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INJECTIVE TENSOR PRODUCTS OF TREE SPACES

CHRISTOPHER BOYD, COSTAS POULIOS, AND MILENA VENKOVA

Abstract. We study tensor products on tree spaces; in particular, we give necessary and sufficient conditions for the $n$-fold injective tensor product of tree spaces to contain a copy of $\ell_1$.

1. Introduction

A Banach space $E$ is said to be Asplund if every separable subspace of $E$ has a separable dual. The space of absolutely convergent sequences, $\ell_1$, is the classical example of a Banach space which is not Asplund. In the early 1970s Stegall asked if every non-Asplund Banach space contains a copy of $\ell_1$. The question was answered in the negative in 1974 when R.C. James, [J], constructed a separable Banach space which does not contain a copy of $\ell_1$ yet has a non-separable dual. This space is now known as the James Tree space and is denoted by $JT$. A further example of such a space was provided by Hagler in 1977, [H], and became known as the James Hagler space $JH$.

Let $E_1, \ldots, E_n$ be Banach spaces over $K$ ($K = \mathbb{R}$ or $\mathbb{C}$). We use $\bigotimes_{j=1}^n E_j$, to denote the tensor product of $E_1, \ldots, E_n$, and define the injective norm on $\bigotimes_{j=1}^n E_j$ by

$$\left\| \sum_{k=1}^m \lambda_k x_k^1 \otimes \cdots \otimes x_k^n \right\| := \sup_{\phi_1 \in B_{E_1^*}, \ldots, \phi_n \in B_{E_n^*}} \left| \sum_{k=1}^m \lambda_k \phi_1(x_k^1) \cdots \phi_n(x_k^n) \right|.$$  

The completion of $\bigotimes_{j=1}^n E_j$ with respect to the injective norm is denoted by $\bigotimes_{j=1}^n E_j$. In the case that $E_1 = E_2 = \cdots = E_n$ we will use the notation $\bigotimes_{n,\epsilon} E$.

Let us see that the containment, or more precisely, the non-containment of copies of $\ell_1$ in injective tensor products of Banach spaces has important consequences. In order to do this we introduce the spaces of $n$-linear integral and nuclear mappings. A mapping $L: E_1 \times \cdots \times E_n \to K$ is said to be $n$-linear


\begin{footnotesize}
2010 Mathematics Subject Classification. Primary 46B28; Secondary 46G25, 46B22.
Key words and phrases. Tree spaces; containment of $\ell_1$; Tensor Product.
\end{footnotesize}
if \( L \) is linear in each variable when the other \( n - 1 \) variables are kept fixed. An \( n \)-linear mapping \( L : E_1 \times \cdots \times E_n \to \mathbb{K} \) is said to be integral if there is a regular Borel measure \( \mu \) on \( (B_{E'_1} \times \cdots \times B_{E'_n}, \sigma(E'_1 \times \cdots \times E'_n, E_1 \times \cdots \times E_n)) \), such that

\[
L(x_1, \ldots, x_n) = \int_{B_{E'_1} \times \cdots \times B_{E'_n}} \phi_1(x_1) \cdots \phi_n(x_n) \, d\mu(\phi_1, \ldots, \phi_n) \quad (*)
\]

for all \( x_1 \in E_1, \ldots, x_n \in E_n \). We denote the space of all \( n \)-linear integral mappings on \( E_1 \times \cdots \times E_n \) by \( \mathcal{L}_I(E_1, \ldots, E_n) \). Endowed with the norm

\[
\|L\|_I := \inf \{ |\mu| : \mu \text{ satisfies (\ast)} \}
\]

the pair \( (\mathcal{L}_I(E_1 \times \cdots \times E_n), \| \cdot \|_I) \) becomes a Banach space. When the representing measure \( \mu \) has countable support we shall say that \( L \) is nuclear. In practice, this means that an \( n \)-linear mapping \( L \) is nuclear if there are sequences \( (\lambda_k)_k \) in \( \mathbb{K} \) and \( (\phi^j_k)_k \) in \( B_{E'_j} \) for \( 1 \leq j \leq n \), with \( \sum_{k=1}^{\infty} |\lambda_k| < \infty \), such that

\[
L(x_1, \ldots, x_n) = \sum_{k=1}^{\infty} \lambda_k \phi^1_k(x_1) \cdots \phi^n_k(x_n)
\]

for all \( x_1, \ldots, x_n \) in \( E_1 \times \cdots \times E_n \). We denote the space of all \( n \)-linear nuclear mappings by \( \mathcal{L}_N(E_1, \ldots, E_n) \). In the case that \( E_1 = E_2 = \cdots = E_n = E \) we will simply use the notation \( \mathcal{L}_I(nE) \) and \( \mathcal{L}_N(nE) \) for the spaces of \( n \)-linear integral and \( n \)-linear nuclear mappings. We note that \( (\mathcal{L}_I(E_1, \ldots, E_n), \| \cdot \|_I) \) is isometrically isomorphic to the dual of \( \bigotimes_{j=1}^{n} \hat{E}_j \).

Alencar, [A], shows that if \( E_1, \ldots, E_n \) are Asplund then the spaces of \( n \)-linear integral and nuclear mappings, \( \mathcal{L}_I(E_1, \ldots, E_n) \) and \( \mathcal{L}_N(E_1, \ldots, E_n) \), coincide.

The results in [BR] and [CD] illustrate the importance in determining whether the injective tensor products of Banach spaces contains a copy of \( \ell_1 \). They show that the condition that a Banach space is Asplund has a weaker incarnation that allows us to conclude that the spaces of integral and nuclear \( n \)-linear mappings coincide. This condition is that its \( n \)-fold injective tensor product does not contain a copy of \( \ell_1 \).

In [R], Ruess shows that there is a copy of \( \ell_1 \) in \( JT \bigotimes_{\epsilon} JT \). On the other hand, Leung, [L], proves that \( JH \bigotimes_{\epsilon} JH \) does not contain a subspace isomorphic to \( \ell_1 \). This leads us to ask the ‘tree’ following questions

(a) Is it true that \( \ell_1 \) is not contained in \( \bigotimes_{n, \epsilon} JH \) for any \( n \)?
(b) Is \( \ell_1 \) contained in \( JT \bigotimes_{\epsilon} JH ? \)
(c) Given a natural number $n$, can we find a Banach space $E$ so that $\ell_1$ is not contained in $\widehat{\bigotimes}_{k,\epsilon} E$ for $k < n$ yet for every $k \geq n$ we have that $\widehat{\bigotimes}_{k,\epsilon} E$ contains a copy of $\ell_1$?

In this paper we define a new property, the branch index, for a large class of tree spaces. We use this index to characterise the containment of $\ell_1$ in the tensor products of these spaces and in particular, answer all of the above questions.

2. Tree Spaces

The dyadic tree $\Upsilon$ is defined as $\bigcup_{n=0}^{\infty}\{0,1\}^n$, the collection of all finite sequences of 0’s and 1’s. Elements of $\Upsilon$ are called nodes. A node $t$ is said to have level $n$ if $t = (\epsilon_i)_{i=1}^n$. When $t$ has level $n$ we write $l(t) = n$. We introduce an ordering, $\leq$, on $\Upsilon$ in the following way. If $t = (\epsilon_i)_{i=1}^n$ and $s = (\delta_j)_{j=1}^m$ we say that $t \leq s$ if $n \leq m$ and $\epsilon_i = \delta_i$ for $1 \leq i \leq n$. We will also say that the empty node, $\emptyset$, has the property that $\emptyset \leq t$ for all $t$ in $\Upsilon$. Under the ordering on $\Upsilon$ every non-empty element, $t$, has an immediate predecessor which we denote by $t^-$. We define an injection $o: \Upsilon \rightarrow \mathbb{N}$ by $o(\emptyset) = 1$ and inductively setting $o(t) = 2o(t^-) + \epsilon_n$ if $t = (\epsilon_i)_{i=1}^n$.

A segment, $S$, is a subset of $\Upsilon$ of the form $S = \{r : t \leq r \leq s\}$, $t, s \in \Upsilon$. A branch is a maximal ordered subset of $\Upsilon$. We denote by $\Gamma$ the set of all branches. Given a function $x: \Upsilon \rightarrow \mathbb{R}$ and a segment $S$ we let $S(x) = \sum_{t \in S} x(t)$. For a segment $S = \{r : t \leq r \leq s\}$ we let $o(S) = o(t)$. Segments $S_1, \ldots, S_n$ are admissible if they are disjoint and begin and end at the same level.

**Definition 2.1.** Let $UT$ be a vector space of functions $x: \Upsilon \rightarrow \mathbb{R}$. A tree space, $VT$, is the completion of $UT$ with respect to some norm $\| \cdot \|$ on $UT$.

For $t$ in $\Upsilon$ we denote by $\eta_t$ the element of $VT$ given by $\eta_t(t) = 1$ and $\eta_t(s) = 0$ for $s \neq t$. Nodes $t$ and $s$ are said to be incomparable if $t \not\leq s$ and $s \not\leq t$. For $\gamma$ in $\Gamma$ we use $\gamma^*$ to denote the linear functional on $VT$ given by $\gamma^*(x) = \sum_{t \in \gamma} x(t)$. We use $(\eta_t^*)_{t \in \Upsilon}$ to denote the system of dual functionals to $(\eta_t)_{t \in \Upsilon}$ and $\Gamma^*$ to denote the dual system of branch functionals $\{\gamma^* : \gamma \in \Gamma\}$. Given a tree space $VT$ and $m$ in $\mathbb{N}$ we define $Q_m: VT \rightarrow VT$ by $Q_m \left( \sum_{t \in \Upsilon} a_t \eta_t \right) = \sum_{l(t) > m} a_t \eta_t$. For $s \in \Upsilon$ we let $Q_s \left( \sum_{t \in \Upsilon} a_t \eta_t \right) = \sum_{l(t) \geq s} a_t \eta_t$. In all the spaces we are interested in, $\gamma^*$ is a norm one linear functional while $Q_m$ is a norm one projection.

We have a number of different ways of constructing tree spaces.
Example 2.2. (Bellenot, Haydon, Odell) [BHO] Let $E$ be a Banach space with a normalised Schauder basis, $(e_i)$. We let $JT(e_i)$ denote the completion of the space of all finitely supported functions $x : \Omega \to \mathbb{R}$ with respect to the norm
\[
\|x\| = \sup \left\{ \left\| \sum_{i=1}^{n} S_i(x)e_{\alpha(S_i)} \right\| : n \in \mathbb{N}, (S_i)_{i=1}^{n} \text{ are disjoint segments in } \Omega \right\}.
\]

A special case of the above construction is the case where we take $E = \ell_p$ with $(e_i)$ equal to the canonical basis in $\ell_p$, $1 < p < \infty$. We will denote the corresponding tree space $JT(e_i)$ by $JT_p$. Note that $JT_p$ is the completion of the space of all finitely supported functions $x : \Omega \to \mathbb{R}$ with respect to the norm
\[
\|x\| = \sup \left\{ \left( \sum_{j=1}^{n} |S_j(x)|^p \right)^{\frac{1}{p}} : (S_j)_{j=1}^{n} \text{ are disjoint segments} \right\}.
\]

In the case where $p = 2$ we obtain the original James Tree space which we denote simply by $JT$. It follows from [BHO, Theorem 6.1 (b)] that $JT_p$ does not contain a copy of $\ell_1$ for any $1 < p < \infty$.

Example 2.3. (Odell) [O] Consider the normed space of all finite sequences, $c_{00}$, with respect to the norm
\[
\|x\| = \sup \left\{ \|x\|_{c_0}, \frac{1}{2} \sum_{i=1}^{n} \|E_i x\| \right\}
\]
with the supremum taken over all finite collections of pairwise disjoint subsets $(E_i)_{i=1}^{n}$ of $\mathbb{N}$ with $n \leq \min(E_i)$ for $1 \leq i \leq n$. Here we let $E_i x = \sum_{j \in E_i} a_j e_j$ for $x = \sum_{j} a_j e_j$ in $c_{00}$. The completion of $c_{00}$ with respect to the norm $\|\cdot\|_M$, denoted by $T_M$, was introduced by Johnson [Jo] and is called the Modified Tsirelson space. In [O, Theorem 2], Odell shows that $\Lambda_T$, the tree space based on $T_M$, does not contain a subspace isomorphic to $\ell_1$ for any $1 < p < \infty$. Since $\Lambda_T$ is non separable, $\Lambda_T$ is a non Asplund space. Moreover, it is shown that $\Lambda_T$ is the dual of the closed linear span of the dual nodes, $\{\eta^* t : t \in \Omega\}$, in $\Lambda_T'$.

