where $A_* = A^* \upharpoonright \text{dom}(\Gamma)$, holds for every $\{f, \hat{l}\}, \{g, \hat{h}\} \in \Gamma$,

- (ii) Γ is maximal in the sense that if $\{\hat{g}, \hat{h}\} \in \mathfrak{H}^2 \oplus \mathcal{H}^2$ satisfies the identity $(A_* f, g) - (f, g') = (l', h) - (l, h')$ for every $\{f, \hat{l}\} \in \Gamma$, then $\{g, \hat{h}\} \in \Gamma$.

Here $f, g \in \text{dom}(\Gamma) (\subset \mathfrak{H}), g' \in \mathfrak{H}, \hat{g} := \{g, g'\}$ and $\hat{h} = \{h, h'\}, \hat{l} = \{l, l'\} \in \text{ran}(\Gamma) (\subset \mathcal{H}^2)$.

Note that in general Γ is multi-valued. If it is single-valued, it splits $\Gamma = \{\Gamma_0, \Gamma_1\}$ and Green's identity (25) takes usual form (24).

2.2. Nonhomogeneous Krein–Stieltjes string

In this subsection, we collect some facts on Jacobi operators of a special form. Namely, consider two sequences with positive elements $m = \{m_n\}_{n=1}^\infty$ and $l = \{l_n\}_{n=1}^\infty$, $m_n, l_n > 0$, $n \in \mathbb{N}$. Next, consider the matrix

$$J_{m,l} = \begin{pmatrix} \frac{1}{m_1} \frac{1}{l_1} & \frac{1}{l_1 \sqrt{m_1 m_2}} & 0 & \dots \\ \frac{1}{l_1 \sqrt{m_1 m_2}} & \frac{1}{m_2} \left(\frac{1}{l_1} + \frac{1}{l_2} \right) & \frac{1}{l_2 \sqrt{m_2 m_3}} & \dots \\ 0 & \frac{1}{l_2 \sqrt{m_2 m_3}} & \frac{1}{m_3} \left(\frac{1}{l_2} + \frac{1}{l_3} \right) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{26}$$

With unilateral shift U in $l_2(\mathbb{N})$, $Ue_n = e_{n+1}$, $n \in \mathbb{N}$, where $\{e_n\}_{n \in \mathbb{N}}$ is the standard orthonormal basis in l_2 , the matrix $J_{m,l}$ can be written as

$$J_{m,l} = M^{-1/2} (I + U) L^{-1} (I + U^*) M^{-1/2}, \quad M = \text{diag}(m_n), \quad L = \text{diag}(l_n). \tag{27}$$

It is known that the difference expression associated with $J_{m,l}$ has a useful mechanical interpretation, related to the Krein string theory (for details we refer the reader to [1, Appendix, pp. 232–236] and [32]). Namely, define the function

$$\mathcal{M}(x) = \sum_{x_{n-1} < x} m_n, \quad x \in [0, \mathcal{L}); \quad \mathcal{L} = \sum_{n=1}^\infty l_n, \quad x_n - x_{n-1} = l_n, \quad x_0 = 0. \tag{28}$$

Then the equation of motion of a nonhomogeneous string with the mass distribution \mathcal{M} is the same as the difference equation associated with the Jacobi matrix $J_{m,l}$ (strings with discrete mass distributions are called Stieltjes strings).

Further, associated with the matrix $J_{m,l}$ one introduces the minimal Jacobi operator in $l_2(\mathbb{N})$ (see [1,6]). We will denote it also by $J_{m,l}$. By *Hamburger's theorem* [1, Theorem 0.5], the operator $J_{m,l}$ is self-adjoint if and only if

$$\sum_n m_{n+1} x_n^2 = \infty. \tag{29}$$

A discreteness criterion for the nonhomogeneous string was obtained by Kac and Krein in [31] (see also [32, §11]). Applying their result to the operator (26), we arrive at the following criterion.

Theorem 2.12. (See [31].) Assume (29). Then $J_{m,l} = J_{m,l}^*$ has discrete spectrum if and only if

- in the case $\mathcal{L} = \infty$, $\lim_{n \rightarrow \infty} x_n \sum_{j=n}^\infty m_j = 0$ (the latter yields $\mathcal{M}(\mathcal{L}) < \infty$);
- in the case $\mathcal{M}(\mathcal{L}) = \infty$, $\lim_{n \rightarrow \infty} (\mathcal{L} - x_n) \sum_{j=1}^n m_j = 0$ (the latter yields $\mathcal{L} < \infty$).

Remark 2.13. If condition (29) does not hold, then $n_\pm(J_{m,l}) = 1$ and hence any self-adjoint extension of $J_{m,l}$ has discrete spectrum. Let us also note that for $J_{m,l}$ to be discrete it is necessary that either $\{m_n\}_{n=1}^\infty \in l_1(\mathbb{N})$ or $\{l_n\}_{n=1}^\infty \in l_1(\mathbb{N})$.

3. Direct sums of symmetric operators and boundary triplets

3.1. Direct sum of boundary triplets as a boundary relation

Let S_n be a densely defined symmetric operator in a Hilbert space \mathfrak{H}_n with equal deficiency indices, $n_+(S_n) = n_-(S_n) \leq \infty$, $n \in \mathbb{N}$. Consider the operator $A := \bigoplus_{n=1}^{\infty} S_n$ acting in a Hilbert direct sum $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$ of spaces \mathfrak{H}_n . By definition, $\mathfrak{H} = \{f = \bigoplus_{n=1}^{\infty} f_n : f_n \in \mathfrak{H}_n, \sum_{n=1}^{\infty} \|f_n\|^2 < \infty\}$. We also denote by \mathfrak{H}^0 the linear manifold consisting of vectors $f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H}$ with finitely many nonzero entries. Clearly,

$$A^* = \bigoplus_{n=1}^{\infty} S_n^*, \quad \text{dom}(A^*) = \left\{ f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H} : f_n \in \text{dom}(S_n^*), \sum_{n=1}^{\infty} \|S_n^* f_n\|^2 < \infty \right\}. \quad (30)$$

We provide the domains $\text{dom}(S_n^*) =: \mathfrak{H}_{n+}$ and $\text{dom}(A^*) =: \mathfrak{H}_+$ with the graph norms $\|f_n\|_{\mathfrak{H}_{n+}}^2 := \|f_n\|^2 + \|S_n^* f_n\|^2$ and $\|f\|_{\mathfrak{H}_+}^2 := \|f\|^2 + \|A^* f\|^2 = \sum_n \|f_n\|_{\mathfrak{H}_{n+}}^2$, respectively.

Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* , $n \in \mathbb{N}$. By $\|\Gamma_j^{(n)}\|$ we denote the norm of the linear mapping $\Gamma_j^{(n)} \in [\mathfrak{H}_{n+}, \mathcal{H}_n]$, $j = 0, 1, n \in \mathbb{N}$.

Let $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ be a Hilbert direct sum of \mathcal{H}_n . Define mappings Γ_0 and Γ_1 by setting

$$\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}, \quad \text{dom}(\Gamma_j) = \left\{ f = \bigoplus_{n=1}^{\infty} f_n \in \text{dom}(A^*) : \sum_{n=1}^{\infty} \|\Gamma_j^{(n)} f_n\|_{\mathcal{H}_n}^2 < \infty \right\}. \quad (31)$$

Clearly $\mathfrak{H}_+ \cap \mathfrak{H}^0 \subset \text{dom}(\Gamma_j) \subset \text{dom}(A^*)$, and $\text{dom}(\Gamma) := \text{dom}(\Gamma_1) \cap \text{dom}(\Gamma_0)$ is dense in \mathfrak{H}_+ since $\mathfrak{H}_+ \cap \mathfrak{H}^0$ is dense in \mathfrak{H}_+ . Define the operators $S_{nj} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $\tilde{A}_j := \bigoplus_{n=1}^{\infty} S_{nj}$, $j = 0, 1$. Then \tilde{A}_0 and \tilde{A}_1 are self-adjoint extensions of A . Note that \tilde{A}_0 and \tilde{A}_1 are disjoint but not necessarily transversal.

Finally, we set

$$A_* = A^* \upharpoonright \text{dom}(\Gamma) \quad \text{and} \quad A_{*j} := A_* \upharpoonright \ker(\Gamma_j), \quad j = 0, 1. \quad (32)$$

Clearly, A_{*j} is symmetric (not necessarily self-adjoint or even closed!) extension of A , $A_{*j} \subset \tilde{A}_j$, $j = 0, 1$, and

$$\text{dom}(A_{*j}) = \left\{ f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H} : f_n \in \ker \Gamma_j^{(n)}, \sum_n (\|S_n^* f_n\|^2 + \|\Gamma_j^{(n)} f_n\|^2) < \infty \right\} \\ (0' := 1, 1' := 0).$$

Definition 3.1. Let Γ_j be defined by (31) and $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a *direct sum of boundary triplets* and will be assigned as $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$.

By Definition 2.1, for a direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ to form a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ it is necessary (but not sufficient!) that

- (a) A_{*0} and A_{*1} are self-adjoint,
- (b) A_{*0} and A_{*1} are transversal,
- (c) $\text{dom}(\Gamma) = \text{dom}(A^*)$,
- (d) Γ_0 and Γ_1 are closed and bounded as mappings from \mathfrak{H}_+ to \mathcal{H} .

It might happen that all of these conditions are violated for the direct sum Π . Nevertheless, we will show that Π is a boundary relation for the operator A^* in the sense of Definition 2.11.

Theorem 3.2. Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* , $M_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. Let also $A^* = \bigoplus_{n=1}^\infty S_n^*$ and $\Pi = \bigoplus_{n=1}^\infty \Pi_n$. Then:

- (i) $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ forms a boundary relation for A^* with single-valued $\Gamma = \{\Gamma_0, \Gamma_1\}$.
- (ii) The corresponding Weyl function is

$$M(z) = \bigoplus_{n=1}^\infty M_n(z). \tag{33}$$

- (iii) $\text{ran}(\Gamma) = \text{ran}(\{\Gamma_0, \Gamma_1\})$ is dense in $\mathcal{H} \oplus \mathcal{H}$.
- (iv) The mapping $\Gamma : \mathfrak{H}_+ \rightarrow \mathcal{H} \oplus \mathcal{H}$ is closed and the mappings $\Gamma_j : \mathfrak{H}_+ \rightarrow \mathcal{H}$ are closable.
- (v) If $\bar{\Gamma}_j$ is a closure of Γ_j , then the following equivalences hold

$$\text{dom}(\bar{\Gamma}_j) = \mathfrak{H}_+ \iff \bar{\Gamma}_j \in [\mathfrak{H}_+, \mathcal{H}] \iff \sup_{n \in \mathbb{N}} \|\Gamma_j^{(n)}\| := C_j < \infty, \quad j = 0, 1. \tag{34}$$

In particular, $\text{dom}(\Gamma) = \text{dom}(\Gamma_0) \cap \text{dom}(\Gamma_1) = \mathfrak{H}_+$ if and only if $\max\{C_0, C_1\} < \infty$.

- (vi) The operator A_{*j} (see (32)) is essentially self-adjoint and $\bar{A}_{*j} = \tilde{A}_j = \bigoplus_{n=1}^\infty S_{nj}$, $j = 0, 1$.
- (vii) A_{*j} is self-adjoint, $A_{*j} = \tilde{A}_j = \bigoplus_{n=1}^\infty S_{nj}$, whenever $C_{j'} = \sup_{n \in \mathbb{N}} \|\Gamma_j^{(n)}\| < \infty$, $j = 0, 1$. If in addition \tilde{A}_0 and \tilde{A}_1 are transversal, then $A_{*j} = (A_{*j})^* \Leftrightarrow C_{j'} = \sup_{n \in \mathbb{N}} \|\Gamma_j^{(n)}\| < \infty$.

Proof. (i) Let us prove Green’s identity (24). By (30)–(32) and Definition 3.1, for $f = \bigoplus_{n=1}^\infty f_n$, $g = \bigoplus_{n=1}^\infty g_n \in \text{dom}(A_*) = \text{dom}(\Gamma)$ we get

$$\begin{aligned} (A_* f, g)_{\mathfrak{H}} - (f, A_* g)_{\mathfrak{H}} &= \sum_{n=1}^\infty [(S_n^* f_n, g_n)_{\mathfrak{H}_n} - (f_n, S_n^* g_n)_{\mathfrak{H}_n}] \\ &= \sum_{n=1}^\infty [(\Gamma_1^{(n)} f_n, \Gamma_0^{(n)} g_n)_{\mathcal{H}_n} - (\Gamma_0^{(n)} f_n, \Gamma_1^{(n)} g_n)_{\mathcal{H}_n}] \\ &= (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}. \end{aligned} \tag{35}$$

Note, that the series in the above equality converge due to (30) and (31).

To prove the maximality assumption assume that Green’s identity

$$(A_* f, g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (\Gamma_1 f, h)_{\mathcal{H}} - (\Gamma_0 f, h')_{\mathcal{H}} \tag{36}$$

holds for every $f \in \text{dom}(A_*)$ and some $g, g' \in \mathfrak{H}$, and $\{h, h'\} \in \mathcal{H} \oplus \mathcal{H}$. Let us show that $g \in \text{dom}(A_*)$ and $\Gamma g = \{\Gamma_0 g, \Gamma_1 g\} = \{h, h'\}$. If $f \in \text{dom}(A)$, equality (36) yields $g \in \text{dom}(A^*)$ and $g' = A^* g$. Hence $g = \bigoplus_{n=1}^\infty g_n$, $g_n \in \text{dom}(S_n^*)$, and $A^* g = \bigoplus_{n=1}^\infty S_n^* g_n$. Setting $f = f_n \in \text{dom}(S_n^*)$ in (36) and noting that $h = \bigoplus_{n=1}^\infty h_n$, $h' = \bigoplus_{n=1}^\infty h'_n \in \mathcal{H}$, we get

$$(S_n^* f_n, g_n)_{\mathfrak{H}_n} - (f_n, S_n^* g_n)_{\mathfrak{H}_n} = (\Gamma_1^{(n)} f_n, h_n)_{\mathcal{H}_n} - (\Gamma_0^{(n)} f_n, h'_n)_{\mathcal{H}_n}, \quad n \in \mathbb{N}. \tag{37}$$

³ $\|\Gamma_j^{(n)}\|$ stands for the norm of $\Gamma_j^{(n)}$ as a bounded linear mapping from \mathfrak{H}_{n+} to \mathcal{H}_n .

Since Π_n is a boundary triplet for S_n^* , $\Gamma_0^{(n)} g_n = h_n$ and $\Gamma_1^{(n)} g_n = h'_n$, $n \in \mathbb{N}$. Moreover, the inclusion $\{h, h'\} \in \mathcal{H} \oplus \mathcal{H}$ yields

$$\sum_{n=1}^{\infty} (\|\Gamma_0^{(n)} g_n\|_{\mathcal{H}_n}^2 + \|\Gamma_1^{(n)} g_n\|_{\mathcal{H}_n}^2) = \sum_{n=1}^{\infty} (\|h_n\|_{\mathcal{H}_n}^2 + \|h'_n\|_{\mathcal{H}_n}^2) < \infty. \tag{38}$$

Inequality (38) means that $g \in \text{dom}(A_*) = \text{dom}(\Gamma)$ and $\Gamma g = \{\Gamma_0 g, \Gamma_1 g\} = \{h, h'\}$. This proves the maximality condition.

(ii) Straightforward.

(iii) Denote by \mathcal{H}^0 the linear manifolds of vectors $h = \bigoplus_{n=1}^{\infty} h_n \in \mathcal{H}$ having finitely many nonzero entries. Clearly \mathcal{H}^0 is dense in \mathcal{H} . It remains to note that $\mathcal{H}^0 = \text{ran}(\Gamma \upharpoonright (\mathfrak{H}_+ \cap \mathfrak{H}^0)) \subset \text{ran}(\Gamma)$, since $\text{ran}(\Gamma^{(n)}) = \mathcal{H}_n \oplus \mathcal{H}_n$, $n \in \mathbb{N}$.

(iv) Let $f_k = \bigoplus_{n=1}^{\infty} f_{kn}$, $\varphi = \bigoplus_{n=1}^{\infty} \varphi_n \in \mathfrak{H}_+$, and $\|f_k - \varphi\|_{\mathfrak{H}_+} \rightarrow 0$ and

$$\lim_{k \rightarrow \infty} \Gamma f_k = \lim_{k \rightarrow \infty} \{\Gamma_0 f_k, \Gamma_1 f_k\} = \{h, h'\} = \left\{ \bigoplus_{n=1}^{\infty} h_n, \bigoplus_{n=1}^{\infty} h'_n \right\} \in \mathcal{H} \oplus \mathcal{H}. \tag{39}$$

Let us prove that $\varphi \in \text{dom}(A_*)$ and $\Gamma \varphi = \{h, h'\}$. Since $\Gamma_j f_k = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)} f_{kn}$, by (39) we get

$$\lim_{k \rightarrow \infty} \Gamma_0^{(n)} f_{kn} = h_n, \quad \lim_{k \rightarrow \infty} \Gamma_1^{(n)} f_{kn} = h'_n, \quad n \in \mathbb{N}. \tag{40}$$

Since $\lim_{k \rightarrow \infty} \|f_{kn} - \varphi_n\|_{\mathfrak{H}_{n+}} = 0$ and the mappings $\Gamma^{(n)} = \{\Gamma_0^{(n)}, \Gamma_1^{(n)}\} : \mathfrak{H}_{n+} \rightarrow \mathcal{H}_n \oplus \mathcal{H}_n$ are closed (in fact, continuous), (40) yields

$$\varphi_n \in \mathfrak{H}_{n+} = \text{dom}(S_n^*) \quad \text{and} \quad \Gamma^{(n)} \varphi_n = \{h_n, h'_n\}. \tag{41}$$

In turn, since $\varphi \in \mathfrak{H}_+ = \text{dom}(A^*)$ and

$$\sum_{n=1}^{\infty} (\|\Gamma_0^{(n)} \varphi_n\|_{\mathcal{H}_n}^2 + \|\Gamma_1^{(n)} \varphi_n\|_{\mathcal{H}_n}^2) = \sum_{n=1}^{\infty} (\|h_n\|_{\mathcal{H}_n}^2 + \|h'_n\|_{\mathcal{H}_n}^2) < \infty, \tag{42}$$

we obtain $\varphi \in \text{dom}(A_*)$ and $\Gamma \varphi = \{\Gamma_0 \varphi, \Gamma_1 \varphi\} = \{h, h'\}$. Hence Γ is closed.

(v) By (iv), the mapping Γ is closed. Hence (v) is implied by the closed graph theorem.

(vi) Clearly, $\mathfrak{H}_+ \cap \mathfrak{H}^0 \subset \text{dom}(\tilde{A}_j)$. Hence $\text{dom}(A_{*j})$ is dense in $\text{dom}(\tilde{A}_j)$ (in the graph topology).

(vii) Let $C_1 < \infty$. Let us prove the self-adjointness of A_{*0} . Since $A_{*0} \subset \tilde{A}_0$, it suffices to show that $\text{dom}(\tilde{A}_0) \subset \text{dom}(A_{*0})$. Let $f = \bigoplus_{n=1}^{\infty} f_n \in \text{dom}(\tilde{A}_0)$. Clearly $f \in \text{dom}(\Gamma_0)$ since $f_n \in \ker \Gamma_0^{(n)}$. Let us show that $f \in \text{dom}(\Gamma_1)$. According to the second J. von Neumann formula,

$$f_n = f_{S_n} + (I + U_n) f_n(i), \quad f_{S_n} \in \text{dom}(S_n), \quad f_n(i) \in \mathfrak{N}_i^{(n)} := \mathfrak{N}_i(S_n), \tag{43}$$

where U_n is an isometry from $\mathfrak{N}_i^{(n)}$ onto $\mathfrak{N}_{-i}^{(n)}$. Since $f \in \text{dom}(A^*)$, it follows from (43) that

$$\begin{aligned} 4 \sum_{n=1}^{\infty} \|f_n(i)\|_{\mathfrak{H}_n}^2 &= \sum_{n=1}^{\infty} \|(I + U_n) f_n(i)\|_{\mathfrak{H}_{n+}}^2 \\ &\leq \sum_{n=1}^{\infty} (\|f_{S_n}\|_{\mathfrak{H}_{n+}}^2 + \|(I + U_n) f_n(i)\|_{\mathfrak{H}_{n+}}^2) = \sum_{n=1}^{\infty} \|f_n\|_{\mathfrak{H}_+}^2 < \infty. \end{aligned}$$

Hence $f(i) := \bigoplus_{n=1}^{\infty} f_n(i) \in \text{dom}(A^*)$. Combining this fact with the assumption $C_1 < \infty$, we get from (43)

$$\sum_{n=1}^{\infty} \|\Gamma_1^{(n)} f_n\|_{\mathcal{H}_n}^2 = \sum_{n=1}^{\infty} \|\Gamma_1^{(n)}(I + U_n) f_n(i)\|_{\mathcal{H}_n}^2 \leq 4C_1^2 \sum_{n=1}^{\infty} \|f_n(i)\|_{\mathfrak{H}_{n+}}^2 \leq 8C_1^2 \sum_{n=1}^{\infty} \|f_n(i)\|_{\mathfrak{H}_n}^2, \tag{44}$$

that is $f \in \text{dom}(\Gamma_1)$. Thus, $f \in \text{dom}(A_*) = \text{dom}(\Gamma) = \text{dom}(\Gamma_0) \cap \text{dom}(\Gamma_1)$.

