2019-1

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Recommended Citation

Venkova, M., Boyd, C. & Poulios, C. (2018) INJECTIVE TENSOR PRODUCTS OF TREE SPACES, *Studia Mathematica* 246(1) · January 2018 DOI: 10.4064/sm170601-11-12

INJECTIVE TENSOR PRODUCTS OF TREE SPACES

CHRISTOPHER BOYD, COSTAS POULIOS, AND MILENA VENKOVA

Abstract. We study tensor products on tree spaces; in particular, we give necessary and sufficient conditions for the n-fold injective tensor product of tree spaces to contain a copy of ℓ_1.

1. Introduction

A Banach space E is said to be Asplund if every separable subspace of E has a separable dual. The space of absolutely convergent sequences, ℓ_1, is the classical example of a Banach space which is not Asplund. In the early 1970s Stegall asked if every non-Asplund Banach space contains a copy of ℓ_1. The question was answered in the negative in 1974 when R.C. James, [J], constructed a separable Banach space which does not contain a copy of ℓ_1 yet has a non-separable dual. This space is now known as the James Tree space and is denoted by JT. A further example of such a space was provided by Hagler in 1977, [H], and became known as the James Hagler space JH.

Let $E_1, \dots, E_n$ be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We use $\bigotimes_{j=1}^{n} E_j$, to denote the tensor product of $E_1, \dots, E_n$, and define the injective norm on $\bigotimes_{j=1}^{n} E_j$ by

$$\left\| \sum_{k=1}^{m} \lambda_k x_k^1 \otimes \cdots \otimes x_k^n \right\|_{\epsilon} := \sup_{\phi_1 \in B_{E_1'}, \dots, \phi_n \in B_{E_n'}} \left| \sum_{k=1}^{m} \lambda_k \phi_1(x_k^1) \dots \phi_n(x_k^n) \right|.$$

The completion of $\bigotimes_{j=1}^{n} E_j$ with respect to the injective norm is denoted by $\widehat{\bigotimes}_{j=1,\epsilon}^{n} E_j$. In the case that $E_1 = E_2 = \cdots = E_n$ we will use the notation $\widehat{\bigotimes}_{n,\epsilon} E$.

Let us see that the containment, or more precisely, the non-containment of copies of ℓ_1 in injective tensor products of Banach spaces has important consequences. In order to do this we introduce the spaces of n-linear integral and nuclear mappings. A mapping $L\colon E_1 \times \cdots \times E_n \to \mathbb{K}$ is said to be n-linear

2010 *Mathematics Subject Classification.* Primary 46B28; Secondary 46G25, 46B22.
Key words and phrases. Tree spaces; containment of ℓ_1; Tensor Product.

if L is linear in each variable when the other $n-1$ variables are kept fixed. An n-linear mapping $L\colon E_1\times\cdots\times E_n\to\mathbb{K}$ is said to be integral if there is a regular Borel measure μ on $(B_{E_1'}\times\cdots\times B_{E_n'},\sigma(E_1'\times\cdots\times E_n',E_1\times\cdots\times E_n))$, such that

$$L(x_1,\ldots,x_n)=\int_{B_{E_1'}\times\cdots\times B_{E_n'}}\phi_1(x_1)\cdots\phi_n(x_n)\,d\mu(\phi_1,\ldots,\phi_n)\qquad(*)$$

for all $x_1\in E_1,\ldots,x_n\in E_n$. We denote the space of all n-linear integral mappings on $E_1\times\cdots\times E_n$ by $\mathcal{L}_I(E_1,\ldots,E_n)$. Endowed with the norm

$$\|L\|_I:=\inf\{|\mu|:\mu\text{ satisfies }(*)\}$$

the pair $(\mathcal{L}_I(E_1\times\cdots\times E_n),\|\cdot\|_I)$ becomes a Banach space. When the representing measure μ has countable support we shall say that L is nuclear. In practice, this means that an n-linear mapping L is nuclear if there are sequences $(\lambda_k)_k$ in $\mathbb{K}$ and $(\phi_k^j)_k$ in $\overline{B}_{E_j'}$ for $1\le j\le n$, with $\sum\limits_{k=1}^{\infty}|\lambda_k|<\infty$, such that

$$L(x_1,\ldots,x_n)=\sum_{k=1}^{\infty}\lambda_k\phi_k^1(x_1)\cdots\phi_k^n(x_n)$$

for all $x_1,\ldots,x_n$ in $E_1\times\cdots\times E_n$. We denote the space of all n-linear nuclear mappings by $\mathcal{L}_N(E_1,\ldots,E_n)$. In the case that $E_1=E_2=\cdots=E_n=E$ we will simply use the notation $\mathcal{L}_I({}^nE)$ and $\mathcal{L}_N({}^nE)$ for the spaces of n-linear integral and n-linear nuclear mappings. We note that $(\mathcal{L}_I(E_1,\ldots,E_n),\|\cdot\|_I)$ is isometrically isomorphic to the dual of $\widehat{\bigotimes}_{j=1,\epsilon}^{n}E_j$.

Alencar, [A], shows that if $E_1,\ldots,E_n$ are Asplund then the spaces of n-linear integral and nuclear mappings, $\mathcal{L}_I(E_1,\ldots,E_n)$ and $\mathcal{L}_N(E_1,\ldots,E_n)$, coincide.

The results in [BR] and [CD] illustrate the importance in determining whether the injective tensor products of Banach spaces contains a copy of ℓ_1. They show that the condition that a Banach space is Asplund has a weaker incarnation that allows us to conclude that the spaces of integral and nuclear n-linear mappings coincide. This condition is that its n-fold injective tensor product does not contain a copy of ℓ_1.

In [R], Ruess shows that there is a copy of ℓ_1 in $JT\widehat{\bigotimes}_\epsilon JT$. On the other hand, Leung, [L], proves that $JH\widehat{\bigotimes}_\epsilon JH$ does not contain a subspace isomorphic to ℓ_1. This leads us to ask the *'tree'* following questions

(a) Is it true that ℓ_1 is not contained in $\widehat{\bigotimes}_{n,\epsilon}JH$ for any n?
(b) Is ℓ_1 contained in $JT\widehat{\bigotimes}_\epsilon JH$?

(c) Given a natural number n, can we find a Banach space E so that ℓ_1 is not contained in $\widehat{\bigotimes}_{k,\epsilon} E$ for $k < n$ yet for every $k \geq n$ we have that $\widehat{\bigotimes}_{k,\epsilon} E$ contains a copy of ℓ_1?

In this paper we define a new property, the branch index, for a large class of tree spaces. We use this index to characterise the containment of ℓ_1 in the tensor products of these spaces and in particular, answer all of the above questions.

2. Tree Spaces

The dyadic tree Υ is defined as $\bigcup_{n=0}^{\infty}\{0,1\}^n$, the collection of all finite sequences of 0's and 1's. Elements of Υ are called nodes. A node t is said to have level n if $t = (\epsilon_i)_{i=1}^n$. When t has level n we write $l(t) = n$. We introduce an ordering, $\leq$, on Υ in the following way. If $t = (\epsilon_i)_{i=1}^n$ and $s = (\delta_j)_{j=1}^m$ we say that $t \leq s$ if $n \leq m$ and $\epsilon_i = \delta_i$ for $1 \leq i \leq n$. We will also say that the empty node, $\emptyset$, has the property that $\emptyset \leq t$ for all t in Υ. Under the ordering on Υ every non-empty element, t, has an immediate predecessor which we denote by t^-. We define an injection $o\colon \Upsilon \to \mathbb{N}$ by $o(\emptyset) = 1$ and inductively setting $o(t) = 2o(t^-) + \epsilon_n$ if $t = (\epsilon_i)_{i=1}^n$.

A segment, S, is a subset of Υ of the form $S = \{r : t \leq r \leq s\}$, $t, s \in \Upsilon$. A branch is a maximal ordered subset of Υ. We denote by Γ the set of all branches. Given a function $x\colon \Upsilon \to \mathbb{R}$ and a segment S we let $S(x) = \sum_{t\in S} x(t)$. For a segment $S = \{r : t \leq r \leq s\}$ we let $o(S) = o(t)$. Segments $S_1, \ldots, S_n$ are admissible if they are disjoint and begin and end at the same level.

Definition 2.1. Let UT be a vector space of functions $x\colon \Upsilon \to \mathbb{R}$. A tree space, VT, is the completion of UT with respect to some norm $\|\cdot\|$ on UT.

For t in Υ we denote by η_t the element of VT given by $\eta_t(t) = 1$ and $\eta_t(s) = 0$ for $s \neq t$. Nodes t and s are said to be incomparable if $t \not\leq s$ and $s \not\leq t$. For γ in Γ we use γ^* to denote the linear functional on VT given by $\gamma^*(x) = \sum_{t\in\gamma} x(t)$. We use $(\eta_t^*)_{t\in\Upsilon}$ to denote the system of dual functionals to $(\eta_t)_{t\in\Upsilon}$ and Γ^* to denote the dual system of branch functionals $\{\gamma^* : \gamma \in \Gamma\}$. Given a tree space VT and m in $\mathbb{N}$ we define $Q_m\colon VT \to VT$ by $Q_m\left(\sum_{t\in\Upsilon} a_t\eta_t\right) = \sum_{l(t)>m} a_t\eta_t$. For $s \in \Upsilon$ we let $Q_s\left(\sum_{t\in\Upsilon} a_t\eta_t\right) = \sum_{t\geq s} a_t\eta_t$. In all the spaces we are interested in, γ^* is a norm one linear functional while Q_m is a norm one projection.

We have a number of different ways of constructing tree spaces.

Example 2.2. (Bellenot, Haydon, Odell)[BHO] Let E be a Banach space with a normalised Schauder basis, $(e_i)_i$. We let $JT(e_i)$ denote the completion of the space of all finitely supported functions $x\colon \Upsilon \to \mathbb{R}$ with respect to the norm

$$\|x\| = \sup\left\{ \left\| \sum_{i=1}^{n} S_i(x) e_{o(S_i)} \right\| : n \in \mathbb{N}, (S_i)_{i=1}^{n} \text{ are disjoint segments in } \Upsilon \right\}.$$

A special case of the above construction is the case where we take $E = \ell_p$ with $(e_i)_i$ equal to the canonical basis in ℓ_p, $1 < p < \infty$. We will denote the corresponding tree space $JT(e_i)_i$ by JT_p. Note that JT_p is the completion of the space of all finitely supported functions $x\colon \Upsilon \to \mathbb{R}$ with respect to the norm

$$\|x\| = \sup\left\{ \left(\sum_{j=1}^{n} |S_j(x)|^p \right)^{\frac{1}{p}} : (S_j)_{j=1}^{n} \text{ are disjoint segments} \right\}.$$

In the case where $p = 2$ we obtain the original James Tree space which we denote simply by JT. It follows from [BHO, Theorem 6.1 (b)] that JT_p does not contain a copy of ℓ_1 for any $1 < p < \infty$.

Example 2.3. (Odell)[O] Consider the normed space of all finite sequences, c_{oo}, with respect to the norm

$$\|x\|_M = \sup\left\{ \|x\|_{c_o}, \frac{1}{2} \sum_{i=1}^{n} \|E_i x\| \right\}$$

with the supremum taken over all finite collections of pairwise disjoint subsets $(E_i)_{i=1}^{n}$ of $\mathbb{N}$ with $n \leq \min(E_i)$ for $1 \leq i \leq n$. Here we let $E_i x = \sum_{j \in E_i} a_j e_j$ for $x = \sum_j a_j e_j$ in c_{00}. The completion of c_{00} with respect to the norm $\|\cdot\|_M$, denoted by T_M, was introduced by Johnson [Jo] and is called the Modified Tsirelson space. In [O, Theorem 2], Odell shows that Λ_T, the tree space based on T_M, does not contain a subspace isomorphic to ℓ_1. Since Λ_T' is non separable, Λ_T is a non Asplund space. Moreover, it is shown that Λ_T is the dual of the closed linear span of the dual nodes, $\{\eta_t^* : t \in \Upsilon\}$, in Λ_T'.

Example 2.4. (Hagler)[H] The space JH is the completion of the space of all finitely supported functions $x\colon \Upsilon \to \mathbb{R}$ with respect to the norm

$$\|x\| = \sup\left\{ \sum_{j=1}^{n} |S_j(x)| : (S_j)_{j=1}^{n} \text{ are admissible segments} \right\}.$$

The space JH is called the James Hagler space.

More generally we have the following definition.

