



2012

On Super and Hereditarily Hopfian and co-Hopfian Abelian Groups

Brendan Goldsmith

Technological University Dublin, brendan.goldsmith@tudublin.ie

Katao Kong

Technological University Dublin

Follow this and additional works at: <https://arrow.dit.ie/scschmatart>



Part of the [Mathematics Commons](#)

Recommended Citation

Goldsmith, B. & Gong, K. (2012). On super and hereditarily hopfian and co-hopfian Abelian groups. *Archiv der Mathematik*, 99(1), pp.1-8.. doi:10.1007/s00013-012-0402-2

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact yvonne.desmond@dit.ie, arrow.admin@dit.ie, brian.widdis@dit.ie.



This work is licensed under a [Creative Commons Attribution-Noncommercial-Share Alike 3.0 License](#)



On Super and Hereditarily Hopfian and co-Hopfian Abelian Groups*

B. Goldsmith

School of Mathematical Sciences
Dublin Institute of Technology
Aungier Street, Dublin 2, Ireland

email: brendan.goldsmith@dit.ie

and

K. Gong

School of Mathematical Sciences
Dublin Institute of Technology
Kevin Street, Dublin 8, Ireland

email: ketao.gong@mydit.ie

March 28, 2012

*2000 Mathematics Subject Classification. Primary: 20K10. Secondary: 20K21. Key words and phrases: Abelian groups, Hopfian groups, co-Hopfian groups.

Abstract

The notions of Hopfian and co-Hopfian groups have been of interest for some time. In this present work we characterize the more restricted classes of hereditarily Hopfian (co-Hopfian) and super Hopfian (co-Hopfian) groups in the case where the groups are Abelian.

1 Introduction

The notions of Hopfian and co-Hopfian groups have been studied for a long time. The terminology “Hopfian” seems to have arisen from the fact that the topologist H. Hopf showed that the fundamental groups of certain closed two-dimensional manifolds have the defining property. The concept, and its dual, were studied by Baer [1], under the names Q -group and S -group. In modern terminology we say that a group G is *Hopfian* if every surjection $\alpha : G \rightarrow G$ is an automorphism; it is said to be *co-Hopfian* if every injection $\beta : G \rightarrow G$ is an automorphism. Finite groups are, of course, the prototypes for both Hopfian and co-Hopfian groups. The existence of infinite co-Hopfian p -groups was first established by Crawley [3]. Abelian Hopfian and co-Hopfian groups have arisen recently in the study of algebraic entropy and its dual, adjoint entropy – see e.g. [4, 7, 9]. Despite the seeming simplicity of their definitions, Hopfian and co-Hopfian groups are notoriously difficult to handle and easily stated problems have remained open for a long time: if G is Hopfian, is the direct product $G \times \mathbb{Z}$ Hopfian? The expository paper [11] gives a nice overview of this problem. In this paper we consider two stronger versions of the concepts, the classes of so-called hereditarily Hopfian (co-Hopfian) groups and the super Hopfian (co-Hopfian) groups. These are the classes obtained by requiring closure under subgroups, respectively homomorphic images. We obtain complete characterizations of these classes when the groups are Abelian. In the sequel, the word group shall mean an additively written Abelian group. The books [5] shall serve as a reference to ideas needed in Abelian group theory. We shall denote the set of rational primes by the symbol \mathbb{P} ; mappings are written on the right.

2 Preliminaries

In this section we record without proof some well-known facts about Hopfian and co-Hopfian groups; proofs may be found in e.g. [8].

It is easy to show that the Hopfian property for G is equivalent to G having no proper isomorphic factor group, while co-Hopficity is equivalent to having no proper isomorphic subgroup. The groups \mathbb{Z} and $\mathbb{Z}(p^\infty)$ show that the notions are independent of each other.