Further, let us prove the converse statement assuming that \tilde{A}_0 and \tilde{A}_1 are transversal. Note that $A_{*0} = \tilde{A}_0$ if $A_{*0} = A_{*0}^*$. Hence (32) yields $\text{dom}(\tilde{A}_0) = \text{dom}(A_{*0}) \subset \text{dom}(A_*) \subset \text{dom}(\Gamma_1)$. On the other hand, $\text{dom}(A^*) = \text{dom}(\tilde{A}_0) + \text{dom}(\tilde{A}_1)$ since \tilde{A}_0 and \tilde{A}_1 are transversal. Thus Γ_1 admits an extension on $\mathfrak{H}_+ = \text{dom}(A^*)$, since $\text{dom}(\tilde{A}_1) \subset \text{dom}(\Gamma_1)$. By (v), $C_1 < \infty$. \square

Next we find a criterion for a direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ to form a generalized boundary triplet.

Proposition 3.3. *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* and $M_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. Then the following conditions are equivalent:*

- (i) A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet for A^* ,
- (ii) $\text{ran}(\Gamma_0) = \mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$,
- (iii) $\sup_n \|M_n(i)\| =: C_3 < \infty$.

Proof. (i) \Rightarrow (ii) This implication is immediate from Definition 2.8.

(ii) \Rightarrow (i) By Theorem 3.2(i), Π is a boundary relation. Therefore, by [12, Lemma 4.10(iii)], A_{*0} is closed since $\text{ran}(\Gamma_0) (= \mathcal{H})$ is closed. On the other hand, by Theorem 3.2(vi), A_{*0} is essentially self-adjoint. Thus $A_{*0} = (A_{*0})^*$ and the assumption (G2) of Definition 2.8 is verified.

(ii) \Rightarrow (iii) Let $\text{ran}(\Gamma_0) = \mathcal{H}$. According to the implication (ii) \Rightarrow (i), Π is a generalized boundary triplet for A^* . Therefore, by [14, Proposition 6.2], the corresponding Weyl function M takes values in $[\mathcal{H}]$. By Theorem 3.2(ii), $M(z) = \bigoplus_{n=1}^{\infty} M_n(z)$ hence $M(i) \in [\mathcal{H}]$ precisely when $C_3 = \sup_n \|M_n(i)\| < \infty$.

(iii) \Rightarrow (ii) Let γ_n be the γ -field of the boundary triplet Π_n . Then (16) implies

$$\text{Im } M_n(i) = (M_n(i) - M_n^*(i))/2i = \gamma_n(i)^* \gamma_n(i), \quad n \in \mathbb{N}. \tag{45}$$

Since $\sup_n \|M_n(i)\| = C_3 < \infty$, equality (45) yields

$$\sup_n \|\gamma_n(i)\|^2 = \sup_n \|\text{Im } M_n(i)\| \leq C_3 < \infty. \tag{46}$$

Let $h = \bigoplus_{n=1}^{\infty} h_n \in \mathcal{H}$. Then $f_n(i) := \gamma_n(i)h_n \in \mathfrak{N}_i(S_n^*)$ and, by (46),

$$\sum_{n=1}^{\infty} \|f_n(i)\|^2 = \sum_{n=1}^{\infty} \|\gamma_n(i)h_n\|^2 \leq C_3 \sum_{n=1}^{\infty} \|h_n\|^2 < \infty. \tag{47}$$

Hence $f(i) := \bigoplus_{n=1}^{\infty} f_n(i) \in \mathfrak{N}_i(A^*) = \bigoplus_{n=1}^{\infty} \mathfrak{N}_i(S_n^*)$ and $\Gamma_0 f(i) = \bigoplus_{n=1}^{\infty} \Gamma_0^{(n)} f_n(i) = \bigoplus_{n=1}^{\infty} h_n = h$. Thus $f(i) \in \text{dom}(\Gamma_0)$ and $\text{ran}(\Gamma_0) = \mathcal{H}$. The proof is completed. \square

Corollary 3.4. *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* , $n \in \mathbb{N}$, and let Γ_1 be defined by (31). Then the following conditions are equivalent:*

- (i) $\sup_n \|M_n(i)^{-1}\| = C_4 < \infty$,
- (ii) $\text{ran}(\Gamma_1) = \mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$.

Proof. Alongside the boundary triplet Π_n we consider a triplet $\tilde{\Pi}_n = \{\mathcal{H}_n, -\Gamma_1^{(n)}, \Gamma_0^{(n)}\}$, $n \in \mathbb{N}$. The corresponding Weyl function is $\tilde{M}_n(\cdot) = -M_n(\cdot)^{-1}$, $n \in \mathbb{N}$. To complete the proof it remains to apply Proposition 3.3. \square

Remark 3.5. By Theorem 3.2(ii), $\ker(\text{Im } M(z)) = \{0\}$, $z \in \mathbb{C}_+$, and hence $M(\cdot) \in R^s(\mathcal{H})$. According to (33), the inequality $\sup_n \|M_n(i)\| < \infty$ is equivalent to the inclusion $M(i) \in [\mathcal{H}]$, that is $M(\cdot) \in R^s[\mathcal{H}]$. Hence, the implication (iii) \Rightarrow (i) in Proposition 3.3 is immediate from [14, Theorem 6.1]. However we prefer a direct proof because of its simplicity.

Here $R^s(\mathcal{H})$ and $R^s[\mathcal{H}]$ are the Nevanlinna subclasses (for definitions see [12, Section 2.6]).

Next we present sufficient conditions for a direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ to be a generalized boundary triplet for A^* . These conditions are formulated only in terms of the mappings Γ_j^n .

Proposition 3.6. *Assume the conditions of Theorem 3.2 hold. Then:*

- (i) *A direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of boundary triplets Π_n is a generalized boundary triplet for A^* provided that $C_1 = \sup_n \|\Gamma_1^{(n)}\| < \infty$.*
- (ii) *If in addition $\tilde{A}_0 = \bigoplus_{n=1}^\infty S_{n0}$ and $\tilde{A}_1 = \bigoplus_{n=1}^\infty S_{n1}$ are transversal, then condition $C_1 < \infty$ is necessary and sufficient for Π to be a generalized boundary triplet for A^* .*

Proof. (i) Condition (G3) of Definition 2.8 is immediate from Theorem 3.2(i). Moreover, by Theorem 3.2(vii), condition $C_1 < \infty$ yields $A_{*0} = (A_{*0})^*$, hence condition (G2) of Definition 2.8. Let us check condition (G1). Since $\gamma_n^*(\bar{z}) = \Gamma_1^{(n)}(S_{n0} - z)^{-1}$ (see (17)), we get that for any $n \in \mathbb{N}$

$$\begin{aligned} \|\gamma_n(\bar{z})^* f\|^2 &= \|\Gamma_1^{(n)}(S_{n0} - z)^{-1} f\|^2 \leq C_1^2 \|(S_{n0} - z)^{-1} f\|_{\mathfrak{H}_+}^2 \\ &= C_1^2 (\|S_{n0}(S_{n0} - z)^{-1} f\|_{\mathfrak{H}_+}^2 + \|(S_{n0} - z)^{-1} f\|_{\mathfrak{H}_+}^2) \\ &\leq 2C_1^2 (1 + (|z|^2 + 1)/|\text{Im } z|^2), \end{aligned} \tag{48}$$

and hence $\|\gamma_n(\pm i)\| = \|\gamma_n^*(\pm i)\| \leq C_1 \sqrt{6}$, $n \in \mathbb{N}$. Since $M_n(z) = \Gamma_1^{(n)} \gamma_n(z)$ (see (14)), we have

$$\|M_n(i)h\| \leq \|\Gamma_1^{(n)}\| \cdot \|\gamma_n(i)h\|_{\mathfrak{H}_+} \leq C_1 \sqrt{2} \|\gamma_n(i)h\| \leq C_1^2 \sqrt{12}, \quad n \in \mathbb{N}.$$

Hence, by Proposition 3.3, $\text{ran}(\Gamma_0) = \mathcal{H}$.

(ii) Follows from Theorem 3.2(vii). \square

Corollary 3.7. *Assume the conditions of Proposition 3.3. Then:*

- (i) *A direct sum $\tilde{\Pi} = \bigoplus_{n=1}^\infty \tilde{\Pi}_n$ of boundary triplets $\tilde{\Pi}_n = \{\mathcal{H}, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\} = \{\mathcal{H}_n, -\Gamma_1^{(n)}, \Gamma_0^{(n)}\}$ is a generalized boundary triplet for A^* whenever $C_0 = \sup_n \|\Gamma_0^{(n)}\| < \infty$.*
- (ii) *If in addition $\tilde{A}_0 = \bigoplus_{n=1}^\infty S_{n0}$ and $\tilde{A}_1 = \bigoplus_{n=1}^\infty S_{n1}$ are transversal, then condition $C_0 < \infty$ is necessary and sufficient for $\tilde{\Pi}$ to be a generalized boundary triplet for A^* .*

3.2. When direct sum of boundary triplets is a boundary triplet?

1. General case. As it was already mentioned, the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is not a boundary triplet without additional restrictions (cf. Theorem 3.2). We start with the following result.

Proposition 3.8. Assume the conditions of Theorem 3.2. Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is an ordinary boundary triplet for A^* if and only if

$$\max\{C_0, C_1\} < \infty, \quad C_j = \sup_{n \in \mathbb{N}} \|\Gamma_j^{(n)}\|. \tag{49}$$

Proof. Necessity is immediate from (31) and Definition 2.1.

Sufficiency. Consider $\mathfrak{H}^2 := \mathfrak{H} \oplus \mathfrak{H}$ and $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$ as Krein spaces with the fundamental symmetries $J_{\mathfrak{H}} = i \begin{pmatrix} 0 & -I_{\mathfrak{H}} \\ I_{\mathfrak{H}} & 0 \end{pmatrix}$ and $J_{\mathcal{H}} = i \begin{pmatrix} 0 & -I_{\mathcal{H}} \\ I_{\mathcal{H}} & 0 \end{pmatrix}$, respectively. Now identity (35) can be rewritten as

$$(J_{\mathfrak{H}} \hat{f}, \hat{g})_{\mathfrak{H}^2} = (J_{\mathcal{H}} \Gamma \hat{f}, \Gamma \hat{g})_{\mathcal{H}^2}, \tag{50}$$

where $\hat{f} := \{f, A^*f\}$, $\hat{g} := \{g, A^*g\}$ and $\Gamma \hat{f} := \Gamma f$. This means that $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is an isometry from the Krein space $\{\mathfrak{H}^2, J_{\mathfrak{H}}\}$ to the Krein space $\{\mathcal{H}^2, J_{\mathcal{H}}\}$. By Theorem 3.2(v), $\text{dom}(\Gamma) = \text{gr}(A^*)$, the graph of A^* . Since $\text{dom}(\Gamma)$ is closed in \mathfrak{H}^2 , $\text{ran}(\Gamma)$ is closed too (see [12, Proposition 2.3]). On the other hand, by Theorem 3.2(iii), $\text{ran}(\Gamma)$ is dense in \mathcal{H}^2 and hence $\text{ran}(\Gamma) = \mathcal{H}^2$. \square

Remark 3.9. Proposition 3.8 shows that condition (49) is sufficient (but not necessary!) for transversality of the extensions A_{*0} and A_{*1} defined by (32). This fact complements Theorem 3.2(vii). Moreover, it shows that in the case of a special boundary relation $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$, condition (d) after Definition 3.1 is sufficient for $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ to be an ordinary boundary triplet. Besides, (d) and (c) are equivalent and yield the previous conditions (a), (b).

Now we are ready to state the main results of this section.

Theorem 3.10. Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* and $M_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if

$$C_3 = \sup_n \|M_n(i)\|_{\mathfrak{H}_n} < \infty \quad \text{and} \quad C_4 = \sup_n \|(\text{Im } M_n(i))^{-1}\|_{\mathfrak{H}_n} < \infty. \tag{51}$$

Proof. By Proposition 3.3, the first inequality in (51) is equivalent to the fact that $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is a generalized boundary triplet for the operator A^* . By Theorem 3.2(ii), the corresponding (generalized) Weyl function is $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$. Therefore, the second inequality in (51) is equivalent to $C_4 = \|(\text{Im } M(i))^{-1}\|_{\mathfrak{H}} < \infty$, that is to the condition $0 \in \rho(\text{Im } M(i))$. To complete the proof it remains to apply Proposition 2.10. \square

Theorem 3.10 makes it possible to construct an ordinary boundary triplet starting with an arbitrary boundary relation $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$.

Theorem 3.11. (See [43].) Let S_n be a symmetric operator in \mathfrak{H}_n with deficiency indices $n_{\pm}(S_k) = n_n \leq \infty$ and $S_{n0} = S_{n0}^* \in \text{Ext } S_n$, $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* such that $\ker \Gamma_0^{(n)} = \text{dom}(S_{n0})$ and $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $\ker \Gamma_0 = \text{dom}(\tilde{A}_0) := \bigoplus_{n=1}^{\infty} S_{n0}$.

Proof. By [22, Chapter III.1.4], there exists a boundary triplet $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ for S_n^* such that $\text{dom}(S_{n0}) = S_n^* \upharpoonright \ker \tilde{\Gamma}_0^{(n)}$, $n \in \mathbb{N}$. Let $\tilde{M}_n(\cdot)$ be the corresponding Weyl function. Denote $Q_n := \text{Re } \tilde{M}_n(i)$

and choose a factorization of $\text{Im } \widetilde{M}_n(i)$, $R_n^* R_n := \text{Im } \widetilde{M}_n(i)$, such that $R_k \in [\mathcal{H}_k]$ and $0 \in \rho(R_k)$. Then we define the mappings $\Gamma_j^{(n)} : \text{dom}(S_n^*) \rightarrow \mathcal{H}_n$ as follows

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} := (R_n^*)^{-1} (\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}), \quad n \in \mathbb{N}. \tag{52}$$

It is easy to check that $\Gamma_j^{(n)}$ are well defined and $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ forms a boundary triplet for S_n^* . Moreover, the Weyl function $M_n(\cdot)$ corresponding to Π_n satisfies $M_n(i) = iI_{\mathcal{H}_n}$, $n \in \mathbb{N}$. Hence, by Theorem 3.10, a triplet $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ forms a boundary triplet for A^* . The required property $\ker \Gamma_0 = \ker \widetilde{\Gamma}_0 = \text{dom}(A_0) := \bigoplus_{n=1}^\infty S_{n0}$ is immediate from (52). \square

Remark 3.12. Note that the regularization (52) of the direct sum $\widetilde{\Pi} = \bigoplus_{n=1}^\infty \widetilde{\Pi}_n = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ was proposed in [43, Theorem 5.3]. However, we emphasize that condition (51) is more flexible than the condition $M_n(i) = iI_{\mathcal{H}_n}$, $n \in \mathbb{N}$, given in [43, Theorem 5.3]. The latter is very important in applications (cf. Remark 3.16 below).

2. The case of operators with common regular real point. Assume the operator $A = \bigoplus_{n=1}^\infty S_n$ has a regular real point, i.e., there exists $a = \bar{a} \in \hat{\rho}(A)$. This is equivalent to the existence of $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset \bigcap_{n=1}^\infty \hat{\rho}(S_n). \tag{53}$$

In particular, (53) holds whenever the operators S_n are nonnegative, $S_n \geq 0$. Assuming condition (53) to be satisfied, we can simplify conditions (51) of Theorem 3.10 as follows.

Theorem 3.13. Let $\{S_n\}_{n=1}^\infty$ be a sequence of symmetric operators satisfying (53). Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$ and $M_n(\cdot)$ the corresponding Weyl function. Then $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is a boundary triplet for $A^* = \bigoplus_{n=1}^\infty S_n^*$ if and only if

$$C_5 := \sup_{n \in \mathbb{N}} \|M_n(a)\| < \infty \quad \text{and} \quad C_6 := \sup_{n \in \mathbb{N}} \|(M_n'(a))^{-1}\| < \infty, \tag{54}$$

where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.

Proof. Necessity is obvious. Indeed, if $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is a boundary triplet, then the corresponding Weyl function $M(\cdot)$ is defined by (33). Moreover, $M(\cdot)$ is an $R_{[\mathcal{H}]}$ -function analytic at $z = a$ and hence $M(a) \in [\mathcal{H}]$. Furthermore, it satisfies $0 \in \rho(M'(a))$ and thus (54) is fulfilled.

Sufficiency. We deduce the proof from Theorem 3.10. Namely, we will show that conditions (51) of Theorem 3.10 are implied by the corresponding conditions in (54).

First note that $M(\cdot) := \bigoplus_{n=1}^\infty M_n(\cdot)$ is a $\mathcal{C}(\mathcal{H})$ -valued Nevanlinna function since for any $z \in \mathbb{C}_+$ the operator $M(z)$ is closed. Further, $M_n(\cdot)$ is regular on $(a - \varepsilon, a + \varepsilon)$ since $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$. Due to condition (53), $M(\cdot)$ is also holomorphic on $(a - \varepsilon, a + \varepsilon)$ in the sense of Kato [33], that is $(M(z) - i)^{-1}$ is bounded and holomorphic at $z_0 = a$, as well as at $z \in \mathbb{C}_+ \cup \mathbb{C}_- \cup (a - \varepsilon, a + \varepsilon)$ (see [33, Theorem 7.1.3]). Moreover, due to the first condition in (54), $M(\cdot)$ is bounded at $z = a$, $M(a) \in [\mathcal{H}]$. By [33, Section 7.1.2], $M(z) \in [\mathcal{H}]$ for $|z - a|$ small enough (see also [33, Theorem 4.2.23(b)]). In turn, the latter yields $M(z) \in [\mathcal{H}]$ for any $z \in \mathbb{C}_+$ (see [12]). In particular, $M(i) \in [\mathcal{H}]$ and the first inequality in (51) is verified.

Further, by (16),

$$M_n'(a) = (dM_n(z)/dz)|_{z=a} = \gamma_n^*(a) \gamma_n(a), \quad n \in \mathbb{N}. \tag{55}$$

According to (15), $\gamma_n(i) = [I - (a - i)(S_{n0} - i)^{-1}]\gamma_n(a)$. Hence

$$\gamma_n^*(i)\gamma_n(i) = \gamma_n^*(a)[I - (a + i)(S_{n0} + i)^{-1}][I - (a - i)(S_{n0} - i)^{-1}]\gamma_n(a). \tag{56}$$

Noting that $(I - (a - i)(S_{n0} - i)^{-1})^{-1} = I + (a - i)(S_{n0} - a)^{-1}$, we get

$$\inf_{f \in \mathcal{H}_n} (\gamma_n^*(i)\gamma_n(i)f, f) \geq \|I + (a - i)(S_{n0} - a)^{-1}\|_{\mathcal{S}_n}^{-2} \inf_{f \in \mathcal{H}_n} (\gamma_n^*(a)\gamma_n(a)f, f).$$

Since $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$, we have $\|I + (a - i)(S_{n0} - a)^{-1}\| \leq 1 + \frac{\sqrt{1+a^2}}{\varepsilon} =: C$. Combining these inequalities with (56) and (45), we obtain

$$\|(\text{Im } M_n(i))^{-1}\|_{\mathcal{H}_n} \leq C^2 \|(M'_n(a))^{-1}\|_{\mathcal{H}_n},$$

and the second inequality in (51) is verified. \square

For operators $A = \bigoplus_{n=1}^{\infty} S_n$ satisfying (53) we complete Theorem 3.13 by presenting a regularization procedure for $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ leading to a boundary triplet.

