R-Hopfian and L-co-Hopfian Abelian Groups

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Abstract

The notions of Hopfian and co-Hopfian groups are well known in both non-commutative and Abelian group theory. In this work we begin a systematic investigation of natural generalizations of these concepts and, in the case of Abelian p-groups, give a complete characterization of the generalizations in terms of the original concepts.

1 Introduction

A standard, and often useful, strategy in mathematics is to seek to investigate notions that are in some sense a generalization of finiteness. Thus, in topology one looks at compactness, in group theory local finiteness is investigated and similarly in many other areas. The starting point is usually to seek some relevant property that finite objects possess and then to look to see if there are non-finite objects possessing the same property. In this paper we seek to employ the

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same strategy; here our setting is the category of all groups $\mathcal{G}$, although we shall focus primarily on the subcategory of Abelian groups $\mathcal{A}b$. Indeed all our comments relating to $\mathcal{A}b$ can be interpreted in the category of modules over a fixed ring R but we shall not carry this out in detail. The property that we wish to explore is the familiar one in the category of sets $\mathcal{S}$: a set S is finite if, and only if, every one-one function $S \to S$ is invertible, if, and only if, every onto function $S \to S$ is invertible. The comparable statements in the category $\mathcal{G}$ would be that G being finite is equivalent to:

(i) every monic endomorphism of a group G is an automorphism;

(ii) every epic endomorphism of a group G is an automorphism.

These equivalences are, of course, not true: multiplication by the prime p in the additive group of integers is a monomorphism which is not an automorphism and the same multiplication in the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is an epimorphism which is not monic. Nevertheless, these conditions can be used to 'select' certain classes of groups which are not necessarily finite. Groups satisfying (i) are usually now referred to as *co-Hopfian groups* and those satisfying (ii) are called *Hopfian groups*. It is well known and easy to establish that the properties (i) and (ii) can be translated into the following equivalent conditions:

(i)' G cannot have a proper isomorphic subgroup and (ii)' G cannot have a proper isomorphic factor group.

There is, of course, a third condition which subsumes both of (i)' and (ii)': (iii)' G cannot have a proper isomorphic subdirect factor (or summand as is the more usual terminology in the Abelian situation). It is straightforward to show that this latter condition is equivalent to

(iii) if ϕ and ψ are endomorphisms of G and $\phi\psi = 1_G$, the identity endomorphism of G, then $\psi\phi = 1_G$.

Groups satisfying condition (iii) are usually referred to as *directly finite groups*.

Hopfian, co-Hopfian and directly finite groups have been the subject of intensive investigation for many years - see, for example, the discussions in [1, 2, 3, 8, 9, 10, 11, 14, 15]. The reader should note that the complements of these notions have also been studied under various names: Abelian groups which are not directly finite have been studied previously by Beaumont and Pierce [2] under the terminology *ID-group* - the context suggests this was intended to mean 'isomorphic direct summand', while in the context of non-Abelian group theory an alternative terminology, due to Peter Neumann [14], is *badly non-Hopfian*.

The conditions (i) and (ii) can be re-formulated to say that a group G is co-Hopfian [Hopfian] if every monomorphism [epimorphism] ϕ of G has a two sided inverse and this leads naturally to the following definition, where in an obvious notation the letters 'R, L' stand for 'right' and 'left' respectively. Note that in this paper, maps are always written on the left.

Definition 1.1. A group G is said to be R-Hopfian [L-Hopfian] if for every surjection $\phi \in \mathrm{End}(G)$, there is an endomorphism ψ of G such that $\phi\psi = 1_G$ [$\psi\phi = 1_G$].

Observe firstly that if G is Hopfian, then certainly G is both R-Hopfian and L-Hopfian. Moreover, if G is L-Hopfian and ϕ is a surjection, then the equation $\psi\phi = 1_G$ implies that ϕ is also an injection, so that ϕ is an automorphism of G. Consequently the class of L-Hopfian groups coincides with the class of Hopfian groups.

We have a dual situation here where Hopficity is replaced by co-Hopficity:

Definition 1.2. A group G is said to be R-co-Hopfian [L-co-Hopfian] if for every injection $\phi \in \mathrm{End}(G)$, there is an endomorphism ψ of G such that $\phi\psi = 1_G$ [$\psi\phi = 1_G$].

Again it is easy to see that a group G is R-co-Hopfian if, and only if, it is co-Hopfian. Thus we concentrate on the concepts of R-Hopficity and L-co-Hopficity. In particular, we look closely at the situation when the groups being considered are also Abelian p-groups for an arbitrary prime p. Our principal result, Theorem 3.11, shall be a classification of R-Hopfian and L-co-Hopfian p-groups in terms of Hopfian and co-Hopfian p-groups.