Definition 2.5. Let E be a Banach space with a normalised Schauder basis, $(e_i)_i$. We let $HT(e_i)$ denote the completion of all finitely supported functions $x\colon \Upsilon \to \mathbb{R}$ with respect to the norm

$$\|x\| = \sup\left\{ \left\| \sum_{i=1}^{n} S_i(x) e_{o(S_i)} \right\| : n \in \mathbb{N}(S_i)_{i=1}^{n} \text{ are admissible segments in } \Upsilon \right\}.$$

When E is equal to ℓ_p we denote the space $HT(e_i)$ by HT_p. Note that HT_1 is James Hagler space JH.

Although the spaces JT_p and HT_p are defined in very similar fashion, we will see that as Banach spaces they behave very differently. The reason for this difference in behaviour is explained by the following definition and lemma of Hagler, [H].

Definition 2.6. We say that the sequence of nodes $(t_n)_n$ is strongly incomparable if

(i) for $n \neq m$, t_n and t_m are incomparable,
(ii) any family of admissible segments passes through no more than two t_n.

Lemma 2.7. *(Hagler)*[H] *Let $(t_n)_n$ be a sequence of nodes in Υ with $l(t_n) < l(t_m)$ if $n < m$. Then there is a subsequence $(t_{n_k})_k$ of $(t_n)_n$ in Υ such that either*

(a) $(t_{n_k})_k$ determines a unique branch; or
(b) $(t_{n_k})_k$ is strongly incomparable.

Consider a strongly incomparable sequence of nodes $(t_n)_n$ in HT_p, it is readily established that $(\eta_{t_n})_n$ is equivalent to the unit basis of c_o. Thus HT_p contains a copy of c_o, while JT_p does not (see Corollary 4.4). Hence the nodes, $(\eta_t)_{t\in\Upsilon}$, form a boundedly complete basis for JT_p, but this is not the case with HT_p. Otherwise, if HT_p had a boundedly complete basis, [LT] would imply that HT_p is a dual Banach space and as it contains a copy of c_o it must also contain a copy of ℓ_∞, contradicting its separability. Moreover, JT_p is a dual space while HT_p is not.

For further reading on the James Tree space see [FG].

3. Branch Index of Tree Spaces

Definition 3.1. Let $1 \leq q \leq \infty$. A tree space VT is said to have branch index q if there is a constant, $C > 0$, such that whenever $(\gamma_j)_{j=1}^{k}$ is a sequence of mutually distinct branches in Υ we are able to find $m \in \mathbb{N}$ so that for any $(\alpha_j)_{j=1}^k$ in $\mathbb{R}^k$ we have

$$\left\|\sum_{j=1}^k \alpha_j Q_m^*(\gamma_j^*)\right\| \le C\left(\sum_{j=1}^k |\alpha_j|^q\right)^{1/q}.$$

In the case there is $m \in \mathbb{N}$ so that

$$\left\|\sum_{j=1}^k \alpha_j Q_m^*(\gamma_j^*)\right\| = \left(\sum_{j=1}^k |\alpha_j|^q\right)^{1/q}$$

we say that VT has branch index q isometrically.

We observe that if VT has branch index q then it will also have branch index $\tilde{q}$ for every $\tilde{q} < q$. It follows from [H, Lemma 9] that JH has branch index ∞. Similarly, in [O, Lemma 11], it is shown that Λ_T has branch index ∞.

Definition 3.2. Let VT be a tree space. We say that VT is quasi-shrinking if $\{(\eta_t^*)_{t\in\Upsilon} \cup \Gamma^*\}$ spans a dense subspace of VT'.

It is shown in [BHO, Theorem 6.1] that $JT(e_i)$ is quasi-shrinking whenever E is a reflexive space. It can also be shown that the James–Hagler space, JH, is also quasi-shrinking, (see [L] and [BHO, page 41]). This will also follow from a more general result (see Theorem 4.11). By [O, Theorem 2(5)] the space Λ_T is quasi-shrinking.

For $1 < p < \infty$ we use q to denote the conjugate index of p. Given a continuous linear operator $T\colon X \to Y$ we use T^* to denote its transpose given by $(T^*\phi)(x) = \phi(T(x))$ for x in X, ϕ in Y'.

Lemma 3.3. *Let $1 < p < \infty$ and $T_j\colon JT_p \to JT_p$, $j = 1,\ldots,k$ be continuous linear operators, and $\{\sigma_j\}_{j=1}^k$ be subsets of Υ. Suppose that*

(a) for every x in JT_p with $\operatorname{supp}(x) \subseteq \Upsilon \setminus \sigma_j$, we have $T_j x = 0$,

(b) each segment in Υ intersects at most one σ_j,

(c) the intersection of any segment of Υ with any σ_j is a segment.

Then for each ϕ in JT_p'

$$\left\|\sum_{j=1}^k T_j^*\phi\right\|^q = \sum_{j=1}^k \|T_j^*\phi\|^q.$$

Proof. By conditions (b) and (c) it follows that if $x_1,\ldots,x_k$ belong to JT_p with $\operatorname{supp}(x_j) \subseteq \sigma_j$ for every $j = 1,2,\ldots,k$, then

$$\left\|\sum_{j=1}^k x_j\right\|^p = \sum_{j=1}^k \|x_j\|^p.$$

In particular, for any $x \in JT_p$ we have

$$\left\|\sum_{j=1}^{k} P_{\sigma_j} x\right\|^p = \sum_{j=1}^{k} \|P_{\sigma_j} x\|^p,$$

where $P_{\sigma_j}\colon JT_p \to JT_p$ is the norm one projection $P_{\sigma_j}(\sum_{t\in\Upsilon} a_t\eta_t) = \sum_{t\in\sigma_j} a_t\eta_t$. Then for each ϕ in JT_p' we have that

$$\begin{aligned}
\langle\sum_{j=1}^{k} T_j^*\phi, x\rangle &= \sum_{j=1}^{k}\langle T_j^*\phi, x\rangle = \sum_{j=1}^{k}\langle \phi, T_j x\rangle = \sum_{j=1}^{k}\langle \phi, T_j P_{\sigma_j} x\rangle \\
&= \sum_{j=1}^{k}\langle T_j^*\phi, P_{\sigma_j} x\rangle \le \sum_{j=1}^{k} \|T_j^*\phi\|\|P_{\sigma_j} x\| \\
&\le \left(\sum_{j=1}^{k} \|T_j^*\phi\|^q\right)^{1/q} \left(\sum_{j=1}^{k} \|P_{\sigma_j} x\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{k} \|T_j^*\phi\|^q\right)^{1/q} \left\|\sum_{j=1}^{k} P_{\sigma_j} x\right\| \\
&\le \left(\sum_{j=1}^{k} \|T_j^*\phi\|^q\right)^{1/q} \|x\|
\end{aligned}$$

and therefore taking the supremum over all x in the unit ball of JT_p we get that

$$\left\|\sum_{j=1}^{k} T_j^*\phi\right\|^q \le \sum_{j=1}^{k} \|T_j^*\phi\|^q.$$

For the reverse inequality let $\phi \in JT_p'$ and $\epsilon > 0$. For each $1 \le j \le k$ use (a) to choose x_j with $\|x_j\| = 1$ and support contained in σ_j so that

$$\|T_j^*\phi\|^q \le |\langle T_j^*\phi, x_j\rangle|^q + \epsilon.$$

Let

$$x = \sum_{j=1}^{k} \operatorname{sgn}(\langle T_j^*\phi, x_j\rangle)|\langle T_j^*\phi, x_j\rangle|^{q-1} x_j.$$

Then, we have $\|x\|^p = \sum_{j=1}^k |\langle T_j^*\phi, x_j\rangle|^q$ and

$$\left(\sum_{j=1}^k \|T_j^*\phi\|^q - \epsilon\right)^{p-1} \|x\|^p \le \left(\sum_{j=1}^k |\langle T_j^*\phi, x_j\rangle|^q\right)^{p-1} \|x\|^p$$
$$= \left(\sum_{j=1}^k |\langle T_j^*\phi, x_j\rangle|^q\right)^p$$
$$= \left|\sum_{j=1}^k \langle T_j^*\phi, x\rangle\right|^p$$
$$\le \left\|\sum_{j=1}^k T_j^*\phi\right\|^p \|x\|^p.$$

As this holds for all $\epsilon > 0$ we get that

$$\left(\sum_{j=1}^k \|T_j^*\phi\|^q\right)^{\frac{p-1}{p}} \le \left\|\sum_{j=1}^k T_j^*\phi\right\|$$

or that

$$\sum_{j=1}^k \|T_j^*\phi\|^q \le \left\|\sum_{j=1}^k T_j^*\phi\right\|^q$$

and the identity is established. □

Given branches $\gamma_1, \ldots, \gamma_k$ in Υ, we choose m in $\mathbb{N}$ sufficiently large so that $\gamma_j \cap \{t \in \Upsilon : l(t) \ge m\}$ are pairwise disjoint, $1 \le j \le k$. Taking $\sigma_j = \gamma_j \cap \{t \in \Upsilon : l(t) \ge m\}$, $T_j = P_{\sigma_j}$ for $1 \le j \le k$ and $\phi = \sum_{j=1}^k a_j \gamma_j^*$ with $a_j \in \mathbb{R}$ in Lemma 3.3, we get the following Corollary.

Corollary 3.4. *The James Tree space, JT_p, has branch index q isometrically, where q is the conjugate index of p.*

4. Biduals of JT_p and HT_p

In this section we will give a description of the biduals of JT_p and HT_p.

4.1. Bidual of JT_p.

Let us begin with JT_p. A description of the bidual of JT is given in [LS].

We introduce the space J_p as the completion of the space of all sequences in c_o, $(a_j)_j$, with respect to the norm

$$|||(a_n)_n||| := \sup\left(\sum_{i=1}^n |a_{k_{2i-1}} - a_{k_{2i}}|^p\right)^{1/p}$$

where the supremum is taken over any choice of n and any choice of positive integers $k_1 < k_2 < \ldots < k_{2n}$. Equivalent norms on J_p are obtained by considering the norm

$$\|(a_n)_n\| = \sup\left(\frac{1}{2}\sum_{i=1}^{n}|a_{k_{i+1}} - a_{k_i}|^p + |a_{k_{n+1}} - a_{k_1}|^p\right)^{1/p}$$

or the norm

$$|||(a_n)_n||| = \sup\left(\frac{1}{2}\sum_{i=0}^{n}|a_{k_{i+1}} - a_{k_i}|^p\right)^{1/p}$$

where $a_0 = 0$ and the supremum is taken over all n and all choices of positive integers $0 = k_0 < k_1 < k_2 < \ldots < k_{n+1}$.

If we denote by $(e_n)_n$ the unit vector basis in J_p then for $p > 1$, $(e_n)_n$ is a shrinking basis for J_p. To see this we assume that there is ϕ in J_p', a block basis sequence, $(x_k)_k$, of $(e_n)_n$ and $\epsilon > 0$ so that $\|x_k\| = 1$ and $\phi(x_k) > \epsilon$ for all k in $\mathbb{N}$. Consider $\sum_{k=1}^{\infty} \frac{x_k}{k}$. Then each term used to calculate the norm of $\sum_{k=1}^{\infty} \frac{x_k}{k}$ is either of the form

$$\left(\left|\frac{e_j^*(x_k)}{k} - \frac{e_l^*(x_k)}{k}\right|\right)^p$$

or of the form

$$\left[\left|\frac{e_j^*(x_k)}{k} - \frac{e_l^*(x_{k+m})}{k+m}\right|\right]^p.$$

It follows from the convexity of the function $f(x) = x^p$ that

$$\left[\left|\frac{e_j^*(x_k)}{k} - \frac{e_l^*(x_{k+m})}{k+m}\right|\right]^p \leq 2^{p-1}\left[\left(\frac{e_j^*(x_k)}{k}\right)^p + \left(\frac{e_l^*(x_{k+m})}{k+m}\right)^p\right].$$

Hence we have that

$$\left\|\sum_{k=1}^{\infty}\frac{x_k}{k}\right\|^p \leq 2^{p-1}\sum_{k=1}^{\infty}\left(\frac{\|x_k\|}{k}\right)^p < \infty.$$

But this contradicts our assumption that $\phi(x_k) > \epsilon$ for all k and thus $(e_n)_n$ is a shrinking basis for J_p.

We denote by $(s_n)_n$ the summing basis for J_p given by $s_n = \sum_{j=1}^{n} e_j$. It is easily checked that the summing basis is a monotone boundedly complete basis for J_p. A routine calculation shows that $s_n^* = e_n^* - e_{n+1}^*$ for all n in $\mathbb{N}$.