The following simple proposition records some well-known, and easily established, facts about Hopficity and co-Hopficity:

Proposition 2.1 (i) *A torsion-free group G is co-Hopfian if, and only if it is divisible of finite rank;*

(ii) *A torsion-free group of finite rank is Hopfian;*

(iii) *Finitely generated groups are Hopfian and finitely co-generated groups are co-Hopfian;*

(iv) *A group G with $\text{End}(G) \cong \mathbb{Z}$ is Hopfian; thus arbitrarily large Hopfian groups exist;*

(v) *Reduced Hopfian (co-Hopfian) p -groups are semi-standard and so have cardinality at most 2^{\aleph_0} ;*

(vi) *A reduced countable Hopfian (co-Hopfian) p -group is finite.*

The classes of Hopfian and co-Hopfian groups exhibit some weak closure properties which are well known:

Proposition 2.2 *Let $0 \rightarrow H \rightarrow G \rightarrow K \rightarrow 0$ be an exact sequence.*

(i) *If H, K are both Hopfian and if H is left invariant by each surjection $\phi : G \rightarrow G$, then G is Hopfian. In particular, extensions of Hopfian torsion groups by torsion-free Hopfian groups are again Hopfian;*

(ii) *If H, K are both co-Hopfian and if H is left invariant by each injection $\psi : G \rightarrow G$, then G is co-Hopfian. In particular, extensions of co-Hopfian torsion groups by torsion-free co-Hopfian groups are again co-Hopfian.*

The classes of Hopfian and co-Hopfian groups are not, however, closed under taking subgroups or forming homomorphic images: for example, if G is an unbounded p -group which is both Hopfian and co-Hopfian (such groups exist, the first example being due to Pierce [12, Theorem 16.4]), then any basic subgroup B of G is an unbounded direct sum of cyclic groups and hence is neither Hopfian nor co-Hopfian; it is, however, both a subgroup and a homomorphic image of G , the latter following from the well-known theorem of Szele [5, Theorem 36.1].

3 Strengthening the concepts

It is clear from our earlier discussions that the classification of both Hopfian and co-Hopfian groups looks extremely difficult. In this section we strengthen both concepts in two natural ways.

Definition 3.1 (i) *A group G is said to be super Hopfian (super co-Hopfian) if every homomorphic image of G is Hopfian (co-Hopfian);*

(ii) A group G is said to be hereditarily Hopfian (hereditarily co-Hopfian) if every subgroup of G is Hopfian (co-Hopfian).

The terminology “super Hopfian” seems to have originated in the paper [10] of Hirshon and we continue to use it here. Clearly, super and hereditarily Hopfian (co-Hopfian) groups are Hopfian (co-Hopfian); indeed the super class is easily seen to be closed under homomorphic images, while the hereditary class is closed under subgroups.

The classification of reduced hereditarily co-Hopfian groups is easily obtained: suppose that G is such a group and observe that if G is not torsion, then it has a subgroup isomorphic to \mathbb{Z} . Since \mathbb{Z} is not co-Hopfian, we conclude that G must be torsion so that there is a primary decomposition $G = \bigoplus_{p \in \mathbb{P}} G_p$. Since G is hereditarily co-Hopfian, each G_p is also hereditarily co-Hopfian and hence a basic subgroup B_p of G_p is both a direct sum of cyclic groups and co-Hopfian. It is therefore finite and it follows easily that G_p is finite for all $p \in \mathbb{P}$. Since any subgroup of a group of the form $H = \bigoplus_{p \in \mathbb{P}} H_p$, where each H_p is finite, is again of this form, we deduce:

Theorem 3.2 *A group G is hereditarily co-Hopfian if, and only if, it is of the form $G = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty) \oplus G_p)$, where each $r_p (p \in \mathbb{P})$ is finite and each G_p is a finite p -group.*

Proof We have already established the form of the reduced part of G , so it suffices to consider divisible hereditarily co-Hopfian groups. As we have seen above, such groups must be torsion and so they have the form $G = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty))$, where each $r_p (p \in \mathbb{P})$ is finite. Since any subgroup of such a group is a direct sum of fully invariant finitely cogenerated groups, it is easily seen to be co-Hopfian. \square