Example 2.4. (Hagler) [H] The space $JH$ is the completion of the space of all finitely supported functions $x : \Omega \to \mathbb{R}$ with respect to the norm
\[
\|x\| = \sup \left\{ \sum_{j=1}^{n} |S_j(x)| : (S_j)_{j=1}^{n} \text{ are admissible segments} \right\}.
\]
The space $JH$ is called the James Hagler space.

More generally we have the following definition.
Definition 2.5. Let $E$ be a Banach space with a normalised Schauder basis, $(e_i)_i$. We let $HT(e_i)$ denote the completion of all finitely supported functions $x : \Omega \to \mathbb{R}$ with respect to the norm

$$\|x\| = \sup \left\{ \left\| \sum_{i=1}^{n} S_i(x) e_{o(S_i)} \right\| : n \in \mathbb{N} (S_i)_{i=1}^{n} \text{ are admissible segments in } \Omega \right\}.$$ 

When $E$ is equal to $\ell_p$ we denote the space $HT(e_i)$ by $HT_p$. Note that $HT_1$ is James Hagler space $JH$.

Although the spaces $JT_p$ and $HT_p$ are defined in very similar fashion, we will see that as Banach spaces they behave very differently. The reason for this difference in behaviour is explained by the following definition and lemma of Hagler, [H].

Definition 2.6. We say that the sequence of nodes $(t_n)_n$ is strongly incomparable if

(i) for $n \neq m$, $t_n$ and $t_m$ are incomparable, 
(ii) any family of admissible segments passes through no more than two $t_n$.

Lemma 2.7. (Hagler)[H] Let $(t_n)_n$ be a sequence of nodes in $\Omega$ with $l(t_n) < l(t_m)$ if $n < m$. Then there is a subsequence $(t_{n_k})_k$ of $(t_n)_n$ in $\Omega$ such that either

(a) $(t_{n_k})_k$ determines a unique branch; or
(b) $(t_{n_k})_k$ is strongly incomparable.

Consider a strongly incomparable sequence of nodes $(t_n)_n$ in $HT_p$, it is readily established that $(\eta_n)_n$ is equivalent to the unit basis of $c_0$. Thus $HT_p$ contains a copy of $c_0$, while $JT_p$ does not (see Corollary 4.4). Hence the nodes, $(\eta_t)_{t \in \Omega}$, form a boundedly complete basis for $JT_p$, but this is not the case with $HT_p$. Otherwise, if $HT_p$ had a boundedly complete basis, [LT] would imply that $HT_p$ is a dual Banach space and as it contains a copy of $c_0$ it must also contain a copy of $\ell_\infty$, contradicting its separability. Moreover, $JT_p$ is a dual space while $HT_p$ is not.

For further reading on the James Tree space see [FG].

3. Branch Index of Tree Spaces

Definition 3.1. Let $1 \leq q \leq \infty$. A tree space $VT$ is said to have branch index $q$ if there is a constant, $C > 0$, such that whenever $(\gamma_j)_{j=1}^{k}$ is a sequence of mutually distinct branches in $\Omega$ we are able to find $m \in \mathbb{N}$ so that for
any \((\alpha_j)_{j=1}^k\) in \(\mathbb{R}^k\) we have
\[
\left\| \sum_{j=1}^k \alpha_j Q_m^*(\gamma_j^*) \right\| \leq C \left( \sum_{j=1}^k |\alpha_j|^q \right)^{1/q}.
\]
In the case there is \(m \in \mathbb{N}\) so that
\[
\left\| \sum_{j=1}^k \alpha_j Q_m^*(\gamma_j^*) \right\| = \left( \sum_{j=1}^k |\alpha_j|^q \right)^{1/q}
\]
we say that \(VT\) has branch index \(q\) isometrically.

We observe that if \(VT\) has branch index \(q\) then it will also have branch index \(\tilde{q}\) for every \(\tilde{q} < q\). It follows from [H, Lemma 9] that \(JT\) has branch index \(\infty\). Similarly, in [O, Lemma 11], it is shown that \(\Lambda_T\) has branch index \(\infty\).

**Definition 3.2.** Let \(VT\) be a tree space. We say that \(VT\) is quasi-shrinking if \(\{(\eta_t^*)_{t \in T} \cup \Gamma^*\}\) spans a dense subspace of \(VT'\).

It is shown in [BHO, Theorem 6.1] that \(JT(e_i)\) is quasi-shrinking whenever \(E\) is a reflexive space. It can also be shown that the James–Hagler space, \(JH\), is also quasi-shrinking, (see [L] and [BHO, page 41]). This will also follow from a more general result (see Theorem 4.11). By [O, Theorem 2(5)] the space \(\Lambda_T\) is quasi-shrinking.

For \(1 < p < \infty\) we use \(q\) to denote the conjugate index of \(p\). Given a continuous linear operator \(T : X \to Y\) we use \(T^*\) to denote its transpose given by \((T^*\phi)(x) = \phi(T(x))\) for \(x \in X\), \(\phi \in Y'\).

**Lemma 3.3.** Let \(1 < p < \infty\) and \(T_j : JT_p \to JT_p\), \(j = 1, \ldots, k\) be continuous linear operators, and \(\{\sigma_j\}_{j=1}^k\) be subsets of \(\Upsilon\). Suppose that

(a) for every \(x\) in \(JT_p\) with \(\text{supp}(x) \subseteq \Upsilon \setminus \sigma_j\), we have \(T_jx = 0\),

(b) each segment in \(\Upsilon\) intersects at most one \(\sigma_j\),

(c) the intersection of any segment of \(\Upsilon\) with any \(\sigma_j\) is a segment.

Then for each \(\phi\) in \(JT_p'\)
\[
\left\| \sum_{j=1}^k T_j^*\phi \right\|_q = \sum_{j=1}^k \|T_j^*\phi\|_q.
\]

**Proof.** By conditions (b) and (c) it follows that if \(x_1, \ldots, x_k\) belong to \(JT_p\) with \(\text{supp}(x_j) \subseteq \sigma_j\) for every \(j = 1, 2, \ldots, k\), then
\[
\left\| \sum_{j=1}^k x_j \right\|_p = \sum_{j=1}^k \|x_j\|_p.
\]
In particular, for any \( x \in JT_p \) we have

\[
\left\| \sum_{j=1}^{k} P\sigma_j x \right\|_p^p = \sum_{j=1}^{k} \| P\sigma_j x \|_p^p,
\]

where \( P\sigma_j : JT_p \to JT_p \) is the norm one projection \( P\sigma_j(\sum_{t \in \Sigma} a_t \eta_t) = \sum_{t \in \sigma_j} a_t \eta_t \).

Then for each \( \phi \) in \( JT_p' \) we have that

\[
\left\langle \sum_{j=1}^{k} T_j^* \phi, x \right\rangle = \sum_{j=1}^{k} \left\langle T_j^* \phi, x \right\rangle = \sum_{j=1}^{k} \langle \phi, T_j P\sigma_j x \rangle \leq \sum_{j=1}^{k} \| T_j^* \phi \| \| P\sigma_j x \|
\]

\[
\leq \left( \sum_{j=1}^{k} \| T_j^* \phi \|^q \right)^{1/q} \left( \sum_{j=1}^{k} \| P\sigma_j x \|^p \right)^{1/p}
\]

\[
= \left( \sum_{j=1}^{k} \| T_j^* \phi \|^q \right)^{1/q} \left( \sum_{j=1}^{k} \| P\sigma_j x \| \right)
\]

\[
\leq \left( \sum_{j=1}^{k} \| T_j^* \phi \|^q \right)^{1/q} \| x \|
\]

and therefore taking the supremum over all \( x \) in the unit ball of \( JT_p \) we get that

\[
\left\| \sum_{j=1}^{k} T_j^* \phi \right\|_q^q \leq \sum_{j=1}^{k} \| T_j^* \phi \|^q.
\]

For the reverse inequality let \( \phi \in JT_p' \) and \( \epsilon > 0 \). For each \( 1 \leq j \leq k \) use (a) to choose \( x_j \) with \( \| x_j \| = 1 \) and support contained in \( \sigma_j \) so that

\[
\| T_j^* \phi \|^q \leq |\langle T_j^* \phi, x_j \rangle|^q + \epsilon.
\]

Let

\[
x = \sum_{j=1}^{k} \text{sgn}(\langle T_j^* \phi, x_j \rangle) |\langle T_j^* \phi, x_j \rangle|^{q-1} x_j.
\]
Then, we have \( \|x\|^p = \sum_{j=1}^k |\langle T_j^* \phi, x_j \rangle|^q \) and

\[
\left( \sum_{j=1}^k \|T_j^* \phi\|^q \right)^{p-1} \|x\|^p \leq \left( \sum_{j=1}^k \|T_j^* \phi, x_j \rangle|^q \right)^{p-1} \|x\|^p
\]

\[
= \left( \sum_{j=1}^k \|T_j^* \phi\|^q \right)^{p-1} \|x\|^p
\]

\[
= \sum_{j=1}^k \|T_j^* \phi\|^q \|x\|^p
\]

As this holds for all \( \epsilon > 0 \) we get that

\[
\left( \sum_{j=1}^k \|T_j^* \phi\|^q \right)^{p-1} \|x\|^p \leq \left( \sum_{j=1}^k \|T_j^* \phi\|^q \right)^{p-1} \|x\|^p
\]

or that

\[
\sum_{j=1}^k \|T_j^* \phi\|^q \leq \left( \sum_{j=1}^k \|T_j^* \phi\|^q \right)^{p-1} \|x\|^p
\]

and the identity is established. \( \square \)

Given branches \( \gamma_1, \ldots, \gamma_k \) in \( \Upsilon \), we choose \( m \) in \( \mathbb{N} \) sufficiently large so that \( \gamma_j \cap \{ t \in \Upsilon : l(t) \geq m \} \) are pairwise disjoint, \( 1 \leq j \leq k \). Taking \( \sigma_j = \gamma_j \cap \{ t \in \Upsilon : l(t) \geq m \} \), \( T_j = P_{\sigma_j} \) for \( 1 \leq j \leq k \) and \( \phi = \sum_{j=1}^k a_j \gamma_j^* \) with \( a_j \in \mathbb{R} \) in Lemma 3.3, we get the following Corollary.

**Corollary 3.4.** The James Tree space, \( JT_p \), has branch index \( q \) isometrically, where \( q \) is the conjugate index of \( p \).

4. **Biduals of \( JT_p \) and \( HT_p \)**

In this section we will give a description of the biduals of \( JT_p \) and \( HT_p \).

4.1. **Bidual of \( JT_p \).** Let us begin with \( JT_p \). A description of the bidual of \( JT \) is given in [LS].

We introduce the space \( J_p \) as the completion of the space of all sequences in \( c_0 \), \( (a_j)_{j} \), with respect to the norm

\[
\|\|(a_n)n\|\| := \sup \left( \sum_{i=1}^n |a_{k_{2i-1}} - a_{k_{2i}}|^p \right)^{1/p}
\]
where the supremum is taken over any choice of \( n \) and any choice of positive integers \( k_1 < k_2 < \ldots < k_{2n} \). Equivalent norms on \( J_p \) are obtained by considering the norm

\[
\| (a_n)_n \| = \sup \left( \frac{1}{2} \sum_{i=1}^{n} |a_{k_{i+1}} - a_{k_i}|^p + |a_{k_{n+1}} - a_{k_1}|^p \right)^{1/p}
\]

or the norm

\[
\|\| (a_n)_n \|\| = \sup \left( \frac{1}{2} \sum_{i=0}^{n} |a_{k_{i+1}} - a_{k_i}|^p \right)^{1/p}
\]

where \( a_0 = 0 \) and the supremum is taken over all \( n \) and all choices of positive integers \( 0 = k_0 < k_1 < k_2 < \ldots < k_{n+1} \).