An important tool in our investigation will be a weakening of the classical notions of (Ker)-direct and (Im)-direct Abelian groups. Recall that an Abelian group G is said to be (Ker)-direct [(Im)-direct] if the kernel [image] of each endomorphism of G is a direct summand of G. Rangaswamy observed in [16], or see [7, Lemma 112.1], a connection between these notions and the (von Neumann) regularity of the endomorphism ring of G: The endomorphism ring of a group G is regular if, and only if, G is both (Ker)-direct and (Im)-direct. In fact, the same observation had been made in the context of module theory by Azumaya in the late 1940s. Recently, in that same context, the notions of (Ker)-direct and (Im)-direct modules have been called

Rickart modules and *dual Rickart modules* respectively - see [12, 13]. We shall come back to this in Section 2.

We finish off this introduction by noting that notation in the paper is standard as in the two volumes of Fuchs [6, 7]; in particular mapping are consistently written on the left and for an Abelian group G, the ring of endomorphisms of G shall be denoted by $\mathrm{End}(G)$. *With the exception of the first two results in Section 2 below, all groups will be additively written Abelian groups.*

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2 Elementary Results

The notion of direct finiteness provides the connection between Hopficity and R-Hopficity (and dually between co-Hopficity and L-co-Hopficity).

Proposition 2.1. *(i) An arbitrary group G is Hopfian if, and only if, it is R-Hopfian and directly finite;*

(ii) An arbitrary group G is co-Hopfian if, and only if, it is L-co-Hopfian and directly finite.

In particular, if the endomorphism monoid of G is commutative, then G is R-Hopfian [L-co-Hopfian] if, and only if, it is Hopfian [co-Hopfian].

Proof. (i) If G is Hopfian then every surjection has an inverse, so G is certainly R-Hopfian. However, if $\alpha\beta = 1_G$ for some endomorphisms α, β, then α is surjective and so, by the Hopficity of G, it has an inverse α^{-1}. It follows immediately that $\beta = \alpha^{-1}$ and so $\beta\alpha = 1_G$, whence G is directly finite.

Conversely, given any surjection ϕ of G, R-Hopficity ensures the existence of an endomorphism ψ such that $\phi\psi = 1_G$. By direct finiteness, we have that $\psi\phi$ is also equal to 1_G and so ϕ is invertible with inverse ψ. Since ϕ was arbitrary, we have that G is Hopfian.

The proof of (ii) runs dually and is left to the reader, while the particular case when the endomorphism monoid of G is commutative, is then immediate. □

Corollary 2.2. *A group which is not a non-trivial semidirect product is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian]. In particular, the group $\mathbb{Z}(p^\infty)$ is not R-Hopfian for any prime p and $\mathbb{Z}$ is not L-co-Hopfian.*

Proof. The necessity is immediate in both cases and doesn't require the semi-direct product condition. Conversely suppose that G is R-Hopfian [L-co-Hopfian]. It suffices by Proposition 2.1 to show that G is directly finite. Suppose then that $\phi\psi = 1_G$ for endomorphisms ϕ, ψ of G. Then $\psi\phi$ is an idempotent endomorphism which cannot be the trivial map and so the fact that G is not a non-trivial semidirect product, forces $\psi\phi = 1_G$, as required. □

From now on all groups will be additively written Abelian groups.

Corollary 2.3. *A reduced group G such that G/pG is finite for all primes p is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian].*

Proof. We show that the hypotheses on G ensure that G cannot have a proper isomorphic direct summand and so G is directly finite and then the result follows from Proposition 2.1. Suppose then that $G \cong H \oplus G$ for some $H \leq G$. Then $G/pG \cong H/pH \oplus G/pG$ and so by the finiteness of the latter term, we conclude that $H/pH = \{0\}$ for all primes p. Thus H is divisible and hence, as G is reduced, $H = \{0\}$. Thus G is directly finite, as required. □

In response to a question of Baumslag [1, Problem 3], Corner [3, Example 1] exhibited a non-Hopfian torsion-free group having automorphism group of order 2. Using Corner's example and Proposition 2.1 we can establish:

Example 2.4. There is a torsion-free group G with automorphism group of order 2, but G is not R-Hopfian.

Proof. Corner's example of a non-Hopfian torsion-free group with automorphism group of order 2 has the property that its full endomorphism ring is isomorphic to the polynomial ring $\mathbb{Z}[X]$; in particular the endomorphism ring is commutative and so the group is directly finite. Since it is non-Hopfian, it cannot be R-Hopfian by Proposition 2.1. □

As noted in [10], the endomorphism ring of a group does not determine its Hopficity since there are also Hopfian groups, hence R-Hopfian groups, with endomorphism ring $\mathbb{Z}[X]$ which can be obtained using Corner's realization theorem. This also applies to R-Hopficity: the group in Example 2.4 is not R-Hopfian but has endomorphism ring $\mathbb{Z}[X]$.