We now consider the problem of determining the super co-Hopfian groups; firstly, we shall suppose that D is such a group and that D is divisible. Since any homomorphic image of a divisible group is divisible, it is an easy exercise to see that a divisible super co-Hopfian group must have the form $D = \bigoplus_{r_0} \mathbb{Q} \oplus \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty))$, where r_0 and all $r_p (p \in \mathbb{P})$ are finite. Conversely every such group is easily seen to be super co-Hopfian. We record this as

Proposition 3.3 *A divisible group D is super co-Hopfian if, and only if, D has the form $D = \bigoplus_{r_0} \mathbb{Q} \oplus \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty))$, where r_0 and all $r_p (p \in \mathbb{P})$ are finite.*

So we now consider reduced super co-Hopfian groups. Clearly, if G is such a group and G is torsion-free, then it must be the trivial group: super co-Hopfian groups are, in particular, co-Hopfian and the only co-Hopfian torsion-free groups are divisible. If G is torsion, then it has the form $G = \bigoplus_{p \in \mathbb{P}} G_p$ where each G_p is a reduced p -group. If B_p is a basic subgroup of G_p , then it is a homomorphic image of G_p and hence of G . As we have seen before, this forces B_p and G_p to be finite. Conversely every image of such a group is again of that form and hence is super

co-Hopfian. Thus the classification of super co-Hopfian groups reduces to the classification of the reduced mixed super co-Hopfian groups.

We shall also have need of the following result of A.L.S. Corner [2] - the result was not published by Corner (see [6]) but the main ideas are outlined in [7, Section 4] - which is an extension of the well-known Szele Theorem on basic subgroups as endomorphic images.

Theorem 3.4 (*Corner*) *Let G be an extension of a torsion group T by a countable torsion-free group, and let B_p be a basic subgroup of the p -component of T . Then B_p is an endomorphic image of G .*

Theorem 3.5 *A group G is super co-Hopfian group if and only if its torsion-free quotient, $G/t(G)$, is divisible of finite rank and its torsion group $t(G)$ has the form $t(G) = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty) \oplus G_p)$, where each r_p is finite and each G_p is a finite p -group.*

Proof Suppose that G has the form that $t(G) = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty) \oplus G_p)$, where each r_p is finite and each G_p is a finite p -group and its torsion-free quotient, $G/t(G)$, is divisible of finite rank, we show that G is super co-Hopfian. Assume that X is an arbitrary homomorphic image of G . Since the divisible part of G is of finite torsion-free rank and finite p -rank for each prime $p \in \mathbb{P}$, the divisible hull of G is of the form $\bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty)) \oplus \bigoplus_{r_0} \mathbb{Q}$, where r_0 and all r_p are finite. Now the divisible subgroup of X is an epimorphic image of the divisible hull of G and hence is of similar form to the divisible hull; in particular, it is co-Hopfian. To establish that X is co-Hopfian there is no loss in generality in assuming that X is reduced.

Let $\phi : G \twoheadrightarrow X$ be the epimorphism, so that $G/Y \cong X/t(X)$, where $Y = \phi^{-1}(t(X)) \geq t(G)$. However, G/Y is a homomorphic image of $G/t(G)$ and hence, since $G/t(G)$ is divisible of finite rank as we assumed, so is G/Y . Thus $X/t(X)$ is torsion-free divisible of finite rank; in particular it is co-Hopfian.