If we denote by \( (e_n)_n \) the unit vector basis in \( J_p \) then for \( p > 1 \), \( (e_n)_n \) is a shrinking basis for \( J_p \). To see this we assume that there is \( \phi \) in \( J'_p \), a block basis sequence, \( (x_k)_k \), of \( (e_n)_n \) and \( \epsilon > 0 \) so that \( \|x_k\| = 1 \) and \( \phi(x_k) > \epsilon \) for all \( k \) in \( \mathbb{N} \). Consider \( \sum_{k=1}^{\infty} \frac{x_k}{k} \). Then each term used to calculate the norm of \( \sum_{k=1}^{\infty} \frac{x_k}{k} \) is either of the form

\[
\left( \frac{e_j^+(x_k)}{k} - \frac{e_j^+(x_k)}{k} \right)^p
\]

or of the form

\[
\left[ \frac{e_j^+(x_k)}{k} - \frac{e_j^+(x_{k+m})}{k+m} \right]^p.
\]

It follows from the convexity of the function \( f(x) = x^p \) that

\[
\left[ \frac{e_j^+(x_k)}{k} - \frac{e_j^+(x_{k+m})}{k+m} \right]^p \leq 2^{p-1} \left[ \left( \frac{e_j^+(x_k)}{k} \right)^p + \left( \frac{e_j^+(x_{k+m})}{k+m} \right)^p \right].
\]

Hence we have that

\[
\left\| \sum_{k=1}^{\infty} \frac{x_k}{k} \right\|^p \leq 2^{p-1} \sum_{k=1}^{\infty} \left( \frac{\|x_k\|}{k} \right)^p < \infty.
\]

But this contradicts our assumption that \( \phi(x_k) > \epsilon \) for all \( k \) and thus \( (e_n)_n \) is a shrinking basis for \( J_p \).

We denote by \( (s_n)_n \) the summing basis for \( J_p \) given by \( s_n = \sum_{j=1}^{n} e_j \). It is easily checked that the summing basis is a monotone boundedly complete basis for \( J_p \). A routine calculation shows that \( s_n^* = e_n^* - e_{n+1}^* \) for all \( n \) in \( \mathbb{N} \).
Moreover, if $\sum_{j=1}^{\infty} a_j s_j$ belongs to $J_p$ then we have that
\begin{align*}
e_1^* \left( \sum_{j=1}^{\infty} a_j s_j \right) &= \left( \sum_{j=1}^{\infty} a_j \right) e_1^* (e_1) + \left( \sum_{j=2}^{\infty} a_j \right) e_1^* (e_2) + \left( \sum_{j=3}^{\infty} a_j \right) e_1^* (e_3) + \cdots \\
&= \sum_{j=1}^{\infty} a_j,
\end{align*}
showing that $e_1^*$ coincides with the summing function of Bellenot, Haydon and Odell [BHO].

Since the canonical basis for $\ell_p$ is shrinking it follows from [BHO, Theorem 4.1] that $\ell_1$ does not embed in $J_p$. Applying [BHO, Theorem 2.2] we get that $J_p'$ is the span of $\{[s_j^*]_{j=1}^{\infty} \cup [e_1^*]\}$. From this we obtain the following Lemma.

**Lemma 4.1.** Let $1 < p < \infty$ and consider $\phi \in J_p'$. Then $\lim_{n \to \infty} \phi(s_n)$ exists. Moreover, it is $0$ if and only if $\phi \in [s_j^*]_{j=1}^{\infty}$.

**Proof.** We have that $J_p' = [s_j^*]_{j=1}^{\infty} \oplus \{e_1^*\}$. Hence we may write $\phi$ in $J_p'$ as $\phi = \sum_{j=1}^{\infty} a_j s_j^* + b e_1^*$. Therefore $\lim_{n \to \infty} \phi(s_n) = b$ and so $\phi \in \text{sp}\{s_j^*\}_{j=1}^{\infty}$ if and only if $\lim_{j \to \infty} \phi(s_j) = 0$. \hfill \Box

**Lemma 4.2.** Consider the subspace $Y$ of $JT_p'$ given by $Y = \{ \phi \in JT_p' : \lim_{t \in \gamma} \phi(\eta_t) = 0 \text{ for all branches } \gamma \}$. Then for each $\phi$ in $Y$
\begin{align*}
\lim_{k \to \infty} \left( \sup_{l(t)=k} \| Q_t^* \phi \| \right) &= 0.
\end{align*}

**Proof.** We suppose that there is $\phi$ in $Y$, a sequence of natural numbers $(n_j)_j$ and a sequence of nodes $(t_{n_j})_j$ with $l(t_{n_j}) = n_j$ such that $\| Q_{t_{n_j}}^* \phi \| > \alpha > 0$. We will first show that only finitely many nodes can be mutually incomparable. To see this suppose that there are $k$ mutually incomparable nodes $t_{n_1}, \ldots, t_{n_k}$. For each $j$, $1 \leq j \leq k$, choose $x_j$ in $Q_{t_{n_j}}(JT_p)$ with $\| x_j \| = 1$ so that $\phi(x_j) \geq \alpha$.

We have that $\left\| \sum_{j=1}^{k} x_j \right\| = k^{1/p}$. This gives us that
\begin{align*}
k \alpha &\leq \phi \left( \sum_{j=1}^{k} x_j \right) \\
&\leq \| \phi \| k^{1/p}
\end{align*}
or that
\begin{align*}
(\| \phi \| / \alpha)^q &\geq k
\end{align*}
and hence \( k \) must be finite. Because of this we may assume without loss of
generality that \((t_{nj})_k\) belong to a single branch \( \gamma \). For each \( \psi \) in \( JT_p^* \) and
any sequence of nodes such that \( l(s_n) < l(s_{n+1}) \) we have that \( \lim_{n \to \infty} \| \psi - Q^*_{s_n} \psi \| = \| \psi \| \). Hence by choosing a subsequence of \((t_{nj})_j\), if necessary, we
may also assume that for all \( j \in \mathbb{N} \) we have
\[
\| Q^*_{t_{nj}} \phi - Q^*_{t_{nj+1}} \phi \| \geq \frac{3}{4} \alpha > 0.
\]
Consider the projection \( P_\gamma : JT_p \to JT_p \) given by
\[
P_\gamma(x) = \sum_{t_n \in \gamma} \langle \eta^*_t, x \rangle \eta_n.
\]
Then the mapping \( T : P_\gamma(JT_p) \to J_p \), given by \( T(\eta_n) = s_n \), is an isometry.
As \( \phi \in Y \) we have that
\[
\lim_{n \to \infty} \phi(T^{-1}s_n) = \lim_{n \to \infty} \phi(\eta_n) = 0
\]
and hence using Lemma 4.1 we have \( \phi \circ T^{-1} \in [s^*_n] \). Write \( \phi \circ T^{-1} \) as
\( \phi \circ T^{-1} = \sum_{i=1}^{\infty} \beta_i s^*_i \). We have \( P_\gamma^* \phi = \sum_{i=1}^{\infty} \beta_i \eta^*_i \) and \( (Q^*_{t_{n_j}} - Q^*_{t_{n_{j+1}}})P_\gamma^* \phi = \sum_{i=n_j}^{n_{j+1}} \beta_i \eta^*_i \) and thus it follows that
\[
P_\gamma^* \phi = \sum_{i=1}^{\infty} \beta_i \eta^*_i = \sum_{j=1}^{\infty} (Q^*_{t_{n_j}} - Q^*_{t_{n_{j+1}}})P_\gamma^* \phi.
\]
It follows that for sufficiently large \( j \)
\[
\left\| (Q^*_{t_{n_j}} - Q^*_{t_{n_{j+1}}})P_\gamma^* \phi \right\| < \frac{1}{2} \alpha.
\]
Let \( R_j = (Q^*_{t_{n_j}} - Q^*_{t_{n_{j+1}}}) - P_\gamma(Q^*_{t_{n_j}} - Q^*_{t_{n_{j+1}}}) \). The image of \( R_j \) consists of
all \( x \) with support \( \sigma_j \) greater than or equal to \( t_{n_j} \) but not greater than or
equal to \( t_{n_{j+1}} \) and not contained in \( \gamma \). Then \((R_j)_j\) and \((\sigma_j)_{j=1}\) satisfy the
conditions of Lemma 3.3 and thus
\[
\sum_{j=1}^{k} \left\| R_j^* \phi \right\|^q = \left\| \sum_{j=1}^{k} R_j^* \phi \right\|^q
\]
for all \( k \). However for each \( j \) we have that \( \left\| R_j^* \phi \right\| > \frac{1}{4} \alpha \) while
\[
\left\| \sum_{j=1}^{k} R_j \right\| = \left\| (Q^*_{t_{n_1}} - Q^*_{t_{n_{j+1}}}) - (Q^*_{t_{n_1}} - Q^*_{t_{n_{j+1}}})P_\gamma \right\| \leq 4
\]
and we have a contradiction. \( \square \)

It follows from the proof of Lemma 4.2 that for each branch \( \gamma \) in \( \Gamma \) and
each \( \phi \) in \( JT_p^* \), \( \lim_{t \in \gamma} \phi(\eta_t) \) exists. Hence the function \( S : JT_p^* \to \mathbb{R}^\Gamma \) given by
\[
S(\phi) = \left( \lim_{t \in \gamma} \phi(\eta_t) \right)_{\gamma \in \Gamma}
\]
is well defined.

We claim that $S(JT''_p) = \ell_q(\Gamma)$. We start by showing that $S(JT''_p) \subseteq \ell_q(\Gamma)$. To see this, let $(\gamma_j)_{j=1}^r$ be distinct branches in $\Gamma$. For $m$ sufficiently large we have that $\gamma_j \cap \{ t \in \Upsilon : l(t) \geq m \}$ are pairwise disjoint, $1 \leq j \leq r$. For $1 \leq j \leq r$ choose $t_j$ in $\gamma_j \cap \{ t : l(t) \geq m \}$. Fix $\phi$ in $JT''_p$ and let $x = \sum_{j=1}^r \text{sgn}(\langle \phi, \eta_{t_j} \rangle) |\langle \phi, \eta_{t_j} \rangle|^{q-1} \eta_{t_j}$. Then we have

\[
\sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q = \phi(x) \leq \|\phi\| \|x\| = \|\phi\| \left( \sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q \right)^{\frac{1}{q}}
\]

and therefore $\left( \sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q \right)^{\frac{1}{q}} \leq \|\phi\|$. Letting $j$ tend to $\infty$, we get that $\|S(\phi)\| \leq \|\phi\|$ and therefore $S$ is continuous and has norm less than or equal to 1.

Conversely, $S(\gamma^*)$ is the vector in $\ell_q(\Gamma)$ which is 1 on $\gamma$ and 0 on every other branch. Since $\|\gamma^*\| = \|S(\gamma^*)\| = 1$ we have that $S$ has norm 1.

To show that $S$ is surjective let $(\gamma_j)_{j=1}^r$ be distinct branches in $\Gamma$ and choose $m$ so that $\gamma_j \cap \{ t \in \Upsilon : l(t) \geq m \}$ are pairwise disjoint, $1 \leq j \leq r$. Given $(a_j)_{j=1}^r$ we define $\phi$ by

\[
\phi \left( \sum_{t \in \Upsilon} b_t \eta_t \right) := \sum_{j=1}^r a_j \left( \sum_{t \in \gamma_j \cap \{ t : l(t) \geq m \}} b_t \right).
\]

Then

\[
\left| \phi \left( \sum_{t \in \Upsilon} b_t \eta_t \right) \right| \leq \left| \sum_{j=1}^r a_j \left( \sum_{t \in \gamma_j \cap \{ t : l(t) \geq m \}} b_t \right) \right| \\
\leq \left( \sum_{j=1}^r |a_j|^q \right)^{\frac{1}{q}} \left( \sum_{j=1}^r \left| \sum_{t \in \gamma_j \cap \{ t : l(t) \geq m \}} b_t \right|^{p} \right)^{\frac{1}{p}} \\
\leq \left( \sum_{j=1}^r |a_j|^q \right)^{\frac{1}{q}} \left\| \sum_{t \in \Upsilon} b_t \eta_t \right\|.
\]

Conversely, we choose $t_j$ in $\gamma_j$ with $l(t_j) > m$, $1 \leq j \leq r$, and we set $x = \sum_{j=1}^r \text{sgn}(a_j)|a_j|^{q-1} \eta_{t_j}$. Then, we have $\|x\| = \left( \sum_{j=1}^r |a_j|^q \right)^{1/p} \phi(\eta_{t_j}) = a_j$ for each $j = 1, 2, \ldots, r$, and $\phi(x) = \sum_{j=1}^r |a_j|^q$. Therefore,

\[
\sum_{j=1}^r |a_j|^q = \phi(x) \leq \|\phi\| \|x\| = \|\phi\| \left( \sum_{j=1}^r |a_j|^q \right)^{\frac{1}{p}}
\]
that is
\[
\left( \sum_{j=1}^{r} |a_j|^q \right)^{1/q} \leq \|\phi\|.
\]

Hence \(\|\phi\| = \left( \sum_{j=1}^{r} |a_j|^q \right)^{1/q}\) and given any \(v\) in \(\ell_q(\Gamma)\) there is \(\phi\) in \(JT_p^r\) with \(S(\phi) = v\) proving that \(S\) is surjective.