In light of Proposition 2.1 we would expect R-Hopfian and L-co-Hopfian groups to share some properties known for Hopfian and co-Hopfian groups. Our first result is an analogue of such a property of Hopfian groups.

Proposition 2.5. *A direct summand of an R-Hopfian [L-co-Hopfian] group G is again R-Hopfian [L-co-Hopfian].*

Proof. We handle the L-co-Hopfian case leaving the analogous proof for R-Hopfian groups to the reader. Suppose then that $G = H \oplus S$ and let α be an arbitrary injection in $\mathrm{End}(H)$. Then $\psi = \alpha \oplus 1_S$ is an injection in $\mathrm{End}(G)$ and so there is a $\phi \in \mathrm{End}(G)$ such that $\phi\psi = 1_G$. Using the standard matrix representation of endomorphisms of a direct sum, this means that

$$\begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 1_S \end{pmatrix} = \begin{pmatrix} 1_H & 0 \\ 0 & 1_S \end{pmatrix}, \quad \text{where } \phi = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}.$$

Thus $\mu\alpha = 1_H$, and so, since α was an arbitrary injection in $\mathrm{End}(H)$, H is L-co-Hopfian. □

Corollary 2.6. *A torsion R-Hopfian group is reduced and an L-co-Hopfian group has trivial dual.*

Proof. The results are immediate from the fact that no quasi-cyclic group $\mathbb{Z}(p^\infty)$ is R-Hopfian while $\mathbb{Z}$ is not L-co-Hopfian. □

It is also possible to relate R-Hopficity [L-co-Hopficity] of a group G to the corresponding properties of subgroups of the form nG:

Proposition 2.7. *If G is R-Hopfian [L-co-Hopfian], then, for each natural number n, the subgroup nG is R-Hopfian [L-co-Hopfian].*

Proof. We only handle with R-Hopfian case, the L-co-Hopfian case is analogous. If $\phi : nG \to nG$ is epic, then it follows from the proof of Proposition 113.3 in [7], that there exists an epic $\psi : G \to G$ such that $\psi \restriction nG = \phi$. Since G is R-Hopfian, ψ must have a right inverse, θ say. But then the restriction $\theta \restriction nG$ is the required right inverse of ϕ. □

Theorem 2.8. *If G is a group which has no n-bounded pure subgroups for a given integer n and nG is R-Hopfian [L-co-Hopfian], then G is R-Hopfian [L-co-Hopfian]. The requirement of no n-bounded pure subgroups cannot be omitted.*

Proof. We only prove the R-Hopfian case, the L-co-Hopfian case is analogous. Suppose that nG is R-Hopfian and $\phi : G \to G$ is a surjection. Then $\alpha = \phi \restriction nG : nG \to nG$ is a surjection and since nG is R-Hopfian , there is an endomorphism of nG, β say, with $\alpha\beta = 1_{nG}$. Now it follows from the proof of [7, Proposition 113.3] (or see [5, Lemma 2.11]) that there is an endomorphism ψ of G with $\psi \restriction nG = \beta$. Now for all $x \in G$, $\phi\psi(nx) = \phi\beta(nx) = \alpha\beta(nx) = nx$ and so $n(\phi\psi - 1_G) = 0$. Thus $\phi\psi$ is an n-map in the sense of Corner [4] and it follows that if G has no nonzero n-bounded pure subgroups, then $\phi\psi$ is an automorphism, θ say, and so $\phi\psi\theta^{-1} = 1_G$. Hence ϕ has a right inverse $\psi\theta^{-1}$ and so G is R-Hopfian.

For the second part of the result take $n = p$, a prime and set $G = \mathbb{Z}(p) \oplus \mathbb{Z}(p^2)^{(\aleph_0)}$. It follows from the discussions in Section 3 below that $pG \cong \mathbb{Z}(p)^{(\aleph_0)}$ is both R-Hopfian and L-co-Hopfian but G itself is neither R-Hopfian nor L-co-Hopfian. □

We also have the easy but useful:

Proposition 2.9. *If $G = \bigoplus_{i \in I} H_i$ and each H_i is fully invariant in G, then G is R-Hopfian [L-co-Hopfian] if, and only if, each H_i is R-Hopfian [L-co-Hopfian].*

Proof. The necessity is immediate from Proposition 2.5 while the sufficiency follows from the fact that every endomorphism of G can be expressed in the form $\bigoplus_{i \in I} \phi_i$, where ϕ_i is an endomorphism of H_i. □

The following notions, which are weaker than the corresponding notions mentioned in the introduction, will play a key role in our investigations.