Moreover, if $\sum_{j=1}^{\infty} a_j s_j$ belongs to J_p then we have that

$$\begin{aligned} e_1^*\left(\sum_{j=1}^{\infty} a_j s_j\right) =& e_1^*\left(a_1 e_1 + a_2(e_1+e_2) + a_3(e_1+e_2+e_3) + \cdots\right) \\ =& \left(\sum_{j=1}^{\infty} a_j\right) e_1^*(e_1) + \left(\sum_{j=2}^{\infty} a_j\right) e_1^*(e_2) + \left(\sum_{j=3}^{\infty} a_j\right) e_1^*(e_3) + \cdots \\ =& \sum_{j=1}^{\infty} a_j, \end{aligned}$$

showing that e_1^* coincides with the summing function of Bellenot, Haydon and Odell [BHO].

Since the canonical basis for ℓ_p is shrinking it follows from [BHO, Theorem 4.1] that ℓ_1 does not embed in J_p. Applying [BHO, Theorem 2.2] we get that J_p' is the span of $\{[s_j^*]_{j=1}^{\infty} \cup [e_1^*]\}$. From this we obtain the following Lemma.

Lemma 4.1. *Let $1 < p < \infty$ and consider $\phi \in J_p'$. Then $\lim_{n\to\infty} \phi(s_n)$ exists. Moreover, it is 0 if and only if $\phi \in [s_j^*]_{j=1}^{\infty}$.*

Proof. We have that $J_p' = [s_j^*]_{j=1}^{\infty} \oplus \{e_1^*\}$. Hence we may write ϕ in J_p' as $\phi = \sum_{j=1}^{\infty} a_j s_j^* + b e_1^*$. Therefore $\lim_{n\to\infty} \phi(s_n) = b$ and so $\phi \in \mathrm{sp}\{s_j^*\}_{j=1}^{\infty}$ if and only if $\lim_{j\to\infty} \phi(s_j) = 0$. □

Lemma 4.2. *Consider the subspace Y of JT_p' given by $Y = \{\phi \in JT_p' : \lim_{t\in\gamma} \phi(\eta_t) = 0$ for all branches $\gamma\}$. Then for each ϕ in Y*

$$\lim_{k\to\infty} \left(\sup_{l(t)=k} \|Q_t^* \phi\|\right) = 0.$$

Proof. We suppose that there is ϕ in Y, a sequence of natural numbers $(n_j)_j$ and a sequence of nodes $(t_{n_j})_j$ with $l(t_{n_j}) = n_j$ such that $\|Q_{t_{n_j}}^* \phi\| > \alpha > 0$. We will first show that only finitely many nodes can be mutually incomparable. To see this suppose that there are k mutually incomparable nodes $t_{n_1}, \ldots, t_{n_k}$. For each j, $1 \le j \le k$, choose x_j in $Q_{t_{n_j}}(JT_p)$ with $\|x_j\| = 1$ so that $\phi(x_j) \ge \alpha$.

We have that $\left\|\sum_{j=1}^{k} x_j\right\| = k^{1/p}$. This gives us that

$$k\alpha \le \phi\left(\sum_{j=1}^{k} x_j\right) \le \|\phi\| k^{1/p}$$

or that

$$(\|\phi\|/\alpha)^q \ge k$$

and hence k must be finite. Because of this we may assume without loss of generality that $(t_{n_k})_k$ belong to a single branch γ. For each ψ in JT_p' and any sequence of nodes such that $l(s_n) < l(s_{n+1})$ we have that $\lim_{n\to\infty} \|\psi - Q_{s_n}^*\psi\| = \|\psi\|$. Hence by choosing a subsequence of $(t_{n_j})_j$, if necessary, we may also assume that for all j in $\mathbb{N}$ we have

$$\|Q_{t_{n_j}}^*\phi - Q_{t_{n_{j+1}}}^*\phi\| \geq \frac{3}{4}\alpha > 0.$$

Consider the projection $P_\gamma\colon JT_p \to JT_p$ given by

$$P_\gamma(x) = \sum_{t_n\in\gamma} \langle \eta_{t_n}^*, x\rangle \eta_{t_n}.$$

Then the mapping $T\colon P_\gamma(JT_p) \to J_p$, given by $T(\eta_{t_n}) = s_n$, is an isometry. As $\phi \in Y$ we have that

$$\lim_{n\to\infty} \phi(T^{-1}s_n) = \lim_{n\to\infty} \phi(\eta_{t_n}) = 0$$

and hence using Lemma 4.1 we have $\phi \circ T^{-1} \in [s_n^*]$. Write $\phi \circ T^{-1}$ as $\phi \circ T^{-1} = \sum_{i=1}^\infty \beta_i s_i^*$. We have $P_\gamma^*\phi = \sum_{i=1}^\infty \beta_{t_i}\eta_{t_i}^*$ and $(Q_{t_{n_j}}^* - Q_{t_{n_{j+1}}}^*)P_\gamma^*\phi = \sum_{i=n_j}^{n_{j+1}-1} \beta_{t_i}\eta_{t_i}^*$ and thus it follows that

$$P_\gamma^*\phi = \sum_{i=1}^\infty \beta_{t_i}\eta_{t_i}^* = \sum_{j=1}^\infty (Q_{t_{n_j}}^* - Q_{t_{n_{j+1}}}^*)P_\gamma^*\phi.$$

It follows that for sufficiently large j

$$\left\|(Q_{t_{n_j}}^* - Q_{t_{n_{j+1}}}^*)P_\gamma^*\phi\right\| < \frac{1}{2}\alpha.$$

Let $R_j = (Q_{t_{n_j}} - Q_{t_{n_{j+1}}}) - P_\gamma(Q_{t_{n_j}} - Q_{t_{n_{j+1}}})$. The image of R_j consists of all x with support σ_j greater than or equal to t_{n_j} but not greater than or equal to $t_{n_{j+1}}$ and not contained in γ. Then $(R_j)_j$ and $(\sigma_j)_{j=1}$ satisfy the conditions of Lemma 3.3 and thus

$$\sum_{j=1}^k \|R_j^*\phi\|^q = \left\|\sum_{j=1}^k R_j^*\phi\right\|^q$$

for all k. However for each j we have that $\|R_j^*\phi\| > \frac{1}{4}\alpha$ while

$$\left\|\sum_{j=1}^k R_j^*\right\| = \left\|(Q_{t_{n_1}}^* - Q_{t_{n_{j+1}}}^*) - (Q_{t_{n_1}}^* - Q_{t_{n_{j+1}}}^*)P_\gamma^*\right\| \leq 4$$

and we have a contradiction. □

It follows from the proof of Lemma 4.2 that for each branch γ in Γ and each ϕ in JT_p', $\lim_{t\in\gamma}\phi(\eta_t)$ exists. Hence the function $S\colon JT_p' \to \mathbb{R}^\Gamma$ given by

$$S(\phi) = \left(\lim_{t\in\gamma}\phi(\eta_t)\right)_{\gamma\in\Gamma}$$

is well defined.

We claim that $S(JT_p') = \ell_q(\Gamma)$. We start by showing that $S(JT_p') \subseteq \ell_q(\Gamma)$. To see this, let $(\gamma_j)_{j=1}^r$ be distinct branches in Γ. For m sufficiently large we have that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint, $1 \leq j \leq r$. For $1 \leq j \leq r$ choose t_j in $\gamma_j \cap \{t : l(t) \geq m\}$. Fix ϕ in JT_p' and let $x = \sum_{j=1}^r \operatorname{sgn}(\langle \phi, \eta_{t_j} \rangle) |\langle \phi, \eta_{t_j} \rangle|^{q-1} \eta_{t_j}$. Then we have

$$\sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q = \phi(x) \leq \|\phi\| \|x\| = \|\phi\| \left(\sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q \right)^{\frac{1}{p}}$$

and therefore $\left(\sum_{j=1}^r |\langle \phi, \eta_{t_j} \rangle|^q \right)^{\frac{1}{q}} \leq \|\phi\|$. Letting j tend to ∞, we get that $\|S(\phi)\| \leq \|\phi\|$ and therefore S is continuous and has norm less than or equal to 1.

Conversely, $S(\gamma^*)$ is the vector in $\ell_q(\Gamma)$ which is 1 on γ and 0 on every other branch. Since $\|\gamma^*\| = \|S(\gamma^*)\| = 1$ we have that S has norm 1.

To show that S is surjective let $(\gamma_j)_{j=1}^r$ be distinct branches in Γ and choose m so that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint, $1 \leq j \leq r$. Given $(a_j)_{j=1}^r$ we define ϕ by

$$\phi \left(\sum_{t \in \Upsilon} b_t \eta_t \right) := \sum_{j=1}^r a_j \left(\sum_{t \in \gamma_j : l(t) \geq m} b_t \right).$$

Then

$$\begin{aligned} \left| \phi \left(\sum_{t \in \Upsilon} b_t \eta_t \right) \right| &= \left| \sum_{j=1}^r a_j \sum_{t \in \gamma_j : l(t) \geq m} b_t \right| \\ &\leq \left(\sum_{j=1}^r |a_j|^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^r \left| \sum_{t \in \gamma_j : l(t) \geq m} b_t \right|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^r |a_j|^q \right)^{\frac{1}{q}} \left\| \sum_{t \in \Upsilon} b_t \eta_t \right\| \end{aligned}$$

Conversely, we choose t_j in γ_j with $l(t_j) > m$, $1 \leq j \leq r$, and we set $x = \sum_{j=1}^r \operatorname{sgn}(a_j) |a_j|^{q-1} \eta_{t_j}$. Then, we have $\|x\| = \left(\sum_{j=1}^r |a_j|^q \right)^{1/p}$, $\phi(\eta_{t_j}) = a_j$ for each $j = 1, 2, \ldots, r$, and $\phi(x) = \sum_{j=1}^r |a_j|^q$. Therefore,

$$\sum_{j=1}^r |a_j|^q = \phi(x) \leq \|\phi\| \|x\| = \|\phi\| \left(\sum_{j=1}^r |a_j|^q \right)^{1/p}$$

that is

$$\left(\sum_{j=1}^{r}|a_j|^q\right)^{1/q} \leq \|\phi\|.$$

Hence $\|\phi\| = \left(\sum_{j=1}^{r}|a_j|^q\right)^{\frac{1}{q}}$ and given any ν in $\ell_q(\Gamma)$ there is ϕ in JT_p' with $S(\phi) = \nu$ proving that S is surjective.

We will next show that $\ker S$, the kernel of S, is equal to $[\eta_t^*]_{t\in\Upsilon}$. We observe that $[\eta_t^*]_{t\in\Upsilon}$, the span of the dual nodes, is contained in the kernel of S. To see that these subspaces actually coincide we assume that $[\eta_t^*]_{t\in\Upsilon}$ is a proper subspace of $\ker S$. Let $\kappa = \frac{1}{2}(2^p - 2^{p-1} - 1) > 0$ and choose $\delta > 0$ so that $\frac{2^{p-1}+\kappa+1}{2^p} < (1-\delta)^p$. Choose ϕ in $\ker S$ of norm 1 so that $d(\phi, [\eta_t^*]_{t\in\Upsilon}) = a > 0$. Let ψ_o in $[\eta_t^*]_{t\in\Upsilon}$ be such that $\|\psi_o + \phi\| < \frac{a}{1-\delta}$. Setting $\nu = \frac{\psi_o+\phi}{\|\psi_o+\phi\|}$ we get that

$$d(\nu, [\eta_t^*]_{t\in\Upsilon}) = \frac{a}{\|\psi_o + \phi\|} > 1 - \delta.$$

Choose x in JT_p of norm 1 so that $\nu(x) > 1 - \delta$ and $r \in \mathbb{N}$ so that $\nu(P_r(x)) > 1-\delta$ where $P_r = I - Q_r$. This, in particular means that $\|P_r x\| > 1 - \delta$. Choose $\epsilon > 0$ so that $2^{r+p-1}\epsilon^{(q-1)p} < (1-\delta)^p\kappa$. It follows from Lemma 4.2 that we can find $r' > r$ so that $\|Q_{u_j}^*\nu\| \leq \epsilon$ for $1 \leq j \leq 2^{r'}$, where u_j, $j = 1, \dots, 2^{r'}$ are the nodes of level r'. Then $(I - Q_{r'})^*\nu \in [\eta_t^*]_{t\in\Upsilon}$ and hence we have that

$$\|Q_{r'}^*\nu\| = \|\nu - (I - Q_{r'})^*\nu\| \geq d(\nu, [\eta_t^*]_{t\in\Upsilon}) > 1 - \delta,$$

and so by Lemma 3.3 we have that

$$\sum_{j=1}^{2^{r'}} \|Q_{u_j}^*\nu\|^q = \|Q_{r'}^*\nu\|^q > (1-\delta)^q.$$

For each $1 \leq j \leq 2^{r'}$ choose x_j in JT_p so that $\|x_j\| = 1$, $Q_{u_j}x_j = x_j$ and

$$A^p := \sum_{j=1}^{2^{r'}} |Q_{u_j}^*\nu(x_j)|^q = \sum_{j=1}^{2^{r'}} |\nu(x_j)|^q > (1-\delta)^q.$$