We will next show that \(\ker S\), the kernel of \(S\), is equal to \([\eta^*_t]_{t \in \Upsilon}\). We observe that \([\eta^*_t]_{t \in \Upsilon}\), the span of the dual nodes, is contained in the kernel of \(S\). To see that these subspaces actually coincide we assume that \([\eta^*_t]_{t \in \Upsilon}\) is a proper subspace of \(\ker S\). Let \(\kappa = \frac{1}{2}(2^p - 2^{p-1} - 1) > 0\) and choose \(\delta > 0\) so that \(\frac{2^{p-1+k+1}}{2p} < (1 - \delta)^{p\kappa}\). Choose \(\phi\) in \(\ker S\) of norm 1 so that \(d(\phi, [\eta^*_t]_{t \in \Upsilon}) = a > 0\). Let \(\psi_o\) in \([\eta^*_t]_{t \in \Upsilon}\) be such that \(\|\psi_o + \phi\| < \frac{a}{1-\delta}\). Setting \(\nu = \frac{\psi_o + \phi}{\|\psi_o + \phi\|}\) we get that
\[
d(\nu, [\eta^*_t]_{t \in \Upsilon}) = \frac{a}{\|\psi_o + \phi\|} > 1 - \delta.
\]

Choose \(x\) in \(JT_p\) of norm 1 so that \(\nu(x) > 1 - \delta\) and \(r \in \mathbb{N}\) so that \(\nu(P_r(x)) > 1 - \delta\) where \(P_r = I - Q_r\). This, in particular means that \(\|P_r x\| > 1 - \delta\). Choose \(\epsilon > 0\) so that \(2^{p+q-1} \epsilon^{-q} < (1 - \delta)^p\). It follows from Lemma 4.2 that we can find \(r' > r\) so that \(\|Q^*_u \nu\| < \epsilon\) for \(1 \leq j \leq 2^{r'}\), where \(u_j, j = 1, \ldots, 2^{r'}\) are the nodes of level \(r'\). Then \((I - Q_r)^* \nu \in [\eta^*_t]_{t \in \Upsilon}\) and hence we have that
\[
\|Q^*_r \nu\| = \|\nu - (I - Q_r)^* \nu\| \geq d(\nu, [\eta^*_t]_{t \in \Upsilon}) > 1 - \delta,
\]
and so by Lemma 3.3 we have that
\[
\sum_{j=1}^{2^{r'}} \|Q^*_u \nu\|^q = \|Q^*_r \nu\|^q > (1 - \delta)^q.
\]

For each \(1 \leq j \leq 2^{r'}\) choose \(x_j\) in \(JT_p\) so that \(\|x_j\| = 1\), \(Q^*_u x_j = x_j\) and
\[
A^p := \sum_{j=1}^{2^{r'}} |Q^*_u \nu(x_j)|^q = \sum_{j=1}^{2^{r'}} |\nu(x_j)|^q > (1 - \delta)^q.
\]

Let
\[
z = \frac{1}{A} \sum_{j=1}^{2^{r'}} \text{sgn}(\nu(x_j)) |\nu(x_j)|^{q-1} x_j.
\]

Then, by definition of the norm on \(JT_p\), we have
\[
\|z\|^p = \frac{1}{A^p} \sum_{j=1}^{2^{r'}} |\nu(x_j)|^{(q-1)p} = \frac{1}{A^p} \sum_{j=1}^{2^{r'}} |\nu(x_j)|^q = 1.
\]
In addition
\[ \nu(z) = \frac{1}{A} \sum_{j=1}^{2^r} |\nu(x_j)|^q = A^{p-1} > 1 - \delta \]
and
\[ \|Q_{u_j}z\| = \frac{1}{A} \|\nu(x_j)|^{q-1} = \frac{1}{A} \|Q_{u_j}^\ast \nu(x_j)|^{q-1} \leq \frac{\epsilon^{q-1}}{(1-\delta)^{q/p}}. \]

So, we have constructed an element, \( z \), of norm 1 in \( JT_p \) with the property that \( \nu(z) > 1 - \delta \). As \( \lim_{n \to \infty} Q_n(z) = 0 \), letting \( \tilde{z} = z - Q_n(z) \) with \( s \) sufficiently large, we get that there is \( s > r \) with \( Q_s(\tilde{z}) = 0 \), \( \|\tilde{z}\| \leq 1 \) and \( \nu(\tilde{z}) > 1 - \delta \).

Since \( \nu(P_r(x)) > 1 - \delta \) we can choose \( y \) in \( JT_p \) with \( \|y\| = 1 \) so that \( Q_r y = 0 \) and \( \nu(y) > 1 - \delta \). Then we have that \( \nu(\tilde{z} + y) > 2(1 - \delta) \) and \( \tilde{z} + y = \sum_{k=0}^s \sum_{t \in \Upsilon} a_t \eta_t \) with \( a_t = 0 \) for \( r < l(t) < r' \).

Let us consider the following three collections of segments,

\[ S_1 = \{S : \text{there is } t \in S \text{ with } l(t) = r \text{ and } t \in S \text{ with } l(t) \neq r'\}, \]
\[ S_2 = \{S : l(t) < r' \text{ for all } t \in S \}, \]
\[ S_3 = \{S : l(t) > r \text{ for all } t \in S \}. \]

Since each segment in \( \Upsilon \) lies in either \( S_1 \), \( S_2 \) or \( S_3 \) by the definition of the norm on \( JT_p \) we have segments \( S_1^1, \ldots, S_l^1 \) in \( S_1 \), \( S_1^2, \ldots, S_l^2 \) in \( S_2 \) and \( S_1^3, \ldots, S_l^3 \) in \( S_3 \),

\[ \|\tilde{z} + y\|^p = \sum_{i=1}^l \left( \sum_{t \in S_i^1} a_t \right)^p + \sum_{i=1}^m \left( \sum_{t \in S_i^2} a_t \right)^p + \sum_{i=1}^n \left( \sum_{t \in S_i^3} a_t \right)^p \]

Firstly, we have
\[ \sum_{i=1}^l \left( \sum_{t \in S_i^1} a_t \right)^p \leq 2^{p-1} \left( \sum_{i=1}^l \left( \sum_{t \in S_i^1 : l(t) < r'} a_t \right) \right)^p + \sum_{i=1}^l \left( \sum_{t \in S_i^1 : l(t) \geq r'} a_t \right)^p = 2^{p-1} (B_1 + B_2). \]

Since \( \tilde{z} \) contains no nodes of level strictly less than \( r' \) we have
\[ 2^{p-1} B_1 + \sum_{i=1}^m \left( \sum_{t \in S_i^2} a_t \right)^p \leq 2^{p-1} \|y\|^p = 2^{p-1}. \]

Secondly, there are at most \( 2^r \) nodes with level less than \( r \). Hence, we have that \( l \leq 2^r \) and we get that
\[ B_2 = \sum_{i=1}^l \left( \sum_{t \in S_i^3 : l(t) \geq r'} a_t \right)^p \leq \sum_{i=1}^l \|Q_{u_j}z\|^p \leq 2^r \frac{\epsilon^{(q-1)p}}{(1-\delta)^q} < \kappa/2^{p-1}. \]
Finally we have that
\[
\sum_{i''=1}^{n} \left( \sum_{t \in S_{i''}} a_t \right)^p \leq \| \tilde{z} \|^p \leq 1
\]
which gives that
\[
\| \tilde{z} + y \| \leq (2^{p-1} + \kappa + 1)^{1/p}.
\]

Thus we get that
\[
1 = \| \nu \| \geq \frac{\| \nu(\tilde{z} + y) \|}{\| \tilde{z} + y \|} > \frac{2(1 - \delta)}{(2^{p-1} + \kappa + 1)^{1/p}},
\]
a contradiction, and therefore ker \( S = [\eta_t^*]_{t \in \Upsilon} \).

We denote by \( j_{[\eta_t]} \) the canonical injection of \([\eta_t^*]\) into \( JT_p' \) and by \( j_{JT_p} \) the canonical inclusion of \( JT_p \) into \( JT''_p \).

We have that

**Theorem 4.3.**

\[
JT''_p = (j_{[\eta_t]}([\eta_t^*]))^\perp \oplus j_{JT_p}(JT_p) \simeq \left( JT'_p / j_{[\eta_t]}([\eta_t^*]) \right)' \oplus JT_p \simeq \ell_p(\Gamma) \oplus JT_p.
\]

**Corollary 4.4.** For \( 1 < p < \infty \), the space \( JT_p \) does not contain a copy of \( c_0 \).

**Proof.** Suppose that \( JT_p \) contains a copy of \( c_0 \). Then \( JT''_p \) has a quotient which is isomorphic to \( \ell'_\infty \). However, as \( JT''_p \) is isomorphic to \( JT'_p \oplus \ell_q(\Gamma) \) it has cardinality equal to the continuum, \( \mathfrak{c} \). The cardinality of \( \ell'_\infty \) is \( 2^\mathfrak{c} \) giving us a contradiction. \( \square \)

### 4.2. Bidual of \( HT_p \)

Let us now consider the space \( HT_p \). Again we will give a description of the bidual of \( HT_p \). This will allow us to show that \( HT_p \) does not contain a copy of \( \ell_1 \) and that \( HT_p \) has branch index \( q \) isometrically.

We define an operator \( Q : HT'_p \to \ell_q(\Gamma) \) by \( Q(\phi)(\gamma) = \lim_{t \in \gamma} \phi(\eta_t) \) for any branch \( \gamma \) in \( \Gamma \) and \( \phi \) in \( HT'_p \).

We first must show that \( Q \) is well-defined. To see this, consider any finite subset \( \Gamma' = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \} \) of \( \Gamma \). Choosing \( m \) sufficiently large we can assume that \( S_i = \gamma_i \cap \{ t \in \Upsilon : l(t) \geq m \}, i = 1, \ldots, n \) are pairwise disjoint segments. For \( i = 1, \ldots, n \) choose \( t_i \) in \( S_i \) and let \( x = \sum_{i=1}^{n} \text{sgn}(\phi(\eta_{t_i}))|\phi(\eta_{t_i})|^{q-1}\eta_{t_i} \).

Then we have
\[
\| x \| = \left( \sum_{i=1}^{n} |\phi(\eta_{t_i})|^{(q-1)p} \right)^{1/p} = \left( \sum_{i=1}^{n} |\phi(\eta_{t_i})|^q \right)^{1/p}.
\]
As $|\phi(x)| \leq \|\phi\| \|x\|$ we get
\[
\sum_{i=1}^{n} |\phi(\eta_i)|^q \leq \|\phi\| \left( \sum_{i=1}^{n} |\phi(\eta_i)|^q \right)^{1/p}
\]
which we rewrite as
\[
\left( \sum_{i=1}^{n} |\phi(\eta_i)|^q \right)^{1/q} \leq \|\phi\|.
\]
Letting $l(t_i)$ tend to infinity we get that
\[
\left( \sum_{i=1}^{n} |Q(\phi)(\gamma_i)|^q \right)^{1/q} \leq \|\phi\|
\]
which proves that $Q(\phi)$ belongs to $\ell_q(\Gamma)$ with $\|Q(\phi)\|_q \leq \|\phi\|$. Hence $Q$ is well defined and bounded with norm no greater than 1. Moreover, taking $\phi = \gamma_o^*$ for any branch $\gamma_o$ in $\Gamma$ we see that
\[
Q(\gamma_o^*)(\gamma) = \begin{cases} 
1 & \text{if } \gamma = \gamma_o, \\
0 & \text{otherwise}
\end{cases}
\]
and therefore $\|Q\| = 1$.