Definition 2.10. A group G is said to be *(sKer)-direct* if the kernel of each surjective endomorphism of G is a direct summand of G; it is said to be *(mIm)-direct* if the image of each monic endomorphism of G is a direct summand of G.

The following theorem gives a complete characterization of R-Hopficity [L-co-Hopficity] in terms of these groups, the proof is well-known and hence omitted.

Theorem 2.11. *A group G is R-Hopfian [L-co-Hopfian] if, and only if, it is (sKer)-direct [(mIm)-direct].*

It follows from the characterization in Theorem 2.11 that the classes of R-Hopfian [L-co-Hopfian] groups are large since they necessarily contain the classes of (Ker)-direct [(Im)-direct] groups. Groups which are (Im)-direct have been classified by Rangaswamy [16] and include groups G where the torsion subgroup $t(G)$ is a direct sum of elementary p-groups for various primes p, $G/t(G)$ is divisible and every endomorphic image of G is maximally disjoint from a pure subgroup of G; the class of (Ker)-direct groups does not seem to have been classified but it is easy to see that in addition to the torsion-free divisible groups and the elementary groups, groups which are free (of arbitrary rank) and torsion-free algebraic compact groups are (Ker)-direct and hence R-Hopfian.

Note that the class of (Im)-direct [(Ker)-direct] groups is strictly contained in the class of L-co-Hopfian [R-Hopfian] groups, indeed containment within the class of co-Hopfian [Hopfian] groups is strict. There are even finite examples: if G is a cyclic group of order p^2, then multiplication by p has both an image and a kernel which are not summands but the group G is both co-Hopfian and Hopfian.

The classification of torsion-free co-Hopfian groups is an easy exercise: they are precisely the class of finite-dimensional $\mathbb{Q}$-vector spaces. Similarly it is easy to classify the torsion-free L-co-Hopfian groups:

Theorem 2.12. *A torsion-free group is L-co-Hopfian if, and only if, it is divisible.*

Proof. The sufficiency is straightforward since all divisible groups are (Im)-direct and hence L-co-Hopfian.

Conversely suppose that G is a torsion-free L-co-Hopfian group. For each natural number n, let ϕ_n denote the endomorphism of G corresponding to multiplication by n. Then ϕ_n is monic and hence there is an endomorphism ψ of G with $\psi\phi_n = 1_G$. However ϕ_n is central in $\mathrm{End}(G)$ and so $\phi_n\psi = 1_G$. Hence ϕ_n is a unit in $\mathrm{End}(G)$ and so G is n-divisible. Since n was arbitrary, G is then divisible. □

Corollary 2.13. *A torsion-free group is both R-Hopfian and L-co-Hopfian if, and only if, it is divisible.*

Proof. This follows immediately from Theorem 2.12 and the fact that torsion-free divisible groups are R-Hopfian since, as observed above, they are (Ker)-direct. □

Our next example shows us that no simple characterization of groups which are both R-Hopfian and L-co-Hopfian is likely to be achieved. We refer to [7, §112] for the notion of π-regularity.

Example 2.14. A group having a left π-regular endomorphism ring is both R-Hopfian and L-co-Hopfian; in fact it is both Hopfian and co-Hopfian.

Proof. If End(G) is π-regular, then it follows from Proposition 112.9 [7], that for any endomorphism ϕ of G, we have a positive integer m and a decomposition $G = \mathrm{Ker}\phi^m \oplus \mathrm{Im}\phi^m$. If ϕ is onto this forces $\mathrm{Ker}\phi^m = 0$, whence $\mathrm{Ker}\phi = 0$ and ϕ is an automorphism. A similar argument using ϕ monic establishes the result. □

3 Torsion Groups

We begin with an example that shows that arbitrary direct sums of R-Hopfian [L-co-Hopfian] groups need not be R-Hopfian [L-co-Hopfian].

Example 3.1. The group $B = \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^{i_n})$, with $i_1 < i_2 < \cdots < i_t < \ldots$, is neither R-Hopfian nor L-co-Hopfian.

Proof. Let e_n denote a generator of the group $\mathbb{Z}(p^{i_n})$ and consider the endomorphism ϕ of B which acts as the left Bernoulli shift: $e_1 \mapsto 0, e_2 \mapsto e_1, \ldots, e_{n+1} \mapsto e_n \ldots$; then ϕ is surjective but, as the kernel is not a summand of B, by Theorem 2.11, B is not R-Hopfian.

The proof that B is not L-co-Hopfian is similar using the right algebraic Bernoulli shift: $e_n \mapsto p^{(i_{n+1}-i_n)}e_{n+1}$, and the fact that its image is not a summand. □

Note that it follows immediately from Example 3.1 that an unbounded direct sum of cyclic p-groups can never be R-Hopfian nor L-co-Hopfian: any such group must contain a summand of the form of B above and summands inherit R-Hopficity and L-co-Hopficity by Proposition 2.5.