Corollary 3.14. *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying (53). Let also $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\tilde{\Gamma}_0^{(n)})$, and $\tilde{M}_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. Then:*

- (i) *The operator $\tilde{M}'_n(a)$ is positively definite, $n \in \mathbb{N}$.*
- (ii) *For any factorization $\tilde{M}'_n(a) = R_n^* R_n$, where $R_n \in [\mathcal{H}_n]$ and $0 \in \rho(R_n)$, a triplet*

$$\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \quad \text{with } \Gamma_0^{(n)} := R_n \tilde{\Gamma}_0^{(n)}, \Gamma_1^{(n)} := (R_n^{-1})^* (\tilde{\Gamma}_1^{(n)} - \tilde{M}_n(a) \tilde{\Gamma}_0^{(n)}), \tag{57}$$

is a boundary triplet for S_n^ .*

- (iii) *A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a boundary triplet for A^* .*

Proof. (i) Let $\tilde{\gamma}_n$ be the γ -field corresponding to the triplet $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$. The functions $\tilde{M}_n(\cdot)$ and $\tilde{\gamma}_n(\cdot)$ are regular within $(a - \varepsilon, a + \varepsilon)$ for every $n \in \mathbb{N}$ since $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$. By (55), $\tilde{M}'_n(a) > 0$ and $0 \in \rho(\tilde{M}'_n(a))$ since $\gamma_n(a)$ isomorphically maps \mathcal{H}_n onto \mathfrak{H}_a .

(ii) By (i), $\tilde{M}'_n(a)$ admits a factorization $\tilde{M}'_n(a) = R_n^* R_n$, where $R_n \in [\mathcal{H}]$ and $0 \in \rho(R_n)$. Therefore, the mappings $\Gamma_0^{(n)}$ and $\Gamma_1^{(n)}$ are defined correctly and Π_n is a boundary triplet for S_n^* .

(iii) Let $M_n(\cdot)$ be the Weyl function corresponding to the triplet Π_n . It follows from (57) and the definition of the Weyl function that

$$M_n(z) = (R_n^{-1})^* [\tilde{M}_n(z) - \tilde{M}_n(a)] R_n^{-1}, \quad n \in \mathbb{N}. \tag{58}$$

Hence $M_n(a) = 0$ and $M'_n(a) = (R_n^{-1})^* \tilde{M}'_n(a) R_n^{-1} = I_{\mathcal{H}_n}$, $n \in \mathbb{N}$. Thus, both conditions in (54) are satisfied and, by Theorem 3.13, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a boundary triplet for A^* . \square

Corollary 3.15. *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying (53). Let also $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\tilde{\Gamma}_0^{(n)})$, and $\tilde{M}_n(\cdot)$ the corresponding Weyl function. If the operators $R_n \in [\mathcal{H}_n]$ satisfy*

$$R_n^{-1} \in [\mathcal{H}_n] \quad \text{and} \quad \sup_n \|(R_n^{-1})^* (\tilde{M}'_n(a))^{-1} R_n^{-1}\| < \infty, \quad n \in \mathbb{N}, \tag{59}$$

then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets (57) is a boundary triplet for $A^ = \bigoplus_{n=1}^{\infty} S_n^*$.*

Proof. Since the Weyl function $M_n(\cdot)$ corresponding to Π_n is given by (58), both conditions (54) are immediate from (59). It remains to apply Theorem 3.13. \square

Remark 3.16. Corollary 3.15 is more useful in applications than Corollary 3.14. The reason is that it is more convenient and easier to select a suitable sequence $\{R_n\}_{n=1}^\infty$ satisfying (59) than to find the operators $(M'_n(a))^{1/2}$. For instance, to construct boundary triplets in Theorems 4.1 and 4.7, we select R_n being diagonal matrices although $M'_n(a)$, hence $(M'_n(a))^{1/2}$, are not diagonal.

3.3. Direct sums of self-similar boundary triplets

In this subsection, we apply Theorem 3.10 to the special case of symmetric operators S_n that are pairwise unitarily equivalent up to multiplicative constants. More precisely, let S_1 be a symmetric operator in \mathfrak{H}_1 , $n_\pm(S_1) = n \leq \infty$. We assume that for any $n \in \mathbb{N}$ there exist a unitary operator U_n from \mathfrak{H}_n onto \mathfrak{H}_1 and a constant $d_n > 0$ such that (to be precise we set $U_1 := I_{\mathfrak{H}_1}$ and $d_1 := 1$)

$$S_n := d_n^{-2} U_n^{-1} S_1 U_n. \tag{60}$$

First we suppose that

$$0 < d_* := \inf_{n \in \mathbb{N}} d_n \leq \sup_{n \in \mathbb{N}} d_n =: d^* < \infty \tag{61}$$

and reprove one result of Kochubei (cf. [34, Theorem 3], [35, Lemma 1]) for this case.

Lemma 3.17. (See [35].) *Let S_n be as above, let $\Pi_1 = \{\mathcal{H}_1, \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ be a boundary triplet for S_1^* , and $A = \bigoplus_{n=1}^\infty S_n$. Assume in addition that condition (61) holds. Then:*

(i) *For any $\alpha \in \mathbb{R}$, a triplet $\Pi_n := \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$, where*

$$\mathcal{H}_n := \mathcal{H}_1, \quad \Gamma_0^{(n)} := d_n^{\alpha-2} \Gamma_0^{(1)} U_n, \quad \Gamma_1^{(n)} := d_n^{-\alpha} \Gamma_1^{(1)} U_n, \quad n \in \mathbb{N}, \tag{62}$$

forms a boundary triplet for the operator S_n^ .*

(ii) *Moreover, $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is an (ordinary) boundary triplet for the operator A^* .*

Proof. (i) Straightforward.

(ii) Let $M_n(\cdot)$ be the Weyl function corresponding to the triplet $\Pi_n = \{\mathcal{H}, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$, $n \in \mathbb{N}$. It follows from (62) that the Weyl functions M_n and M_1 are connected by

$$M_n(z) = d_n^{2-2\alpha} M_1(d_n^2 z), \quad z \in \mathbb{C}_\pm, \quad n \geq 2. \tag{63}$$

Hence

$$\|M_n(i)\| = d_n^{2-2\alpha} \|M_1(id_n^2)\|, \quad \|(\operatorname{Im} M_n(i))^{-1}\| = d_n^{2\alpha-2} \|(\operatorname{Im} M_1(id_n^2))^{-1}\|. \tag{64}$$

Combining (64) with (61), we obtain that $\{M_n\}_{n=1}^\infty$ satisfies (51) since M_1 is continuous on $[i(d_*)^2, i(d^*)^2] \subset \mathbb{C}_+$. Theorem 3.10 completes the proof. \square

The following results demonstrate importance of both inequalities in (61) for the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ to be an (ordinary) boundary triplet for A^* .

Lemma 3.18. *Let S_1 be a closed densely defined symmetric operator in \mathfrak{H}_1 with $n_{\pm}(S_1) = n < \infty$, let $\Pi_1 = \{\mathcal{H}_1, \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ be a boundary triplet for S_1^* and $M_1(\cdot)$ the corresponding Weyl function. Let also $S_n, n \in \mathbb{N}$, be defined by (60) and suppose that $\{d_n\}_{n=1}^{\infty}$ satisfies $d_* = 0$ and $d^* < \infty$. Then:*

(i) *A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of triplets $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$, where*

$$\mathcal{H}_n = \mathcal{H}_1, \quad \Gamma_0^{(n)} = \Gamma_0^{(1)}U_n, \quad \Gamma_1^{(n)} = d_n^{-2}\Gamma_1^{(1)}U_n, \tag{65}$$

forms an ordinary boundary triplet for the operator $A^ = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if*

$$C_+ := -\lim_{y \downarrow 0} \frac{M_1(iy)}{iy} \in [\mathcal{H}_1] \quad \text{and} \quad 0 \in \rho(C_+). \tag{66}$$

(ii) *A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of triplets $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$, where*

$$\mathcal{H} = \mathcal{H}_1, \quad \Gamma_0^{(n)} = d_n^{-2}\Gamma_0^{(1)}U_n, \quad \Gamma_1^{(n)} = \Gamma_1^{(1)}U_n, \tag{67}$$

forms an ordinary boundary triplet for $A^ = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if*

$$C_- := -\lim_{y \downarrow 0} iyM_1(iy) \in [\mathcal{H}_1] \quad \text{and} \quad 0 \in \rho(C_-). \tag{68}$$

Proof. (i) By (63), we get $M_n(i) = d_n^{-2}M_1(id_n^2)$. Since $d_* = 0$, by Proposition 3.3, Π is a generalized boundary triplet for A^* if and only if $C_+ \in [\mathcal{H}_1]$. Moreover, by Theorem 3.10, Π is an ordinary boundary triplet precisely if in addition $0 \in \rho(C_+)$.

(ii) The proof is similar to that of (i) if one notices that $M_n(i) = d_n^2M_1(id_n^2)$. \square

Remark 3.19. Let $\Sigma_{M_1}(\cdot)$ be the spectral measure of $M_1(\cdot)$ (see Section 2.1.2). Then the operators C_+ and C_- can easily be expressed in terms of $\Sigma_{M_1}(\cdot)$. Namely, condition (66) means that the limit $M_1(0) := M_1(+i0)$ exists, moreover, $M_1(0) = 0$, and the following integral converges

$$C_+ = \int_{\mathbb{R}} \frac{d\Sigma_{M_1}(t)}{t^2} \in [\mathcal{H}_1].$$

Besides, we note that $C_- = \Sigma_{M_1}(\{0\})$.

Corollary 3.20. *Let S_n be as in Lemma 3.18, let $\Pi_1 = \{\mathcal{H}_1, \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ be a boundary triplet for S_1^* and $S_{10} := S_1^* \upharpoonright \ker(\Gamma_0^{(1)})$. Assume that $d_* = 0$ and $d^* < \infty$. Assume also that S_1 is a simple symmetric operator. The direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets defined by (67) (by (65)) is an ordinary boundary triplet for A^* if and only if*

$$\dim(\ker S_{10}) = n_{\pm}(S_1) \quad (\text{respectively, } \dim(\ker S_{11}) = n_{\pm}(S_1)). \tag{69}$$

Proof. Let us prove the first equality in (69) assuming that the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets (67) forms a boundary triplet. By Remark 3.19, $C_- = \Sigma_{M_1}(\{0\})$ where $\Sigma_{M_1}(\cdot)$ is a nonorthogonal spectral measure of $M_1(\cdot)$. The latter implies

$$\dim(\ker S_{10}) = \text{rank}(\Sigma_{M_1}(\{0\})) = \text{rank} C_-.$$

Since Π is an ordinary boundary triplet for A^* , Lemma 3.18(ii) yields $0 \in \rho(C_-)$, that is, C_- is of maximal rank. Combining these relations, we get $\dim(\ker S_{10}) = \text{rank}(C_-) = \dim \mathcal{H}_1 = n_{\pm}(S_1)$.

To prove sufficiency, let us note that $C_- = \Sigma_{M_1}(\{0\})$. Since $n_{\pm}(S_1) = \dim \mathcal{H}_1 < \infty$, we obviously get that relations (69) and (68) are equivalent. \square

We complete this subsection by considering the situation when $d^* = \infty$.

Lemma 3.21. *Let $d_* > 0$ and $d^* = \infty$. Then:*

- (i) *The direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of triplets defined by (65) is a generalized boundary triplet for A^* , but not an ordinary boundary triplet for A^* ,*
- (ii) *$\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is not a generalized boundary triplet for A^* if Π_n is defined by (67).*

Proof. (i) Since S_1 is densely defined, the Weyl function $M_1(\cdot)$ corresponding to the triplet Π_1 satisfies (cf. (19))

$$s - \lim_{y \uparrow \infty} M_1(iy)/y = 0. \tag{70}$$

Let $\Pi_n, n \in \mathbb{N}$, be the boundary triplet for S_n^* defined by (65) and $M_n(\cdot)$ the corresponding Weyl function. Setting in (62) and (63) $\alpha = 2$ and combining these relations with (65), we get $M_n(z) = d_n^{-2} M_1(d_n^2 z)$. Combining these relations with (70), we obtain

$$\sup_n \|M_n(i)\| = \sup_n d_n^{-2} \|M_1(id_n^2)\| < \infty. \tag{71}$$

By Proposition 3.3, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a generalized boundary triplet for A^* .

Further, the above relations yield $\text{Im } M_n(z) = d_n^{-2} \text{Im } M_1(d_n^2 z)$. Hence and from (70) we get

$$\sup_n \|(\text{Im } M_n(i))^{-1}\| = \sup_n d_n^2 \|(\text{Im } M_1(id_n^2))^{-1}\| = \infty. \tag{72}$$

By Theorem 3.10, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is not an ordinary boundary triplet for A^* .

(ii) Since S_1 is densely defined, the Weyl function $M_1(\cdot)$ satisfies (cf. (70))

$$s - \lim_{y \uparrow \infty} y^{-1} M_1(iy)^{-1} = 0. \tag{73}$$

Let $\Pi_n, n \in \mathbb{N}$, be a boundary triplet for S_n^* defined by (67) and $M_n(\cdot)$ the corresponding Weyl function. It follows from (62) and (63) (with $\alpha = 0$) that $M_n(z) = d_n^2 M_1(d_n^2 z), n \geq 2$. Hence $\sup_n \|M_n(i)\| = \sup_n d_n^2 \|M_1(id_n^2)\| = \infty$. By Proposition 3.3, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is not a generalized boundary triplet for A^* . \square

4. Boundary triplets for the operator H_{\min}^*

In what follows we assume that $\mathcal{I} = [0, b) \subseteq \mathbb{R}_+, 0 < b \leq +\infty$, is either a bounded interval or positive semi-axis, $X = \{x_n\}_{n=0}^{\infty} \subset \mathcal{I}$ is a strictly increasing sequence,

$$0 = x_0 < x_1 < x_2 < \dots < x_n < \dots < b \leq +\infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = b. \tag{74}$$

We denote $d_n := x_n - x_{n-1}$. Consider the following symmetric operator in $L^2(\mathcal{I})$

$$H_{\min} = -\frac{d^2}{dx^2}, \quad \text{dom}(H_{\min}) = W_0^{2,2}(\mathcal{I} \setminus X). \tag{75}$$

Clearly, H_{\min} is closed and

$$H_{\min} = \bigoplus_{n=1}^{\infty} H_n, \quad \text{where } H_n = -\frac{d^2}{dx^2}, \quad \text{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n]. \quad (76)$$

1. Note that $H_{\min} \geq 0$. It is known (see for instance [22]) that Friedrichs' extension H_n^F of H_n is defined by the Dirichlet boundary conditions, i.e., $\text{dom}(H_n^F) = \{f \in W^{2,2}[x_{n-1}, x_n] : f(x_{n-1}+) = f(x_n-) = 0\}$. Therefore, the Friedrichs' extension H^F of H_{\min} is $H^F = \bigoplus_{n=1}^{\infty} H_n^F$, that is

$$H_F = -\frac{d^2}{dx^2}, \quad \text{dom}(H_F) = \{f \in W_2^2(\mathcal{I} \setminus X) : f(0) = f(x_n+) = f(x_n-) = 0, n \in \mathbb{N}\}. \quad (77)$$

It is easily seen that a triplet $\tilde{\Pi}_n = \{\mathbb{C}^2, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ given by

$$\tilde{\Gamma}_0^{(n)} f := \begin{pmatrix} f(x_{n-1}+) \\ -f(x_n-) \end{pmatrix}, \quad \tilde{\Gamma}_1^{(n)} f := \begin{pmatrix} f'(x_{n-1}+) \\ f'(x_n-) \end{pmatrix}, \quad f \in W_2^2[x_{n-1}, x_n], \quad (78)$$

forms a boundary triplet for H_n^* satisfying $\ker(\tilde{\Gamma}_0^{(n)}) = \text{dom}(H_n^F)$. Moreover, $H_n = d_n^{-2}U_n^{-1}S_1U_n$, where $S_1 := -\frac{d^2}{dx^2}$, $\text{dom}(S_1) = W_0^{2,2}[0, 1]$, and $(U_n f)(x) := \sqrt{d_n}f(d_n x + x_{n-1})$. Clearly, U_n isometrically maps $L^2[x_{n-1}, x_n]$ onto $L^2[0, 1]$. As it follows from Lemma 3.17, a triplet $\tilde{\Pi} = \bigoplus_{n \in \mathbb{N}} \tilde{\Pi}_n$ forms a boundary triplet for the operator $H_{\min}^* := (H_{\min})^* = H_{\max}$ whenever

$$0 < d_* = \inf_{n \in \mathbb{N}} d_n \leq d^* = \sup_{n \in \mathbb{N}} d_n < +\infty. \quad (79)$$

If $d_* = 0$, then the direct sum $\tilde{\Pi} = \bigoplus_{n=1}^{\infty} \tilde{\Pi}_n$ of triplets (78) is not a boundary triplet for H_{\max} . We regularize the triplet $\tilde{\Pi}$ by applying Corollary 3.15 in order to obtain a direct sum triplet $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ for the operator H_{\min}^* , assuming only that

$$d^* = \sup_{n \in \mathbb{N}} d_n < +\infty. \quad (80)$$

Theorem 4.1. Assume condition (80) and define the mappings $\Gamma_j^{(n)} : W_2^2[x_{n-1}, x_n] \rightarrow \mathbb{C}^2$, $n \in \mathbb{N}$, $j = 0, 1$, by setting

$$\begin{aligned} \Gamma_0^{(n)} f &:= \begin{pmatrix} d_n^{1/2} f(x_{n-1}+) \\ -d_n^{1/2} f(x_n-) \end{pmatrix}, \\ \Gamma_1^{(n)} f &:= \begin{pmatrix} d_n^{-1/2} f'(x_{n-1}+) + d_n^{-3/2}(f(x_{n-1}+) - f(x_n-)) \\ d_n^{-1/2} f'(x_n-) + d_n^{-3/2}(f(x_{n-1}+) - f(x_n-)) \end{pmatrix}. \end{aligned} \quad (81)$$

Then:

- (i) For any $n \in \mathbb{N}$ the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is a boundary triplet for H_n^* .
- (ii) The direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is a boundary triplet for the operator H_{\min}^* .

Proof. (i) Straightforward.

(ii) The Weyl function $\tilde{M}_n(\cdot)$ corresponding to the triplet $\tilde{\Pi}_n$ of the form (78) is

$$\tilde{M}_n(z) = -\frac{\sqrt{z}}{\sin(\sqrt{z}d_n)} \begin{pmatrix} \cos(\sqrt{z}d_n) & 1 \\ 1 & \cos(\sqrt{z}d_n) \end{pmatrix}, \quad z \in \mathbb{C}_+. \quad (82)$$

Comparing definitions (78) and (81) of triplets Π_n and $\tilde{\Pi}_n$, respectively, we get

$$\Gamma_0^{(n)} = R_n \tilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} = R_n^{-1} (\tilde{\Gamma}_1^{(n)} - Q_n \tilde{\Gamma}_0^{(n)}), \quad \text{and} \quad M_n(z) = R_n^{-1} (\tilde{M}_n(z) - Q_n) R_n^{-1}, \quad (83)$$

where

$$R_n = R_n^* := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{1/2} \end{pmatrix} \quad \text{and} \quad Q_n = \frac{1}{d_n} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \tilde{M}_n(0). \quad (84)$$

It follows from (83), (84), and (82) that

$$M_n(0) = 0, \quad M'_n(0) = R_n^{-1} \tilde{M}'_n(0) R_n^{-1} = R_n^{-1} \begin{pmatrix} d_n/3 & -d_n/6 \\ -d_n/6 & d_n/3 \end{pmatrix} R_n^{-1} = \begin{pmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{pmatrix}. \quad (85)$$

Relations (85) yield conditions (59). One completes the proof by applying Corollary 3.15. \square

Remark 4.2. Let $d_* = 0$. Hence both families $\{\tilde{M}_n(i)\}_{n \in \mathbb{N}}$ and $\{\tilde{M}_n(i)\}_{n \in \mathbb{N}}^{-1}$ (see (82)) are unbounded. By Proposition 3.3, neither $\tilde{\Pi} = \bigoplus_{n=1}^{\infty} \tilde{\Pi}_n$ nor $\widetilde{\Pi}^{(1)} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}^{(1)}_n$, where $\tilde{\Pi}_n = \{\mathbb{C}^2, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ is defined by (78) and $\widetilde{\Pi}^{(1)}_n := \{\mathbb{C}^2, -\tilde{\Gamma}_1^{(n)}, \tilde{\Gamma}_0^{(n)}\}$, forms a generalized boundary triplet for H_{\min}^* . Moreover, by Proposition 3.6(i), the mappings $\tilde{\Gamma}_0 = \bigoplus_{n=1}^{\infty} \tilde{\Gamma}_0^{(n)}$ and $\tilde{\Gamma}_1 = \bigoplus_{n=1}^{\infty} \tilde{\Gamma}_1^{(n)}$ are unbounded. Note that, the latter might be checked by restricting the mappings $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ on $\mathfrak{H}_i(H_{\min})$.