We claim that $Q$ is surjective. To see this, let $\gamma_1, \ldots, \gamma_n$ be $n$ distinct branches in $\Gamma$ and $\alpha_1, \ldots, \alpha_n$ belong to $\mathbb{R}$. Choose $m \in \mathbb{N}$ so that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint, $1 \leq j \leq n$. Let $\phi = \sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*)(x)$. For any $x$ in $HT_p$ we have
\[
|\phi(x)| = \left| \sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*)(x) \right| \\
\leq \sum_{i=1}^{n} |\alpha_i| |Q_m^*(\gamma_i^*)(x)| \\
\leq \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \left( \sum_{i=1}^{n} |Q_m^*(\gamma_i^*)(x)|^p \right)^{1/p} \\
\leq \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \|x\|
\]
and therefore
\[
\|\phi\| \leq \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q}.
\]
For \( i = 1, \ldots, n \) choose nodes \( t_i \) in \( \gamma_i \cap \{ t \in \Upsilon : l(t) \geq m \} \) and let \( x = \sum_{i=1}^{n} \text{sgn}(\alpha_i)|\alpha_i|^{q-1}\eta_{t_i} \). Then we have

\[
\|x\| = \left( \sum_{i=1}^{n} |\alpha_i|^{(q-1)p} \right)^{1/p} = \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/p}.
\]

Hence

\[
\|\phi\| \geq \frac{\|\phi(x)\|}{\|x\|} = \frac{1}{(\sum_{i=1}^{n} |\alpha_i|^q)^{1/p}} \left| \sum_{i=1}^{n} \text{sgn}(\alpha_i)|\alpha_i|^{q-1} \phi(\eta_{t_i}) \right| \\
= \frac{1}{(\sum_{i=1}^{n} |\alpha_i|^q)^{1/p}} \sum_{i=1}^{n} |\alpha_i|^q \\
= \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q}.
\]

Thus, we have \( \|\phi\| = \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \). Furthermore,

\[
Q(\phi)(\gamma) = \begin{cases} 
\alpha_i & \text{if } \gamma = \gamma_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence \( Q \) is a quotient mapping and we have shown that \( Q \) is a bounded, linear mapping of norm 1 from \( HT_p' \) onto \( \ell_q(\Gamma) \).

Let \( G = \ker Q \). Then we have that \( HT_p'/G \) is isomorphic to \( \ell_q(\Gamma) \). We claim that \( G = [\eta_t^*]_{t \in \Upsilon} \). We use the following Lemma of Hagler.

**Lemma 4.5.** (Hagler) \[H, \text{Lemma 8}\] For \( \phi \) in \( G = \ker Q \),

\[
\lim_{n \to \infty} \left( \max_{l(t) = n} \|Q_t^*(\phi)\| \right) = 0.
\]

**Theorem 4.6.** If \( 1 < p < \infty \) then \( \ker Q = [\eta_t^*]_{t \in \Upsilon} \).

**Proof.** Let us use \( F \) to denote \( [\eta_t^*]_{t \in \Upsilon} \), the closed linear span of \( \{\eta_t^*\}_{t \in \Upsilon} \). Clearly we have that \( F \subseteq \ker Q \). Assume that \( F \not\subseteq \ker Q \). Choose \( \delta \in (0,1) \) so that

\[
1 - (1 - \delta)^p < p + \frac{1}{3}
\]

and

\[
(3 - 4\delta)^p > 2 \cdot 3^{p-1} + 3^{p-1}(3p + 1)\delta.
\]

Choose \( \phi \) in \( \ker Q \) with \( \|\phi\| = 1 \) and \( \inf\{\|\phi - \psi\| : \psi \in F\} > 1 - \delta \). We now choose \( x, y \) and \( z \) as follows:

(i) Choose \( x \) in \( HT_p \) with \( \|x\| = 1 \) so that \( Q_m(x) = 0 \), for some \( m \in \mathbb{N} \) and \( \phi(x) > 1 - \delta \).
(ii) Choose $\epsilon > 0$ so that $2^m \epsilon < \delta$. By Lemma 4.5 we can find $n \geq 2^{m+1}$ so that $\|\phi \circ Q_t\| \leq \epsilon$ for every node $t$ with $l(t) = n$. Pick $y$ in $HT_p$ with $\|y\| = 1$ so that $Q_n(y) = y$, $\phi(y) > 1 - \delta$ and $Q_k(y) = 0$ for some $k > n$.

(iii) Choose $z$ in $HT_p$ so that $\|z\| = 1$, $Q_k(z) = z$ and $\phi(z) > 1 - \delta$.

Then we have

$$\|x + y + z\| \geq \phi(x + y + z) = 3(1 - \delta).$$

We will now consider two cases and in each we arrive at a contradiction.

**Case I:** We assume that for any admissible family of segments $S_1, S_2, \ldots, S_{2^m}$ passing through the support of $y$ we have

$$\sum_{j=1}^{2^m} |S_j(y)|^p \leq 1 - (3p + 1)\delta.$$ 

Then if $S_1, S_2, \ldots, S_r$ are admissible segments which do not pass through the support of $y$ then either of the two mutually exclusive events occurs

(a) $S_1, \ldots, S_r$ intersect the support of $x$ and the support of $y$,
(b) $S_1, \ldots, S_r$ intersect the support of $y$ and the support of $z$.

If for instance, (b) occurs we have

$$\left(\sum_{j=1}^{r} |S_j(x + y + z)|^p \right)^{1/p} = \left(\sum_{j=1}^{r} |S_j(y + z)|^p \right)^{1/p} \leq \|y + z\| \leq 2.$$

However, as $\|x + y + z\| > 3(1 - \delta)$ it follows that there must exist admissible segments, $S_1, \ldots, S_r$, passing through the support of $y$ and which give the norm of $x + y + z$. Now,

$$\|x + y + z\|^p = \sum_{j=1}^{r} |S_j(x + y + z)|^p$$

$$\leq \sum_{j=1}^{r} (|S_j(x)| + |S_j(y)| + |S_j(z)|)^p$$

$$\leq \sum_{j=1}^{r} 3^{p-1} (|S_j(x)|^p + |S_j(y)|^p + |S_j(z)|^p)$$

$$\leq 3^{p-1} (\|x\|^p + 1 - (3p + 1)\delta + \|z\|^p)$$

$$= 3^{p-1} (3 - (3p + 1)\delta).$$

It follows that

$$3^p(1 - \delta)^p \leq \|x + y + z\|^p \leq 3^{p-1}(3 - (3p + 1)\delta)$$
which implies that

\[ 3(1 - \delta)^p \leq 3 - (3p + 1)\delta \]

or that

\[ p + \frac{1}{3} \leq \frac{1 - (1 - \delta)^p}{\delta}. \]

But this contradicts our choice of \( \delta \).

**Case II:** We assume that for some admissible family of segments \( S_1, S_2, \ldots, S_{2^m} \) passing through the support of \( y \) we have

\[ \sum_{j=1}^{2^m} |S_j(y)|^p > 1 - (3p + 1)\delta. \]

For \( j = 1, 2, 3, \ldots, 2^m \) let \( t_j \) be the node of \( S_j \) with level \( n \). Let \( y_1 = \sum_{j=1}^{2^m} Q_{t_j}(y) \) and \( y_2 = y - y_1 \).

Then for any family of admissible segments, \( R_1, \ldots, R_{2^m} \), passing through the support of \( y \) but disjoint from \( S_1, \ldots, S_{2^m} \) we have

\[ \sum_{j=1}^{2^m} |R_j(y)|^p = \sum_{j=1}^{2^m} |R_j(y_2)|^p < (3p + 1)\delta \]

as otherwise \( y \) would have norm strictly greater than 1. Hence, for any family of admissible segments, \( R_1, \ldots, R_{2^m} \), passing through the support of \( y_2 \) we have

\[ \sum_{j=1}^{2^m} |R_j(y_2)|^p < (3p + 1)\delta. \]

Furthermore,

\[ |\phi(y_1)| = \left| \sum_{j=1}^{2^m} \phi \circ Q_{t_j}(y) \right| \]
\[ \leq \sum_{j=1}^{2^m} \|\phi \circ Q_{t_j}\| \|y\| \]
\[ < \epsilon 2^m < \delta \]

Hence,

\[ \phi(y_2) = \phi(y) - \phi(y_1) > 1 - \delta - \delta = 1 - 2\delta. \]

Repeating the argument of Case I, using \( y_2 \) instead of \( y \) we get

\[ \|x + y_2 + z\| \geq |\phi(x + y_2 + z)| > 1 - \delta + 1 - 2\delta + 1 - \delta = 3 - 4\delta \]
For some admissible family, $R_1, \ldots, R_s$ passing through the support of $y_2$ we have

$$\|x + y_2 + z\|^p = \sum_{j=1}^{s} |R_j(x + y_2 + z)|^p$$

$$\leq \sum_{j=1}^{s} (|R_j(x)| + |R_j(y_2)| + |R_j(z)|)^p$$

$$\leq \sum_{j=1}^{s} 3^{p-1} (|R_j(x)|^p + |R_j(y_2)|^p + |R_j(z)|^p)$$

$$= 3^{p-1} \left[ \sum_{j=1}^{s} |R_j(x)|^p + \sum_{j=1}^{s} |R_j(y_2)|^p + \sum_{j=1}^{s} |R_j(z)|^p \right]$$

$$\leq 3^{p-1} \left( \|x\|^p + (3p + 1)\delta + \|z\|^p \right)$$

$$= 3^{p-1} (2 + (3p + 1)\delta)$$

This implies that

$$(3 - 4\delta)^p \leq 2 \cdot 3^{p-1} + 3^{p-1}(3p + 1)\delta$$

which contradicts our choice of $\delta$.

Thus we see that Cases I and II give a contradiction and so we have $\ker Q = F$. □

We also have shown that $HT_p'/\ker Q = HT_p'/F$ is isometrically isomorphic to $\ell_q(\Gamma)$. We use this to obtain the following theorem.

**Theorem 4.7.** $HT^*_p$ is isomorphic to $F' \oplus \ell_p(\Gamma)$ where $F = [\eta^*_i]_{i \in \Upsilon}$.

**Proof.** The mapping $Q : HT^*_p \to \ell_q(\Gamma)$ is a quotient map. Its adjoint, $Q^*$, is a mapping from $\ell_q(\Gamma)' = \ell_p(\Gamma)$ into $HT^*_p$.

We claim that $Q^*(\ell_p(\Gamma))$ is complemented in $HT^*_p$. To see this let $\gamma_1, \ldots, \gamma_n$ be distinct branches of $\Upsilon$. Choose $m$ so that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint for $1 \leq j \leq n$. Consider the subspace of $HT^*_p$ spanned by $Q^*_m(\gamma_1^*), Q^*_m(\gamma_2^*), \ldots, Q^*_m(\gamma_n^*)$. For $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$ we have

$$\left\| \sum_{i=1}^{n} \alpha_i Q^*_m(\gamma_i^*) \right\| = \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q}$$

proving that the span of $Q^*_m(\gamma_1^*), Q^*_m(\gamma_2^*), \ldots, Q^*_m(\gamma_n^*)$ is isometrically isomorphic to $\ell_q^n$. Furthermore,

$$Q \left( \sum_{i=1}^{n} \alpha_i Q^*_m(\gamma_i^*) \right)(\gamma) = \begin{cases} \alpha_i & \text{if } \gamma = \gamma_i, \\ 0 & \text{if } \gamma \neq \gamma_1, \ldots, \gamma_n. \end{cases}$$
Therefore the restriction of $Q$ to the span of $Q^*_m(\gamma^*_1), Q^*_m(\gamma^*_2), \ldots, Q^*_m(\gamma^*_n)$ maps the span of $Q^*_m(\gamma^*_1), Q^*_m(\gamma^*_2), \ldots, Q^*_m(\gamma^*_n)$ isometrically onto the span of $e_{\gamma_1}, e_{\gamma_2}, \ldots, e_{\gamma_n}$ in $\ell_q(\Gamma)$. So if $G = [Q^*_m(\gamma^*_1), Q^*_m(\gamma^*_2), \ldots, Q^*_m(\gamma^*_n)]$ we have a local selection $S: G \to HT_p'$ given by $S = (Q(\gamma_{m,n}))^{-1}$. This implies that $Q$ admits an approximate local selection (see [S]) and so, by Lemma 1 of [S], $Q^*(\ell_p(\Gamma))$ is complemented in $HT_p''$. □

Examining the proof of Theorem 4.7 we get the following Corollary.

**Corollary 4.8.** For $1 < p < \infty$, $HT_p$ has branch index $q$ isometrically, where $q$ is the conjugate index of $p$.

**Theorem 4.9.** For $1 \leq p < \infty$, $HT_p$ does not contain a copy of $\ell_1$.

**Proof.** First note that $HT_1 = JH$ which does not contain a copy of $\ell_1$ by [H]. For $1 < p < \infty$, $HT_p''$ is isometrically isomorphic to $F' \oplus \ell_p(\Gamma)$ and therefore has cardinality $c$. It therefore follows from [P] that $HT_p$ does not contain a copy of $\ell_1$. □

Let us see that unlike $JT_p$, whose bidual, $JT_p''$, is isomorphic to $JT_p \oplus \ell_1(\Gamma)$, the bidual of $HT_p$, $HT_p''$, is not isomorphic to $HT_p \oplus \ell_1(\Gamma)$. Suppose that $HT_p''$ was isomorphic to $HT_p \oplus \ell_1(\Gamma)$. It follows as in [Po, Proposition 2.1] that $HT_p$ contains a copy of $c_0$ and therefore that $HT_p''$ contains a copy of $\ell_\infty$. As every separable Banach space is isometrically isomorphic to a subspace of $\ell_\infty$ we have that $HT_p''$ and hence $HT_p \oplus \ell_1(\Gamma)$ contains a copy of $\ell_1$. However, if we now apply [D, Theorem 7] we see that $HT_p$ or $\ell_1(\Gamma) \cong (HT_p \oplus \ell_1(\Gamma))/HT_p$ contains a copy of $\ell_1$. This is impossible, as we know that $HT_p$ does not contain a copy of $\ell_1$ and $\ell_1(\Gamma)$ is reflexive.