Let

$$z = \frac{1}{A}\sum_{j=1}^{2^{r'}} \operatorname{sgn}(\nu(x_j))|\nu(x_j)|^{q-1}x_j.$$

Then, by definition of the norm on JT_p, we have

$$\|z\|^p = \frac{1}{A^p}\sum_{j=1}^{2^{r'}} |\nu(x_j)|^{(q-1)p} = \frac{1}{A^p}\sum_{j=1}^{2^{r'}} |\nu(x_j)|^q = 1.$$

In addition

$$\nu(z) = \frac{1}{A}\sum_{j=1}^{2^{r'}} |\nu(x_j)|^q = A^{p-1} > 1-\delta$$

and

$$\|Q_{u_j} z\| = \frac{1}{A}|\nu(x_j)|^{q-1} = \frac{1}{A}|Q^*_{u_j}\nu(x_j)|^{q-1} \le \frac{1}{A}\|Q^*_{u_j}\nu\|^{q-1} \le \frac{\epsilon^{q-1}}{(1-\delta)^{q/p}}.$$

So, we have constructed an element, z, of norm 1 in JT_p with the property that $\nu(z) > 1-\delta$. As $\lim_{n\to\infty} Q_n(z) = 0$, letting $\tilde{z} = z - Q_s(z)$ with s sufficiently large, we get that there is $s > r'$ with $Q_s(\tilde{z}) = 0$, $\|\tilde{z}\| \le 1$ and $\nu(\tilde{z}) > 1-\delta$.

Since $\nu(P_r(x)) > 1-\delta$ we can choose y in JT_p with $\|y\| = 1$ so that $Q_r y = 0$ and $\nu(y) > 1-\delta$. Then we have that $\nu(\tilde{z}+y) > 2(1-\delta)$ and $\tilde{z}+y = \sum_{k=0}^{s}\sum_{t\in\Upsilon:l(t)=k} a_t\eta_t$ with $a_t = 0$ for $r < l(t) < r'$.

Let us consider the following three collections of segments,

$$\mathcal{S}_1 = \{S : \text{there is } t\in S \text{ with } l(t) = r \text{ and } t \in S \text{ with } l(t) = r'\},$$

$$\mathcal{S}_2 = \{S : l(t) < r' \text{ for all } t \in S\},$$

$$\mathcal{S}_3 = \{S : l(t) > r \text{ for all } t \in S\}.$$

Since each segment in Υ lies in either S_1, S_2 or S_3 by the definition of the norm on JT_p we have segments $S^1_1,\ldots,S^1_l$ in $\mathcal{S}_1$, $S^2_1,\ldots,S^2_m$ in $\mathcal{S}_2$ and $S^3_1,\ldots,S^3_n$ in $\mathcal{S}_3$,

$$\|\tilde{z}+y\|^p = \sum_{i=1}^{l}\left(\sum_{t\in S^1_i} a_t\right)^p + \sum_{i'=1}^{m}\left(\sum_{t\in S^2_{i'}} a_t\right)^p + \sum_{i''=1}^{n}\left(\sum_{t\in S^3_{i''}} a_t\right)^p$$

Firstly, we have

$$\begin{aligned}\sum_{i=1}^{l}\left(\sum_{t\in S^1_i} a_t\right)^p \le& 2^{p-1}\left(\sum_{i=1}^{l}\left(\sum_{t\in S^1_i:l(t)<r'} a_t\right)^p + \sum_{i=1}^{l}\left(\sum_{t\in S^1_i:l(t)\ge r'} a_t\right)^p\right)\\ =:& 2^{p-1}(B_1+B_2).\end{aligned}$$

Since $\tilde{z}$ contains no nodes of level strictly less than r' we have

$$2^{p-1}B_1 + \sum_{i'=1}^{m}\left(\sum_{t\in S^2_{i'}} a_t\right)^p \le 2^{p-1}\|y\|^p = 2^{p-1}.$$

Secondly, there are at most 2^r nodes with level less than r. Hence, we have that $l \le 2^r$ and we get that

$$B_2 = \sum_{i=1}^{l}\left(\sum_{t\in S^1_i:l(t)\ge r'} a_t\right)^p \le \sum_{i=1}^{l}\|Q_{u_{j_i}}\tilde{z}\|^p \le 2^r\frac{\epsilon^{(q-1)p}}{(1-\delta)^q} < \kappa/2^{p-1}.$$

Finally we have that

$$\sum_{i''=1}^{n}\left(\sum_{t\in S^3_{i''}} a_t\right)^p \le \|\tilde{z}\|^p \le 1$$

which gives that

$$\|\tilde{z}+y\| \le (2^{p-1}+\kappa+1)^{\frac{1}{p}}.$$

Thus we get that

$$1=\|\nu\| \ge \frac{|\nu(\tilde{z}+y)|}{\|\tilde{z}+y\|} > \frac{2(1-\delta)}{(2^{p-1}+\kappa+1)^{\frac{1}{p}}},$$

a contradiction, and therefore $\ker S = [\eta_t^*]_{t\in\Upsilon}$.

We denote by $j_{[\eta_t^*]}$ the canonical injection of $[\eta_t^*]$ into JT_p' and by j_{JT_p} the canonical inclusion of JT_p into JT_p''.

We have that

Theorem 4.3.

$$JT_p'' = \left(j_{[\eta_t^*]}([\eta_t^*])\right)^\perp \oplus j_{JT_p}(JT_p) \simeq \left(JT_p'/j_{[\eta_t^*]}([\eta_t^*])\right)' \oplus JT_p \simeq \ell_p(\Gamma)\oplus JT_p.$$

Corollary 4.4. *For $1<p<\infty$, the space JT_p does not contain a copy of c_o.*

Proof. Suppose that JT_p contains a copy of c_o. Then JT_p''' has a quotient which is isomorphic to ℓ_∞'. However, as JT_p''' is isomorphic to $JT_p'\oplus\ell_q(\Gamma)$ it has cardinality equal to the continuum, $\mathfrak{c}$. The cardinality of ℓ_∞' is $2^{\mathfrak{c}}$ giving us a contradiction. □

4.2. **Bidual of HT_p.** Let us now consider the space HT_p. Again we will give a description of the bidual of HT_p. This will allow us to show that HT_p does not contain a copy of ℓ_1 and that HT_p has branch index q isometrically.

We define an operator $Q\colon HT_p' \to \ell_q(\Gamma)$ by $Q(\phi)(\gamma) = \lim_{t\in\gamma}\phi(\eta_t)$ for any branch γ in Γ and ϕ in HT_p'.

We first must show that Q is well-defined. To see this, consider any finite subset $\Gamma' = \{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ of Γ. Choosing m sufficiently large we can assume that $S_i = \gamma_i \cap \{t\in\Upsilon : l(t)\ge m\}$, $i=1,\ldots,n$ are pairwise disjoint segments. For $i=1,\ldots,n$ choose t_i in S_i and let $x=\sum_{i=1}^{n}\operatorname{sgn}(\phi(\eta_{t_i}))|\phi(\eta_{t_i})|^{q-1}\eta_{t_i}$. Then we have

$$\|x\| = \left(\sum_{i=1}^{n}|\phi(\eta_{t_i})|^{(q-1)p}\right)^{1/p} = \left(\sum_{i=1}^{n}|\phi(\eta_{t_i})|^{q}\right)^{1/p}.$$

As $|\phi(x)| \leq \|\phi\| \|x\|$ we get

$$\sum_{i=1}^{n} |\phi(\eta_{t_i})|^q \leq \|\phi\| \left(\sum_{i=1}^{n} |\phi(\eta_{t_i})|^q \right)^{1/p}$$

which we rewrite as

$$\left(\sum_{i=1}^{n} |\phi(\eta_{t_i})|^q \right)^{1/q} \leq \|\phi\|.$$

Letting $l(t_i)$ tend to infinity we get that

$$\left(\sum_{i=1}^{n} |Q(\phi)(\gamma_i)|^q \right)^{1/q} \leq \|\phi\|,$$

which proves that $Q(\phi)$ belongs to $\ell_q(\Gamma)$ with $\|Q(\phi)\|_q \leq \|\phi\|$. Hence Q is well defined and bounded with norm no greater than 1. Moreover, taking $\phi = \gamma_o^*$ for any branch γ_o in Γ we see that

$$Q(\gamma_o^*)(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_o, \\ 0 & \text{otherwise.} \end{cases}$$

and therefore $\|Q\| = 1$.

We claim that Q is surjective. To see this, let $\gamma_1, \ldots, \gamma_n$ be n distinct branches in Γ and $\alpha_1, \ldots, \alpha_n$ belong to $\mathbb{R}$. Choose $m \in \mathbb{N}$ so that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint, $1 \leq j \leq n$. Let $\phi = \sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*)$. For any x in HT_p we have

$$\begin{aligned} |\phi(x)| &= \left| \sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*)(x) \right| \\ &\leq \sum_{i=1}^{n} |\alpha_i| |Q_m^*(\gamma_i^*)(x)| \\ &\leq \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \left(\sum_{i=1}^{n} |Q_m^*(\gamma_i^*)(x)|^p \right)^{1/p} \\ &\leq \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \|x\| \end{aligned}$$

and therefore

$$\|\phi\| \leq \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q}.$$

For $i = 1, \ldots, n$ choose nodes t_i in $\gamma_i \cap \{t \in \Upsilon : l(t) \geq m\}$ and let $x = \sum_{i=1}^n \operatorname{sgn}(\alpha_i)|\alpha_i|^{q-1}\eta_{t_i}$. Then we have

$$\|x\| = \left(\sum_{i=1}^n |\alpha_i|^{(q-1)p}\right)^{1/p} = \left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/p}.$$

Hence

$$\begin{aligned}\|\phi\| \geq & \frac{|\phi(x)|}{\|x\|} = \frac{1}{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/p}} \left|\sum_{i=1}^n \operatorname{sgn}(\alpha_i)|\alpha_i|^{q-1}\phi(\eta_{t_i})\right| \\ = & \frac{1}{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/p}} \sum_{i=1}^n |\alpha_i|^q \\ = & \left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}.\end{aligned}$$

Thus, we have $\|\phi\| = \left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}$. Furthermore,

$$Q(\phi)(\gamma) = \begin{cases} \alpha_i & \text{if } \gamma = \gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence Q is a quotient mapping and we have shown that Q is a bounded, linear mapping of norm 1 from HT_p' onto $\ell_q(\Gamma)$.

Let $G = \ker Q$. Then we have that HT_p'/G is isomorphic to $\ell_q(\Gamma)$. We claim that $G = [\eta_t^*]_{t\in\Upsilon}$. We use the following Lemma of Hagler.

Lemma 4.5. *(Hagler)*[H, Lemma 8] *For ϕ in $G = \ker Q$,*

$$\lim_{n\to\infty}\left(\max_{l(t)=n} \|Q_t^*(\phi)\|\right) = 0.$$

Theorem 4.6. *If $1 < p < \infty$ then* $\ker Q = [\eta_t^*]_{t\in\Upsilon}$.

Proof. Let us use F to denote $[\eta_t^*]_{t\in\Upsilon}$, the closed linear span of $\{\eta_t^*\}_{t\in\Upsilon}$. Clearly we have that $F \subseteq \ker Q$. Assume that $F \subsetneq \ker Q$. Choose $\delta \in (0,1)$ so that

$$\frac{1-(1-\delta)^p}{\delta} < p + \frac{1}{3}$$

and

$$(3-4\delta)^p > 2 \cdot 3^{p-1} + 3^{p-1}(3p+1)\delta.$$

Choose ϕ in $\ker Q$ with $\|\phi\| = 1$ and $\inf\{\|\phi - \psi\| : \psi \in F\} > 1 - \delta$. We now choose x, y and z as follows:

(i) Choose x in HT_p with $\|x\| = 1$ so that $Q_m(x) = 0$, for some $m \in \mathbb{N}$ and $\phi(x) > 1 - \delta$.