Note also that $\tilde{\Gamma}_0$ coincides with the mapping Γ^2 in [44, Theorem 1]. Hence the triplet Π constructed in [44, Theorem 1] is not an ordinary boundary triplet.

Remark 4.3. Let us sketch another proof of Theorem 4.1. Simple calculations with account of (80) yield that the family $\{M_n(i)\}_{n=1}^{\infty}$ is bounded. Moreover, it follows from (82) that

$$\lim_{n_k \rightarrow \infty} M_{n_k}(i) = i \lim_{n_k \rightarrow \infty} \text{Im} M_{n_k}(i) = i \begin{pmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{pmatrix} \quad \text{whenever} \quad \lim_{n_k \rightarrow \infty} d_{n_k} = 0.$$

Hence, by Theorem 3.10, $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ defined by (81) forms a boundary triplet for H_{\min}^* .

Proposition 4.4. Let Π be the boundary triplet defined in Theorem 4.1 and $M(\cdot)$ the corresponding Weyl function. If condition (80) is satisfied, then

$$M(-a^2) \rightrightarrows -\infty \quad \text{as} \quad a \rightarrow +\infty. \quad (86)$$

Proof. By Theorem 3.2(ii), the Weyl function $M(\cdot)$ has the form $M(z) = \bigoplus_{n=1}^{\infty} M_n(z)$, where $M_n(\cdot)$ is defined by (83), (82) and (84). Consider the following matrix-function

$$M(-a^2, x) := \begin{pmatrix} F_a(x) & G_a(x) \\ G_a(x) & F_a(x) \end{pmatrix}, \quad x > 0, \quad (87)$$

where

$$F_a(x) := \frac{1}{x^2} - a \frac{\cosh ax}{x \sinh ax}, \quad G_a(x) := \frac{1}{x^2} - \frac{a}{x \sinh ax}.$$

It is easy to check that

$$F_a(x) < 0 \quad \text{and} \quad G_a(x) > 0 \quad \text{for} \quad x > 0.$$

Since $\sigma(M(-a^2, x)) = \{F_a(x) + G_a(x), F_a(x) - G_a(x)\}$, we get

$$M(-a^2, x) \leq (F_a(x) + G_a(x))I_2, \quad x > 0.$$

Further, consider the function

$$f(x) = \frac{2}{x^2} - \frac{1 + \cosh x}{x \sinh x} (= F_1(x) + G_1(x)).$$

Note that $f(x) < 0$ if $x > 0$. Moreover, f is continuous on \mathbb{R}_+ and

$$\lim_{x \rightarrow +0} f(x) = -\frac{1}{6}, \quad \lim_{x \rightarrow +\infty} f(x) = 0.$$

Note also that $\lim_{x \rightarrow +\infty} x^2 f'(x) = 1$. Hence $f'(x) > 0$ for $x \geq x_0$ with sufficiently large $x_0 \in \mathbb{R}_+$. Since $F_a(x) + G_a(x) = a^2 f(ax)$, for $a \geq a_0 > 0$ large enough we obtain

$$\sup_{x \in (0, d^*)} (F_a(x) + G_a(x)) = \frac{2}{(d^*)^2} - \frac{a}{d^*} \cdot \frac{1 + \cosh ad^*}{\sinh ad^*} \leq -2\frac{a}{d^*} + \frac{2}{(d^*)^2}.$$

Note that $M_n(-a^2) = M(-a^2, d_n)$. Combining this fact with the last inequality, we obtain

$$M(-a^2) = \bigoplus_{n=1}^{\infty} M_n(-a^2) \leq -\frac{a}{d^*} I_2, \quad a \geq \max\{a_0, 2/d^*\}. \tag{88}$$

This completes the proof. \square

Combining Theorem 4.1 with Proposition 2.2, we arrive at the following parametrization of the set $\text{Ext } H_{\min}$ of closed proper extensions of the operator H_{\min} :

$$\tilde{H} = H_{\Theta} := H_{\min}^* \upharpoonright \text{dom}(H_{\Theta}), \quad \text{dom}(H_{\Theta}) = \{f \in \text{dom}(H_{\min}^*): \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}, \tag{89}$$

where $\Theta \in \tilde{\mathcal{C}}(l_2)$ and Γ_0, Γ_1 are defined by (81).

Theorem 4.5. Let $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ be a boundary triplet for H_{\min}^* defined in Theorem 4.1, $\Theta, \tilde{\Theta} \in \tilde{\mathcal{C}}(\mathcal{H})$, and $H_{\Theta}, H_{\tilde{\Theta}} \in \text{Ext } H_{\min}$ proper extensions of H_{\min} defined by (89). Then:

- (i) The operator H_{Θ} is symmetric (self-adjoint) if and only if so is Θ , and $n_{\pm}(H_{\min}) = n_{\pm}(\Theta)$.
- (ii) The self-adjoint (symmetric) operator H_{Θ} is lower semibounded if and only if so is Θ .
- (iii) For any $p \in (0, \infty]$, $z \in \rho(H_{\Theta}) \cap \rho(H_{\tilde{\Theta}})$, and $\zeta \in \rho(\Theta) \cap \rho(\tilde{\Theta})$ the following equivalence holds

$$(H_{\Theta} - z)^{-1} - (H_{\tilde{\Theta}} - z)^{-1} \in \mathfrak{S}_p \iff (\Theta - \zeta)^{-1} - (\tilde{\Theta} - \zeta)^{-1} \in \mathfrak{S}_p.$$

- (iv) The operator $H_{\Theta} = H_{\Theta}^*$ has discrete spectrum if and only if $d_n \rightarrow 0$ and Θ has discrete spectrum.

Proof. (i) is immediate from Proposition 2.2.

(ii) Combining Proposition 2.7 with Proposition 4.4 yields the first statement. Then the second one is implied by estimate (88).

(iii) is implied by Proposition 2.5.

(iv) First we show that conditions are sufficient. Indeed, the operator

$$H_0 := H_{\min}^* \upharpoonright \ker(\Gamma_0) = \bigoplus_{n \in \mathbb{N}} H_{n0}, \quad H_{n0} := H_n^* \upharpoonright \ker(\Gamma_0^{(n)}), \tag{90}$$

has discrete spectrum if $\lim_{n \rightarrow \infty} d_n = 0$. Moreover, the Krein resolvent formula and discreteness of $\sigma(\Theta)$ implies $\mathcal{R}_{H_\Theta}(z) - \mathcal{R}_{H_0}(z) \in \mathfrak{S}_\infty$, $z \in \mathbb{C}_+$, and hence $\mathcal{R}_{H_\Theta}(z) \in \mathfrak{S}_\infty$.

Let us show that condition $d_n \searrow 0$ is necessary for discreteness of $\sigma(H_\Theta)$. Without loss of generality we can assume that $0 \in \rho(H_\Theta)$. Assume also that $\limsup_{n \rightarrow \infty} d_n > 0$ and H_Θ has discrete spectrum. Then there exists a sequence $\{d_{n_k}\}_{k=1}^\infty$ such that $d_{n_k} \geq d_*/2 > 0$. For $\varepsilon \in (0, d_*/2)$, define the function

$$\varphi_\varepsilon(\cdot) \in W_2^2(\mathbb{R}), \quad \varphi_\varepsilon(x) = \begin{cases} 1, & \varepsilon \leq x \leq d_* - \varepsilon, \\ 0, & x \notin [0, d_*]. \end{cases}$$

Note that $\varphi_k(x) := P_{\mathcal{I}}\varphi_\varepsilon(x + x_{n_k}) \in \text{dom}(H_\Theta)$, where $P_{\mathcal{I}}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathcal{I})$. Moreover, $\|\varphi_k\|_{L^2} \equiv \text{const}$ and $\|H_\Theta \varphi_k\|_{L^2} \equiv \text{const}$. Since the functions $\varphi_k(\cdot)$ have disjoint supports, the operator $(H_\Theta)^{-1}$ is not compact. Contradiction. \square

Corollary 4.6. *H_Θ is nonnegative if and only if the linear relation Θ is nonnegative. Moreover, if a is large enough, then $H_\Theta \geq -a^2$ whenever $\Theta \geq -\frac{a}{d_*} I_2$.*

Proof. Since $M(0) = 0$, by [13, Theorem 4], we get the first part. Moreover, we have the estimate $M(-a^2) \leq -a/d_* I$ (see the proof of Proposition 4.4), and Krein’s formula (20) completes the proof. \square

2. Alongside boundary triplet (81) consider another boundary triplet. Namely, define $\tilde{\Pi}_n = \{\mathcal{H}, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ for the operator H_n^* , $n \in \mathbb{N}$, by setting

$$\mathcal{H} = \mathbb{C}^2, \quad \tilde{\Gamma}_0^{(n)} f := \begin{pmatrix} f(x_{n-1}+) \\ f'(x_{n-}) \end{pmatrix}, \quad \tilde{\Gamma}_1^{(n)} f := \begin{pmatrix} f'(x_{n-1}+) \\ f(x_{n-}) \end{pmatrix}, \quad f \in W_2^2[x_{n-1}, x_n]. \tag{91}$$

In the following theorem we regularize the family $\{\tilde{\Pi}_n\}_{n=1}^\infty$ in such a way that the direct sum of new boundary triplets Π_n is already a boundary triplet for $H_{\min}^* = \bigoplus_{n=1}^\infty H_n^*$ if $d^* < \infty$.

Theorem 4.7. *Assume condition (80) and define the mappings $\Gamma_j^{(n)} : W_2^2[x_{n-1}, x_n] \rightarrow \mathbb{C}^2$, $n \in \mathbb{N}$, $j = 0, 1$, by setting*

$$\Gamma_0^{(n)} f := \begin{pmatrix} d_n^{1/2} f(x_{n-1}+) \\ d_n^{3/2} f'(x_{n-}) \end{pmatrix},$$

$$\Gamma_1^{(n)} f := \begin{pmatrix} d_n^{-1/2} (f'(x_{n-1}+) - f'(x_{n-})) \\ d_n^{-3/2} (f(x_{n-}) - f(x_{n-1}+)) - d_n^{-1/2} f'(x_{n-}) \end{pmatrix}. \tag{92}$$

Then:

- (i) For any $n \in \mathbb{N}$ the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is a boundary triplet for H_n^* .
- (ii) The direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is a boundary triplet for the operator $H_{\max} = H_{\min}^*$.

Proof. (i) Straightforward.

(ii) The Weyl function of H_n^* corresponding to the triplet $\tilde{\Pi}_n$ defined by (91) is

$$\tilde{M}_n(z) = \frac{1}{\cos(\sqrt{z}d_n)} \begin{pmatrix} \sqrt{z} \sin(\sqrt{z}d_n) & 1 \\ 1 & \sin(\sqrt{z}d_n)/\sqrt{z} \end{pmatrix}. \tag{93}$$

Comparing definitions (91) and (92), we get that the triplets Π_n and $\tilde{\Pi}_n$ are connected by (83), where the matrices R_n and Q_n are given by

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2} \end{pmatrix} \quad \text{and} \quad Q_n := \tilde{M}_n(0) = \begin{pmatrix} 0 & 1 \\ 1 & d_n \end{pmatrix}. \tag{94}$$

Hence $M_n(z) = R_n^{-1}(\tilde{M}_n(z) - Q_n)R_n^{-1}$ is the Weyl function corresponding to the triplet Π_n . It follows from (93) and (94) that

$$M_n(0) = 0, \quad M'_n(0) = R_n^{-1}\tilde{M}'_n(0)R_n^{-1} = R_n^{-1} \begin{pmatrix} d_n & d_n^2/2 \\ d_n^2/2 & d_n^3/3 \end{pmatrix} R_n^{-1} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}. \tag{95}$$

One completes the proof by applying Theorem 3.13. \square

Remark 4.8. Clearly, all statements of Theorem 4.5 with exception of (ii) remain valid for the boundary triplet $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ with Π_n defined by (92) in place of (81).

Corollary 4.9. Let $\tilde{\Pi}_n$ be a boundary triplet for H_n^* defined by (91) and $\widetilde{\Pi}^{(1)}_n := \{\mathbb{C}^2, -\tilde{T}_1^{(n)}, \tilde{T}_0^{(n)}\}$. Let also $\tilde{\Pi} = \bigoplus_{n=1}^{\infty} \tilde{\Pi}_n$ and $\widetilde{\Pi}^{(1)} := \bigoplus_{n=1}^{\infty} \widetilde{\Pi}^{(1)}_n$ be direct sums of boundary triplets and $d_* = 0$. Then:

- (i) $\tilde{\Pi}$ and $\widetilde{\Pi}^{(1)}$ are generalized boundary triplets for H_{\min}^* .
- (ii) $\tilde{\Pi}$ and $\widetilde{\Pi}^{(1)}$ are not ordinary boundary triplets for H_{\min}^* .
- (iii) The operators $(H_{\min})_{*0}$ and $(H_{\min})_{*1}$ (see (32)) are self-adjoint and $(H_{\min})_{*j} = \bigoplus_{n=1}^{\infty} H_{nj}$.
- (iv) The mappings \tilde{T}_0 and \tilde{T}_1 are closed and unbounded on $\mathfrak{H}_+ = \text{dom}(H_{\min}^*)$.
- (v) $(H_{\min})_{*0}$ and $(H_{\min})_{*1}$ are not transversal.

Proof. (i) It follows from (93) that the families $\{\tilde{M}_n(i)\}_{n=1}^{\infty}$ and $\{\tilde{M}_n^{-1}(i)\}_{n=1}^{\infty}$ are bounded if $d^* < \infty$. It remains to apply Proposition 3.3.

(ii) If $\lim_{k \rightarrow \infty} d_{nk} = 0$, then $\lim_{k \rightarrow \infty} \text{Im} \tilde{M}_{nk}(i) = \text{Im} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, the second of conditions (51) is violated, hence neither $\tilde{\Pi}$ nor $\widetilde{\Pi}^{(1)}$ forms a boundary triplet for H_{\min}^* .

(iii) follows from (i) and Theorem 3.2(vi).

(iv) Clearly, \tilde{T}_0 and \tilde{T}_1 are unitarily equivalent. Hence \tilde{T}_0 and \tilde{T}_1 might be bounded only simultaneously. Combining (ii) with Proposition 3.8, we conclude that both \tilde{T}_0 and \tilde{T}_1 are unbounded. Further, by Theorem 3.2(iv), \tilde{T}_j is closable. Since, by (iii), $\ker(\tilde{T}_j) = \bigoplus_{n=1}^{\infty} \text{dom}(H_{nj})$ is closed in \mathfrak{H}_+ and $\text{ran}(\tilde{T}_j) = \mathcal{H}$ is closed, the mapping \tilde{T}_j is closed.

(v) follows from (iii) and Proposition 3.6(ii). \square

Remark 4.10. Corollary 4.9 shows that condition $C_1 < \infty$ in Proposition 3.6 is only sufficient for $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n$ to form a generalized boundary triplet.

5. Schrödinger operators with δ -interactions

Let $\mathcal{I} = [0, b)$ and let $X = \{x_n\}_{n=1}^{\infty}$ be defined by (74). In what follows we will always assume that condition (80) is satisfied, i.e. $d^* = \sup_n d_n < \infty$.

The main object of this section is the differential expression

$$\ell_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n), \quad \alpha_n \in \mathbb{R}. \tag{96}$$

In $L^2(\mathcal{I})$, one associates with (96) a symmetric differential operator

$$H_{X,\alpha}^0 := -\frac{d^2}{dx^2}, \quad \text{dom}(H_{X,\alpha}^0) = \left\{ f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) : \begin{array}{l} f'(0) = 0, \quad f(x_{n+}) = f(x_{n-}) \\ f'(x_{n+}) - f'(x_{n-}) = \alpha_n f(x_n) \end{array} \right\}. \quad (97)$$

Denote by $H_{X,\alpha}$ the closure of $H_{X,\alpha}^0$, $H_{X,\alpha} = \overline{H_{X,\alpha}^0}$.

5.1. Parametrization of the operator $H_{X,\alpha}$

Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ and $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplets defined in Theorems 4.1 and 4.7, respectively. By Proposition 2.2, the extension $H_{X,\alpha} (\in \text{Ext } H_{\min})$ admits two representations

$$H_{X,\alpha} = H_{\Theta_j} := H_{\min}^* \upharpoonright \text{dom}(H_{\Theta_j}),$$

$$\text{dom}(H_{\Theta_j}) = \{ f \in \text{dom}(H_{\min}^*) : \{ \Gamma_0^j f, \Gamma_1^j f \} \in \Theta_j \}, \quad j = 1, 2 \quad (98)$$

(cf. (89)) with closed symmetric linear relation $\Theta_j \in \tilde{\mathcal{C}}(\mathcal{H})$, $j = 1, 2$. We show that Θ_2 as well as the operator part Θ_1' of Θ_1 is a Jacobi matrix.

1. *The first parametrization.* We begin with the triplet $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ constructed in Theorem 4.7. For any α the operators $H_{X,\alpha}$ and $H_0^{(2)} := H_{\min}^* \upharpoonright \ker(\Gamma_0^2)$ are disjoint. Hence Θ_2 in (98) is a (closed) operator in $\mathcal{H} = l_2(\mathbb{N})$, $\Theta_2 \in \mathcal{C}(l_2)$. More precisely, consider the Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} 0 & -d_1^{-2} & 0 & 0 & 0 & \dots \\ -d_1^{-2} & -d_1^{-2} & d_1^{-3/2} d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2} d_2^{-1/2} & \alpha_1 d_2^{-1} & -d_2^{-2} & 0 & \dots \\ 0 & 0 & -d_2^{-2} & -d_2^{-2} & d_2^{-3/2} d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2} d_3^{-1/2} & \alpha_2 d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (99)$$

Let $\tau_{X,\alpha}$ be a second order difference expression associated with (99). One defines the corresponding minimal symmetric operator in l_2 by (see [1,6])

$$B_{X,\alpha}^0 f := \tau_{X,\alpha} f, \quad f \in \text{dom}(B_{X,\alpha}^0) := l_{2,0}, \quad \text{and} \quad B_{X,\alpha} = \overline{B_{X,\alpha}^0}. \quad (100)$$

Recall that $B_{X,\alpha}^4$ has equal deficiency indices and $n_+(B_{X,\alpha}) = n_-(B_{X,\alpha}) \leq 1$.

Note that $B_{X,\alpha}$ admits a representation

$$B_{X,\alpha} = R_X^{-1} (\tilde{B}_\alpha - Q_X) R_X^{-1}, \quad \text{where} \quad \tilde{B}_\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (101)$$

and $R_X = \bigoplus_{n=1}^\infty R_n$, $Q_X = \bigoplus_{n=1}^\infty Q_n$ are defined by (94).

⁴ Usually we will identify the Jacobi matrix with (closed) minimal symmetric operator associated with it. Namely, we denote by $B_{X,\alpha}$ the Jacobi matrix (99) as well as the minimal closed symmetric operator (100).

Proposition 5.1. Let $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplet for H_{\min}^* constructed in Theorem 4.7 and let $B_{X,\alpha}$ be the minimal Jacobi operator defined by (99)–(100). Then $\Theta_2 = B_{X,\alpha}$, i.e.,

$$H_{X,\alpha} = H_{B_{X,\alpha}} = H_{\min}^* \upharpoonright \text{dom}(H_{B_{X,\alpha}}), \quad \text{dom}(H_{B_{X,\alpha}}) = \{f \in W^{2,2}(\mathcal{I} \setminus X) : \Gamma_1^2 f = B_{X,\alpha} \Gamma_0^2 f\}.$$

Proof. Let $f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X)$. Then $f \in \text{dom}(H_{X,\alpha})$ if and only if $\tilde{\Gamma}_1^2 f = \tilde{B}_\alpha \tilde{\Gamma}_0^2 f$. Here $\tilde{\Gamma}_j^2 := \bigoplus_{n \in \mathbb{N}} \tilde{\Gamma}_j^{(n)}$ where $\tilde{\Gamma}_j^{(n)}$, $j = 0, 1$, are defined by (91), and \tilde{B}_α is defined by (101). Combining (83), (94) with (101), we rewrite the equality $\tilde{\Gamma}_1^2 f = \tilde{B}_\alpha \tilde{\Gamma}_0^2 f$ as $\Gamma_1^2 f = B_{X,\alpha} \Gamma_0^2 f$.