For $1 \leq p < \infty$ we consider the continuous map, $x \to \hat{x}$, from $HT_p$ into $C(\Gamma^*)$, where $\hat{x}(\gamma^*) = \gamma^*(x)$ and $\Gamma^*$ is endowed with the weak* topology.

The proof of [BHO, Lemma 6.2] is easily adapted to give the following result.

**Lemma 4.10.** Let $(x_i)_i$ be a normalised block basis sequence of $(\eta_t)_{t \in \mathcal{T}}$ in $HT_p$ such that $(\hat{x}_i)_i$ converges weakly to 0 in $C(\Gamma^*)$. Then $(x_i)_i$ is weakly null on $HT_p$.

**Proposition 4.11.** For $1 < p < \infty$, $HT_p$ is quasi-shrinking.

**Proof.** Suppose that $HT_p' \neq [\eta^*_t, \gamma^*: t \in \mathcal{T}, \gamma \in \Gamma]$. Then there is $x^{**}$ in $HT_p''$ with $\|x^{**}\| = 1$ and $x^{**}|_{[\eta^*_t]} = 0$. Since $HT_p$ does not contain a copy of $\ell_1$ we can find a sequence $(x_n)_n$ in $HT_p$ which converges weak* to $x^{**}$. Since $x^{**}|_{[\eta^*_t]} = 0$, by passing to a subsequence if necessary we may assume that
(x_n)_n is a block basis of (η_t)_{t∈Λ}. As x^{**} |_{Γ^*} = 0 we have that (x_n)_n converges weakly to 0 in C(Γ^*). Lemma 4.10 now implies that (x_n)_n is weakly null in HT and therefore that x^{**} = 0, a contradiction to our assumption. □

5. Containment of ℓ_1 in Injective Tensor Products of Tree Spaces

**Theorem 5.1.** For 1 ≤ i ≤ n let 1 < p_i < ∞. For i = 1, ..., n let q_i be such that 1/p_i + 1/q_i = 1. If ∑_{i=1}^{n} 1/q_i ≥ 1 then ℓ_1 ↪ \( \widehat{\bigotimes}_{i=1}^{n} JT_{p_i} \) and ℓ_1 ↪ \( \widehat{\bigotimes}_{i=1}^{n} HT_{p_i} \).

**Proof.** We prove the result for \( \widehat{\bigotimes}_{i=1}^{n} JT_{p_i} \), the proof for \( \widehat{\bigotimes}_{i=1}^{n} HT_{p_i} \) is identical. For k ∈ N let \( A_k := \{(ε_j)_{j=1}^{k} : (ε_j)_{j=1}^{k} ∈ Υ, ε_k = 0\} \). Given k ∈ N we define \( U_k \) in \( \widehat{\bigotimes}_{i=1}^{n} JT_{p_i} \) by \( U_k := \sum_{t ∈ A_k} η_t ⊗ η_t ⊗ \cdots ⊗ η_t \). We show that each \( U_k \) has norm 1. Let q be such that \( \frac{1}{q} = \sum_{i=1}^{n} \frac{1}{q_i} \). For 1 ≤ i ≤ n choose \( φ_i \) in \( JT_{p_i} \). For \( φ \) in \( JT_{p_i} \), t in Υ we use \( φ_t \) to denote \( φ(η_t) \). Then by Hölder’s Inequality we have that

\[
\left| \langle φ_1 ⊗ φ_2 ⊗ \cdots ⊗ φ_n, \sum_{t ∈ A_k} η_t ⊗ η_t ⊗ \cdots ⊗ η_t \rangle \right|
\]

\[
= \left| \sum_{t ∈ A_k} (φ_1)_t(φ_2)_t \cdots (φ_n)_t \right|
\]

\[
\leq \left( \sum_{t ∈ A_k} |(φ_1)_t|^{q_1/q} \right)^{\frac{q}{q_1}} \left( \sum_{t ∈ A_k} |(φ_2)_t|^{q_2/q} \right)^{\frac{q}{q_2}} \cdots \left( \sum_{t ∈ A_k} |(φ_n)_t|^{q_n/q} \right)^{\frac{q}{q_n}}
\]

\[
\leq \left( \sum_{t ∈ A_k} |(φ_1)_t|^{q_1} \right)^{\frac{1}{q_1}} \left( \sum_{t ∈ A_k} |(φ_2)_t|^{q_2} \right)^{\frac{1}{q_2}} \cdots \left( \sum_{t ∈ A_k} |(φ_n)_t|^{q_n} \right)^{\frac{1}{q_n}}
\]

Setting

\[
x_i = \sum_{t ∈ A_k} \text{sgn}(φ_i)_t|(φ_i)_t|^{q_i-1}η_t / \left( \sum_{t ∈ A_k} |(φ_i)_t|^{q_i} \right)^{\frac{1}{q_i}}
\]

we get that \( ||x_i|| = 1 \) and that

\[
\left( \sum_{t ∈ A_k} |(φ_1)_t|^{q_1} \right)^{\frac{1}{q_1}} \cdots \left( \sum_{t ∈ A_k} |(φ_n)_t|^{q_n} \right)^{\frac{1}{q_n}} = |φ_1(x_1)| |φ_2(x_2)| \cdots |φ_n(x_n)|
\]

\[
\leq ||φ_1||_q ||φ_2||_q \cdots ||φ_n||_q.
\]
Hence, using the injectivity of the epsilon tensor product, we get that

$$\|U_k\|_\epsilon = \sup_{\|\phi_1\|_q \leq 1} \cdots \sup_{\|\phi_n\|_q \leq 1} \left| \langle \phi_1 \otimes \cdots \otimes \phi_n, \sum_{t \in A_k} \eta_t \otimes \eta_t \otimes \cdots \otimes \eta_t \rangle \right| \leq 1.$$  

However, if $s$ belongs to $A_k$ we have that

$$\langle \eta_s^* \otimes \eta_s^* \otimes \cdots \otimes \eta_s^*, U_k \rangle = 1$$

and hence $\|U_k\|_\epsilon = 1$.

Let us show that $(U_k)_k$ is equivalent to the $\ell_1$ basis. Consider $(a_k)_k$ in $\ell_1$. Let $N_+ = \{ k : a_k \geq 0 \}$. We may suppose without loss of generality that $\sum_{n \in N_+} a_k \geq \frac{1}{2} \sum_{k \in \mathbb{N}} |a_k|$. We now choose a branch $\gamma$ as follows. Suppose that $(\epsilon_j)_{j=1}^k$ is the $k^{th}$ node of $\gamma$. If $k+1$ belongs to $N_+$, we set $\epsilon_{k+1} = 0$ and $(\epsilon_j)_{j=1}^{k+1} \in \gamma$. If $k+1$ does not belong to $N_+$, we set $\epsilon_{k+1} = 1$ and $(\epsilon_j)_{j=1}^{k+1} \in \gamma$.

By the choice of $\gamma$ it follows immediately that $\langle \gamma^* \otimes \gamma^* \otimes \cdots \otimes \gamma^*, U_k \rangle = 1$ if $k$ is in $N_+$ and is 0 otherwise. Therefore

$$\left\| \sum_{k \in \mathbb{N}} a_k U_k \right\|_\epsilon \geq \left| \sum_{k \in \mathbb{N}} a_k \langle \gamma^* \otimes \gamma^* \otimes \cdots \otimes \gamma^*, U_k \rangle \right| = \sum_{k \in \mathbb{N}} a_k \left| \sum_{m \in N_+} \gamma_m \right| \geq \frac{1}{2} \sum_{k \in \mathbb{N}} \sum_{m \in N_+} a_k.$$  

Clearly, we also have that $\left\| \sum_{k \in \mathbb{N}} a_k U_k \right\|_\epsilon \leq \sum_{k \in \mathbb{N}} |a_k|$ and hence $(U_k)_k$ is equivalent to the $\ell_1$ vector basis. \qed

When $q = q_1 = q_2 = \cdots = q_n$, we obtain the following result.

**Theorem 5.2.** Let $1 < p < \infty$. If $n \geq q$ then $\ell_1 \hookrightarrow \widehat{\otimes}_{n,\epsilon} J_{T_p}$ and $\ell_1 \hookrightarrow \widehat{\otimes}_{n,\epsilon} H_{T_p}$.

Given Banach spaces $X_1, \ldots, X_n$ and $R = \sum_{j=1}^\infty x^1_j \otimes \cdots \otimes x^i_j$ in $\widehat{\otimes}_{1 \leq j \leq n, \epsilon} X_j$ for $1 \leq i \leq n$, we let $R^i : X'_i \to \widehat{\otimes}_{1 \leq j \leq n, j \neq i, \epsilon} X_j$ be given by

$$R^i(\xi_i) = \sum_{j=1}^\infty \xi_i(x^1_i) x^1_j \otimes \cdots \otimes x^i_i \otimes x^{i+1}_j \cdots \otimes x^j_n$$

for $\xi_i$ in $X'_i$, $1 \leq i \leq n$.

**Lemma 5.3.** Let $n$ be a positive integer and $X_1, \ldots, X_n$ be Banach spaces with finite dimensional decompositions $(X^m_i)_{m=1}^\infty$. For $x$ in $X_i$ write $x$ as $x =$
\[ \sum_{j=1}^{\infty} x_j \text{ with } x_j \text{ in } X_i^j. \]

Let \( Q_m \) be defined as \( Q_m \left( \sum_{j=1}^{\infty} x_j \right) = \sum_{j=m+1}^{\infty} x_j. \)

Suppose that for each \( i, 1 \leq i \leq n, \) we have \( \ell_1 \not\hookrightarrow \widehat{\bigotimes}_{j=1, \epsilon}^{n} X_j \) yet \( \ell_1 \hookrightarrow \widehat{\bigotimes}_{j=1, \epsilon}^{n} X_j. \) Then there is a sequence \((S_k)_{k}\) in \( \widehat{\bigotimes}_{j=1, \epsilon}^{n} X_j \) such that

(a) \((S_k)_{k}\) is equivalent to the \( \ell_1 \) basis.

(b) For every \( 1 \leq i \leq n, \) \( \xi \) in \( X_i^1 \) and \( m \) in \( \mathbb{N} \) we have

\[
\lim_{k \to \infty} (S_k)^i ((I - Q_m^*) (\xi)) = 0
\]

with respect to the weak topology.