(ii) Choose $\epsilon > 0$ so that $2^m\epsilon < \delta$. By Lemma 4.5 we can find $n \geq 2^{m+1}$ so that $\|\phi \circ Q_t\| \leq \epsilon$ for every node t with $l(t) = n$. Pick y in HT_p with $\|y\| = 1$ so that $Q_n(y) = y$, $\phi(y) > 1 - \delta$ and $Q_k(y) = 0$ for some $k > n$.

(iii) Choose z in HT_p so that $\|z\| = 1$, $Q_k(z) = z$ and $\phi(z) > 1 - \delta$.

Then we have

$$\|x + y + z\| \geq \phi(x + y + z) = 3(1 - \delta).$$

We will now consider two cases and in each we arrive at a contradiction.

Case I: We assume that for any admissible family of segments $S_1.S_2, \dots, S_{2^m}$ passing through the support of y we have

$$\sum_{j=1}^{2^m} |S_j(y)|^p \leq 1 - (3p + 1)\delta.$$

Then if $S_1, S_2, \dots, S_r$ are admissible segments which do not pass through the support of y then either of the two mutually exclusive events occurs

(a) $S_1, \dots, S_r$ intersect the support of x and the support of y,

(b) $S_1, \dots, S_r$ intersect the support of y and the support of z.

If for instance, (b) occurs we have

$$\left(\sum_{j=1}^{r} |S_j(x + y + z)|^p\right)^{1/p} = \left(\sum_{j=1}^{r} |S_j(y + z)|^p\right)^{1/p} \leq \|y + z\| \leq 2.$$

However, as $\|x+y+z\| > 3(1-\delta)$ it follows that there must exist admissible segments, $S_1, \dots, S_r$, passing through the support of y and which give the norm of $x + y + z$. Now,

$$\begin{aligned}
\|x + y + z\|^p =& \sum_{j=1}^{r} |S_j(x + y + z)|^p \\
\leq& \sum_{j=1}^{r} (|S_j(x)| + |S_j(y)| + |S_j(z)|)^p \\
\leq& \sum_{j=1}^{r} 3^{p-1} (|S_j(x)|^p + |S_j(y)|^p + |S_j(z)|^p) \\
\leq& 3^{p-1} (\|x\|^p + 1 - (3p + 1)\delta + \|z\|^p) \\
=& 3^{p-1} (3 - (3p + 1)\delta) .
\end{aligned}$$

It follows that

$$3^p(1 - \delta)^p \leq \|x + y + z\|^p \leq 3^{p-1}(3 - (3p + 1)\delta)$$

which implies that

$$3(1-\delta)^p \leq 3-(3p+1)\delta$$

or that

$$p+\frac{1}{3} \leq \frac{1-(1-\delta)^p}{\delta}.$$

But this contradicts our choice of δ.

Case II: We assume that for some admissible family of segments $S_1, S_2, \ldots, S_{2^m}$ passing through the support of y we have

$$\sum_{j=1}^{2^m} |S_j(y)|^p > 1-(3p+1)\delta.$$

For $j = 1, 2, 3, \ldots 2^m$ let t_j be the node of S_j with level n. Let $y_1 = \sum_{j=1}^{2^m} Q_{t_j}(y)$ and $y_2 = y - y_1$.

Then for any family of admissible segments, $R_1, \ldots, R_{2^m}$, passing through the support of y but disjoint from $S_1, \ldots, S_{2^m}$ we have

$$\sum_{j=1}^{2^m} |R_j(y)|^p = \sum_{j=1}^{2^m} |R_j(y_2)|^p < (3p+1)\delta$$

as otherwise y would have norm strictly greater than 1. Hence, for any family of admissible segments, $R_1, \ldots, R_{2^m}$, passing through the support of y_2 we have

$$\sum_{j=1}^{2^m} |R_j(y_2)|^p < (3p+1)\delta.$$

Furthermore,

$$\begin{aligned} |\phi(y_1)| &= \left| \sum_{j=1}^{2^m} \phi \circ Q_{t_j}(y) \right| \\ &\leq \sum_{j=1}^{2^m} \|\phi \circ Q_{t_j}\| \|y\| \\ &< \epsilon 2^m < \delta \end{aligned}$$

Hence,

$$\phi(y_2) = \phi(y) - \phi(y_1) > 1-\delta-\delta = 1-2\delta.$$

Repeating the argument of Case I, using y_2 instead of y we get

$$\|x+y_2+z\| \geq |\phi(x+y_2+z)| > 1-\delta+1-2\delta+1-\delta = 3-4\delta$$

For some admissible family, $R_1, \dots, R_s$ passing through the support of y_2 we have

$$\begin{aligned}
\|x + y_2 + z\|^p =& \sum_{j=1}^{s} |R_j(x + y_2 + z)|^p \\
\leq& \sum_{j=1}^{s} (|R_j(x)| + |R_j(y_2)| + |R_j(z)|)^p \\
\leq& \sum_{j=1}^{s} 3^{p-1} (|R_j(x)|^p + |R_j(y_2)|^p + |R_j(z)|^p) \\
=& 3^{p-1} \left[\sum_{j=1}^{s} |R_j(x)|^p + \sum_{j=1}^{s} |R_j(y_2)|^p + \sum_{j=1}^{s} |R_j(z)|^p \right] \\
\leq& 3^{p-1} (\|x\|^p + (3p+1)\delta + \|z\|^p) \\
=& 3^{p-1} (2 + (3p+1)\delta)
\end{aligned}$$

This implies that

$$(3 - 4\delta)^p \leq 2 \cdot 3^{p-1} + 3^{p-1}(3p+1)\delta$$

which contradicts our choice of δ.

Thus we see that Cases I and II give a contradiction and so we have $\ker Q = F$. □

We also have shown that $HT_p' / \ker Q = HT_p'/F$ is isometrically isomorphic to $\ell_q(\Gamma)$. We use this to obtain the following theorem.

Theorem 4.7. *HT_p'' is isomorphic to $F' \oplus \ell_p(\Gamma)$ where $F = [\eta_t^*]_{t \in \Upsilon}$.*

Proof. The mapping $Q \colon HT_p' \to \ell_q(\Gamma)$ is a quotient map. Its adjoint, Q^*, is a mapping from $\ell_q(\Gamma)' = \ell_p(\Gamma)$ into HT_p''.

We claim that $Q^*(\ell_p(\Gamma))$ is complemented in HT_p''. To see this let $\gamma_1, \dots, \gamma_n$ be distinct branches of Υ. Choose m so that $\gamma_j \cap \{t \in \Upsilon : l(t) \geq m\}$ are pairwise disjoint for $1 \leq j \leq n$. Consider the subspace of HT_p' spanned by $Q_m^*(\gamma_1^*), Q_m^*(\gamma_2^*), \dots, Q_m^*(\gamma_n^*)$. For $\alpha_1, \dots, \alpha_n$ in $\mathbb{R}$ we have

$$\left\| \sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*) \right\| = \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q}$$

proving that the span of $Q_m^*(\gamma_1^*), Q_m^*(\gamma_2^*), \dots, Q_m^*(\gamma_n^*)$ is isometrically isomorphic to ℓ_q^n. Furthermore,

$$Q\left(\sum_{i=1}^{n} \alpha_i Q_m^*(\gamma_i^*) \right)(\gamma) = \begin{cases} \alpha_i & \text{if } \gamma = \gamma_i, \\ 0 & \text{if } \gamma \neq \gamma_1, \dots, \gamma_n. \end{cases}$$

Therefore the restriction of Q to the span of $Q_m^*(\gamma_1^*), Q_m^*(\gamma_2^*), \ldots, Q_m^*(\gamma_n^*)$ maps the span of $Q_m^*(\gamma_1^*), Q_m^*(\gamma_2^*), \ldots, Q_m^*(\gamma_n^*)$ isometrically onto the span of $e_{\gamma_1}, e_{\gamma_2}, \ldots, e_{\gamma_n}$ in $\ell_q(\Gamma)$. So if $G = [Q_m^*(\gamma_1^*), Q_m^*(\gamma_2^*), \ldots, Q_m^*(\gamma_n^*)]$ we have a local selection $S\colon G \to HT_p'$ given by $S = (Q_{\langle Q_m'(\gamma_i)\rangle_{i=1}^n})^{-1}$. This implies that Q admits an approximate local selection (see [S]) and so, by Lemma 1 of [S], $Q^*(\ell_p(\Gamma))$ is complemented in HT_p''. □

Examining the proof of Theorem 4.7 we get the following Corollary.

Corollary 4.8. *For $1 < p < \infty$, HT_p has branch index q isometrically, where q is the conjugate index of p.*

Theorem 4.9. *For $1 \le p < \infty$, HT_p does not contain a copy of ℓ_1.*

Proof. First note that $HT_1 = JH$ which does not contain a copy of ℓ_1 by [H]. For $1 < p < \infty$, HT_p'' is isometrically isomorphic to $F' \oplus \ell_p(\Gamma)$ and therefore has cardinality $\mathfrak{c}$. It therefore follows from [P] that HT_p does not contain a copy of ℓ_1. □

Let us see that unlike JT_p, whose bidual, JT_p'', is isomorphic to $JT_p \oplus \ell_p(\Gamma)$, the bidual of HT_p, HT_p'', is not isomorphic to $HT_p \oplus \ell_p(\Gamma)$. Suppose that HT_p'' was isomorphic to $HT_p \oplus \ell_p(\Gamma)$. It follows as in [Po, Proposition 2.1] that HT_p contains a copy of c_o and therefore that HT_p'' contains a copy of ℓ_∞. As every separable Banach space is isometrically isomorphic to a subspace of ℓ_∞ we have that HT_p'' and hence $HT_p \oplus \ell_p(\Gamma)$ contains a copy of ℓ_1. However, if we now apply [D, Theorem 7] we see that HT_p or $\ell_p(\Gamma) \cong (HT_p \oplus \ell_p(\Gamma))/HT_p$ contains a copy of ℓ_1. This is impossible, as we know that HT_p does not contain a copy of ℓ_1 and $\ell_p(\Gamma)$ is reflexive.

For $1 \le p < \infty$ we consider the continuous map, $x \to \hat{x}$, from HT_p into $C(\Gamma^*)$, where $\hat{x}(\gamma^*) = \gamma^*(x)$ and Γ^* is endowed with the weak* topology,

The proof of [BHO, Lemma 6.2] is easily adapted to give the following result.

Lemma 4.10. *Let $(x_i)_i$ be a normalised block basis sequence of $(\eta_t)_{t\in\Upsilon}$ in HT_p such that $(\hat{x}_i)_i$ converges weakly to 0 in $C(\Gamma^*)$. Then $(x_i)_i$ is weakly null on HT_p.*

Proposition 4.11. *For $1 < p < \infty$, HT_p is quasi-shrinking.*

Proof. Suppose that $HT_p' \neq [\eta_t^*, \gamma^* : t \in \Upsilon, \gamma \in \Gamma]$. Then there is x^{**} in HT_p'' with $\|x^{**}\| = 1$ and $x^{**}|_{[\{\eta_t^*\}\cup\Gamma^*]} = 0$. Since HT_p does not contain a copy of ℓ_1 we can find a sequence $(x_n)_n$ in HT_p which converges weak* to x^{**}. Since $x^{**}|_{[\{\eta_t^*\}]} = 0$, by passing to a subsequence if necessary we may assume that

$(x_n)_n$ is a block basis of $(\eta_t)_{t\in\Upsilon}$. As $x^{**}|_{\Gamma^*} = 0$ we have that $(\hat{x}_n)_n$ converges weakly to 0 in $C(\Gamma^*)$. Lemma 4.10 now implies that $(x_n)_n$ is weakly null in HT_p and therefore that $x^{**} = 0$, a contradiction to our assumption. $\square$

5. Containment of ℓ_1 in Injective Tensor Products of Tree Spaces

Theorem 5.1. *For $1 \le i \le n$ let $1 < p_i < \infty$. For $i = 1, \ldots, n$ let q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. If $\sum_{i=1}^n \frac{1}{q_i} \ge 1$ then $\ell_1 \hookrightarrow \widehat{\bigotimes}_{i=1,\epsilon}^n JT_{p_i}$ and $\ell_1 \hookrightarrow \widehat{\bigotimes}_{i=1,\epsilon}^n HT_{p_i}$.*