Taking the closures one completes the proof. \square

Remark 5.2. Note that the matrix (99) has negative off-diagonal entries, although, in the classical theory of Jacobi operators, off-diagonal entries are assumed to be positive. But it is known (see, for instance, [58]) that the (minimal) operator $B_{X,\alpha}$ is unitarily equivalent to the minimal Jacobi operator associated with the matrix

$$B'_{X,\alpha} := \begin{pmatrix} 0 & d_1^{-2} & 0 & 0 & 0 & \dots \\ d_1^{-2} & -d_1^{-2} & d_1^{-3/2} d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2} d_2^{-1/2} & \alpha_1 d_2^{-1} & d_2^{-2} & 0 & \dots \\ 0 & 0 & d_2^{-2} & -d_2^{-2} & d_2^{-3/2} d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2} d_3^{-1/2} & \alpha_2 d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{102}$$

In the sequel we will identify the operators $B_{X,\alpha}$ and $B'_{X,\alpha}$ when investigating those spectral properties of the operator $H_{X,\alpha}$, which are invariant under unitary transformations.

2. The second parametrization. Let us consider the boundary triplet $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ constructed in Theorem 4.1. Now the operators $H_{X,\alpha}$ and $H_0^{(1)} := H_{\min}^* \upharpoonright \ker(\Gamma_0^1)$ are not disjoint, hence by Proposition 2.2(ii), the corresponding linear relation Θ_1 in (98) is not an operator, i.e. has a nontrivial multi-valued part, $\text{mul } \Theta_1 := \{f \in \mathcal{H} : \{0, f\} \in \Theta_1\} \neq \{0\}$.

Let $f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X)$. Then $\Gamma_0^1 f, \Gamma_1^1 f \in l_{2,0}$ and $f \in \text{dom}(H_{X,\alpha})$ if and only if $C_{X,\alpha} \Gamma_1 f = D_{X,\alpha} \Gamma_0 f$, where

$$C_{X,\alpha} := CR_X, \quad D_{X,\alpha} := (D_\alpha - CQ_X)R_X^{-1}, \tag{103}$$

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad D_\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{104}$$

and $R_X = \bigoplus_{n=1}^\infty R_n$, $Q_X = \bigoplus_{n=1}^\infty Q_n$ are defined by (84).

Define a linear relation Θ_1^0 by

$$\Theta_1^0 = \{\{f, g\} \in l_{2,0} \oplus l_{2,0} : D_{X,\alpha} f = C_{X,\alpha} g\}. \tag{105}$$

Hence we obviously get

$$H_{X,\alpha}^0 = H_{\min}^* \upharpoonright \text{dom}(H_{X,\alpha}^0), \tag{106}$$

$$\text{dom}(H_{X,\alpha}^0) = \{f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) : \{\Gamma_0^1 f, \Gamma_1^1 f\} \in \Theta_1^0\}.$$

Straightforward calculations show that Θ_1^0 is symmetric. Moreover, (106) implies that the closure of Θ_1^0 is Θ_1 . Hence Θ_1 is a closed symmetric linear relation. Therefore (see Section 2.1.1), Θ_1 admits the representation

$$\begin{aligned} \Theta_1 &= \Theta_1^{\text{op}} \oplus \Theta_1^\infty, & \mathcal{H} &= \mathcal{H}_{\text{op}} \oplus \mathcal{H}_\infty, \\ \mathcal{H}_{\text{op}} &= \overline{\text{dom}(\Theta_1)} = \overline{\text{dom}(\Theta_1^{\text{op}})}, & \mathcal{H}_\infty &:= \text{mul } \Theta_1, \end{aligned} \tag{107}$$

where $\Theta_1^{\text{op}} (\in \mathcal{C}(\mathcal{H}_{\text{op}}))$ is the operator part of Θ_1 . Moreover, it follows from (103) that

$$\text{mul } \Theta_1 = \ker(C_{X,\alpha}) = \overline{R_X^{-1}(\ker C)}, \quad \Theta_1^\infty = \{ \{0, f\} : f \in \text{mul } \Theta_1 \}. \tag{108}$$

Since $\mathcal{H}_{\text{op}} = \overline{\text{ran}(R_X C^*)}$, the system $\{\mathbf{f}_n\}_{n=1}^\infty$, $\mathbf{f}_n := \frac{\sqrt{d_n}e_{2n} - \sqrt{d_{n+1}}e_{2n+1}}{\sqrt{d_n+d_{n+1}}}$, forms the orthonormal basis in \mathcal{H}_{op} . Next we show that the operator part Θ_1^{op} of Θ_1 is unitarily equivalent to the minimal Jacobi operator

$$B_{X,\alpha} = \begin{pmatrix} r_1^{-2}(\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}) & -(r_1 r_2 d_2)^{-1} & 0 & \dots \\ -(r_1 r_2 d_2)^{-1} & r_2^{-2}(\alpha_2 + \frac{1}{d_2} + \frac{1}{d_3}) & -(r_2 r_3 d_3)^{-1} & \dots \\ 0 & -(r_2 r_3 d_3)^{-1} & r_3^{-2}(\alpha_3 + \frac{1}{d_3} + \frac{1}{d_4}) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{109}$$

where $r_n := \sqrt{d_n + d_{n+1}}$, $n \in \mathbb{N}$. Firstly, observe that

$$B_{X,\alpha} = \tilde{R}_X^{-1} (B_X + \mathcal{A}_\alpha) \tilde{R}_X^{-1}, \tag{110}$$

where

$$\tilde{R}_X = \text{diag}(r_n), \quad \mathcal{A}_\alpha := \text{diag}(\alpha_n), \quad B_X = \begin{pmatrix} \frac{1}{d_1} + \frac{1}{d_2} & -\frac{1}{d_2} & 0 & \dots \\ -\frac{1}{d_2} & \frac{1}{d_2} + \frac{1}{d_3} & -\frac{1}{d_3} & \dots \\ 0 & -\frac{1}{d_3} & \frac{1}{d_3} + \frac{1}{d_4} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{111}$$

Further, let us show that $\{\mathbf{f}_n\}_{n=1}^\infty \subset \text{dom}(\Theta_1^{\text{op}})$. Assume that there exists \mathbf{g}_n such that $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta_1^{\text{op}}$, i.e., $\mathbf{g}_n = \Theta_1^{\text{op}} \mathbf{f}_n$. The latter yields $\mathbf{g}_n \in \mathcal{H}_{\text{op}}$ and hence $\mathbf{g}_n = \sum_{k=1}^\infty g_{n,k} \mathbf{f}_k$. Moreover, after straightforward calculations we obtain

$$\begin{aligned} D_{X,\alpha} \mathbf{f}_1 &= r_1^{-1} (-(\alpha_1 + d_1^{-1} + d_2^{-1})e_3 + d_2^{-1}e_5), \\ D_{X,\alpha} \mathbf{f}_n &= r_n^{-1} (d_n^{-1}e_{2n-1} - (\alpha_n + d_n^{-1} + d_{n+1}^{-1})e_{2n+1} + d_{n+1}^{-1}e_{2n+3}), \quad n \geq 2, \\ C_{X,\alpha} \mathbf{g}_n &= -\sum_{k=1}^\infty g_{n,k} r_k e_{2k+1}, \quad n \geq 1. \end{aligned}$$

Hence $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta$, i.e., equality $D_{X,\alpha} \mathbf{f}_n = C_{X,\alpha} \mathbf{g}_n$ holds, if and only if

$$g_{n,n-1} = -\frac{1}{d_n r_{n-1} r_n}, \quad g_{n,n} = \frac{1}{r_n^2} \left(\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} \right), \quad g_{n,n+1} = -\frac{1}{d_{n+1} r_n r_{n+1}}, \quad n \geq 2,$$

and $g_{n,k} = 0$ for all $k \notin \{n - 1, n, n + 1\}$. Hence $\mathbf{f}_n \in \text{dom}(\Theta_1^{\text{op}})$ and in the basis $\{\mathbf{f}_n\}_{n=1}^\infty$ the matrix representation of the operator Θ_1^{op} coincides with the matrix $B_{X,\alpha}$ defined by (109). Since the operator $B_{X,\alpha}$ of the form (100) and (109) is closed, we conclude that Θ_1^{op} and $B_{X,\alpha}$ are unitarily equivalent.

Let us summarize the above considerations in the following proposition.

Proposition 5.3. *Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ be the boundary triplet constructed in Theorem 4.1 and let the linear relation Θ_1 be defined by (98). Then Θ_1 admits representation (107), where the “pure” relation Θ_1^∞ is determined by (108) and (104), and the operator part Θ_1^{op} is unitarily equivalent to the minimal Jacobi operator $B_{X,\alpha}$ of the form (100) and (109).*

5.2. Self-adjointness

1. We start with a result that reduces the property of $H_{X,\alpha}$ to be self-adjoint to that of the corresponding Jacobi matrices $B_{X,\alpha}$.

Theorem 5.4. *The operator $H_{X,\alpha}$ has equal deficiency indices and $n_+(H_{X,\alpha}) = n_-(H_{X,\alpha}) \leq 1$. Moreover, $n_\pm(H_{X,\alpha}) = n_\pm(B_{X,\alpha})$, where $B_{X,\alpha}$ is the minimal operator associated with the Jacobi matrix either (99) or (109). In particular, $H_{X,\alpha}$ is self-adjoint if and only if $B_{X,\alpha}$ is.*

Proof. Combining Theorem 4.5(i) with Propositions 5.1 and 5.3, we arrive at the equality $n_\pm(H_{X,\alpha}) = n_\pm(B_{X,\alpha})$. It remains to note that for Jacobi matrices $n_\pm(B_{X,\alpha}) \leq 1$ (see [1,6]). \square

The following result is immediate from Theorem 5.4 though we don’t know its direct proof.

Corollary 5.5. *Let $B_{X,\alpha}^{(1)}$ and $B_{X,\alpha}^{(2)}$ be the minimal Jacobi operators associated with (109) and (99), respectively. Then $n_\pm(B_{X,\alpha}^{(1)}) = n_\pm(B_{X,\alpha}^{(2)})$. In particular, $B_{X,\alpha}^{(1)}$ is self-adjoint if and only if so is $B_{X,\alpha}^{(2)}$.*

Remark 5.6. It was found out by Shubin Christ and Stolz [53] that the operator $H_{X,\alpha}$ may be symmetric with $n_\pm(H_{X,\alpha}) = 1$ even if $\mathcal{I} = \mathbb{R}_+$. In this case the set of self-adjoint extensions of $H_{X,\alpha}$ can be described in terms of the classical Sturm–Liouville theory (for details see [9]). Theorem 5.4 enables us to describe self-adjoint extensions of $H_{X,\alpha}$ in a different way. More precisely, consider the boundary triplet Π^2 defined in Theorem 4.7. By Theorem 5.4, $H_{X,\alpha}$ is symmetric if and only if the Jacobi operator $B_{X,\alpha}$ of the form (99)–(100) is also symmetric. By Proposition 2.2, the mapping

$$\tilde{B}_{X,\alpha} \rightarrow H_{\tilde{B}_{X,\alpha}} := H_{\min}^* \upharpoonright \text{dom } H_{\tilde{B}_{X,\alpha}}, \quad \text{dom } H_{\tilde{B}_{X,\alpha}} := \ker(\Gamma_1^2 - \tilde{B}_{X,\alpha} \Gamma_0^2)$$

establishes a bijective correspondence between the sets of self-adjoint extensions of $B_{X,\alpha}$ and $H_{X,\alpha}$.

Using various criteria of self-adjointness of Jacobi matrices (see e.g. [1,6,38,39]), we obtain necessary and sufficient conditions for the operator $H_{X,\alpha}$ to be self-adjoint (symmetric) in $L^2(\mathcal{I})$. We emphasize that different parameterizations (99) and (109) of $H_{X,\alpha}$ lead to different criteria.

Proposition 5.7. *The Hamiltonian $H_{X,\alpha}$ is self-adjoint for any $\alpha = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}$ whenever*

$$\sum_{n=1}^\infty d_n^2 = \infty. \tag{112}$$

Proof. Let $B_{X,\alpha}$ be the minimal Jacobi operator of the form (102), (100). By Carleman’s theorem [1, [6, Chapter VII.1.2], $B_{X,\alpha}$ is self-adjoint provided that

$$\sum_{n=1}^{\infty} (d_n^2 + d_n^{3/2} d_{n+1}^{1/2}) = \infty. \tag{113}$$

Clearly, $d_n^2 < d_n^2 + d_n^{3/2} d_{n+1}^{1/2} \leq \frac{7}{4} d_n^2 + \frac{1}{4} d_{n+1}^2$ and hence relations (112) and (113) are equivalent.

One completes the proof by applying Theorem 5.4. \square

Remark. Let us note that for Schrödinger operators with continuous strongly oscillating potentials the condition similar to (112) has been appeared in [27, Theorem 1].

If $\limsup_n d_n > 0$, then condition (112) is obviously satisfied and Proposition 5.7 yields the following improvement of the result of Gesztesy and Kirsch (cf. [19, Theorem 3.1]).

Corollary 5.8. (See [19].) *If $\limsup_n d_n > 0$ (in particular, $\liminf_n d_n > 0$), then $H_{X,\alpha}$ is self-adjoint.*

In fact, Gesztesy and Kirsch [19] established self-adjointness for the operator $H_{X,\alpha,q}$ (see (1)–(3)) for a wide class of unbounded potentials assuming only $d_* > 0$. Note also that under assumption $d_* > 0$ Corollary 5.8 was reproved by Kochubei [35] in the framework of boundary triplets approach.

2. If $\mathcal{I} = \mathbb{R}_+$ and condition (112) is violated, then the operator $H_{X,\alpha}$ might be symmetric with nontrivial deficiency indices $n_{\pm}(H_{X,\alpha}) = 1$. In particular, this is the case when $\mathcal{I} = \mathbb{R}_+$, $d_n = 1/n$, and $\alpha_n = -(2n + 1)$ (see [53, Remark on pp. 495–496]). Our next result is partially inspired by the example of C. Shubin Christ and G. Stolz, and it also shows that Proposition 5.7 is sharp.

Proposition 5.9. *Let $\{d_n\}_{n=1}^{\infty} \in l_2$, $d_n \geq 0$, and*

$$d_{n-1} d_{n+1} \geq d_n^2, \quad n \in \mathbb{N}. \tag{114}$$

If, in addition, the strengths α_n of δ -interactions satisfy

$$\sum_{n=1}^{\infty} d_{n+1} \left| \alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} \right| < \infty, \tag{115}$$

then the operator $H_{X,\alpha}$ is symmetric with $n_{\pm}(H_{X,\alpha}) = 1$.

Proof. Consider the Jacobi matrix (109). To apply [39, Theorem 1] we denote $a_n := r_n^{-2}(\alpha_n + 1/d_n + 1/d_{n+1})$ and $b_n := (r_n r_{n+1} d_{n+1})^{-1}$, $n \in \mathbb{N}$, and define the sequence $\{c_n\}_{n=1}^{\infty}$ as follows

$$c_1 := b_1, \quad c_2 := 1, \quad c_{n+1} := -\frac{b_{n-1}}{b_n} c_{n-1}, \quad n \in \mathbb{N}.$$

It is easily seen that

$$c_{n+1} = (-1)^{n+1} r_{n+1} \frac{d_{n+1} d_{n-1} \cdots}{d_n d_{n-2} \cdots} \cdot \tilde{c}, \quad n \in \mathbb{N}; \quad \tilde{c} := \begin{cases} c_1 r_1^{-1}, & n = 2k + 1, \\ c_2 r_2^{-1}, & n = 2k. \end{cases}$$

Due to (114), we obtain

$$\frac{d_{n+1} d_{n-1} \cdots}{d_n d_{n-2} \cdots} = \sqrt{d_{n+2}} \frac{d_{n+1}}{\sqrt{d_{n+2} d_n}} \frac{d_{n-1} \cdots}{\sqrt{d_n d_{n-2} \cdots}} \leq C \sqrt{d_{n+2}}, \quad n \in \mathbb{N}. \tag{116}$$

Therefore,

$$|c_{n+1}| \leq \tilde{c}Cr_{n+1}\sqrt{d_{n+2}},$$

and hence $\{c_n\}_{n=1}^\infty \in l_2$. On the other hand, it follows from (115) and (116) that $\sum_{n=1}^\infty |a_n|c_n^2 < \infty$. By [39, Theorem 1], this inequality together with the inclusion $\{c_n\}_{n=1}^\infty \in l_2$ yields $n_\pm(B_{X,\alpha}) = 1$. It remains to apply Theorem 5.4. \square

Remark 5.10. Note that in the case $\mathcal{I} = \mathbb{R}_+$ the self-adjointness of $H_{X,\alpha}$ for arbitrary $\alpha \subset \mathbb{R}$ was erroneously stated in [44,45].

Let us present sufficient conditions for self-adjointness in the case when (112) does not hold.

Proposition 5.11. Assume that (112) does not hold. Let also $\alpha = \{\alpha_n\}_{n=1}^\infty$ and $X = \{x_n\}_{n=1}^\infty$ satisfy one of the following conditions:

(i)
$$\sum_{n=1}^\infty |\alpha_n|d_n d_{n+1}r_{n-1}r_{n+1} = \infty, \quad r_n = \sqrt{d_n + d_{n+1}}. \tag{117}$$

(ii) There exists a positive constant $C_1 > 0$ such that

$$\alpha_n + \frac{1}{d_n} \left(1 + \frac{r_n}{r_{n-1}}\right) + \frac{1}{d_{n+1}} \left(1 + \frac{r_n}{r_{n+1}}\right) \leq C_1(d_n + d_{n+1}), \quad n \in \mathbb{N}. \tag{118}$$

(iii) There exists a positive constant $C_2 > 0$ such that

$$\alpha_n + \frac{1}{d_n} \left(1 - \frac{r_n}{r_{n-1}}\right) + \frac{1}{d_{n+1}} \left(1 - \frac{r_n}{r_{n+1}}\right) \geq -C_2(d_n + d_{n+1}), \quad n \in \mathbb{N}. \tag{119}$$

Then the operator $H_{X,\alpha}$ is self-adjoint in $L^2(\mathcal{I})$.

Proof. (i) Since $\{d_n\}_{n=1}^\infty \in l_2$, we get $\sum_{n=1}^\infty (d_n + d_{n+1})r_{n-1}r_{n+1} < C \sum_{n=1}^\infty d_n^2 < \infty$. Applying the Dennis–Wall test [1, p. 25, Problem 2] to matrix (109), we obtain that (117) yields self-adjointness of the minimal operator $B_{X,\alpha}$ associated with (109). By Theorem 5.4, $H_{X,\alpha} = H_{X,\alpha}^*$.

(ii)–(iii) Applying [6, Theorem VII.1.4] (see also [1, Problem 3, p. 37]) to the Jacobi matrix (109), we obtain that conditions (118) and (119) guarantee self-adjointness of $B_{X,\alpha}$. Theorem 5.4 completes the proof. \square

Conditions (i)–(iii) show that if $H_{X,\alpha}$ is self-adjoint, then the coefficients α_n cannot tend to ∞ very fast. Let us demonstrate this by considering an example.

Example 5.12. Let $\mathcal{I} = \mathbb{R}_+$, $x_0 = 0$, $x_n - x_{n-1} = d_n := 1/n$, $n \in \mathbb{N}$. Consider the operator

$$H_A := -\frac{d^2}{dx^2} + \sum_{n=1}^\infty \alpha_n \delta(x - x_n). \tag{120}$$

Clearly, $\{d_n\}_{n=1}^\infty \in l_2$, i.e., condition (112) is violated. Applying Propositions 5.9 and 5.11, after straightforward calculations we obtain:

- (i) If $\sum_{n=1}^\infty |\alpha_n|n^{-3} = \infty$, then the operator H_A is self-adjoint (cf. Proposition 5.11(i)).
- (ii) If $\alpha_n \leq -4(n + \frac{1}{2}) + O(n^{-1})$, then H_A is self-adjoint (cf. Proposition 5.11(ii)).
- (iii) If $\alpha_n \geq -Cn^{-1}$, $n \in \mathbb{N}$, $C \equiv \text{const} > 0$, then H_A is self-adjoint (cf. Proposition 5.11(iii)).
- (iv) If $\alpha_n = -2n - 1 + O(n^{-\varepsilon})$ with some $\varepsilon > 0$, then $n_\pm(H_A) = 1$ (cf. Proposition 5.9).

Conditions (ii) and (iii) show that there is a gap between conditions of self-adjointness. Moreover, (iii) shows that for the case of positive interactions α_n the operator H_A is self-adjoint. We can extend (iv) as follows.

Proposition 5.13. *Let the Hamiltonian H_A be the same as in Example 5.12. If*

$$\alpha_n = a \left(n + \frac{1}{2} \right) + O(n^{-1}), \quad a \in (-4, 0), \tag{121}$$

then the operator H_A is symmetric with $n_{\pm}(H_A) = 1$.