Our first result establishes the unsurprising fact that homocyclic p-groups are both R-Hopfian and L-co-Hopfian. There are two possible different approaches to proving this: a direct, and possibly more insightful, approach and an approach utilizing Theorem 2.8. Considering the merits of both approaches we use the direct proof for establishing L-co-Hopficity and use Theorem 2.8 for showing R-Hopficity.

Proposition 3.2. *A homocyclic p-group A is both R-Hopfian and L-co-Hopfian.*

Proof. First note that in a homocyclic p-group A of exponent n, an element $a \in A$ is divisible by p^k if and only if $p^{n-k}a = 0$ (for $0 \leq k \leq n$). We deal with the L-co-Hopfian case first. It follows from Theorem 2.11 that it will suffice to show that A is (mIm)-direct. Let $A = \bigoplus_{i \in I} \langle e_i \rangle$, where the order of each e_i is p^n. Let ϕ be an injective endomorphism of A; since A is bounded it will suffice to show that the image $\phi(A)$ is pure in A.

Pick an element $a \in \phi(A) \cap p^k A$, then $a = \phi(x)$ for some $x \in A$, and $a = p^k a'$ for some $a' \in A$. Multiplying by p^{n-k}, we have $p^{n-k}a = p^{n-k}\phi(x) = p^n a' = 0$. Hence, $\phi(p^{n-k}x) = 0$ and since ϕ is injective, we have $p^{n-k}x = 0$, and thus x is divisible by p^k. Therefore, $a = \phi(x) = \phi(p^k y) = p^k \phi(y) \in p^k \phi(A)$ and $\phi(A)$ is pure in A, as required.

For R-Hopficity observe that if the exponent of A equals 1, then A is elementary and thus is certainly R-Hopfian. If the exponent of A is $n > 1$, then $p^{n-1}A$ is R-Hopfian and, as A clearly has no p^{n-1}-bounded pure subgroups, it follows from Theorem 2.8 that A is R-Hopfian.

□

Our next result shows that there are considerable restrictions on the p-groups which can be R-Hopfian or L-co-Hopfian.

Proposition 3.3. *A direct sum of an infinite rank homocyclic p-group and a cyclic p-group of smaller exponent is neither R-Hopfian nor L-co-Hopfian. Consequently a direct sum of two homocyclic p-groups of infinite rank and of different exponents is neither R-Hopfian nor L-co-Hopfian.*

Proof. The arguments for R-Hopficity and L-co-Hopficity are broadly similar so we give details of just the L-co-Hopficity case.

Since a direct summand of an L-co-Hopfian group is again L-co-Hopfian, it suffices to show that a direct sum of a countable rank homocyclic p-group and a cyclic p-group of smaller exponent is not L-co-Hopfian. Suppose then that $G = \langle e \rangle \oplus \bigoplus_{i=1}^{\infty} \langle f_i \rangle$, where $o(e) = p^n, o(f_i) = p^{n+k}$ for each i and $k > 0$. Consider the map $\phi : G \to G$ as follows(similar to the forward shift):

$$e \mapsto p^k f_1, f_i \mapsto f_{i+1} (i \geq 1).$$

It is easy to see that ϕ is a monomorphism. Now suppose on the contrary that there is an endomorphism ψ with $\psi\phi = 1_G$. So on the one hand, $\psi\phi(e) = e$, on the other hand,$\psi\phi(e) = \psi(p^k f_1) = p^k\phi(f_1)$, hence $e = p^k\phi(f_1)$, this is not possible since the height of e in G is 0, but the height of $p^k\phi(f_1)$ in G is $\geq k > 0$.

The final statement follows immediately from the fact that a direct sum of two homocyclic p-groups of infinite rank and of different exponents has a summand which is a direct sum of an infinite rank homocyclic p-group and a cyclic p-group of smaller exponent. $\square$

The following technical lemma will simplify arguments we require later.

Lemma 3.4. *Let A be an R-Hopfian [L-co-Hopfian group] and B an arbitrary group. If a surjective endomorphism [monic endomorphism] Φ of $A \oplus B$ has a matrix representation of the form $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$, where β is an automorphism of B, then Φ has a right [left] inverse Ψ, i.e., $\Phi\Psi = 1$ [$\Psi\Phi = 1$].*

Proof. We give only the argument for L-co-Hopficity, the argument for R-Hopficity is dual. So assume that Φ represents a monic endomorphism of $A \oplus B$.