Proof. We prove the result for $\widehat{\bigotimes}_{i=1,\epsilon}^n JT_{p_i}$, the proof for $\widehat{\bigotimes}_{i=1,\epsilon}^n HT_{p_i}$ is identical. For $k \in \mathbb{N}$ let $A_k := \big\{(\epsilon_j)_{j=1}^k : (\epsilon_j)_{j=1}^k \in \Upsilon, \epsilon_k = 0\big\}$. Given $k \in \mathbb{N}$ we define U_k in $\widehat{\bigotimes}_{i=1,\epsilon}^n JT_{p_i}$ by $U_k := \sum_{t\in A_k} \eta_t \otimes \eta_t \otimes \cdots \otimes \eta_t$. We show that each U_k has norm 1. Let q be such that $\frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i}$. For $1 \le i \le n$ choose ϕ_i in JT'_{p_i}. For ϕ in JT'_{p_i}, t in Υ we use ϕ_t to denote $\phi(\eta_t)$. Then by Hölder's Inequality we have that

$$
\begin{aligned}
&\Big|\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n, \sum_{t\in A_k} \eta_t \otimes \eta_t \otimes \cdots \otimes \eta_t\rangle\Big| \\
&\qquad = \left|\sum_{t\in A_k} (\phi_1)_t (\phi_2)_t \cdots (\phi_n)_t\right| \\
&\qquad \le \left(\sum_{t\in A_k} |(\phi_1)_t|^{q_1/q}\right)^{\frac{q}{q_1}} \left(\sum_{t\in A_k} |(\phi_2)_t|^{q_2/q}\right)^{\frac{q}{q_2}} \cdots \left(\sum_{t\in A_k} |(\phi_n)_t|^{q_n/q}\right)^{\frac{q}{q_n}} \\
&\qquad \le \left(\sum_{t\in A_k} |(\phi_1)_t|^{q_1}\right)^{\frac{1}{q_1}} \left(\sum_{t\in A_k} |(\phi_2)_t|^{q_2}\right)^{\frac{1}{q_2}} \cdots \left(\sum_{t\in A_k} |(\phi_n)_t|^{q_n}\right)^{\frac{1}{q_n}}
\end{aligned}
$$

Setting

$$
x_i = \sum_{t\in A_k} \operatorname{sgn}(\phi_i)_t |(\phi_i)_t|^{q_i - 1} \eta_t \Big/ \left(\sum_{t\in A_k} |(\phi_i)_t|^{q_i}\right)^{\frac{1}{p_i}}
$$

we get that $\|x_i\| = 1$ and that

$$
\begin{aligned}
\left(\sum_{t\in A_k} |(\phi_1)_t|^{q_1}\right)^{\frac{1}{q_1}} \cdots \left(\sum_{t\in A_k} |(\phi_n)_t|^{q_n}\right)^{\frac{1}{q_n}} &= |\phi_1(x_1)||\phi_2(x_2)| \ldots |\phi_n(x_n)| \\
&\le \|\phi_1\|_{q_1} \|\phi_2\|_{q_2} \ldots \|\phi_n\|_{q_n}
\end{aligned}
$$

Hence, using the injectivity of the epsilon tensor product, we get that

$$\|U_k\|_\epsilon = \sup_{\|\phi_1\|_{q_1}\le 1} \dots \sup_{\|\phi_n\|_{q_n}\le 1} \left| \left\langle \phi_1 \otimes \cdots \otimes \phi_n, \sum_{t\in A_k} \eta_t \otimes \eta_t \otimes \cdots \otimes \eta_t \right\rangle \right| \le 1.$$

However, if s belongs to A_k we have that

$$\langle \eta_s^* \otimes \eta_s^* \otimes \cdots \otimes \eta_s^*, U_k\rangle = 1$$

and hence $\|U_k\|_\epsilon = 1$.

Let us show that $(U_k)_k$ is equivalent to the ℓ_1 basis. Consider $(a_k)_k$ in ℓ_1. Let $\mathbb{N}_+ = \{k : a_k \ge 0\}$. We may suppose without loss of generality that $\sum_{n\in\mathbb{N}_+} a_k \ge \frac{1}{2}\sum_{k\in\mathbb{N}} |a_k|$. We now choose a branch γ as follows. Suppose that $(\epsilon_j)_{j=1}^k$ is the k^{th} node of γ. If $k+1$ belongs to $\mathbb{N}_+$, we set $\epsilon_{k+1} = 0$ and $(\epsilon_j)_{j=1}^{k+1} \in \gamma$. If $k+1$ does not belong to $\mathbb{N}_+$, we set $\epsilon_{k+1} = 1$ and $(\epsilon_j)_{j=1}^{k+1} \in \gamma$. By the choice of γ it follows immediately that $\langle \gamma^* \otimes \gamma^* \otimes \cdots \otimes \gamma^*, U_k\rangle = 1$ if k is in $\mathbb{N}_+$ and is 0 otherwise. Therefore

$$\begin{aligned}\left\| \sum_{k\in\mathbb{N}} a_k U_k \right\|_\epsilon &\ge \left| \langle \gamma^* \otimes \cdots \otimes \gamma^*, \sum_{k\in\mathbb{N}} a_k U_k\rangle \right| \\ &= \left| \sum_{k\in\mathbb{N}} a_k \langle \gamma^* \otimes \cdots \otimes \gamma^*, U_k\rangle \right| \\ &= \sum_{k\in\mathbb{N}_+} a_k \\ &\ge \frac{1}{2} \sum_{n\in\mathbb{N}} |a_k|.\end{aligned}$$

Clearly, we also have that $\left\|\sum_{k\in\mathbb{N}} a_k U_k\right\|_\epsilon \le \sum_{k\in\mathbb{N}} |a_k|$ and hence $(U_k)_k$ is equivalent to the ℓ_1 vector basis. □

When $q = q_1 = q_2 = \cdots = q_n$, we obtain the following result.

Theorem 5.2. *Let $1 < p < \infty$. If $n \ge q$ then $\ell_1 \hookrightarrow \widehat{\bigotimes}_{n,\epsilon} JT_p$ and $\ell_1 \hookrightarrow \widehat{\bigotimes}_{n,\epsilon} HT_p$.*

Given Banach spaces $X_1, \dots, X_n$ and $R = \sum_{j=1}^\infty x_1^j \otimes \cdots \otimes x_n^j$ in $\widehat{\bigotimes}_{1\le j\le n,\epsilon} X_j$ for $1 \le i \le n$, we let $R^i \colon X_i' \to \widehat{\bigotimes}_{1\le j\le n, j\ne i,\epsilon} X_j$ be given by

$$R^i(\xi_i) = \sum_{j=1}^\infty \xi_i(x_i^j) x_1^j \otimes \cdots x_{i-1}^j \otimes x_{i+1}^j \cdots \otimes x_n^j$$

for ξ_i in X_i', $1 \le i \le n$.

Lemma 5.3. *Let n be a positive integer and $X_1, \dots, X_n$ be Banach spaces with finite dimensional decompositions $(X_i^m)_{m=1}^\infty$. For x in X_i write x as $x =$*

$\sum_{j=1}^{\infty} x_j$ with x_j in X_i^j. Let Q_m be defined as $Q_m\left(\sum_{j=1}^{\infty} x_j\right) = \sum_{j=m+1}^{\infty} x_j$. Suppose that for each i, $1 \le i \le n$, we have $\ell_1 \not\hookrightarrow \widehat{\bigotimes}^n_{\substack{j=1,\epsilon\\ j\neq i}} X_j$ yet $\ell_1 \hookrightarrow \widehat{\bigotimes}^n_{j=1,\epsilon} X_j$. Then there is a sequence $(S_k)_k$ in $\widehat{\bigotimes}^n_{j=1,\epsilon} X_j$ such that

(a) $(S_k)_k$ is equivalent to the ℓ_1 basis.

(b) For every $1 \le i \le n$, ξ in X_i' and m in $\mathbb{N}$ we have

$$\lim_{k\to\infty} (S_k)^i((I - Q_m^*)(\xi)) = 0$$

with respect to the weak topology.

Proof. We argue as in Section 3 of [L]. We start with $i = 1$ and let ξ belong to X_1'. Let $(R_j)_j$ be a sequence in $\widehat{\bigotimes}^n_{j=1,\epsilon} X_j$ which is equivalent to the ℓ_1 basis. Since $\ell_1 \not\hookrightarrow \widehat{\bigotimes}^n_{j=2,\epsilon} X_j$, for each ξ in X_1' we have that $(R_j^1(\xi))_j$ has a weak Cauchy subsequence. Fix m in $\mathbb{N}$ and choose $\phi_1, \ldots, \phi_{l_m}$ so that $(I - Q_m)^*\phi_1, \ldots, (I - Q_m)^*\phi_{l_m}$ is a basis for $(I - Q_m)^*(X_1')$. We can find a subsequence N_1 of $\mathbb{N}$ so that $(R_j^1(I - Q_m)^*(\phi_1))_{j\in N_1}$ is weakly Cauchy. We then obtain a subsequence N_2 of N_1 so that $(R_j^1(I - Q_m)^*(\phi_2))_{j\in N_2}$ is weakly Cauchy. Continuing like this, after l_m steps, we have a subset N_{l_m} of $\mathbb{N}$ so that $(R_j^1(I - Q_m)^*(\phi))_{j\in N_{l_m}}$ is weakly Cauchy for all ϕ in X_1'. Starting with $m = 1$ and repeating the above process we then obtain a decreasing chain of subsets $(L_m)_m$ so that $(R_j^1(I - Q_m)^*(\phi))_{j\in L_m}$ is weakly Cauchy for all ϕ in X_1' and all m in $\mathbb{N}$. Choose n_k in L_k for each k in $\mathbb{N}$. Then the sequence $(R_{n_{2k-1}} - R_{n_{2k}})_{k=1}^{\infty}$ is also equivalent to the unit basis of ℓ_1. From the above we have that $\lim_{k\to\infty}(R^1_{n_{2k-1}} - R^1_{n_{2k}})(I - Q_m)^*(\phi) = 0$ for all m in $\mathbb{N}$ and all ϕ in X_1'. Repeat the argument with $(R_j)_j = (R_{1,j})_j$ replaced with $(R_{i,k})_k = (R_{i-1,n_{2k-1}} - R_{i-1,n_{2k}})_k$ and 1 with $i = 2, \ldots, n$ in turn. Finally, set $S_k = R_{n,n_{2k-1}} - R_{n,n_{2k}}$. □

We define the oscillation of a bounded sequence of real numbers $(a_n)_n$ by $\mathrm{osc}((a_n)_n) = \limsup_n(a_n) - \liminf_n(a_n)$.

Theorem 5.4. *Let $VT_1, \ldots, VT_v$ be quasi-shrinking tree spaces which have branch index $q_1, \ldots, q_v$ respectively and none of which contain a copy of ℓ_1. Let $n \ge v$ and $1 < p_{v+1}, \ldots, p_n < \infty$. If $\displaystyle\sum_{i=1}^{n} \frac{1}{q_i} < 1$ then*

$$\ell_1 \not\hookrightarrow \left(\widehat{\bigotimes}^v_{i=1,\epsilon} VT_i\right) \hat{\otimes}_\epsilon \left(\widehat{\bigotimes}^n_{i=v+1,\epsilon} \ell_{p_i}(\Gamma)\right).$$

Proof. We use E_i to denote VT_i if $1 \le i \le v$ and $\ell_{p_i}(\Gamma)$ if $v < i \le n$. Our proof is by complete induction on n. For $n = 1$ the statement is clearly true. Let us assume that for each $l < n$ we have shown that if $1 < p_i < \infty$

for $1 \leq i \leq l$ and $\displaystyle\sum_{i=1}^{l} \frac{1}{q_i} < 1$ then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{i=1,\epsilon}^{l} E_i$. Now suppose we have $1 < p_i < \infty$ for $1 \leq i \leq n$ with $\displaystyle\sum_{i=1}^{n} \frac{1}{q_i} < 1$. Then for each i, $1 \leq i \leq n$ we have $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{\substack{j=1,\epsilon \\ j\neq i}}^{n} E_j$.

Since VT_i has branch index q_i we can find a constant, $C > 0$, such that whenever $(\gamma_j)_{j=1}^k$ is a sequence of mutually distinct branches in Υ we are able to find $m \in \mathbb{N}$ so that for any $(\alpha_j)_{j=1}^k$ in $\mathbb{R}^k$ we have

$$\left\| \sum_{j=1}^{k} \alpha_j Q_m^*(\gamma_j^*) \right\| \leq C \left(\sum_{j=1}^{k} |\alpha_j|^{q_i} \right)^{1/q_i}.$$

Suppose that $\ell_1 \hookrightarrow \widehat{\bigotimes}_{j=1,\epsilon}^{n} E_j$. Choose $(S_w)_w$ in $\widehat{\bigotimes}_{j=1,\epsilon}^{n} E_j$ to be equivalent to the unit basis of ℓ_1 which satisfies the conditions of Lemma 5.3.