Proof. Define the sequence

$$\tilde{r}_{n+1} := -\frac{d_{n+1}}{\tilde{r}_n}, \quad \tilde{r}_1 := 1, \quad d_n = \frac{1}{n}, \quad n \in \mathbb{N}. \tag{122}$$

Then

$$\tilde{r}_{n+1} = (-1)^n \frac{n(n-2) \cdots}{(n+1)(n-1) \cdots} = \frac{n!!}{(n+1)!!}. \tag{123}$$

Let us estimate \tilde{r}_n . Observe that

$$(2k-1)!! = 2^k \frac{\Gamma(k + \frac{1}{2})}{\Gamma(1/2)}, \quad (2k)!! = 2^k \Gamma(k+1),$$

where $\Gamma(\cdot)$ is the classical Γ -function. Using the asymptotic of $\Gamma(\cdot)$, we get

$$(4k+1)\tilde{r}_{2k}^2 = \frac{4}{\pi}(1 + O(k^{-2})), \quad (4k+3)\tilde{r}_{2k+1}^2 = \pi(1 + O(k^{-2})), \quad k \rightarrow \infty. \tag{124}$$

Indeed, consider the first equality in (124). Since $\Gamma(1/2) = \sqrt{\pi}$ and

$$\Gamma(k) = \sqrt{2\pi} e^{-k} k^{k-1/2} \left(1 + \frac{1}{12k} + O(k^{-2}) \right), \quad \left(1 + \frac{1}{k} \right)^k = e \left(1 - \frac{1}{2k} + O(k^{-2}) \right),$$

we obtain

$$\begin{aligned} (4k+1)\tilde{r}_{2k}^2 &= (4k+1) \frac{\Gamma(k+1/2)^2}{\pi \Gamma(k+1)^2} = (4k+1) \frac{e}{\pi} \frac{(k+1/2)^{2k} (1 + \frac{1}{6(2k+1)} + O(k^{-2}))^2}{(k+1)^{2k+1} (1 + \frac{1}{6(2k+2)} + O(k^{-2}))^2} \\ &= \frac{e}{\pi} \frac{4k+1}{k+1/2} \left(1 + \frac{1}{2k+1} \right)^{-(2k+1)} (1 + O(k^{-2})) = \frac{4}{\pi} (1 + O(k^{-2})), \quad k \rightarrow \infty. \end{aligned}$$

Further, define $\alpha^0 := \{\alpha_n^0\}_{n=1}^\infty$ by setting

$$\alpha_n^0 := \begin{cases} -(4k+1) + \frac{4}{\pi} (1 + \frac{a}{2}) \tilde{r}_n^{-2}, & n = 2k, \\ -(4k+3) + \pi (1 + \frac{a}{2}) \tilde{r}_n^{-2}, & n = 2k+1. \end{cases}$$

Clearly, by (124), α^0 satisfies (121). Moreover, for this choice of α^0 we get

$$B_X + \mathcal{A}_{\alpha^0} = \tilde{R}_1^{-1} J_a \tilde{R}_1^{-1},$$

where B_X is defined by (111), $\mathcal{A}_{\alpha^0} = \text{diag}(\alpha_n^0)$, $\tilde{R}_1 := \text{diag}(\tilde{r}_n)$, and

$$J_a := \begin{pmatrix} \pi(1 + \frac{a}{2}) & 1 & 0 & 0 & \dots \\ 1 & \frac{4}{\pi}(1 + \frac{a}{2}) & 1 & 0 & \dots \\ 0 & 1 & \pi(1 + \frac{a}{2}) & 1 & \dots \\ 0 & 0 & 1 & \frac{4}{\pi}(1 + \frac{a}{2}) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The Floquet determinant (see, for instance, [58, §7.1]) of the periodic Jacobi matrix J_a is $\Delta_a(\lambda) = -2 + (\lambda - \frac{4}{\pi}(1 + \frac{a}{2}))(\lambda - \pi(1 + \frac{a}{2}))$. Note that all solutions of $\tau_a f = 0$ are bounded if $|\Delta_a(0)| < 2$ (here τ_a is the difference expression associated with the matrix J_a). The latter is equivalent to the inequality $0 < |1 + \frac{a}{2}| < 1$. Moreover, all solutions of $\tau_{-2} f = 0$ are bounded too. Therefore, all solutions of $\tau_a f = 0$ are bounded if

$$|2 + a| < 2.$$

Furthermore, g solves $\tau_{X,\alpha} y = 0$ precisely when $\tilde{R}_X \tilde{R}_1 g$ solves $\tau_a f = 0$. By (123)–(124) and (111), we get $\{\tilde{r}_n \tilde{r}_n\}_{n \in \mathbb{N}} \in l_2$. Hence all solutions of the equation $\tau_{X,\alpha} y = 0$ are l_2 solutions, that is the operator B_{X,α^0} is symmetric with $n_{\pm}(B_{X,\alpha^0}) = 1$. Since bounded perturbations do not change the deficiency indices of $B_{X,\alpha}$, we complete the proof by applying Theorem 4.5(i). \square

5.3. Resolvent comparability

Let us fix $X = \{x_n\}_1^\infty \subset \mathcal{I}$ and consider the Hamiltonians H_{X,α_1} and H_{X,α_2} with strengths $\alpha_1 = \{\alpha_n^{(1)}\}_{n=1}^\infty$ and $\alpha_2 = \{\alpha_n^{(2)}\}_{n=1}^\infty$, respectively.

Proposition 5.14. *Suppose H_{X,α_1} and H_{X,α_2} are self-adjoint and B_{X,α_1} and B_{X,α_2} are the corresponding (self-adjoint) Jacobi operators defined either by (99) or by (109). Then for any $z \in \rho(H_{X,\alpha_1}) \cap \rho(H_{X,\alpha_2})$ and $p \in (0, \infty]$ the inclusion*

$$(H_{X,\alpha_1} - z)^{-1} - (H_{X,\alpha_2} - z)^{-1} \in \mathfrak{S}_p \tag{125}$$

is equivalent to the inclusion

$$(B_{X,\alpha_1} - i)^{-1} - (B_{X,\alpha_2} - i)^{-1} \in \mathfrak{S}_p. \tag{126}$$

Proof. Combining Theorem 4.5 with Proposition 5.3, we get the result with B_{X,α_j} defined by (109). The result with the matrices defined by (99) is implied by combining Proposition 5.1 with Remark 4.8. \square

Next we present simple sufficient condition.

Corollary 5.15. *If $\{\frac{\alpha_n^{(1)} - \alpha_n^{(2)}}{d_{n+1}}\}_{n=1}^\infty \in l_p$, $p \in (0, \infty)$ ($\in c_0$, $p = \infty$), then inclusion (125) holds.*

Proof. Clearly, $l_{2,0} \subset \text{dom}(B_{X,\alpha_1}) \cap \text{dom}(B_{X,\alpha_2})$. On the other hand, for any $f \in l_{2,0}$ (101) yields

$$B_{X,\alpha_2} f - B_{X,\alpha_1} f = R_X^{-1} (\tilde{B}_{\alpha_1} - \tilde{B}_{\alpha_2}) R_X^{-1} f = \bigoplus_{n=1}^\infty \begin{pmatrix} \frac{\alpha_n^{(1)} - \alpha_n^{(2)}}{d_{n+1}} & 0 \\ 0 & 0 \end{pmatrix} f.$$

Hence and due to the assumption, $\overline{B_{X,\alpha_2} - B_{X,\alpha_1}} \in \mathfrak{S}_p \subset [\mathcal{H}]$ and $\text{dom}(B_{X,\alpha_1}) = \text{dom}(B_{X,\alpha_2})$. It remains to apply Proposition 2.5. \square

In the case $d_* > 0$, the resolvent comparability criterion was obtained in [35]. We omit the corresponding proof, though it can be extracted from Proposition 5.14.

Corollary 5.16. (See [35].) *If $0 < d_* \leq d^* < \infty$, then (125) is equivalent to the inclusion*

$$(\alpha_n^{(1)} - i)^{-1} - (\alpha_n^{(2)} - i)^{-2} \in l_p, \quad p \in (0, \infty) (\in c_0, \text{ if } p = \infty). \tag{127}$$

Moreover, if $\{\alpha_n^{(j)}\}_{n=1}^\infty \in l_\infty$, then (127) holds precisely when $\{\alpha_n^{(1)} - \alpha_n^{(2)}\}_{n=1}^\infty \in l_p (\in c_0)$.

5.4. Operators with discrete spectrum

Combining the results of Section 5.1 with Theorem 4.5, we obtain the discreteness criterion for the Hamiltonian $H_{X,\alpha}$.

Theorem 5.17. *Let $B_{X,\alpha}$ be the minimal Jacobi operator defined either by (99) or by (109).*

- (i) *If $n_\pm(B_{X,\alpha}) = 1$, then self-adjoint extensions of $H_{X,\alpha}$ have discrete spectrum.*
- (ii) *If $B_{X,\alpha} = B_{X,\alpha}^*$, then the Hamiltonian $H_{X,\alpha} (= H_{X,\alpha}^*)$ has discrete spectrum if and only if*
 - $\lim_{n \rightarrow \infty} d_n = 0$, and
 - $B_{X,\alpha}$ has discrete spectrum.

Proof. (i) To be precise, let $B_{X,\alpha}$ be defined by (99). Since $n_\pm(B_{X,\alpha}) = 1$, any self-adjoint extension of $B_{X,\alpha}$ has discrete spectrum (see [1,6]). Moreover, by Corollary 5.8, $\lim_{n \rightarrow \infty} d_n = 0$. Hence the operator H_0 defined by (90) has discrete spectrum too. The Krein resolvent formula (20) implies that any self-adjoint extension of $H_{X,\alpha}$ is discrete.

(ii) follows from Theorem 4.5(iv) and Remark 4.8. \square

Next we present some sufficient conditions for the self-adjoint Hamiltonian $H_{X,\alpha}$ to be discrete.

Proposition 5.18. *Let the minimal Jacobi operator $B_{X,\alpha}$ of the form (99) be self-adjoint and $\lim_{n \rightarrow \infty} d_n = 0$. If*

$$\lim_{n \rightarrow \infty} \frac{|\alpha_n|}{d_n} = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{d_n \alpha_n} > -\frac{1}{4}, \tag{128}$$

then the operator $H_{X,\alpha}$ has discrete spectrum.

Proof. Applying [10, Theorem 8] to the operator $B'_{X,\alpha}$ of the form (102), we obtain that the spectrum of $B'_{X,\alpha}$ is discrete provided that $\lim_{n \rightarrow \infty} d_n = 0$ and conditions (128) are satisfied. Theorem 5.17 completes the proof. \square

Proposition 5.18 enables us to construct Hamiltonians $H_{X,\alpha}$ with discrete spectrum, which are not lower semibounded.

Example 5.19. (a) Let $\mathcal{I} = \mathbb{R}_+$, $x_n = \sqrt{n}$, $n \in \mathbb{N}$. Then $d_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} \approx \frac{1}{2\sqrt{n}}$ and, by Proposition 5.7, the operator $H_{X,\alpha}$ is self-adjoint for arbitrary $\alpha = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}$. Consider the operator

$$H_\varepsilon := -\frac{d^2}{dx^2} + \sum_{n=1}^\infty n^{-\varepsilon} \delta(x - \sqrt{n}), \quad \varepsilon \in (0, 1/2).$$

Clearly, conditions (128) hold and hence the operator H_ε is discrete if $\varepsilon \in (0, 1/2)$.

(b) Again, let $\mathcal{I} = \mathbb{R}_+$, $x_n = \sqrt{n}$, $n \in \mathbb{N}$. Define $\alpha_n = -C\sqrt{n}$, $C \equiv \text{const} \in \mathbb{R}$. By Proposition 5.18, the operator

$$H_C := -\frac{d^2}{dx^2} - \sum_{n=1}^{\infty} C\sqrt{n} \delta(x - \sqrt{n})$$

has discrete spectrum if $C > 8$. Moreover, the operator H_C is not lower semibounded since so is the operator considered in Proposition 5.28 (see below).

Remark 5.20. It was stated in [46] that the spectrum $\sigma(H_{X,\alpha})$ of $H_{X,\alpha}$ is not discrete whenever $\alpha \in l^\infty$. However, Example 5.19(a) shows that $\sigma(H_{X,\alpha})$ may be discrete even if $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Proposition 5.21. *Let the operator $B_{X,\alpha}$ defined by (109) be self-adjoint and $\lim_{n \rightarrow \infty} d_n = 0$. If*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|\alpha_n + 1/d_n + 1/d_{n+1}|}{d_n + d_{n+1}} &= \infty, \\ \lim_{n \rightarrow \infty} \left(\alpha_n d_{n+1} + 1 + \frac{d_{n+1}}{d_n} \right)^{-1} \left(\alpha_{n+1} d_{n+1} + 1 + \frac{d_{n+1}}{d_{n+2}} \right)^{-1} &< \frac{1}{4}, \end{aligned} \tag{129}$$

then the operator $H_{X,\alpha}$ has discrete spectrum.

Proof. Applying [10, Theorem 8] to the Jacobi matrix $B_{X,\alpha}$ of the form (109) we get that $B_{X,\alpha}$ is discrete. Since $\lim_{n \rightarrow \infty} d_n = 0$, by Theorem 5.17 so is $H_{X,\alpha}$. \square

Remark 5.22. In the case $\lim_{n \rightarrow \infty} \frac{d_n}{d_{n+1}} = 1$, Proposition 5.18 follows from Proposition 5.21. Let us also note that the second condition in (128) (in (129)) is sharp. In [57], under additional mild assumptions on coefficients it is shown that the operator $B_{X,\alpha}$ has absolutely continuous spectrum if the limit in (128) is less than $-\frac{1}{4}$ (resp. greater than $\frac{1}{4}$) and $\{d_n\}_{n \in \mathbb{N}} \notin l_2$.

Proposition 5.23. *Assume that $\lim_{n \rightarrow \infty} d_n = 0$ and*

$$\lim_{n \rightarrow \infty} \frac{1}{(d_n + d_{n+1})} \left(\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} - \frac{r_{n-1}}{d_n r_n} - \frac{r_{n+1}}{d_{n+1} r_n} \right) = +\infty, \tag{130}$$

where $r_n = \sqrt{d_n + d_{n+1}}$. Then the operator $H_{X,\alpha}$ is self-adjoint and has discrete spectrum.

Proof. By Proposition 5.11(iii), the operator $B_{X,\alpha}$ defined by (109) is self-adjoint. By [11, Theorem 3.1], (130) yields discreteness of $B_{X,\alpha}$. It remains to apply Theorem 5.17. \square

5.5. Semiboundedness

We start with the general criterion of semiboundedness.

Theorem 5.24. *Let the minimal Jacobi operator $B_{X,\alpha}$ be defined by (100) and (109). Then the operator $H_{X,\alpha}$ is lower semibounded if and only if $B_{X,\alpha}$ is lower semibounded.*

Proof. According to (98), $H_{X,\alpha} = H_{\Theta_1}$. By Theorem 4.5(ii), the operator $H_{X,\alpha} = H_{\Theta_1}$ is lower semibounded if and only if Θ_1 is lower semibounded. It remains to note that by Proposition 5.3, the operator part Θ_1^{op} of Θ_1 is unitarily equivalent to the operator $B_{X,\alpha}$ defined by (100) and (109). \square

Let us present several conditions for semiboundedness in terms of $X = \{x_n\}_1^\infty$ and $\alpha = \{\alpha_n\}_1^\infty$. The following result has been obtained in [7] using the form method.

Corollary 5.25. (See [7].) *Let $d_* > 0$. Then the operator $H_{X,\alpha}$ is lower semibounded if and only if*

$$\inf_{n \rightarrow \infty} \alpha_n > -\infty. \tag{131}$$

Proof. Since $d_* > 0$, the operators B_X, R_X, R_X^{-1} in (110) are bounded. Therefore, $B_{X,\alpha}$ is semibounded if and only if so is \mathcal{A}_α , that is the sequence $\alpha = \{\alpha_n\}_{n=1}^\infty$. \square

In the case $d_* = 0$ the situation becomes more complicated. Indeed, condition (131) is no longer necessary for lower semiboundedness (see [7, Example 2]). Moreover, we will show that (131) is no longer sufficient (cf. [44, Corollary 2] where the opposite statement is announced). Moreover, $H_{X,\alpha}$ might be non-semibounded below even if $\inf_{n \rightarrow \infty} \alpha_n = 0$.

We begin with the following sufficient condition.

Corollary 5.26. *The Hamiltonian $H_{X,\alpha}$ is semibounded below whenever*

$$\inf_{n \rightarrow \infty} \frac{\alpha_n}{d_n + d_{n+1}} > -\infty. \tag{132}$$

Proof. The matrix B_X in (111) admits the representation $B_X = (I - U)D_X^{-1}(I - U^*)$, where $D_X := \text{diag}(d_n)$ and U is unilateral shift in l_2 . Hence B_X is nonnegative, $B_X \geq 0$, and we get

$$B_{X,\alpha} = \tilde{R}_X^{-1}(B_X + \mathcal{A}_\alpha)\tilde{R}_X^{-1} \geq \tilde{R}_X^{-1}\mathcal{A}_\alpha\tilde{R}_X^{-1}.$$

Since $\tilde{R}_X = \text{diag}(r_n)$ and $\mathcal{A}_\alpha = \text{diag}(\alpha_n)$ we obtain lower semiboundedness of $B_{X,\alpha}$ by combining the last inequality with condition (132). Theorem 5.24 completes the proof. \square

Remark 5.27. In the case $d_* > 0$, condition (132) is equivalent to (131) and hence is also necessary for semiboundedness of $H_{X,\alpha}$. If $d_* = 0$, then (132) is only sufficient (see [7, Example 2]).

Note that condition (132) may be violated even if $\alpha_n \rightarrow 0$. Next example shows that in this case the operator $H_{X,\alpha}$ might be non-semibounded below.

Proposition 5.28. *Let $\mathcal{I} = \mathbb{R}_+$ and $x_n = \sqrt{n}$. If $\alpha_n = -n^{-\varepsilon}$ with $\varepsilon \in [0, 1/2)$, then the operator $H_{X,\alpha}$ is self-adjoint and not semibounded below in $L^2(\mathcal{I})$.*

Proof. Note that $d_n = \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n-1} + \sqrt{n}} \asymp \frac{1}{2\sqrt{n}}$ as $n \rightarrow \infty$. Hence, by Proposition 5.7, the operator $H_{X,\alpha}$ is self-adjoint.

By Proposition 5.3, $H_{X,\alpha} = H_{\Theta_1}$, where the operator part Θ_1' of Θ_1 is unitarily equivalent to the Jacobi matrix $B_{X,\alpha}$ of the form (109). Clearly, $B_{X,\alpha}$ admits the following representation

$$B_{X,\alpha} = \tilde{R}_X^{-1}(B_X + \mathcal{A}_\alpha)\tilde{R}_X^{-1} = \tilde{R}_X^{-1}[D_X^{-1/2}(J_{\text{per}} + UK^2U^* - UK - KU^* + \tilde{\mathcal{A}}_\alpha)D_X^{-1/2}]\tilde{R}_X^{-1},$$

where $D_X = \text{diag}(d_n)$, U is unilateral shift in l_2 , and

$$J_{\text{per}} = \begin{pmatrix} 2 & 1 & 0 & \dots \\ 1 & 2 & 1 & \dots \\ 0 & 1 & 2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad K = \text{diag}(k_n), \quad k_n := \frac{\sqrt{d_{n-1}}}{\sqrt{d_n}} - 1,$$

$$\tilde{\mathcal{A}}_\alpha = \mathcal{A}_\alpha D_X = \text{diag}(\tilde{\alpha}_n).$$

Note that $\tilde{\alpha}_n = \alpha_n d_n \asymp -n^{-(1/2+\varepsilon)}$ and $k_n = O(n^{-1})$ as $n \rightarrow \infty$. Since $\varepsilon \in [0, 1/2)$, the sum $\frac{1}{2}\tilde{\alpha}_n + k_n$ is negative for n large enough. Therefore, $\tilde{B}_{X,\alpha} := \tilde{R}_X^{-1} [D_X^{-1/2} (J_{\text{per}} + \frac{1}{2}\tilde{A}_\alpha) D_X^{-1/2}] \tilde{R}_X^{-1}$ is lower semibounded if so is $B_{X,\alpha}$.