To establish our claim it is sufficient to show that $t(X)$ is co-Hopfian. Let $t(X) = X_p \oplus \bigoplus_{q \neq p} X_q$ be the primary decomposition of $t(X)$ and set $X'_p = \bigoplus_{q \neq p} X_q$. Then $X/X'_p \cong G/\phi^{-1}(X'_p)$ and note that $\phi^{-1}(X'_p) \geq G'_p = \bigoplus_{q \neq p} G_q$. Thus $G/\phi^{-1}(X'_p)$ is a homomorphic image of G/G'_p and this latter satisfies $G/G'_p/t(G)/G'_p \cong G/t(G)$. Hence G/G'_p is an extension of the finite p -group $t(G)/G'_p$ by $\mathbb{Q}^{(n)}$ for some finite n . However, such an extension necessarily splits and so $X_p \cong X/X'_p \cong G/\phi^{-1}(X'_p)$ is a homomorphic image of a group of the form $F \oplus \mathbb{Q}^{(n)}$. It follows from Lemma 3.7 below that X_p is again of the form $F_1 \oplus D_1$, where F_1 is finite and D_1 is divisible. Since we assumed that X was reduced, we have that X_p is finite and hence co-Hopfian. It follows immediately that $t(X)$ is also co-Hopfian, as required.

Now suppose that G is a super co-Hopfian group and let $G = D_t \oplus G_0$, where D_t is the maximal divisible subgroup of $T(G)$; note that G_0 is then a group with a reduced torsion subgroup. Now

the divisible part, D_t , of $t(G)$ is super co-Hopfian; thus, by Proposition 3.3 D_t is of the form $D = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty))$, where each r_p is finite. On the other hand, $G/t(G) \cong G_0/t(G_0)$, as an epimorphic image of G , is super co-Hopfian; thus, $G_0/t(G_0)$ is divisible of finite rank. Hence G_0 is a countable torsion-free extension of the reduced torsion group $t(G_0)$ and so by Corner's Theorem 3.4, any basic subgroup B_p of the p -component G_p of $t(G_0)$, is an endomorphic image of G_0 and consequently must be a co-Hopfian group; since it is also a direct sum of cyclic groups, it is finite. This in turn implies that G_p is finite and hence $t(G_0) = \bigoplus_{p \in \mathbb{P}} G_p$ is a direct sum of finite groups, as required. \square

Some specific cases are recorded as:

Corollary 3.6 *Let G be a super co-Hopfian group then*

- (i) *if G is torsion-free, it has the form $G = \bigoplus_{r_0} \mathbb{Q}$, where r_0 is finite;*
- (ii) *if G is torsion, it has the form $G = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty) \oplus G_p)$, where each r_p is finite and each G_p is a finite p -group;*
- (iii) *if G is splitting mixed, it has the form $G = \bigoplus_{r_0} \mathbb{Q} \oplus \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty) \oplus G_p)$, where r_0 and each r_p are finite and each G_p is a finite p -group;*

Conversely, every group of the form (i) - (iii) is super co-Hopfian.

Lemma 3.7 *Every homomorphic image of a group of the form $F \oplus D$, where F is finite and D is divisible of finite rank, is again of the form $F_1 \oplus D_1$, where F_1 is finite and D_1 is divisible of finite rank.*

Proof Suppose that G is of the desired form and $\phi : G \rightarrow X$ is an epimorphism. Then $D\phi$ is a divisible subgroup of X and so $X = D\phi \oplus X_1$. But then $X_1 \cong X/D\phi = G\phi/D\phi \cong G/(D + \text{Ker}\phi)$ and, since $G/(D + \text{Ker}\phi)$ is a homomorphic image of G/D , it is finite. Thus $X = D_1 \oplus F_1$ where $F_1 \cong X_1$ is finite and $D_1 \cong D\phi$ is divisible of finite rank. \square

We now turn our attention to hereditarily Hopfian groups.

Theorem 3.8 *A group G is hereditarily Hopfian if and only if it is an extension of $\bigoplus_{p \in \mathbb{P}} G_p$ by a torsion-free finite rank group, where each G_p is a finite p -group; in other words, G is hereditarily Hopfian if and only if, the torsion subgroup, $t(G)$, has the form $\bigoplus_{p \in \mathbb{P}} G_p$, and the torsion-free quotient $G/t(G)$ is of finite rank.*

Proof Suppose that G is a hereditarily Hopfian group, then its divisible part D , is also hereditarily Hopfian; then D must have the form $D = \bigoplus_{p \in \mathbb{P}} (\bigoplus_{r_p} \mathbb{Z}(p^\infty))$. However, as the groups $\mathbb{Z}(p^\infty)$ are never Hopfian, so we deduce that the torsion subgroup $t(G)$ is reduced. Moreover, a basic subgroup

B_p of G_p - the p -component of $t(G)$ - is also Hopfian, so it must be finite, then so is G_p . Therefore, $t(G)$, has the form $\bigoplus_{p \in \mathbb{P}} G_p$, where each G_p is a finite p -group.