Pre-multiplying Φ by the invertible matrix $\Delta = \left(\begin{smallmatrix} 1 & -\gamma\beta^{-1} \\ 0 & \beta^{-1} \end{smallmatrix}\right)$ and post-multiplying it by the invertible matrix $\Sigma = \left(\begin{smallmatrix} 1 & -\gamma\beta^{-1} \\ 0 & \beta^{-1} \end{smallmatrix}\right)$ reduces Φ to a diagonal matrix $\Delta\Phi\Sigma = \left(\begin{smallmatrix} \alpha-\gamma\beta^{-1}\delta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ which is again injective.

Claim that $\alpha - \gamma\beta^{-1}\delta$ is injective. Suppose, on the contrary, that there is a non-zero element $a \in A$ with $(\alpha - \gamma\beta^{-1}\delta)(a) = 0$, then the injection $\Delta\Phi\Sigma$ maps the non-zero element $(a, 0)$ to $(0, 0)$ – contradiction.

Now since A is L-co-Hopfian, there is an endomorphism μ of A such that $\mu(\alpha - \gamma\beta^{-1}\delta) = 1$. If $\Gamma = \left(\begin{smallmatrix} \mu & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $\Gamma\Delta\Phi\Sigma = 1$. Hence $\Sigma\Gamma\Delta$ is the required left inverse of Φ. $\square$

Proposition 3.5. *(i) If A is an R-Hopfian [L-co-Hopfian] group, B a Hopfian [co-Hopfian] group and* $\mathrm{Hom}(A, B) = 0$ *[*Hom*$(B, A) = 0$], then $A \oplus B$ is R-Hopfian [L-co-Hopfian];*

(ii) If A is an R-Hopfian [L-co-Hopfian] p-group of exponent n and B is a Hopfian [co-Hopfian] p-group that has no p^n-bounded pure subgroups, then $A \oplus B$ is R-Hopfian [L-co-Hopfian].

Proof. We deal first with the R-Hopficity.

(i) An arbitrary surjection of $A \oplus B$ has the form $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ 0 & \sigma \end{smallmatrix}\right)$ and this forces σ to be a surjection of B. Since B is Hopfian this implies that σ is an automorphism of B. It follows from Lemma 3.4 that Ψ has a right inverse and so $A \oplus B$ is R-Hopfian, as required.

(ii) Let $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ \rho & \sigma \end{smallmatrix}\right)$ be an arbitrary surjective endomorphism of $A \oplus B$. Then $\rho(A) + \sigma(B) = B$, and so $p^n\rho(A) + p^n\sigma(B) = p^nB$, implying that $\sigma(p^nB) = p^nB \neq 0$. Thus $\sigma \restriction p^nB$ is a surjection of the nontrivial Hopfian group p^nB and so $\sigma \restriction p^nB$ is an automorphism of p^nB. By [7, Proposition 113.3], there is an automorphism ϕ of B with $\phi \restriction p^nB = \sigma \restriction p^nB$. Hence $p^n(\sigma - \phi) = 0$, and as ϕ is an automorphism of B, σ is a p^n-map of B in the sense of Corner [4]. Since B has no p^n-bounded pure subgroups, it follows from [4, Theorem 3] that σ is an automorphism of B. It follows immediately from Lemma 3.4 that Δ has a right inverse and so $A \oplus B$ is R-Hopfian.

The argument for L-co-Hopficity in part (i) follows a similar argument to that used for R-Hopficity noting that in this case endomorphisms of $A \oplus B$ have matrix representations of the form $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ \rho & \sigma \end{smallmatrix}\right)$ with $\nu = 0$.

For part (ii) the argument is entirely dual to that used for R-Hopficity. □

We can now classify those direct sums of cyclic p-groups which are R-Hopfian [L-co-Hopfian]; the situation parallels that in Hopfian [co-Hopfian] groups where the only direct sums of cyclic groups which are Hopfian [co-Hopfian] are the finite groups and so Hopficity and co-Hopficity coincide for such groups.

Theorem 3.6. *A direct sum of cyclic p-groups G is R-Hopfian [L-co-Hopfian] if, and only if, it has the form $G = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, where $B_1 = \bigoplus_{\kappa_1} \mathbb{Z}(p^{n_1})$ for some cardinal κ_1 which may be infinite and each B_i $(2 \leq i \leq k)$ is of the form $B_i = \bigoplus_{\kappa_i} \mathbb{Z}(p^{n_i})$ with κ_i finite and $n_1 < n_2 < \cdots < n_k$. In particular, a direct sum of cyclic groups is R-Hopfian if, and only if, it is L-co-Hopfian.*

Proof. For the sufficiency note that B_1 is R-Hopfian [L-co-Hopfian] and $B_2 \oplus \cdots \oplus B_k$ is a Hopfian [co-Hopfian] p-group which has no p^{n_1}-bounded pure subgroups and thus the result follows from Proposition 3.5(ii) above.