Let $f_N = (f_1, \dots, f_{2N}, 0, 0, \dots)$, where $f_{2n} = 1, f_{2n-1} = -1, n \in \{1, \dots, N\}$. Then we get

$$\begin{aligned}
 (\tilde{A}_\alpha f_N, f_N) &= \sum_{n=1}^{2N} \alpha_n d_n = - \sum_{n=1}^{2N} \frac{n^{-\varepsilon}}{\sqrt{n} + \sqrt{n-1}} \geq - \sum_{n=1}^{2N} n^{-\varepsilon-1/2}, \\
 (J_{\text{per}} f_N, f_N) &= 2, \quad \|\tilde{R}_X D_X^{1/2} f_N\|^2 = \sum_{n=1}^{2N} d_n (\sqrt{d_n} + \sqrt{d_{n+1}})^2 \geq \sum_{n=1}^{2N} \frac{1}{n+1} = \sum_{n=2}^{2N+1} \frac{1}{n}.
 \end{aligned}$$

Therefore,

$$\inf_{f \neq 0} \frac{(\tilde{B}_{X,\alpha} f, f)}{\|f\|^2} \leq \frac{((J_{\text{per}} + \frac{1}{2}\tilde{A}_\alpha) f_N, f_N)}{\|\tilde{R}_X D_X^{1/2} f_N\|^2} \leq - \frac{\sum_{n=1}^{2N} n^{-\varepsilon-1/2}}{\sum_{n=2}^{2N+1} n^{-1}} \asymp - \frac{(2N)^{1/2-\varepsilon}}{\log(2N+1)}, \quad N \rightarrow \infty.$$

Since $\varepsilon \in [0, 1/2)$, the operator $\tilde{B}_{X,\alpha}$ is not lower semibounded and hence so is $B_{X,\alpha}$. By Theorem 5.24, $H_{X,\alpha}$ is not lower semibounded too. \square

Remark 5.29. The matrix $B_{X,\alpha}$ in Proposition 5.28 can be considered as an unbounded Jacobi matrix with periodically modulated entries [29,30]. But in the above situation we cannot apply the criteria of Janas and Naboko [30, §2] since $\sigma_{ac}(J_{\text{per}}) = [0, 2]$. In the proof of Proposition 5.28 we follow the line of [30, Example 3.2].

Remark 5.30. (i) In [44, Theorem 3.2], it was announced (without proof) that $H_{X,\alpha}$ is lower semibounded if $\mathcal{I} = \mathbb{R}_+$ and (131) holds. However, by Proposition 5.28, $H_{X,\alpha}$ may be not lower semibounded even in the case $\lim_{n \rightarrow \infty} \alpha_n = 0$.

(ii) Using the form method, semiboundedness of the operator $H_{X,\alpha}$ has been studied by Brasche (see [7] and references therein). In the case when all strength α_n are negative, he obtained a criterion for the operator $H_{X,\alpha}$ to be lower semibounded [7, Theorem 3]. Note also that Proposition 5.28 can be extracted from [7, Theorem 3].

Semiboundedness and discreteness of the operator $H_{X,\alpha}$ will be treated by using the form method in our forthcoming paper.

6. Operators with δ' -interactions

Let \mathcal{I} and X be as in Section 4 and let $\beta = \{\beta_n\}_{n=1}^\infty \subset \mathbb{R}$. Consider the following operator in $L^2(\mathcal{I})$

$$\begin{aligned}
 H_{X,\beta}^0 &:= -\frac{d^2}{dx^2}, \\
 \text{dom}(H_{X,\alpha}^0) &= \left\{ f \in W_{\text{comp}}^{2,2}(\mathcal{I} \setminus X) : \begin{aligned} &f'(0+) = 0, \quad f'(x_n+) = f'(x_n-), \quad x_n \in X \\ &f(x_n+) - f(x_n-) = \beta_n f'(x_n) \end{aligned} \right\}. \quad (133)
 \end{aligned}$$

Note that $H_{X,\beta}^0$ is symmetric in $L^2(\mathcal{I})$. Denote its closure by $H_{X,\beta}, H_{X,\beta} = \overline{H_{X,\beta}^0}$. The Hamiltonian $H_{X,\beta}$ is known in the literature as the Hamiltonian of δ' -interactions with strengths β_n at points x_n (see [3,18,54] and also [21]) and it is associated with the formal differential expression

$$\ell_{X,\beta} := -\frac{d^2}{dx^2} + \sum_{n=1}^\infty \beta_n (\cdot, \delta'_n) \delta'_n, \quad \beta_n \in \mathbb{R}, \quad (134)$$

where $\delta'_n := \delta'(x - x_n)$. In the following we always assume that $\beta_n \neq 0, n \in \mathbb{N}$, and $d^* < \infty$.

6.1. Parametrization of the operator $H_{X,\beta}$

Following the line of reasoning of Section 5.1, we treat $H_{X,\beta}$ as an extension of H_{\min} defined by (75). As in Section 5.1 we consider two parameterizations of $H_{X,\beta}$ corresponding to the boundary triplets constructed in Theorems 4.1 and 4.7.

1. The first parametrization. We begin with the triplet $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ constructed in Theorem 4.1. Since $\beta_n \neq 0, n \in \mathbb{N}$, the operator $H_{X,\beta}$ is disjoint with the operator H_0 defined by (90) and (81). Therefore, the linear relation Θ_1 associated with $H_{X,\beta}$ is a closed operator, $\Theta_1 \in \mathcal{C}(\mathcal{H})$.

Consider the following Jacobi matrix

$$B_{X,\beta} := \begin{pmatrix} d_1^{-2} & d_1^{-2} & 0 & 0 & 0 & \dots \\ d_1^{-2} & \frac{d_1^{-1}}{\beta_1} + d_1^{-2} & \frac{d_1^{-1/2}d_2^{-1/2}}{\beta_1} & 0 & 0 & \dots \\ 0 & \frac{d_1^{-1/2}d_2^{-1/2}}{\beta_1} & \frac{d_2^{-1}}{\beta_1} + d_2^{-2} & d_2^{-2} & 0 & \dots \\ 0 & 0 & d_2^{-2} & \frac{d_2^{-1}}{\beta_2} + d_2^{-2} & \frac{d_2^{-1/2}d_3^{-1/2}}{\beta_2} & \dots \\ 0 & 0 & 0 & \frac{d_2^{-1/2}d_3^{-1/2}}{\beta_2} & \frac{d_3^{-1}}{\beta_2} + d_3^{-2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{135}$$

Note that

$$B_{X,\beta} = R_X^{-1}(\tilde{B}_\beta - Q_X)R_X^{-1}, \quad \tilde{B}_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\beta_1} & \frac{1}{\beta_1} & 0 & \dots \\ 0 & \frac{1}{\beta_1} & \frac{1}{\beta_1} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\beta_2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{136}$$

and $R_X = \bigoplus_{n=1}^\infty R_n, Q_X = \bigoplus_{n=1}^\infty Q_n$ are determined by (84). Arguing as in the proof of Proposition 5.1, we arrive at the following representation.

Proposition 6.1. *Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ be the boundary triplet constructed in Theorem 4.1 and let $B_{X,\beta}$ be the minimal closed symmetric operator associated with the matrix (135). Then*

$$H_{X,\beta} = H_{B_{X,\beta}} := H_{\min}^* \upharpoonright \text{dom}(H_{B_{X,\beta}}), \quad \text{dom } H_{B_{X,\beta}} := \{f \in \text{dom}(H_{\min}^*): \Gamma_1^1 = B_{X,\beta} \Gamma_0^1\}. \tag{137}$$

2. The second parametrization. Consider now the boundary triplet $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ constructed in Theorem 4.7. Further, consider the matrix

$$B_{X,\beta} = \begin{pmatrix} 0 & -d_1^{-2} & 0 & 0 & 0 & \dots \\ -d_1^{-2} & -(\beta_1 + d_1)d_1^{-3} & d_1^{-3/2}d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2}d_2^{-1/2} & 0 & -d_2^{-2} & 0 & \dots \\ 0 & 0 & -d_2^{-2} & -(\beta_2 + d_2)d_2^{-3} & d_2^{-3/2}d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2}d_3^{-1/2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{138}$$

Using the boundary triplet $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$, after straightforward calculations we arrive at the following representation of $H_{X,\beta}$.

Proposition 6.2. Let $\Gamma^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplet constructed in Theorem 4.1 and let $B_{X,\beta}$ be the minimal closed symmetric operator associated with the Jacobi matrix (138). Then

$$H_{X,\beta} = H_{B_{X,\beta}} := H_{\min}^* \upharpoonright \text{dom } H_{B_{X,\beta}}, \quad \text{dom } H_{B_{X,\beta}} = \{f \in \text{dom}(H_{\min}^*): \Gamma_1^2 = B_{X,\beta}\Gamma_0^2\}. \quad (139)$$

6.2. Self-adjointness

The following result gives a self-adjointness criterion for the operator with δ' -interactions on X .

Theorem 6.3. The operator $H_{X,\beta}$ has equal deficiency indices and $n_{\pm}(H_{X,\beta}) \leq 1$. Moreover, $H_{X,\beta}$ is self-adjoint if and only if at least one of the following conditions holds:

- (i) $\sum_{n=1}^{\infty} d_n = \infty$, i.e., $\mathcal{I} = \mathbb{R}_+$.
- (ii) $\sum_{n=1}^{\infty} [d_{n+1} | \sum_{i=1}^n (\beta_i + d_i)|^2] = \infty$.

Proof. Combining Theorem 4.5(i) with Proposition 6.1, we get $n_{\pm}(H_{X,\beta}) = n_{\pm}(B_{X,\beta})$, and hence $n_+(H_{X,\beta}) = n_-(H_{X,\beta}) \leq 1$.

Further, consider the operator $B_{X,\beta}$ defined by (135). One can check that the matrix $B_{X,\beta}$ admits the representation (26). Namely,

$$B_{X,\beta} = R_X^{-1}(I + U)D_{X,\beta}^{-1}(I + U^*)R_X^{-1}, \quad D_{X,\beta} := \begin{pmatrix} d_1 & 0 & 0 & 0 & \dots \\ 0 & \beta_1 & 0 & 0 & \dots \\ 0 & 0 & d_2 & 0 & \dots \\ 0 & 0 & 0 & \beta_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (140)$$

where $R_X = \bigoplus_{n=1}^{\infty} R_n$ is defined by (84). In other words, $B_{X,\beta}$ coincides with $J_{m,l}$ if we set

$$l_{2n-1} := d_n, \quad l_{2n} := \beta_n, \quad m_{2n-1} = m_{2n} := d_n, \quad n \in \mathbb{N}. \quad (141)$$

Therefore, the corresponding difference equation $\tau_{X,\beta}y = 0$ has the following linearly independent solutions (cf. [1, formulas (0.9), p. 236])

$$P(0) := \{p_n\}_{n=1}^{\infty}, \quad p_{2n-1} = -p_{2n} = \sqrt{d_n},$$

$$Q(0) := \{q_n\}_{n=1}^{\infty}, \quad q_{2n-1} = -\sqrt{d_n} \sum_{k=1}^{n-1} (\beta_k + d_k), \quad q_{2n} = -q_{2n-1} + d_n^{3/2}, \quad n \in \mathbb{N}.$$

The operator $B_{X,\beta}$ is symmetric with $n_{\pm}(B_{X,\beta}) = 1$ precisely when $P(0), Q(0) \in l_2$ (cf. [1,6]). The latter holds if and only if both conditions (i) and (ii) are not satisfied. \square

Condition (i) of Theorem 6.3 immediately yields the following result of Buschmann, Stolz and Weidmann [9, Theorem 4.7].

Corollary 6.4. (See [9].) If $\mathcal{I} = \mathbb{R}_+$, then the operator $H_{X,\beta}$ with δ' -interactions is self-adjoint.

Remark 6.5. In the case $d_* = 0$, the structure of the boundary matrices $B_{X,\alpha}$ and $B_{X,\beta}$ that correspond to operators with δ - and δ' -interactions, respectively, is completely different. Therefore, the spectral properties of operators $H_{X,\alpha}$ and $H_{X,\beta}$ are substantially different if $d_* = 0$. For instance, in the case of δ' -interactions we have simple self-adjointness criterion for $H_{X,\beta}$ formulated in terms of X and β .

6.3. Resolvent comparability

Let us fix $X \subset \mathcal{I}$ and consider the Hamiltonians $H_{X,\beta^{(1)}}$ and $H_{X,\beta^{(2)}}$ (133) with strengths $\beta = \beta^{(1)}$ and $\beta = \beta^{(2)}$, respectively.

Proposition 6.6. *Let $d^* < \infty$ and $p \in (0, \infty]$. Suppose $H_{X,\beta^{(1)}}$ and $H_{X,\beta^{(2)}}$ are self-adjoint. Let also $B_{X,\beta^{(1)}}$ and $B_{X,\beta^{(2)}}$ be the corresponding (self-adjoint) Jacobi operators defined either by (135) or by (138). Then:*

(i) *For any $z \in \rho(H_{X,\beta^{(1)}}) \cap \rho(H_{X,\beta^{(2)}})$ the inclusion*

$$(H_{X,\beta^{(1)}} - z)^{-1} - (H_{X,\beta^{(2)}} - z)^{-1} \in \mathfrak{S}_p \tag{142}$$

is equivalent to the inclusion

$$(B_{X,\beta^{(1)}} - i)^{-1} - (B_{X,\beta^{(2)}} - i)^{-1} \in \mathfrak{S}_p. \tag{143}$$

(ii) *If*

$$\left\{ \left(\frac{1}{\beta_n^{(1)}} - \frac{1}{\beta_n^{(2)}} \right) \left(\frac{1}{d_n} + \frac{1}{d_{n+1}} \right) \right\}_{n=1}^{\infty} \in l_p, \quad p \in (0, \infty) (\in c_0, p = \infty),$$

then (142) holds.

(iii) *If*

$$\left\{ \frac{\beta_n^{(1)} - \beta_n^{(2)}}{d_n^3} \right\}_{n=1}^{\infty} \in l_p, \quad p \in (0, \infty) (\in c_0, p = \infty),$$

then (142) holds.

Proof. (i) follows from Theorem 4.5 and Propositions 6.1 and 6.2.

(ii) Clearly, $l_{2,0} \subset \text{dom}(B_{X,\beta_1}) \cap \text{dom}(B_{X,\beta_2})$. On the other hand, for any $f \in l_{2,0}$ (136) yields

$$B_{X,\beta_2} f - B_{X,\beta_1} f = R_X^{-1} (\tilde{B}_{\beta_1} - \tilde{B}_{\beta_2}) R_X^{-1} f = \bigoplus_{n=1}^{\infty} (1/\beta_n^{(1)} - 1/\beta_n^{(2)}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} f.$$

Hence and due to the assumption, $\overline{B_{X,\beta_2} - B_{X,\beta_1}} \in \mathfrak{S}_p \subset [\mathcal{H}]$ and $\text{dom}(B_{X,\beta_1}) = \text{dom}(B_{X,\beta_2})$. It remains to apply Proposition 2.5(ii).

Proof of (iii) is analogous to the proof of (ii). We only emphasize that for proving (iii) we use parametrization (138) of the Hamiltonians $H_{X,\beta^{(1)}}$ and $H_{X,\beta^{(2)}}$. \square

In the case $d_* > 0$, the resolvent comparability criterion was obtained in [47].

Corollary 6.7. *(See [47].) If $0 < d_* \leq d^* < \infty$, then (142) is equivalent to the inclusion*

$$(\beta_n^{(1)} - i)^{-1} - (\beta_n^{(2)} - i)^{-1} \in l_p, \quad p \in (0, \infty) (\in c_0, p = \infty). \tag{144}$$

The proof of Corollary 6.7 can be extracted from Proposition 6.6(i) and we omit it.

6.4. Operators with discrete spectrum

Following the line of Section 5.4, we begin with the criterion for the operator $H_{X,\beta}$ to have purely discrete spectrum.

Theorem 6.8. *Let $B_{X,\beta}$ be the minimal Jacobi operator defined either by (135) or by (138).*

- (i) *If $n_{\pm}(B_{X,\alpha}) = 1$, i.e., both conditions of Theorem 6.3 are not satisfied, then self-adjoint extensions of $H_{X,\beta}$ have discrete spectrum.*
- (ii) *If $B_{X,\beta} = B_{X,\beta}^*$, then the Hamiltonian $H_{X,\beta}(= H_{X,\beta}^*)$ has discrete spectrum if and only if*
 - $\lim_{n \rightarrow \infty} d_n = 0$, and
 - $B_{X,\beta}$ has discrete spectrum.

Proof. Easily follows from Theorem 4.5 and the results of Section 6.1. \square

Let us first present several simple necessary conditions for the operator $H_{X,\beta}$ to have purely discrete spectrum.

Proposition 6.9. *Assume $\mathcal{I} = \mathbb{R}_+$, $d_n \rightarrow 0$. If there exists a positive constant $C > 0$ such that one of the following conditions is satisfied:*

- (i) $\beta_n \geq -Cd_n^3, n \in \mathbb{N}$,
- (ii) $\beta_n^- \leq -C(d_n^{-1} + d_{n+1}^{-1}), n \in \mathbb{N}$ ($\beta_n^- := \beta_n$ if $\beta_n < 0$ and $\beta_n^- := -\infty$ if $\beta_n > 0$),

then the spectrum of the operator $H_{X,\beta}$ is not discrete.

Proof. Assume that $\beta_n > 0$ for all $n \in \mathbb{N}$. Consider the matrix (135). Since $B_{X,\beta}$ admits the representation (140), we can apply the discreteness criterion of Kac and Krein (Theorem 2.12). However, by (141), neither $\{m_n\}_{n=1}^\infty$ nor $\{l_n\}_{n=1}^\infty$ is in l_1 if $\{d_n\}_{n=1}^\infty \notin l_1$. Hence, by Remark 2.13, the spectrum of $B_{X,\beta}$ is not discrete. Applying Theorem 6.8, we conclude that the spectrum of $H_{X,\beta}$ is not discrete if $\beta_n > 0, n \in \mathbb{N}$.

Assume that condition (i) is satisfied, i.e., $\beta_n \geq -Cd_n^3, n \in \mathbb{N}$, with some positive constant $C > 0$. Then there exists $\tilde{\beta}_n > 0$ such that $\{(\beta_n - \tilde{\beta}_n)d_n^{-3}\}_{n=1}^\infty \in l_\infty$ and hence $B_{X,\beta}$ of the form (138) is a bounded perturbation of $B_{X,\tilde{\beta}}$. Therefore, the spectrum of $B_{X,\beta}$ is not discrete and hence the spectrum of the operator $H_{X,\beta}$ is not discrete too.

Assume now that condition (ii) holds. Then the matrix $B_{X,\beta}$ of the form (135) is a bounded perturbation of the matrix $B_{X,|\beta|}$, where $|\beta| := \{|\beta_n|\}_{n=1}^\infty$. Indeed, in this case

$$\left\{ \left(\frac{1}{\beta_n} - \frac{1}{|\beta_n|} \right) \left(\frac{1}{d_n} + \frac{1}{d_{n+1}} \right) \right\}_{n=1}^\infty \in l_\infty$$

clearly holds. Hence the spectrum of $B_{X,\beta}$ is not discrete and hence the spectrum of $H_{X,\beta}$ is not discrete too. \square

Corollary 6.10. *If $\mathcal{I} = \mathbb{R}_+$ and $\beta_n > 0$ for all $n \in \mathbb{N}$, then the spectrum of $H_{X,\beta}$ is not discrete.*

The following result gives sufficient condition for the operator $H_{X,\beta}$ to have discrete spectrum.

Proposition 6.11. Assume $\beta_n + d_n \geq 0$ for all $n \in \mathbb{N}$.