We now claim that there is $\epsilon > 0$ such that for all subsets N of $\mathbb{N}$ there are branches $\gamma_1, \gamma_2, \cdots, \gamma_n$ such that $\operatorname{osc}(\langle S_w, \gamma_1 \otimes \cdots \otimes \gamma_\nu \otimes e_{\gamma_{\nu+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{w\in N} > \epsilon$. If not there is a subset N_o of $\mathbb{N}$ such that $\operatorname{osc}(\langle S_j, \gamma_1 \otimes \cdots \otimes \gamma_\nu \otimes e_{\gamma_{\nu+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{j\in N_o} = 0$ for all branches $\gamma_1, \gamma_2, \cdots, \gamma_n$. Since $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{j=2,\epsilon}^{n} E_j$, for each node t in Υ the sequence $((S_w^1)(\eta_t^*))_{w\in N_o}$ has a weak Cauchy subsequence. Using a diagonal argument on the set of nodes we get that there is a subset N_1 of N_o so that $((S_w^1)(\eta_t^*))_{w\in N_1}$ is weakly Cauchy for all t in Υ. We then obtain a subset N_2 of N_1 so that $((S_w^2)(\eta_t^*))_{w\in N_2}$ is weakly Cauchy for all t in Υ. Repeating the argument a further $v-2$ times we get subset $\tilde{N}$ of N_o so that $((S_w^i)(\eta_t^*))_{w\in \tilde{N}}$ is weakly Cauchy for all t in Υ, all $1 \leq i \leq \nu$. Hence we have that $\operatorname{osc}(\langle S_w, \psi_1 \otimes \cdots \otimes \psi_\nu \otimes e_{\gamma_{v+1}} \otimes \cdots \otimes e_{\gamma_n} \rangle)_{w\in\tilde{N}} = 0$ for all ψ_i, $1 \leq i \leq v$, in the span of the union of the nodes and the branches and all γ_i in Γ, $v+1 \leq i \leq n$. However, since VT_j is quasi-shrinking, the union of the node and branch functionals spans a dense subspace of VT_j', $1 \leq j \leq v$, and $(e_\gamma)_{\gamma\in\Gamma}$ span a dense subset of $\ell_{p_l}(\Gamma)$, $v+1 \leq l \leq n$, hence $(S_w)_{w\in\tilde{N}}$ is weakly Cauchy which contradicts the assumption that $(S_w)_{w\in\tilde{N}}$ is equivalent to the ℓ_1 basis and hence our claim is proven.

Starting with $J_1 = \mathbb{N}$ we inductively choose decreasing subsets $(J_k)_k$ of $\mathbb{N}$ and n-tuples of branches $(\gamma_1^k, \gamma_2^k, \ldots, \gamma_n^k)_k$ such that

(i) $(\langle S_w, \gamma_1^{k-1} \otimes \cdots \otimes \gamma_v^{k-1} \otimes e_{\gamma_{v+1}^{k-1}} \ldots \otimes e_{\gamma_n^{k-1}} \rangle)_{w\in J_k}$ converges and

$$\left| \lim_{w\in J_k} \langle S_w, \gamma_1^{k-1} \otimes \cdots \otimes \gamma_v^{k-1} \otimes e_{\gamma_{v+1}^{k-1}} \ldots \otimes e_{\gamma_n^{k-1}} \rangle \right| > \epsilon/2$$

for all $k > 1$,

(ii) $(S^i_w(\gamma_i^{k-1}))_{w\in J_k}$ and $(S^j_w(e_{\gamma_j^{k-1}}))_{w\in J_k}$ are weakly Cauchy for all $1\leq i\leq v$, $v<j\leq n$ and all $k\geq 2$,

(iii) $\mathrm{osc}(\langle S_w, \gamma_1^k\otimes\cdots\otimes\gamma_v^k\otimes e_{\gamma_{v+1}^k}\ldots\otimes e_{\gamma_n^k}\rangle)_{w\in J_k}>\epsilon$ for all k in $\mathbb{N}$.

Given $\underline{i}=(i_1,\ldots,i_n)$ in $\mathbb{N}^n$ let $l(\underline{i})=\max_{1\leq k\leq n} i_k$. Let $\mathcal{I}_n$ denote the subsets of $\{1,\ldots,n\}$ ordered by set inclusion. Given $\mathbf{j}$ in $\mathcal{I}_n$ we denote the cardinality of $\mathbf{j}$ by $|\mathbf{j}|$ and the complement of $\mathbf{j}$ in $\mathcal{I}_n$ by $\mathbf{j}^c$. We wish to consider the situation where we fix some of the indices $i_1,\ldots,i_n$ and where we let the others tend to infinity.

Given Banach spaces $X_1,\ldots,X_n$ and $R=\sum_{j=1}^\infty x_1^j\otimes\cdots\otimes x_n^j$ in $\widehat{\bigotimes}_{1\leq j\leq n,\epsilon}X_j$, $\mathbf{j}=\{1\leq j_1<j_2<\ldots j_t\leq n\}$ in $\mathcal{I}_n$, let $\mathbf{j}^c=\{1\leq l_1<l_2<\ldots<l_{n-t}\leq n\}$. We define $R^{\mathbf{j}}\colon X'_{j_1}\times X'_{j_2}\times\cdots\times X'_{j_t}\to X_{l_1}\widehat{\bigotimes}_\epsilon X_{l_2}\widehat{\bigotimes}_\epsilon\cdots\widehat{\bigotimes}_\epsilon X_{l_{n-t}}$ by

$$R^{\mathbf{j}}(\phi_1,\phi_2\ldots,\phi_t)=\sum_{j=1}^\infty \phi_1(x^j_{j_1})\phi_2(x^j_{j_2})\ldots\phi_t(x^j_{j_t})x^j_{l_1}\otimes x^j_{l_2}\otimes\cdots\otimes x^j_{l_{n-t}}$$

for ϕ_i in X'_{j_i}, $i=1\ldots t$.

For S in $\widehat{\bigotimes}_{i=1,\epsilon}^v VT_i\hat{\otimes}_\epsilon\widehat{\bigotimes}_{i=v+1,\epsilon}^n\ell_{p_i}(\Gamma)$ we use $\tilde{S}$ to denote the mapping from Γ^n to $\mathbb{R}$ given by $\tilde{S}(\gamma_1,\gamma_2,\ldots,\gamma_n)=S(\gamma_1\otimes\gamma_2\otimes\ldots\otimes\gamma_v\otimes e_{\gamma_{v+1}}\otimes\ldots\otimes e_{\gamma_n})$. We will adapt our notation and write $(\tilde{S})^{\mathbf{j}}(\gamma_{j_1},\gamma_{j_2},\ldots,\gamma_{j_t})$ as a function on $\Gamma^{n-|\mathbf{j}|}$.

By our above choice of $(J_k)_k$ and n-tuples $(\gamma_1^k,\gamma_2^k,\ldots,\gamma_n^k)_k$ we have that

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\emptyset}(\gamma_1^k,\ldots,\gamma_n^k)\right|\geq\frac{\epsilon}{2}$$

for each k in $\mathbb{N}$.

We consider the set, $\mathcal{J}$, of all $\mathbf{j}=(j_1,\ldots,j_t)$ in $\mathcal{I}_n$ for which there is $\overline{s}=(s_1,\ldots,s_t)$ in $\mathbb{N}^t$ and $\delta>0$ so that

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2},\ldots,\gamma_{j_t}^{s_t})(\gamma_{l_1}^k,\ldots,\gamma_{l_{n-t}}^k)\right|>\delta$$

for all k sufficiently large. We consider two cases.

(a) No $\mathbf{j}$ in $\mathcal{J}$ has length $n-1$,

(b) $\mathcal{J}$ contains a $\mathbf{j}$ with $|\mathbf{j}|=n-1$.

If (a) occurs we consider the largest value t_o of $|\mathbf{j}|$ in $\mathcal{J}$. Note that $|\mathbf{j}|<n-1$. Choose $\mathbf{j}_o$ in $\mathcal{J}$ with $|\mathbf{j}_o|=t_o$ and $\overline{s}=(s_1,\ldots,s_{t_o})$ in $\mathbb{N}^{t_o}$ so that

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\ldots,\gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^k,\ldots,\gamma_{l_{n-t_o}}^k)\right|>\delta$$

for some $\delta>0$, all k sufficiently large. Then, we can inductively choose a subsequence of $(\gamma_1^k,\gamma_2^k,\cdots,\gamma_n^k)_k$, which we also denote by $(\gamma_1^k,\gamma_2^k,\cdots,\gamma_n^k)_k$, so that for all $\mathbf{j}$ with $|\mathbf{j}|=d>t_o$, all $\mathbf{j}_o\subset\mathbf{j}$ and $\overline{u}=(u_1,\ldots,u_d)$ we have that

$$\lim_{k\to\infty}\left(\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}}(\gamma_{j_1}^{u_1},\gamma_{j_2}^{u_2}\dots,\gamma_{j_d}^{u_d})(\gamma_{l_1}^{k},\gamma_{l_2}^{k},\dots,\gamma_{l_{n-d}}^{k})\right)=0.$$

Then, by inductively choosing another subsequence of $(\gamma_1^k,\dots,\gamma_n^k)_k$, if necessary, we have that

$$\lim_{r\in J_{\max\{k_c:1\le c\le n-t_o\}+1}}\left((\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^{k_1},\gamma_{l_2}^{k_2},\dots,\gamma_{l_{n-t_o}}^{k_{n-t_o}})\right) <\delta 2^{-(k_1+\dots+k_{n-t_o}+n)}$$

whenever at least two of $k_1,\dots,k_{n-t_o}$ are distinct. Note that since

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^{k},\gamma_{l_2}^{k},\dots,\gamma_{l_{n-t_o}}^{k})\right|>\delta$$

for all k sufficiently large we may also assume that

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j_o}}(\gamma_{j_1}^{s_1}\gamma_{j_2}^{s_2}\dots,\gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^{k},\dots,\gamma_{l_{n-t_o}}^{k})\right|>\delta$$

all k in $\mathbb{N}$.

Let us write the set $\mathbf{j}_o^c$ as the union of $\mathbf{j}_1$ and $\mathbf{j}_2$ where $\mathbf{j}_1=\{l_1,\dots,l_{s_o}\}=\{l_i\in\mathbf{j}_o^c: E_i=VT_i\}$ and $\mathbf{j}_2=\{l_{s_o+1},\dots,l_{n-t_o}\}=\{l_i\in\mathbf{j}_o^c: E_i=\ell_{p_i}(\Gamma)\}$.

Note that by construction, for each i, $1\le i\le v$, we have that the sequence $(\gamma_i^k)_k$ is a sequence of mutually distinct branches. Let us fix u_o in $\mathbb{N}$. Choose m sufficiently large so that $(Q_m^*(\gamma_i^k))_{k=1}^{u_o}$ are disjoint. Since VT_i has branch index q_i we may also suppose that m is chosen so that we can find $C>0$ such that for any $(\alpha_j)_{j=1}^k$ in $\mathbb{R}^k$ we have

$$\left\|\sum_{j=1}^k\alpha_jQ_m^*(\gamma_j^*)\right\|\le C\left(\sum_{j=1}^k|\alpha_j|^{q_i}\right)^{1/q_i}.$$

Using Lemma 5.3 we then have that

$$\begin{aligned}\lim_{r\in J_{\max\{k_c:1\le c\le n-t_o\}+1}}&(\tilde{S}_r)^{\mathbf{j_o}}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{t_o}}^{s_{t_o}})(\gamma_{l_1}^{k_1},\gamma_{l_2}^{k_2},\dots,\gamma_{l_{n-t_o}}^{k_{n-t_o}})=\\ \lim_{r\in J_{\max\{k_c:1\le c\le n-t_o\}+1}}&(\tilde{S}_r)^{\mathbf{j_o}}(\gamma_{j_1}^{s_1},\dots,\gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{k_1}),\dots,Q_m^*(\gamma_{l_{s_o}}^{k_{s_o}}),\gamma_{l_{s_o+1}}^{k_{s_o+1}}\dots,\\ &\gamma_{l_{n-t_o}}^{k_{n-t_o}}).\end{aligned}$$

Therefore we can find a positive integer r_o so that

$$|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{k}),Q_m^*(\gamma_{l_2}^{k}),\dots,Q_m^*(\gamma_{l_{s_o}}^{k}),\gamma_{l_{s_o+1}}^{k}\dots,\gamma_{l_{n-t_o}}^{k})| >\delta/2$$

for all $1 \le k \le u_o$ and

$$|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \dots, \gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{k_1}), \dots, Q_m^*(\gamma_{l_{s_o}}^{k_{s_o}}), \gamma_{l_{s_o+1}}^{k_{l_{s_o+1}}} \dots, \gamma_{l_{n-t_o}}^{k_{n-t_o}})| < \delta 2^{-(k_1+\dots+k_{n-t_o}+n)}$$

whenever at least two of $k_1, \dots, k_{n-t_o}$ are distinct, $1 \le k_c \le u_o$ for $1 \le c \le n - t_o$.