**Proof.** We argue as in Section 3 of [L]. We start with \( i = 1 \) and let \( \xi \) belong to \( X_i^1. \) Let \((R_j)_{j}\) be a sequence in \( \bigotimes_{j=1, \epsilon}^{n} X_j \) which is equivalent to the \( \ell_1 \) basis. Since \( \ell_1 \not\hookrightarrow \bigotimes_{j=1, \epsilon}^{n} X_j, \) for each \( \xi \) in \( X_i^1 \) we have that \((R_j^1 (\xi))_{j}\) has a weak Cauchy subsequence. Fix \( m \) in \( \mathbb{N} \) and choose \( \phi_1, \ldots, \phi_m \) so that \((I - Q_m)^* \phi_1, \ldots, (I - Q_m)^* \phi_m \) is a basis for \((I - Q_m)^*(X_i^1). \) We can find a subsequence \( N_1 \) of \( \mathbb{N} \) so that \((R_j^1 (I - Q_m)^* (\phi_1))_{j \in N_1} \) is weakly Cauchy. We then obtain a subsequence \( N_2 \) of \( N_1 \) so that \((R_j^1 (I - Q_m)^* (\phi_2))_{j \in N_2} \) is weakly Cauchy. Continuing like this, after \( l_m \) steps, we have a subset \( N_m \) of \( \mathbb{N} \) so that \((R_j^1 (I - Q_m)^* (\phi))_{j \in N_m} \) is weakly Cauchy for all \( \phi \) in \( X_i^1. \) Starting with \( m = 1 \) and repeating the above process we then obtain a decreasing chain of subsets \((L_m)_{m}\) so that \((R_j^1 (I - Q_m)^* (\phi))_{j \in L_m} \) is weakly Cauchy for all \( \phi \) in \( X_i^1 \) and all \( m \) in \( \mathbb{N}. \) Choose \( n_k \) in \( L_k \) for each \( k \) in \( \mathbb{N}. \) Then the sequence \((R_{n_{2k-1}} - R_{n_{2k}})_{k=1}^{\infty} \) is also equivalent to the unit basis of \( \ell_1 \). From the above we have that \( \lim_{k \to \infty} (R_{n_{2k-1}} - R_{n_{2k}})_{k=1}^{\infty} (I - Q_m)^*(\phi) = 0 \) for all \( m \) in \( \mathbb{N} \) and all \( \phi \) in \( X_i^1. \) Repeat the argument with \((R_j)_{j} = (R_{1,j})_{j} \) replaced with \((R_{i,k})_{k} = (R_{i-1,n_{2k-1}} - R_{i-1,n_{2k}})_{k} \) and 1 with \( i = 2, \ldots, n \) in turn. Finally, set \( S_k = R_{n,n_{2k-1}} - R_{n,n_{2k}}. \)

We define the oscillation of a bounded sequence of real numbers \((a_n)_{n}\) by \( \text{osc}((a_n)_{n}) = \lim \sup_{n} (a_n) - \lim \inf_{n} (a_n). \)

**Theorem 5.4.** Let \( VT_1, \ldots, VT_v \) be quasi-shrinking tree spaces which have branch index \( q_1, \ldots, q_v \) respectively and none of which contain a copy of \( \ell_1. \)

Let \( n \geq v \) and \( 1 < p_{v+1}, \ldots, p_n < \infty. \) If \( \sum_{i=1}^{n} \frac{1}{q_i} < 1 \) then

\[
\ell_1 \not\hookrightarrow \left( \bigotimes_{i=1, \epsilon}^{v} VT_i \right) \hat{\otimes} \left( \bigotimes_{i=v+1, \epsilon}^{n} \ell_{p_i}(\Gamma) \right).
\]

**Proof.** We use \( E_i \) to denote \( VT_i \) if \( 1 \leq i \leq v \) and \( \ell_{p_i}(\Gamma) \) if \( v < i \leq n. \) Our proof is by complete induction on \( n. \) For \( n = 1 \) the statement is clearly true. Let us assume that for each \( l < n \) we have shown that if \( 1 < p_i < \infty \)
for $1 \leq i \leq l$ and $\sum_{i=1}^{l} \frac{1}{q_i} < 1$ then $\ell_1 \not\rightarrow \bigotimes_{i=1}^{l} E_i$. Now suppose we have

$1 < p_i < \infty$ for $1 \leq i \leq n$ with $\sum_{i=1}^{n} \frac{1}{q_i} < 1$. Then for each $i$, $1 \leq i \leq n$ we have $\ell_1 \not\rightarrow \bigotimes_{j=1}^{n} E_j$.

Since $VT_j$ has branch index $q_i$ we can find a constant, $C > 0$, such that whenever $(\gamma_j)_{j=1}^{k}$ is a sequence of mutually distinct branches in $\Upsilon$ we are able to find $m \in \mathbb{N}$ so that for any $(\alpha_j)_{j=1}^{k}$ in $\mathbb{R}^k$ we have

$$\left\| \sum_{j=1}^{k} \alpha_j Q_m^* (\gamma_j^*) \right\| \leq C \left( \sum_{j=1}^{k} |\alpha_j|^q \right)^{1/q}.$$ 

Suppose that $\ell_1 \hookrightarrow \bigotimes_{j=1}^{n} E_j$. Choose $(S_w)_w$ in $\bigotimes_{j=1}^{n} E_j$ to be equivalent to the unit basis of $\ell_1$ which satisfies the conditions of Lemma 5.3.

We now claim that there is $\epsilon > 0$ such that for all subsets $N$ of $\mathbb{N}$ there are branches $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\text{osc}(\langle S_w, \gamma_1 \otimes \cdots \otimes \gamma_v \otimes e_{\gamma_{v+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{w \in N} > \epsilon$. If not then there is a subset $N_o$ of $\mathbb{N}$ such that $\text{osc}(\langle S_j, \gamma_1 \otimes \cdots \otimes \gamma_v \otimes e_{\gamma_{v+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{j \in N_o} = 0$ for all branches $\gamma_1, \gamma_2, \ldots, \gamma_n$. Since $\ell_1 \not\rightarrow \bigotimes_{j=2,\ell}^{n} E_j$, for each node $t$ in $\Upsilon$ the sequence $((S^1_w)(\eta^*_t))_{w \in N_a}$ has a weak Cauchy subsequence. Using a diagonal argument on the set of nodes we get that there is a subset $N_1$ of $N_o$ so that $((S^1_w)(\eta^*_t))_{w \in N_1}$ is weakly Cauchy for all $t$ in $\Upsilon$. We then obtain a subset $N_2$ of $N_1$ so that $((S^2_w)(\eta^*_t))_{w \in N_2}$ is weakly Cauchy for all $t$ in $\Upsilon$. Repeating the argument a further $v - 2$ times we get a subset $N_3$ of $N_2$ so that $((S^3_w)(\eta^*_t))_{w \in N_3}$ is weakly Cauchy for all $t$ in $\Upsilon$, all $1 \leq i \leq \nu$. Hence we have that $\text{osc}(\langle S_w, \psi_1 \otimes \cdots \otimes \psi_v \otimes e_{\gamma_{v+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{w \in N_3} = 0$ for all $\psi_i$, $1 \leq i \leq v$, in the span of the union of the nodes and the branches and all $\gamma_i$ in $\Gamma$, $v + 1 \leq i \leq n$. However, since $VT_j$ is quasi-shrinking, the union of the node and branch functionals spans a dense subspace of $VT_j', 1 \leq j \leq v$, and $(e_{\gamma})_{\gamma \in \Gamma}$ span a dense subset of $\ell_p(\Gamma)$, $v + 1 \leq l \leq n$, hence $(S_w)_{w \in \tilde{N}}$ is weakly Cauchy which contradicts the assumption that $(S_w)_{w \in \tilde{N}}$ is equivalent to the $\ell_1$ basis and hence our claim is proven.

Starting with $J_1 = \mathbb{N}$ we inductively choose decreasing subsets $(J_k)_k$ of $\mathbb{N}$ and $n$-tuples of branches $(\gamma_1^k, \gamma_2^k, \ldots, \gamma_n^k)_k$ such that

(i) $((S_w, \gamma_1^{k-1} \otimes \cdots \otimes \gamma_v^{k-1} \otimes e_{\gamma_{v+1}}^{k-1} \cdots \otimes e_{\gamma_n}^{k-1}))_{w \in J_k}$ converges and

$$\left| \lim_{w \in J_k} \langle S_w, \gamma_1^{k-1} \otimes \cdots \otimes \gamma_v^{k-1} \otimes e_{\gamma_{v+1}}^{k-1} \cdots \otimes e_{\gamma_n}^{k-1} \rangle \right| > \epsilon/2$$

for all $k > 1$. 

(ii) \((S^i_w (\gamma^k_1))_{w \in J_k}\) and \((S^j_w (e_1^{k-1}))_{w \in J_k}\) are weakly Cauchy for all \(1 \leq i \leq j, v < j \leq n\) and all \(k \geq 2,\)

(iii) osc\(((S_w, \gamma^k_1 \otimes \cdots \otimes \gamma^k_v \otimes e_{1^{k+1}}, \cdots e_{1^n}))_{w \in J_k} > \epsilon\) for all \(k \in \mathbb{N}.

Given \(i = (i_1, \ldots, i_n)\) in \(\mathbb{N}^n\) let \(l(i) = \max_{1 \leq k \leq n} i_k.\) Let \(\mathcal{I}_n\) denote the subsets of \(\{1, \ldots, n\}\) ordered by set inclusion. Given \(j\) in \(\mathcal{I}_n\) we denote the cardinality of \(j\) by \(|j|\) and the complement of \(j\) in \(\mathcal{I}_n\) by \(j^c.\) We wish to consider the situation where we fix some of the indices \(i_1, \ldots, i_n\) and where we let the others tend to infinity.

Given Banach spaces \(X_1, \ldots, X_n\) and \(R = \sum_{j=1}^{\infty} x_1^j \otimes \cdots \otimes x_n^j\) in \(\mathcal{T}_{\mathbb{N}}\) \(j = \{1 \leq j_1 < j_2 < \cdots < j_t \leq n\}\) in \(\mathcal{I}_n\), let \(j^c = \{1 \leq l_1 < l_2 < \cdots < l_{n-t} \leq n\}\).

We define \(R^j: X_{j_1}^t \times X_{j_2}^t \times \cdots \times X_{j_t}^t \to X_{l_1} \otimes X_{l_2} \otimes \cdots \otimes X_{l_{n-t}}\) by

\[
R^j(\phi_1, \phi_2, \ldots, \phi_t) = \sum_{j=1}^{\infty} \phi_1(x_{j_1}^j) \phi_2(x_{j_2}^j) \cdots \phi_t(x_{j_t}^j) x_{i_1}^j \otimes x_{i_2}^j \otimes \cdots \otimes x_{i_{n-t}}^j
\]

for \(\phi_i\) in \(X_{j_i}^t, i = 1 \ldots t.\)

For \(S\) in \(\mathcal{T}_{\mathbb{N}} \otimes \mathcal{T}_{\mathbb{N}} \otimes \mathcal{T}_{\mathbb{N}} \otimes \cdots \otimes \mathcal{T}_{\mathbb{N}}\) we use \(\hat{S}\) to denote the mapping from \(\Gamma^n\) to \(\mathbb{R}\) given by \(\hat{S}(\gamma_1, \gamma_2, \ldots, \gamma_n) = S(\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_v \otimes e_{v+1} \otimes \cdots \otimes e_{n}).\) We will adapt our notation and write \((\hat{S})^j(\gamma_{j_1}, \gamma_{j_2}, \ldots, \gamma_{j_t})\) as a function on \(\Gamma^{n-|j|}.)

By our above choice of \((J_k)\) and \(n\)-tuples \((\gamma^k_1, \gamma^k_2, \ldots, \gamma^k_n)\) we have that

\[
\left| \lim_{r \in J_{k+1}} (\hat{S}_r)^j(\gamma^k_1, \ldots, \gamma^k_n) \right| \geq \frac{\epsilon}{2}
\]

for each \(k \in \mathbb{N}.

We consider the set, \(\mathcal{J},\) of all \(j = (j_1, \ldots, j_t)\) in \(\mathcal{I}_n\) for which there is \(s = (s_1, \ldots, s_t)\) in \(\mathbb{N}^t\) and \(\delta > 0\) so that

\[
\left| \lim_{r \in J_{k+1}} (\hat{S}_r)^j(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_t}^{s_t})(\gamma_{l_1}^k, \ldots, \gamma_{l_{n-t}}^k) \right| > \delta
\]

for all \(k\) sufficiently large. We consider two cases.

(a) No \(j\) in \(\mathcal{J}\) has length \(n - 1,\)

(b) \(\mathcal{J}\) contains a \(j\) with \(|j| = n - 1.\)

If (a) occurs we consider the largest value \(t_o\) of \(|j|\) in \(\mathcal{J}.)\) Note that \(|j| < n - 1.\) Choose \(j_o\) in \(\mathcal{J}\) with \(|j_o| = t_o\) and \(\bar{s} = (s_1, \ldots, s_{t_o})\) in \(\mathbb{N}^{t_o}\) so that

\[
\left| \lim_{r \in J_{k+1}} (\hat{S}_r)^{j_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^k, \ldots, \gamma_{l_{n-1-t_o}}^k) \right| > \delta
\]

for some \(\delta > 0,\) all \(k\) sufficiently large. Then, we can inductively choose a subsequence of \((\gamma^k_1, \gamma^k_2, \ldots, \gamma^k_n)\), which we also denote by \((\gamma^k_1, \gamma^k_2, \ldots, \gamma^k_n)\), so that for all \(j\) with \(|j| = d > t_o,\) all \(j_o \subset j\) and \(\bar{s} = (u_1, \ldots, u_d)\) we have
that
\[
\lim_{k \to \infty} \left( \lim_{r \in J_{k+1}} (\tilde{S}_r)^{(\gamma_j^{u_1}, \gamma_j^{u_2}, \ldots, \gamma_j^{u_d})(\gamma_{l_1}, \gamma_{l_2}, \ldots, \gamma_{l_{n-d}})} = 0.
\]

Then, by inductively choosing another subsequence of \((\gamma_j^k, \ldots, \gamma_n^k)_k\), if necessary, we have that
\[
\lim_{r \in J_{\max\{k+1 \leq n-t_0\}+1}} \left( (\tilde{S}_r)^{(\gamma_j^{u_1}, \gamma_j^{u_2}, \ldots, \gamma_j^{s_{i_0}})(\gamma_{l_1}, \gamma_{l_2}, \ldots, \gamma_{l_{n-t_0}})} < \delta 2^{-(k_1+\ldots+k_{n-t_0}+n)}
\]

whenever at least two of \(k_1, \ldots, k_{n-t_0}\) are distinct. Note that since
\[
\lim_{r \in J_{k+1}} (\tilde{S}_r)^{(\gamma_j^{s_1}, \gamma_j^{s_2}, \ldots, \gamma_j^{s_{i_0}})(\gamma_{l_1}, \gamma_{l_2}, \ldots, \gamma_{l_{n-t_0}})} > \delta
\]

for all \(k\) sufficiently large we may also assume that
\[
\lim_{r \in J_{k+1}} (\tilde{S}_r)^{(\gamma_j^{s_1}, \gamma_j^{s_2}, \ldots, \gamma_j^{s_{i_0}})(\gamma_{l_1}, \gamma_{l_2}, \ldots, \gamma_{l_{n-t_0}})} > \delta
\]

all \(k\) in \(\mathbb{N}\).