(i) Let \mathcal{I} be a bounded interval and let X and β be such that the Hamiltonian $H_{X,\beta}$ is self-adjoint. Then $H_{X,\beta}$ has discrete spectrum if and only if

$$\lim_{n \rightarrow \infty} (b - x_n) \sum_{j=1}^n (\beta_j + d_j) = 0. \tag{145}$$

(ii) Let $\mathcal{I} = \mathbb{R}_+$. Then the Hamiltonian $H_{X,\beta} (= H_{X,\beta}^*)$ has discrete spectrum if and only if

$$\lim_{n \rightarrow \infty} x_n \sum_{j=n}^{\infty} d_j^3 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n \sum_{j=n}^{\infty} (\beta_j + d_j) = 0. \tag{146}$$

Proof. Consider the minimal symmetric operator associated with (138). First note that it is unitary equivalent to the operator with positive off-diagonal entries,

$$B'_{X,\beta} = \begin{pmatrix} 0 & d_1^{-2} & 0 & 0 & 0 & \dots \\ d_1^{-2} & -(\beta_1 + d_1)d_1^{-3} & d_1^{-3/2}d_2^{-1/2} & 0 & 0 & \dots \\ 0 & d_1^{-3/2}d_2^{-1/2} & 0 & d_2^{-2} & 0 & \dots \\ 0 & 0 & d_2^{-2} & -(\beta_2 + d_2)d_2^{-3} & d_2^{-3/2}d_3^{-1/2} & \dots \\ 0 & 0 & 0 & d_2^{-3/2}d_3^{-1/2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{147}$$

Consider the decomposition

$$l_2 = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \mathcal{H}_1 = \text{span}\{e_{2n-1}\}_{n \in \mathbb{N}}, \quad \mathcal{H}_2 = \text{span}\{e_{2n}\}_{n \in \mathbb{N}},$$

and define the unitary operators

$$V_j : \mathcal{H}_j \rightarrow l_2 \quad (j = 1, 2), \quad V_1(e_{2n-1}) = e_n \quad \text{and} \quad V_2(e_{2n}) = e_n, \quad n \in \mathbb{N}. \tag{148}$$

Then the operator $\tilde{B}_{X,\beta} := V B'_{X,\beta} V^{-1}$ with $V := V_1 \oplus V_2$ admits the representation

$$\tilde{B}_{X,\beta} = \begin{pmatrix} D_X^{-1/2} & 0 \\ 0 & D_X^{-3/2} \end{pmatrix} \begin{pmatrix} 0_{\mathcal{H}_1} & I + U \\ I + U^* & -(\mathcal{B}_\beta + D_X) \end{pmatrix} \begin{pmatrix} D_X^{-1/2} & 0 \\ 0 & D_X^{-3/2} \end{pmatrix},$$

where

$$\mathcal{B}_\beta = \text{diag}(\beta_n), \quad D_X = \text{diag}(d_n).$$

Since $B'_{X,\beta}$ is symmetric and $\dim \ker B'_{X,\beta} \leq 1$, then the inverse operator $(B'_{X,\beta})^{-1}$ is closed on $\mathcal{H} \ominus \ker(B'_{X,\beta})$ and is given by the following matrix

$$(\tilde{B}_{X,\beta})^{-1} = \begin{pmatrix} D_X^{1/2} & 0 \\ 0 & D_X^{3/2} \end{pmatrix} \begin{pmatrix} -(I + U^*)^{-1}(\mathcal{B}_\beta + D_X)(I + U)^{-1} & (I + U^*)^{-1} \\ (I + U)^{-1} & 0 \end{pmatrix} \begin{pmatrix} D_X^{1/2} & 0 \\ 0 & D_X^{3/2} \end{pmatrix}.$$

It is clear that the operator $(B'_{X,\beta})^{-1}$ is compact precisely when the spectra of operators

$$J_X := D_X^{-1/2}(I + U)D_X^{-3}(I + U^*)D_X^{-1/2}, \tag{149}$$

$$J_\beta := D_X^{-1/2}(I + U)(\mathcal{B}_\beta + D_X)^{-1}(I + U^*)D_X^{-1/2} \tag{150}$$

are purely discrete. Without loss of generality we can assume that $\beta_n + d_n > 0$ for all $n \in \mathbb{N}$. Indeed, in the opposite case we can choose $\tilde{\beta}_n$ satisfying the assumption of Proposition 6.11 and such that $\tilde{\beta}_n + d_n > 0$ for all $n \in \mathbb{N}$ and $\{(\tilde{\beta}_n - \beta_n)d_n^{-3}\}_{n=1}^\infty \in c_0$. Therefore, $B_{X,\tilde{\beta}}$ is a bounded perturbation of $B_{X,\beta}$ and hence $H_{X,\beta}$ is discrete precisely when so is $H_{X,\tilde{\beta}}$.

As in Section 2.2, with J_X and J_β we associate the functions, respectively,

$$\mathcal{M}_X(x) = \sum_{y_{n-1} < x} d_n, \quad y_n - y_{n-1} = d_n^3, \quad \mathcal{M}_\beta(x) = \sum_{z_{n-1} < x} d_n, \quad z_n - z_{n-1} = \beta_j + d_j, \tag{151}$$

where $x > 0$ and $y_0 = z_0 = 0$.

We begin with the case of a finite interval \mathcal{I} , i.e., $\sum_{n \in \mathbb{N}} d_n < \infty$. Then $\sum_{n \in \mathbb{N}} d_n^3 < \infty$ and hence the string with the mass \mathcal{M}_X is regular. Therefore, $\sigma(J_X)$ is discrete. Moreover, by Theorem 2.12, the string with the mass distribution \mathcal{M}_β has discrete spectrum precisely when (145) holds.

Assume now that $\mathcal{I} = \mathbb{R}_+$, i.e., $\sum_{n \in \mathbb{N}} d_n = \infty$. Then J_X and J_β are discrete if and only if $\{d_n^3\}_{n=1}^\infty \in l_1$, $\{\beta_j + d_j\}_{n=1}^\infty \in l_1$ (see Remark 2.13), and the functions \mathcal{M}_X , \mathcal{M}_β also satisfy the second condition in (2.12).

Theorem 6.8 completes the proof. \square

Corollary 6.12. *Let $\mathcal{I} = \mathbb{R}_+$. If*

$$\lim_{n \rightarrow \infty} x_n \sum_{j=n}^\infty d_j^3 > 0, \tag{152}$$

then the spectrum of the operator $H_{X,\beta}$ is not discrete for any β .

Proof. It easily follows from the proof of Proposition 6.11 that the operator J_X has discrete spectrum if $H_{X,\beta}$ has discrete spectrum too. However, by Theorem 2.12, J_X has discrete spectrum if and only if the limit in (152) equals 0. \square

Corollary 6.13. *If $\{d_n^3\}_{n=1}^\infty \notin l_1$, then the spectrum of $H_{X,\beta}$ is not discrete for any β .*

Let us illustrate the above results by the following example.

Example 6.14. Let $\mathcal{I} = \mathbb{R}_+$. Consider the Hamiltonian

$$H_\beta = -\frac{d^2}{dx^2} + \sum_{n=1}^\infty \beta_n(\cdot, \delta'(x - n^\varepsilon))\delta'(x - n^\varepsilon), \quad 0 < \varepsilon < 1.$$

First note that, by Theorem 6.3 (see also [9, Theorem 4.7]), the operator H_β is self-adjoint for any $\beta = \{\beta_n\}_{n=1}^\infty \subset \mathbb{R}$. Further, $d_n \asymp n^{\varepsilon-1}$ and hence $\sum_{j=1}^n d_j^3 \asymp n^{3\varepsilon-2}$. Therefore, we get:

- (i) If $\varepsilon \geq 1/2$, then the spectrum of H_β is not discrete for any β .
- (ii) If $\varepsilon < 1/2$ and either $\beta_n^- \geq -Cn^{3\varepsilon-3}$, $n \in \mathbb{N}$, or $\beta_n^- \leq -Cn^{1-\varepsilon}$, $n \in \mathbb{N}$, with some positive constant $C > 0$, then the spectrum of H_β is not discrete.

(iii) Assume $\varepsilon < 1/2$ and $\beta_n + n^\varepsilon - (n - 1)^\varepsilon \geq 0, n \in \mathbb{N}$. Then the operator H_β has discrete spectrum if and only if

$$\lim_{n \rightarrow \infty} n^\varepsilon \sum_{j=n}^{\infty} (\beta_j + j^\varepsilon - (j - 1)^\varepsilon) = 0.$$

6.5. Semiboundedness

Theorem 4.5(iii) and Proposition 6.1 yield the following result.

Theorem 6.15. *The operator $H_{X,\beta}$ with δ' -interactions on X is lower semibounded if and only if the operator $B_{X,\beta}$ of the form (135) is lower semibounded.*

Proof. Straightforward. \square

Proposition 6.16. *For the operator $H_{X,\beta}$ to be lower semibounded it is necessary that*

$$\frac{1}{\beta_n} \geq -C_1 d_n - \frac{1}{d_n} \quad \text{and} \quad \frac{1}{\beta_n} \geq -C_1 d_{n+1} - \frac{1}{d_{n+1}}, \quad n \in \mathbb{N}, \tag{153}$$

and sufficient that

$$\frac{1}{\beta_n} \geq -C_2 \min\{d_n, d_{n+1}\}, \quad n \in \mathbb{N}, \tag{154}$$

with some positive constants $C_1, C_2 > 0$ independent of $n \in \mathbb{N}$.

Proof. By Theorem 6.15, $H_{X,\beta}$ is lower semibounded if and only if the matrix (135) is lower semibounded. First, consider the representation (140). Let V_1 and V_2 be the unitary mappings defined by (148) and $V := V_1 \oplus V_2$. Then it is easy to check that

$$V R_X V^{-1} = \begin{pmatrix} D_X & 0 \\ 0 & D_X \end{pmatrix}, \quad V(I + U)V^{-1} = \begin{pmatrix} I & U \\ I & I \end{pmatrix}, \quad V D_{X,\beta} V^{-1} = \begin{pmatrix} D_X & 0 \\ 0 & B_\beta \end{pmatrix},$$

where $D_X := \text{diag}(d_n), B_\beta = \text{diag}(\beta_n), I = I_{l_2}$, and U is unilateral shift in l_2 . After straightforward calculations we obtain

$$\tilde{B}_{X,\beta} := V B'_{X,\beta} V^{-1} = \begin{pmatrix} D_X^{-2} + D_X^{-1/2} U B_\beta^{-1} U^* D_X^{-1/2} & D_X^{-1/2} U B_\beta^{-1} D_X^{-1/2} + D_X^{-2} \\ D_X^{-1/2} B_\beta^{-1} U^* D_X^{-1/2} + D_X^{-2} & D_X^{-2} + B_\beta^{-1} D_X^{-1} \end{pmatrix},$$

where U_+ is unilateral shift in \mathcal{H}_+ . Therefore, inequalities

$$D_X^{-2} + B_\beta^{-1} D_X^{-1} \geq -C_1 I, \quad D_X^{-2} + D_X^{-1/2} U B_\beta^{-1} U^* D_X^{-1/2} \geq -C_1 I$$

are necessary for the operator $B_{X,\beta}$ to be lower semibounded. The latter is equivalent to (153).

Further, let us consider the representation (136). By (84), $Q_X \leq 0$ and hence the operator $B_{X,\beta}$ is lower semibounded if so is the operator $R_X^{-1} \tilde{B}_\beta R_X^{-1}$. The latter is equivalent to the following inequalities

$$\begin{pmatrix} \frac{1}{\beta_n} & \frac{1}{\beta_n} \\ \frac{1}{\beta_n} & \frac{1}{\beta_n} \end{pmatrix} \geq -2C_2 \begin{pmatrix} d_n & 0 \\ 0 & d_{n+1} \end{pmatrix}, \quad n \in \mathbb{N},$$

where $C_2 > 0$ is a constant independent of $n \in \mathbb{N}$. Thus condition (154) is sufficient for lower semiboundedness.

The proof is completed. \square

Corollary 6.17. *Let $0 < d_* \leq d^* < \infty$. Then the Hamiltonian $H_{X,\beta}$ is lower semibounded if and only if $\{\frac{1}{\beta_n}\}_{n=1}^\infty$ is lower semibounded.*

7. Operators with δ -interactions and semibounded potentials

The results of Section 5 are stable under perturbations by L^∞ potentials q since deficiency indices, discreteness, and lower semiboundedness are stable under bounded perturbation. In particular, the results of Section 5 hold true for operators

$$H_{X,\alpha,q} = -\frac{d^2}{dx^2} + q(x) + \sum_{n=1}^\infty \alpha_n \delta(x - x_n), \quad q \in L^\infty(\mathcal{I}). \tag{155}$$

Moreover, it follows from [19, Theorem 3.1] that self-adjointness is stable under perturbations by lower semibounded potentials if $d_* > 0$.

The main aim of this section is to show that in the case $d_* = 0$ the situation is substantially different. Namely, we will show that self-adjointness of the operators with δ -interactions is not stable under perturbations by positive potentials q if $d_* = 0$.

Let $\mathcal{I} = \mathbb{R}_+$, $x_0 = 0$, $x_n - x_{n-1} = d_n := \frac{1}{n}$, $n \in \mathbb{N}$. Set

$$q_a(x) := a^2 \sum_{n=1}^\infty n^2 \chi_{(x_{n-1}, x_n)}(x), \quad a \in \mathbb{R}_+. \tag{156}$$

Consider the operator

$$H_{X,\alpha,q_a} = -\frac{d^2}{dx^2} + q_a(x) + \sum_{n=1}^\infty \alpha_n \delta(x - x_n). \tag{157}$$

The corresponding minimal symmetric operator H_{\min} has the form

$$H_{\min} = \bigoplus_{n=1}^\infty H_n, \quad H_n := -\frac{d^2}{dx^2} + a^2 n^2, \quad \text{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n]. \tag{158}$$

In the following proposition we construct a boundary triplet for H_{\min}^* .

Proposition 7.1. *For $f \in W_2^2[x_{n-1}, x_n]$, define the mappings $\Gamma_j^{(n)} : W_2^2[x_{n-1}, x_n] \rightarrow \mathbb{C}^2$,*

$$\Gamma_0^{(n)} f := \begin{pmatrix} d_n^{1/2} f(x_{n-1}+) \\ -d_n^{1/2} f(x_n-) \end{pmatrix}, \quad \Gamma_1^{(n)} f := \begin{pmatrix} \frac{d_n f'(x_{n-1}+) + (\varepsilon_1 f(x_{n-1}+) - \varepsilon_2 f(x_n-))}{d_n^{3/2}} \\ \frac{d_n f'(x_n-) + (\varepsilon_1 f(x_{n-1}+) - \varepsilon_2 f(x_n-))}{d_n^{3/2}} \end{pmatrix}, \tag{159}$$

where

$$d_n = \frac{1}{n}, \quad \varepsilon_1 = \varepsilon_1(a) := a \frac{\cosh a}{\sinh a}, \quad \varepsilon_2 = \varepsilon_2(a) := \frac{a}{\sinh a}. \tag{160}$$

Then:

- (i) For any $n \in \mathbb{N}$ the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is a boundary triplet for H_n^* .
- (ii) The direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is a boundary triplet for the operator H_{\min}^* .

Proof. (i) Straightforward.

(ii) Note that the triplet $\tilde{\Pi}_n = \{\mathbb{C}^2, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ defined by (78) forms a boundary triplet for the operator H_n^* defined by (158). The corresponding Weyl function $\tilde{M}_n(\cdot)$ is

$$\tilde{M}_n(z) = -\frac{\sqrt{z - a^2 n^2}}{\sin \sqrt{z/n^2 - a^2}} \begin{pmatrix} \cos \sqrt{z/n^2 - a^2} & 1 \\ 1 & \cos \sqrt{z/n^2 - a^2} \end{pmatrix}, \quad z \in \mathbb{C}_+. \tag{161}$$

It is easily seen that $\tilde{\Pi} := \bigoplus_{n=1}^\infty \tilde{\Pi}_n$ is not an ordinary boundary triplet for H_{\min}^* . On the other hand, triplets $\tilde{\Pi}_n$ and Π_n of the form (78) and (159), respectively, are connected by

$$\Gamma_0^{(n)} = R_n \tilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} = R_n^{-1} (\tilde{\Gamma}_1^{(n)} - Q_n \tilde{\Gamma}_0^{(n)}), \tag{162}$$

where

$$Q_n := \tilde{M}_n(0) = \begin{pmatrix} -n\varepsilon_1(a) & -n\varepsilon_2(a) \\ -n\varepsilon_2(a) & -n\varepsilon_1(a) \end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} n^{-1/2} & 0 \\ 0 & n^{-1/2} \end{pmatrix}.$$

The corresponding Weyl functions $M_n(\cdot)$ and $\tilde{M}_n(\cdot)$ are connected by $M_n(z) = R_n^{-1} (\tilde{M}_n(z) - Q_n) R_n^{-1}$. Clearly, relations (162) coincide with (57). Moreover, direct calculations show that

$$M_n(0) = 0, \quad M_n'(0) = R_n^{-1} \tilde{M}_n'(0) R_n^{-1} = a^{-2} \begin{pmatrix} (a - \varepsilon_1(a))(\varepsilon_1(a) - 1) & \varepsilon_2(a) - \varepsilon_1(a)\varepsilon_2(a) \\ \varepsilon_2(a) - \varepsilon_1(a)\varepsilon_2(a) & (a - \varepsilon_1(a))(\varepsilon_1(a) - 1) \end{pmatrix}.$$

Therefore, by Corollary 3.15, the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ forms a boundary triplet for H_{\min}^* . \square

Arguing as in Section 5.1, we obtain that the operator H_{X, α, q_a} admits the representation

$$H_{X, \alpha, q_a} = H_\Theta := H_{\min}^* \upharpoonright \text{dom}(H_\Theta), \quad \text{dom}(H_\Theta) := \{f \in \text{dom}(H_{\min}^*) : \{\Gamma_0, \Gamma_1\} \in \Theta\},$$

where $\Gamma_0 = \bigoplus_{n=1}^\infty \Gamma_0^{(n)}$ and $\Gamma_1 = \bigoplus_{n=1}^\infty \Gamma_1^{(n)}$ are defined by (159) and the operator part Θ_{op} of the linear relation $\Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ is unitary equivalent to the following Jacobi matrix

$$B_{X, \alpha, q_a} = \tilde{R}_X^{-1} (B_X(a) + \mathcal{A}_\alpha) \tilde{R}_X^{-1},$$

$$B_X(a) = \begin{pmatrix} 3\varepsilon_1(a) & 2\varepsilon_2(a) & 0 & 0 & \dots \\ 2\varepsilon_2(a) & 5\varepsilon_1(a) & 3\varepsilon_2(a) & 0 & \dots \\ 0 & 3\varepsilon_2(a) & 7\varepsilon_1(a) & 4\varepsilon_2(a) & \dots \\ 0 & 0 & 4\varepsilon_2(a) & 9\varepsilon_1(a) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

and

$$\tilde{R}_X = \text{diag}(\tilde{r}_n), \quad \tilde{r}_n := \sqrt{\frac{1}{n} + \frac{1}{n+1}}, \quad \mathcal{A}_\alpha = \text{diag}(\alpha_n). \tag{163}$$

Thus we arrive at the following result.

Proposition 7.2. Let q_a be defined by (156) and let B_{X,α,q_a} be the minimal symmetric operator associated with the Jacobi matrix (163). Then the operator H_{X,α,q_a} has equal deficiency indices and $n_{\pm}(H_{X,\alpha,q_a}) = n_{\pm}(B_{X,\alpha,q_a}) \leq 1$. In particular, H_{X,α,q_a} is self-adjoint if and only if so is B_{X,α,q_a} .

Proof is straightforward and we omit it.

Let us consider $\varepsilon_1(a)$, $a > 0$. Since $\lim_{a \rightarrow 0} \varepsilon_1(a) = 1$ and $\varepsilon_1(a) \approx a$ as $a \rightarrow +\infty$, there exists $a_0 > 0$ such that

$$\varepsilon_1(a_0) = 2. \quad (164)$$

Corollary 7.3. Let $\mathcal{I} = \mathbb{R}_+$, $d_n = 1/n$, and $\alpha_n = -4n - 2$, $n \in \mathbb{N}$.

(i) The Hamiltonian

$$H_{X,\alpha,0} = -\frac{d^2}{dx^2} - \sum_{n=1}^{\infty} (4n+2)\delta(x-x_n)$$

is self-adjoint.

(ii) Let a_0 be defined by (164). Then the Hamiltonian

$$H_{X,\alpha,q_a} = -\frac{d^2}{dx^2} + a_0^2 \sum_{n=1}^{\infty} n^2 \chi_{(x_{n-1}, x_n)} - \sum_{n=1}^{\infty} (4n+2)\delta(x-x_n)$$

is symmetric with $n_{\pm}(H_{X,\alpha,q_a}) = 1$.

Proof. (i) follows from Example 5.12(ii).

(ii) Consider the matrix B_{X,α,q_a} with $a = a_0$. Clearly, $\alpha_n = -\varepsilon_1(a_0)(2n+1)$ and hence the diagonal entries of B_{X,α,q_a} equal zero. The off diagonal entries $b_n = n \frac{\varepsilon_2(a_0)}{r_n r_{n+1}}$ satisfy $b_n \approx \varepsilon_2(a_0)n^2/4$ and hence $\{b_n^{-1}\}_{n=1}^{\infty} \in l_1$. Moreover, $b_{n-1}b_{n+1} \leq b_n^2$ holds for all $n \in \mathbb{N}$. Therefore, Berezanskii's test [6, Theorem VII.1.5] implies $n_{\pm}(B_{X,\alpha,q_a}) = 1$. By Proposition 7.2, $n_{\pm}(H_{X,\alpha,q_a}) = 1$. \square

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