For $1 \le k \le u_o$ we let

$$b_k = \operatorname{sgn}(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \dots, \gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{k}), \dots, Q_m^*(\gamma_{l_{s_o}}^{k}), e_{\gamma_{l_{s_o+1}}^k}, \dots, e_{\gamma_{l_{n-t_o}}^k}),$$

set $\phi_{k_1} = u_o^{\frac{-1}{q_{l_1}}} \sum_{v_1=1}^{u_o} b_{v_1} Q_m^*(\gamma_{l_1}^{v_1})$ and $\phi_{k_s} = u_o^{\frac{-1}{q_{l_s}}} \sum_{v_s=1}^{u_o} Q_m^*(\gamma_{l_s}^{v_s})$ for $2 \le s \le s_o$ and $\phi_{k_s} = u_o^{\frac{-1}{q_{l_s}}} \sum_{v_s=1}^{u_o} e_{\gamma_{l_s}^{v_s}}$ for $s_o < s \le n - t_o$. Since VT_i has branch index q_i we have $\|\phi_{k_1}\| \le C, \cdots, \|\phi_{k_s}\| \le C$ independent of u_o. We have

$$\left|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2} \dots, \gamma_{j_{t_o}}^{s_{t_o}}) \ (\phi_{k_1}, \dots, \phi_{k_{n-t_o}})\right|$$
$$= u_o^{-\sum_{s=1}^{n-t_o} \frac{1}{q_{l_s}}} \left| \sum_{v_1=1}^{u_o} \cdots \sum_{v_{n-t_o}=1}^{u_o} b_{v_1} (\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \dots, \gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{v_1}), \dots, \right.$$
$$\left. Q_m^*(\gamma_{l_{s_o}}^{v_{s_o}}), e^{v_{s_o+1}}_{\gamma_{l_{s_o+1}}}, \dots, e_{\gamma_{l_{n-t_o}}^{v_{n-t_o}}}) \right|$$
$$\ge u_o^{-\sum_{s=1}^{n-t_o} \frac{1}{q_{l_s}}} \left(\sum_{v=1}^{u_o} b_v (\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \dots, \gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{v}), \dots, Q_m^*(\gamma_{l_{s_o}}^{v}), \right.$$
$$e_{\gamma_{l_{s_o+1}}^v}, \dots, e_{\gamma_{l_{n-t_o}}^v})$$
$$\sum_{v_1}^{u_o} \cdots \sum_{\substack{v_{n-t_o} \\ (v_1,\dots,v_{n-t}) \in \mathbb{N}^{n-t_o} \setminus D}}^{u_o} |(\tilde{S}_{r_o})^{\mathbf{j}_o} - (\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2}, \dots, \gamma_{j_{t_o}}^{s_{t_o}})(Q_m^*(\gamma_{l_1}^{v_1}), \dots, Q_m^*(\gamma_{l_{s_o}}^{v_{s_o}})$$
$$\left., e_{\gamma_{l_{s_o+1}}^{v_{s_o+1}}}, \dots, e_{\gamma_{l_{n-t_o}}^{v_{n-t_o}}})| \right)$$
$$\ge u_o^{1-\sum_{s=1}^{n-t_o} \frac{1}{q_{l_s}}} \left(\delta - \frac{\delta}{u_o 2^n} \right).$$

And this proves that S_{r_o} is unbounded, contradicting the fact that it is equivalent to an element of the unit basis of ℓ_1. We have proved the result.

If (b) occurs we proceed as in (a) to choose $\mathbf{j}_o$ in $\mathcal{I}_n$ with $|\mathbf{j}_o| = n - 1$ and $\overline{s} = (s_1, \dots, s_{n-1})$ in $\mathbb{N}^{n-1}$ so that

$$\left| \lim_{r \in J_{k+1}} (\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1}, \gamma_{j_2}^{s_2} \dots, \gamma_{j_{n-1}}^{s_{n-1}})(\gamma_{l_1}^k) \right| > \delta$$

for some $\delta > 0$. Fix u_o in $\mathbb{N}$ and choose m so that $(Q_m^*(\gamma_{l_1}^k))_{k=1}^{u_o}$ is equivalent to the unit basis vector of $\ell_{q_{l_1}}^{u_o}$. We first assume that $E_{l_1} = VT_{l_1}$. By

Lemma 5.3 we have

$$\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(\gamma_{l_1}^{k})\right|$$
$$=\left|\lim_{r\in J_{k+1}}(\tilde{S}_r)^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(Q_m^*(\gamma_{l_1}^{k}))\right|.$$

Since, taking r_o sufficiently large we may assume that

$$\left|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(Q_m^*(\gamma_{l_1}^{k}))\right|>\delta/2$$

for all k sufficiently large, we may assume that

$$\left|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(Q_m^*(\gamma_{l_1}^{k}))\right|>\delta/2$$

for all k in $\mathbb{N}$.

Let

$$b_k=\operatorname{sgn}(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2},\dots,\gamma_{j_{n-1}}^{s_{n-1}})(Q_m^*(\gamma_{l_1}^{k})),$$

and

$$\psi=u_o^{\frac{-1}{q_{l_1}}}\sum_{v=1}^{u_o}b_vQ_m^*(\gamma_{l_1}^{v}).$$

Since $(Q_m^*(\gamma_{l_s}^{k}))_{k=1}^{u_o}$ is isometrically equivalent to the unit vector basis of $\ell_{q_{l_1}}^{u_o}$ we have that $\|\psi\|\leq M$. We now have that

$$\left|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(\psi)\right|$$
$$=u_o^{\frac{-1}{q_{l_1}}}\sum_{v=1}^{u_o}\left|(\tilde{S}_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(Q_m^*(\gamma_{l_1}^{v}))\right|$$
$$\geq u_o^{1-\frac{1}{q_{l_1}}}\delta/2.$$

Which shows that S_{r_o} is unbounded.

Finally if $E_{l_1}=\ell_{q_{l_1}}$ a similar but somewhat simpler argument, setting

$$b_k=\operatorname{sgn}(S_{r_o})^{\mathbf{j}_o}(\gamma_{j_1}^{s_1},\gamma_{j_2}^{s_2}\dots,\gamma_{j_{n-1}}^{s_{n-1}})(e_{\gamma_{1_1}^{k}})$$

and $\rho=u_o^{-\frac{1}{q_{l_1}}}\sum_{v=1}^{u_o}b_ve_{\gamma_{l_1}^{v}}$, shows that S_{r_o} is unbounded. □

If we take $v=n$ in Theorem 5.4 we get the following Corollary.

Corollary 5.5. *Let $VT_1,\dots,VT_n$ be quasi-shrinking tree spaces which have branch indices $q_1,\dots,q_n$ respectively and which do not contain a copy of ℓ_1. If $\sum_{i=1}^{n}\frac{1}{q_i}<1$ then $\ell_1\not\hookrightarrow\widehat{\bigotimes}_{i=1,\epsilon}^{n}VT_i$.*

Taking VT_i equal to JT_{p_i} or HT_{p_i} for $1 \leq i \leq n$ we get the following Corollaries to Theorems 5.1 and 5.4.

Corollary 5.6. *Let n be a positive integer and $1 < p_i < \infty$ for $1 \leq i \leq n$. Then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{i=1,\epsilon}^n JT_{p_i}$ if and only if $\sum_{i=1}^n \frac{1}{q_i} < 1$.*

Corollary 5.7. *Let n be a positive integer and $1 < p_i < \infty$ for $1 \leq i \leq n$. Then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{i=1,\epsilon}^n HT_{p_i}$ if and only if $\sum_{i=1}^n \frac{1}{q_i} < 1$.*

Taking $p_i = p$ for $1 \leq i \leq n$ we get.

Corollary 5.8. *Let n be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,\epsilon} JT_p$ if and only if $n < q$.*

Corollary 5.9. *Let n be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,\epsilon} HT_p$ if and only if $n < q$.*

As $JT_p'' = JT_p \oplus \ell_p(\Gamma)$ we get the the following corollary from Theorem 5.4.

Corollary 5.10. *Let n be a positive integer and $p > 1$. Then $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,\epsilon} JT_p''$ if and only if $n < q$.*

If we take $VT_1 = JH$ and $VT_2 = JT$ we get that

Theorem 5.11. *The Banach space $JH \widehat{\bigotimes}_\epsilon JT$ does not contain a copy of ℓ_1.*

While taking $VT_i = JH$ or Λ_T for $1 \leq i \leq n$ we get that

Theorem 5.12. *For any n in $\mathbb{N}$, $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,\epsilon} JH$ and $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,\epsilon} \Lambda_T$.*

References

[A] R. Alencar *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. 94 (1985), 33–38

[AF] A. Arias and J. Farmer, *On the structure of tensor products of ℓ_p–spaces*, Pacific J. Math. 175 (1) (1996), 13–37.

[BHO] S. Bellenot, R. Haydon and E. Odell, *Quasi-reflexive and tree spaces constructed in the spirit of R. C. James*, Banach space theory (Iowa City, IA, 1987), Contemp. Math. 85 Amer. Math. Soc., Providence, RI, (1989), 19–43.

[BR] C. Boyd and R. Ryan, *Geometric theory of spaces of integral polynomials and symmetric tensor products*, Jour. Functional Analysis 179 (2001), 18–42.

[CD] D. Carando and V. Dimant, *Duality in spaces of nuclear and integral polynomials*, Jour. Math. Anal. Appl. 241 (1) (2000), 107–121.

[D] J.C. Díaz, *On the three-space property: non-containment of ℓ_p, $1 \leqslant p < \infty$, or c_0 subspaces*, Publicacions Matemàtiques 37 (1), (1993), 127–132.

[Din] S. Dineen, *Complex analysis on infinite dimensional spaces*, Monographs in Mathematics, Springer-Verlag, 1999.

[FG] H. Fetter and B. Gamboa de Buen, *The James Forest*, London Mathematical Society Lecture Note Series 236, Cambridge University Press, Cambridge, 1997.

[H] J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. 60 (3) (1977), 289–308.

[J] R.C. James, *A separable somewhat reflexive Banach space with non-separable dual*, Bull. Amer. Math. Soc. 80 (1974), 738–743.

[Jo] W.B. Johnson, *A reflexive Banach space which is not sufficiently Euclidean*, Studia Math. 55 (2) (1976), 201–205.

[L] D. Leung, *Embedding ℓ^1 into tensor products of Banach spaces*, Functional Analysis (Austin, TX, 1987/1989), Lecture Notes in Math. 1470 Springer, Berlin (1991), 171–176.

[LS] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain ℓ_1 and whose duals are non-separable*, Studia Math. 54 (1975), 81–105.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I.*, Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 92 Springer-Verlag, Berlin-New York, 1977. xiii+188 pp.

[O] E. Odell, *A nonseparable Banach space not containing a subsymmetric basic sequence*, Israel J. Math. 52 1-2 (1985), 97–109

[P] A. Pełczyński *On Banach spaces containing $L^1(\mu)$*, Studia Math. 30 (1968), 231–246.

[Po] C. Poulios, *Non-separable tree-like Banach spaces and Rosenthal's ℓ_1-theorem*, Studia Math. 221 no. 1 (2014), 1–12.

[R] W. Ruess *Duality and geometry of spaces of compact operators*, Functional analysis: surveys and recent results, III (Paderborn (1983), North-Holland Math. Stud. 90 North-Holland, Amsterdam, (1984), 59–78.

[S] C. Stegall *Banach spaces whose duals contain $\ell_1(\Gamma)$ with applications to the study of dual $L_1(\mu)$ spaces*, Trans. Amer. Math. Soc. 176 (1973), 463–477.

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