Let us write the set \(j^c_o\) as the union of \(j_1\) and \(j_2\) where \(j_1 = \{l_1, \ldots, l_{s_0}\} = \{l_i \in j^c_o : E_i = VT_i\}\) and \(j_2 = \{l_{s_0+1}, \ldots, l_{n-t_0}\} = \{l_i \in j^c_o : E_i \in E_{p_i}(\Gamma)\}\).

Note that by construction, for each \(i, 1 \leq i \leq v\), we have that the sequence \((\gamma_i^k)_k\) is a sequence of mutually distinct branches. Let us fix \(u_0\) in \(\mathbb{N}\). Choose \(m\) sufficiently large so that \((Q_m^*\gamma_i^k)_k\) are disjoint. Since \(VT_i\) has branch index \(q_i\) we may also suppose that \(m\) is chosen so that we can find \(C > 0\) such that for any \((\alpha_j^k)_{j=1}^k\) in \(\mathbb{R}^k\) we have
\[
\left\| \sum_{j=1}^k \alpha_j Q_m^* (\gamma_j^k) \right\| \leq C \left( \sum_{j=1}^k |\alpha_j|^{q_i} \right)^{1/q_i}.
\]

Using Lemma 5.3 we then have that
\[
\lim_{r \in J_{\max\{k+1 \leq n-t_0\}+1}} (\tilde{S}_r)^{(\gamma_j^{s_1}, \gamma_j^{s_2}, \ldots, \gamma_j^{s_{i_0}})(\gamma_{l_1}, \gamma_{l_2}, \ldots, \gamma_{l_{n-t_0}})} =
\]
\[
\lim_{r \in J_{\max\{k+1 \leq n-t_0\}+1}} (\tilde{S}_r)^{(\gamma_j^{s_1}, \gamma_j^{s_2}, \ldots, \gamma_j^{s_{i_0}})(Q_m^*(\gamma_{l_1}^{k_1}), \ldots, Q_m^*(\gamma_{l_{s_0}}^{k_{s_0}}), \gamma_{l_{s_0+1}}^{k_{n-t_0}})}.
\]

Therefore we can find a positive integer \(r_o\) so that
\[
|(\tilde{S}_r)^{(\gamma_j^{s_1}, \gamma_j^{s_2}, \ldots, \gamma_j^{s_{i_0}})(Q_m^*(\gamma_{l_1}^{k_1}), Q_m^*(\gamma_{l_2}^{k_2}), \ldots, Q_m^*(\gamma_{l_{s_0}}^{k_{s_0}}), \gamma_{l_{s_0+1}}^{k_{n-t_0}})| > \delta/2
\]
for all $1 \leq k \leq u_o$ and

$$| \langle \tilde{S}_{r_o} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{k_o}}^{s_{k_o}}) (Q_m^* (\gamma_{l_1}^k), \ldots, Q_m^* (\gamma_{l_{q_o}}^{k_{q_o}}), \gamma_{l_{k_o+1}}^{k_{k_o+1}}, \ldots, \gamma_{l_{n-t_o}}^{k_{n-t_o}}) |$$

$$< \delta 2^{-(k_1 + \cdots + k_{n-t_o} + n)}$$

whenever at least two of $k_1, \ldots, k_{n-t_o}$ are distinct, $1 \leq k_c \leq u_o$ for $1 \leq c \leq n - t_o$.

For $1 \leq k \leq u_o$ we let

$$b_k = \text{sgn} (\tilde{S}_{r_o} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{k_o}}^{s_{k_o}}) (Q_m^* (\gamma_{l_1}^k), \ldots, Q_m^* (\gamma_{l_{q_o}}^{k_{q_o}}), e_{\gamma_{l_{k_o+1}}}^{k_{k_o+1}}, \ldots, e_{\gamma_{l_{n-t_o}}}^{k_{n-t_o}}),$$

set $\phi_{k_1} = \frac{-1}{u_o^c} \sum_{v_1 = 1}^{u_o} b_{v_1} Q_m^* (\gamma_{l_1}^{v_1})$ and $\phi_{k_s} = \frac{-1}{u_o^c} \sum_{v_s = 1}^{u_o} Q_m^* (\gamma_{l_s}^{v_s})$ for $2 \leq s \leq s_o$ and $\phi_{k_s} = \frac{-1}{u_o^c} \sum_{v_s = 1}^{u_o} e_{\gamma_{l_s}^{v_s}}$ for $s_o < s \leq n - t_o$. Since $VT_i$ has branch index $q_i$ we have $\| \phi_{k_1} \| \leq C, \ldots, \| \phi_{k_s} \| \leq C$ independent of $u_o$. We have

$$\left| \langle \tilde{S}_{r_o} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{k_o}}^{s_{k_o}}) (\phi_{k_1}, \ldots, \phi_{k_{n-t_o}}) \right|$$

$$= u_o^{-n-t_o} \frac{1}{\eta_s} \sum_{v_1 = 1}^{u_o} \ldots \sum_{v_{n-t_o} = 1}^{u_o} b_{v_1} \langle \tilde{S}_{r_o} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{k_o}}^{s_{k_o}}) (Q_m^* (\gamma_{l_1}^{v_1}), \ldots, Q_m^* (\gamma_{l_{q_o}}^{v_{q_o}}), \gamma_{l_{k_o+1}}^{v_{k_o+1}}, \ldots, \gamma_{l_{n-t_o}}^{v_{n-t_o}})$$

$$\geq u_o^{-n-t_o} \frac{1}{\eta_s} \left( \sum_{v_1 = 1}^{u_o} \ldots \sum_{v_{n-t_o} = 1}^{u_o} \left| \langle \tilde{S}_{r_o} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{k_o}}^{s_{k_o}}) (Q_m^* (\gamma_{l_1}^{v_1}), \ldots, Q_m^* (\gamma_{l_{q_o}}^{v_{q_o}}), \gamma_{l_{k_o+1}}^{v_{k_o+1}}, \ldots, \gamma_{l_{n-t_o}}^{v_{n-t_o}}) \right| \right)$$

$$\geq u_o^{-n-t_o} \frac{1}{\eta_s} \left( \delta - \frac{\delta}{u_0 2^n} \right).$$

And this proves that $S_{r_o}$ is unbounded, contradicting the fact that it is equivalent to an element of the unit basis of $\ell_1$. We have proved the result.

If (b) occurs we proceed as in (a) to choose $j_o$ in $\mathcal{I}_n$ with $|j_o| = n - 1$ and $\bar{s} = (s_1, \ldots, s_{n-1})$ in $\mathbb{N}^{n-1}$ so that

$$\lim_{r \in \ell_{q_i}^{u_i}} \langle \tilde{S}_{r} \rangle^0 (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \ldots, \gamma_{j_{n-1}}^{s_{n-1}}) (\gamma_{l_1}^{k}) > \delta$$

for some $\delta > 0$. Fix $u_o$ in $\mathbb{N}$ and choose $m$ so that $(Q_m^* (\gamma_{l_1}^{k}))_{k=1}^{u_o}$ is equivalent to the unit basis vector of $\ell_{q_i}^{u_i}$. We first assume that $E_{l_1} = VT_{l_1}$. By
Lemma 5.3 we have
\[
\lim_{r \to 0} \left( \tilde{S}_r \right)^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(\nu_k)
\] = \lim_{r \to 0} \left( \tilde{S}_r \right)^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(Q_m^*(\nu_k)).
\]
Since, taking \( r_0 \) sufficiently large we may assume that
\[
\left( \tilde{S}_{r_0} \right)^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(Q_m^*(\nu_k)) > \delta/2
\]
for all \( k \) sufficiently large, we may assume that
\[
\left( \tilde{S}_{r_0} \right)^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(Q_m^*(\nu_k)) > \delta/2
\]
for all \( k \) in \( \mathbb{N} \).

Let
\[
b_k = \text{sgn}(\tilde{S}_{r_0})^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(Q_m^*(\nu_k)),
\]
and
\[
\psi = u_0^{m_1} \sum_{v=1}^{u_0} b_v Q_m^*(\nu_k).
\]
Since \( (Q_m^*(\nu_k))_{k=1}^{u_0} \) is isometrically equivalent to the unit vector basis of \( \ell_{u_0}^{q_1} \) we have that \( \|\psi\| \leq M \). We now have that
\[
\left| (\tilde{S}_{r_0})^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(\psi) \right| = u_0^{m_1} \sum_{v=1}^{u_0} \left| (\tilde{S}_{r_0})^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(Q_m^*(\nu_k)) \right| \geq u_0^{m_1} \delta/2.
\]
Which shows that \( S_{r_0} \) is unbounded.

Finally if \( E_{l_1} = \ell_{q_1} \), a similar but somewhat simpler argument, setting
\[
b_k = \text{sgn}(S_{r_0})^{k_0} (\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n})(e_{s_1}^k),
\]
and \( \rho = u_0^{-m_1} \sum_{v=1}^{u_0} b_v e_{s_1}^k \), shows that \( S_{r_0} \) is unbounded. \( \square \)

If we take \( v = n \) in Theorem 5.4 we get the following Corollary.

**Corollary 5.5.** Let \( VT_1, \ldots, VT_n \) be quasi-shrinking tree spaces which have branch indices \( q_1, \ldots, q_n \) respectively and which do not contain a copy of \( \ell_1 \).

If \( \sum_{i=1}^{n} \frac{1}{q_i} < 1 \) then \( \ell_1 \not\cong \bigotimes_{i=1}^{n} VT_i \).
Taking $VT_i$ equal to $JT_{p_i}$ or $HT_{p_i}$ for $1 \leq i \leq n$ we get the following Corollaries to Theorems 5.1 and 5.4.

**Corollary 5.6.** Let $n$ be a positive integer and $1 < p_i < \infty$ for $1 \leq i \leq n$. Then $\ell_1 \not\hookrightarrow \bigotimes_{i=1}^{n} JT_{p_i}$ if and only if $\sum_{i=1}^{n} \frac{1}{q_i} < 1$.

**Corollary 5.7.** Let $n$ be a positive integer and $1 < p_i < \infty$ for $1 \leq i \leq n$. Then $\ell_1 \not\hookrightarrow \bigotimes_{i=1}^{n} HT_{p_i}$ if and only if $\sum_{i=1}^{n} \frac{1}{q_i} < 1$.

Taking $p_i = p$ for $1 \leq i \leq n$ we get.

**Corollary 5.8.** Let $n$ be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \bigotimes_{n,e} JT_p$ if and only if $n < q$.

**Corollary 5.9.** Let $n$ be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \bigotimes_{n,e} HT_p$ if and only if $n < q$.

As $JT_p'' = JT_p \oplus \ell_p(\Gamma)$ we get the the following corollary from Theorem 5.4.

**Corollary 5.10.** Let $n$ be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \bigotimes_{n,e} JT_p''$ if and only if $n < q$.

If we take $VT_1 = JH$ and $VT_2 = JT$ we get that

**Theorem 5.11.** The Banach space $JH \bigotimes_{\epsilon} JT$ does not contain a copy of $\ell_1$.

While taking $VT_i = JH$ or $\Lambda_T$ for $1 \leq i \leq n$ we get that

**Theorem 5.12.** For any $n$ in $\mathbb{N}$, $\ell_1 \not\hookrightarrow \bigotimes_{n,e} JH$ and $\ell_1 \not\hookrightarrow \bigotimes_{n,e} \Lambda_T$.

**References**


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