Conversely, suppose that $G = \bigoplus_{i=1}^{\infty} B_i$ is R-Hopfian [L-co-Hopfian] and each B_n is homocyclic of exponent n. It follows, as noted after Example 3.1, that almost all the B_n are zero. Let B_r be the first homocyclic component of infinite rank; if no such exists then G is a finite group and clearly has the desired form. It follows from Proposition 3.3 that each B_i $(1 \leq i < r)$ must be zero since summands of R-Hopfian [L-co-Hopfian] groups are again R-Hopfian [L-co-Hopfian]. Furthermore, it follows from the same proposition that no B_j with $j > r$ can be of infinite rank. Thus G is of the claimed form.

The final statement follows from the fact that the classifications of R-Hopficity and L-co-Hopficity coincide for the class of direct sums of cyclic groups. □

Recall that a reduced p-group G is said to be *semi-standard* if for each $n < \omega$, the Ulm invariant $f_n(G)$ is finite; it is well known that both Hopfian and co-Hopfian p-groups are necessarily semi-standard.

Our next result shows that Hopficity and R-Hopficity [co-Hopficity and L-co-Hopficity] coincide for semi-standard p-groups.

Proposition 3.7. *A semi-standard p-group is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian].*

Proof. The necessity is clear in both cases. For the sufficiency, by Proposition 2.1, it is enough to prove that every semi-standard p-group G is directly finite. Suppose $G \cong G \oplus K$. Then $f_\sigma(G) = f_\sigma(G) + f_\sigma(H)$ for all ordinals σ. If $\sigma < \omega$, we must have $f_\sigma(K) = 0$ as the cardinals in question are finite. Hence a basic subgroup of K is the zero subgroup; since K is reduced, we are forced to have $K = 0$. □

We can now classify R-Hopficity [L-co-Hopficity] for reduced p-groups in terms of Hopficity [co-Hopficity].

Proposition 3.8. *A reduced p-group G is R-Hopfian [L-co-Hopfian] if, and only if, it is of the form $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$, where κ is a cardinal which may be infinite and H is Hopfian [co-Hopfian] and all Ulm invariants $f_i(H) (i < m)$ are zero.*

Proof. The condition on the Ulm invariants of H ensure that H has no p^m-bounded pure subgroups and so the sufficiency follows from Proposition 3.5 (ii).

Conversely suppose that G is a reduced R-Hopfian [L-co-Hopfian] p-group. Let B_{i_k} be the first nonzero infinite homogeneous component of a basic subgroup of G; if no such component exists then G is semi-standard and hence Hopfian [co-Hopfian] by Proposition 3.7 above, so we are finished in that case. It follows from Proposition

3.3 that B_{i_n} $(n > k)$ cannot be infinite since $B_{i_k} \oplus B_{i_n}$ is a summand of G. Furthermore, B_{i_j} $(j < k)$ cannot be nonzero by the same proposition. Simplifying notation by writing $i_n = m$, we conclude that $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$ and that H is semi-standard and all Ulm invariants $f_i(G)$ $(i < m)$ are zero. □

Proposition 3.8 can be re-phrased to say that a reduced R-Hopfian [L-co-Hopfian] p-group G differs from a reduced Hopfian [co-Hopfian] p-group in that it may have at most one infinite homogeneous component and this corresponds to the summand of G of least exponent. Notice also that although a reduced R-Hopfian [L-co-Hopfian] p-group G can be of arbitrarily large cardinality, there is an integer n such that the cardinality of p^nG is at most $2^{\aleph_0}$, the cardinality of the continuum.

Note that it is not necessary to specify that the group be reduced in the case of R-Hopficity: the group $\mathbb{Z}(p^\infty)$ is not R-Hopfian for any prime p. For L-co-Hopfian groups we need to some further work to handle the situation where the group may have a divisible summand.

Lemma 3.9. *The group $G = \mathbb{Z}(p^n) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^\infty)$ is not L-co-Hopfian.*

Proof. Write G as $G = \langle e \rangle \oplus \mathbb{Z}(p^\infty) f_1 \oplus \mathbb{Z}(p^\infty) f_2 \oplus \mathbb{Z}(p^\infty) f_3 \oplus \cdots$, where the order of e is p^n. Consider the forward shift mapping $\phi : G \to G$, $e \mapsto 1/p^n f_1, f_1 \mapsto f_2, f_2 \mapsto f_3, \cdots$. Then ϕ is an injective endomorphism of G. Suppose on the contrary that there is an endomorphism ψ with $\psi\phi = 1_G$. Then $\psi\phi(e) = e$, that is, $\psi(1/p^n f_1) = e$, but $1/p^n f_1 = p(1/p^{n+1} f_1)$, so $\psi(1/p^n f_1) = p\psi(1/p^{n+1} f_1) = px$ for some $x \in G$, this is impossible since e is not divisible by p. □