Choose an independent system L which contains only torsion-free elements and which is maximal with respect to this property; so the subgroup $\langle L \rangle$ generated by L is a free group of rank the cardinality of L . Since G is hereditarily Hopfian, so is $\langle L \rangle$; which implies $\langle L \rangle$ is of finite rank. Hence the torsion-free rank of G , which is equal to the cardinality of L , is finite. Thus the torsion-free quotient, $G/t(G)$ is of finite rank.

Conversely, if G has the conditions stated in the theorem, we show that it is a hereditarily Hopfian group. For an arbitrary subgroup H , we show that it is Hopfian. As the torsion subgroup $t(H)$ of H is exactly $t(G) \cap H$, so $t(H)$ has the form $\bigoplus_{p \in \mathbb{P}} H_p$ where each H_p is a finite p -group; in particular, it is a Hopfian group. Furthermore, $H/t(H) = H/(t(G) \cap H) \cong (t(G) + H)/t(G)$ which is a subgroup of $G/t(G)$ and so we see that $H/t(H)$ is torsion-free of finite rank, in particular, it is Hopfian. By (i) in Proposition 2.2, H is a Hopfian group. \square

We can obtain more specific information in some cases:

Corollary 3.9 *Let G be a hereditarily Hopfian group. Then*

- (i) *if G is torsion-free, then G is of finite rank;*
- (ii) *if G is torsion, then $G \cong \bigoplus_{p \in \mathbb{P}} G_p$, where each G_p is a finite p -group;*
- (iii) *if G is splitting mixed, then $G \cong F \oplus \bigoplus_{p \in \mathbb{P}} G_p$, where F is torsion-free of finite rank and each G_p is a finite p -group;*

Conversely, every group of the form (i) - (iii) is hereditarily Hopfian.

We now consider super Hopfian groups. Our first observation is elementary:

Proposition 3.10 *A super Hopfian group has no non-trivial divisible epimorphic image; in particular, it is reduced and if it is torsion-free, then it is of finite rank.*

Proof Since an epimorphic image of a super Hopfian group is super Hopfian, it suffices for the first part to show a super Hopfian group is reduced. So suppose that G is super Hopfian and that the divisible part of G is non-trivial. Since a super Hopfian group is necessarily Hopfian, it follows that the divisible part of G must be torsion-free. However, since \mathbb{Q} has the quasi-cyclic groups $\mathbb{Z}(p^\infty)$ as homomorphic images, this would imply that each $\mathbb{Z}(p^\infty)$ is Hopfian - contradiction. Thus a super Hopfian group must be reduced.

The final remark follows from Reid's work [13]: a torsion-free group which has no non-zero divisible epimorphic image is necessarily of finite rank. \square

A super Hopfian p -group G is easily seen to be finite: a basic subgroup of G is, by Szele's theorem, an epimorphic image of G and hence must be simultaneously Hopfian and a direct sum of cyclic groups, and as we have seen, this forces G to be finite. It follows immediately that a torsion super Hopfian group is a direct sum of finite p -groups ($p \in \mathbb{P}$).

We are now in a position to establish the classification of super Hopfian groups.

Theorem 3.11 *A group G is super Hopfian if and only if it is an extension of a free group of finite rank by a torsion group which is a direct sum of finite primary components; in other words, G is super Hopfian if and only if there exists a finite rank free subgroup F of G such that the quotient group G/F is torsion and has a primary decomposition with finite primary components.*

Proof For the necessity, suppose that G is super Hopfian. Then its torsion-free quotient, $G/t(G)$, is also super Hopfian and by Proposition 3.10 must be of finite rank. Therefore, there is a free subgroup F of G having finite rank such that G/F is torsion. Again, G/F is super Hopfian, thus it is a direct sum of finite primary components as observed above.