Theorem 3.10. *If A is a non-trivial, reduced L-co-Hopfian p-group and D is a divisible p-group, then the direct sum $A \oplus D$ is L-co-Hopfian if, and only if, D is of finite rank, $D \cong \bigoplus_n \mathbb{Z}(p^\infty)$, for some finite n.*

Proof. The sufficiency follows from the fact that a finite rank divisible p-group D is actually co-Hopfian: any injective endomorphism of D has image whose rank is equal to that of D and, since the image is a summand, it must be the whole of D, so that the injection is an automorphism. Now apply Proposition 3.5 (i) and it follows immediately that $A \oplus D$ is L-co-Hopfian.

Conversely, suppose for a contradiction, that $A \oplus D$ is L-co-Hopfian but that D has infinite rank. Then there is a summand of $A \oplus D$ of the form $\mathbb{Z}(p^n) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^\infty)$ and this summand is also L-co-Hopfian. This, however, contradicts Lemma 3.9. Thus D has finite rank, as required. □

We summarize the preceding results as:

Theorem 3.11. *A p-group G is R-Hopfian if, and only if, it is of the form* $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$, *where* κ *is a cardinal which may be infinite, H is Hopfian and all Ulm invariants* $f_i(H)(i < m)$ *are zero and D is a finite direct sum of copies of* $\mathbb{Z}(p^\infty)$.

A p-group G is L-co-Hopfian if, and only if, it is of the form (i) $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H \oplus D$, $\bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$ *non-trivial, where* κ *is a cardinal which may be infinite, H is reduced co-Hopfian and all Ulm invariants* $f_i(H)(i < m)$ *are zero and D is a finite direct sum of copies of* $\mathbb{Z}(p^\infty)$*; or of the form (ii)* $G = \bigoplus_\kappa \mathbb{Z}(p^\infty)$, *where* κ *is an arbitrary cardinal.*

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References

[1] G. Baumslag, *Hopficity and Abelian Groups,* in Topics in Abelian Groups, Scott Foresman, Chicago, Illinois (1963) 331–335.

[2] R.A. Beaumont and R.S. Pierce, *Isomorphic Direct Summands of Abelian Groups,* Math. Ann. **153** (1964) 21–37.

[3] A.L.S. Corner, *Three Examples on Hopficity in Torsion-Free Abelian Groups,* Acta Math. Acad. Sci. Hungar. **16** (1965) 303–310.

[4] A.L.S. Corner, *Near Automorphisms of an Abelian Group,* to appear in Mülheim Proceedings.

[5] P. Danchev and B. Goldsmith, On commutator socle-regular Abelian p-groups, J. Group Theory, **17** (2014) 781–803.

[6] L. Fuchs, Infinite Abelian Groups, Vol.I, Academic Press 1970.

[7] L. Fuchs, Infinite Abelian Groups, Vol.II, Academic Press 1973.

[8] B. Goldsmith and K. Gong, *Algebraic entropies, Hopficity and co-Hopficity of direct sums of Abelian Groups,* Topol. Algebra Appl **3** (2015) 75 – 85.

[9] B. Goldsmith and K. Gong, *A Note on Hopfian and co-Hopfian Abelian Groups,* in Groups and Model Theory, Contemp. Math. **576** (2012) Amer Math. Soc., Providence R.I. 124 –136.

[10] B. Goldsmith and P. Vámos, *The Hopfian Exponent of an Abelian Group,* Period. Math. Hung. **69** (2014) 21–31.

[11] R. Hirshon, *On Hopfian groups,* Pacific J. Math. **32** (1970) 753–766.

[12] G. Lee, S.T. Rizvi and C.S. Roman, *Rickart Modules,* Comm. Algebra, **38** (2010) 4005–4027.

[13] G. Lee, S.T. Rizvi and C.S. Roman, *Dual Rickart Modules,* Comm. Algebra, **39** (2011) 4036–4058.

[14] P.M. Neumann, *Pathology in the Representation Theory of Infinite Soluble Groups,* in Proc. 'Groups-Korea 1988' Lecture Notes in Mathematics 1398 (Eds A.C. Kim and B.H. Neumann) 124–139.

[15] R.S. Pierce, *Homomorphisms of Primary Abelian Groups*, in Topics in Abelian Groups, Scott Foresman, Chicago, Illinois (1963), 215–310.

[16] K.M. Rangaswamy, *Abelian Groups with Endomorphic Images of Special Types,* J. Algebra **6** (1967) 271–280.