For the sufficiency, suppose that there exists a finite rank free subgroup F of G such that the quotient group G/F is a direct sum of finite primary components. First we note that the torsion subgroup, $t(G)$, of G , is also a direct sum of finite primary components since we have the isomorphism $(t(G) + F)/F \cong t(G)$. The torsion-free quotient, $G/t(G)$ is of finite rank, since it is an extension of the free group $(F \oplus t(G))/t(G)$ by a torsion group.

Suppose that $X = G\phi$ is a homomorphic image of G for a given homomorphism ϕ . Now $X/t(X)$ is a homomorphic image of $G/t(G)$ and so $X/t(X)$ is a group of finite rank. We need to show that X is a Hopfian group. It suffices, by Proposition 2.2 (i), to show that $t(X)$ is Hopfian. We show that for any $p \in \mathbb{P}$, the p -component, X_p of $t(X)$, is finite. So in this case, $t(X)$ is a direct sum of finite primary components and so is Hopfian.

The group $X/F\phi$ is an epimorphic image of G/F and hence is a torsion group with finite primary components. Moreover, as F is free of finite rank, the group $t(X) \cap F\phi$ is both torsion and finitely generated, whence it is finite. Also we have that $X_p/(X_p \cap F\phi)$ is a finite p -group since it is isomorphic to a subgroup of $X/F\phi$. Since $(X_p \cap F\phi) \leq (t(X) \cap F\phi)$, it is also finite and so X_p is finite, as required. □

Finally observe that, unlike Hopfian groups which may have arbitrarily large cardinality, super Hopfian groups are necessarily countable.

References

- [1] R. Baer, Groups without Proper Isomorphic Quotient Groups, *Bull. Amer. Math. Soc.* **50** (1944) 267 - 278.
- [2] A. L. S. Corner, *Extensions of Torsion Groups by Countable Torsion-free Groups*, unpublished manuscript [U14] listed in [6]; to appear at <http://arrow.dit.ie/cgi/siteview.cgi/corner>
- [3] P. Crawley, An infinite primary group without proper isomorphic subgroups, *Bull. Amer. Math. Soc.* **68** (1962) 463 - 467.
- [4] D. Dikranian, B. Goldsmith, L. Salce and P. Zanardo, Algebraic entropy for Abelian groups, *Trans Amer. Math. Soc.* **361** (2009) 3401 - 3434.
- [5] L. Fuchs, *Infinite Abelian Groups*, Vol. I and II, Academic Press, 1970 and 1973.
- [6] B. Goldsmith *Anthony Leonard Southern Corner 1934-2006*, in *Models, Modules and Abelian Groups*, (editors R. Göbel and B. Goldsmith) Walter de Gruyter, Berlin (2008), pp 1 - 7.
- [7] B. Goldsmith and K. Gong, On adjoint entropy of Abelian groups, *Comm. Algebra* **40**(3) (2012) 972 - 987.
- [8] B. Goldsmith and K. Gong, A note on Hopfian and co-Hopfian Abelian groups, *to appear*.
- [9] K. Gong, *Entropy in Abelian Groups*, Ph.D Thesis, Dublin Institute of Technology, to be examined in Spring, 2012.
- [10] R. Hirshon *Some Theorems on Hopficity*, *Trans. Amer. Math. Soc.* Vol. **141**, (Jul., 1969) 229-244.
- [11] R. Hirshon, Misbehaved Direct Products, *Expo. Math.* **20** (2002) 365 - 374.
- [12] R. S. Pierce, Homomorphisms of primary Abelian groups, in *Topics in Abelian Groups*, *Scott Foresman* (1963), pp 215 - 310.
- [13] J. D. Reid, A note on torsion-free Abelian groups of infinite rank, *Proc. Amer. Math. Soc.* **13** (1962) 222 - 225.