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R. Gobel
Fachbereich 6, Mathematik und Informatik, Universiriit Essen GM, D-451 I7 Essen I, Germany

Brendan Goldsmith
Technological University Dublin, brendan.goldsmith@tudublin.ie

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The Kaplansky Test Problems – An approach via radicals

R. Göbel[a,*,1], B. Goldsmith[b,c,1]

[a] Fachbereich 6, Mathematik und Informatik, Universität Essen GHS, D-45117 Essen 1, Germany
[b] Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
[c] Dublin Institute for Advanced Studies, Dublin 4, Ireland

Dedicated to the memory of Alan H. Mekler who made so many important contributions to this subject

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Abstract

The existence of non-free, κ-free Abelian groups and modules (over some non-left perfect rings R) having prescribed endomorphism algebra is established within ZFC + $\diamond$ set theory. The principal technique used exploits free resolutions of non-free R-modules X and is similar to that used previously by Griffith and Eklof; much stronger results than have been obtained heretofore are obtained by coding additional information into the module X. As a consequence we can show, inter alia, that the Kaplansky Test Problems have negative answers for strongly $\aleph_1$-free Abelian groups of cardinality $\aleph_1$ in ZFC and assuming the weak Continuum Hypothesis.

1. Introduction

In problem 10 of [11], Fuchs asked for which cardinals κ are there Abelian groups which are κ-free but not free. This problem has attracted considerable interest and much insight has been obtained by a number of authors working in both ZFC set theory and other stronger formulations such as ZFC + (V = L) (see e.g. [6, 8, 9, 11, 15, 19]). Surprisingly from the viewpoint of an algebraist, few additional algebraic properties have been obtained for these "almost-free' groups when working in ZFC set theory; in [8] the additional algebraic property of having trivial dual was obtained for $\aleph_1$-free groups. The present work is based on an idea of Griffith [15] which also has been exploited by Eklof in [9]. The central idea in [15] and [9] is to use free resolutions of a non-free group X to code this non-freeness into an almost-free group. In this paper we use the same idea for modules but code additional information into

* Corresponding author.

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the module X (e.g. prescribing its endomorphism algebra) and see how this is reflected in the outcoming almost-free module. The success of such an approach depends on how much additional set theory one wants to assume but it is worth noting that in ZFC + CH we are able to show that the Kaplansky Test Problems [18, p. 12] fail for strongly $\aleph_1$-free groups of cardinality $\aleph_1$. Recall that the notions of κ-freeness and strong κ-freeness of a module M are defined in terms of properties of certain subsets of the power set of M; see [10, Chapter IV]. Since we shall always be dealing with these notions in a context where κ is a regular infinite cardinal and the module M is generated by a set of cardinality at most κ, it is convenient to use the following known equivalents [10, IV 1.5, 1.11] of the usual definitions: if A is any ring, then a $\leq \kappa$-generated A-module M is κ-free if and only if M has a κ-filtration $\{M_\nu \mid \nu < \kappa\}$ consisting of free modules. Similarly M is strongly κ-free if it has a filtration $\{M_\nu \mid \nu < \kappa\}$ such that for all $\mu < \nu$, $M_{\nu+1}$, and $M_{\nu+1}/M_{\mu+1}$ are free; if $\kappa > |A|$, then this condition is also necessary. It is worth remarking that we shall always construct modules M which are simultaneously κ-free and strongly κ-free. In the situation where $A = \mathbb{Z}$, the ring of integers, the above notions reduced to the familiar: M is κ-free if every subgroup of cardinality $< \kappa$ is free and M is strongly κ-free if it is κ-free and every subgroup of cardinality $< \kappa$ is contained in a free subgroup U, of cardinality $< \kappa$, with G/U κ-free.

In order to give a precise statement of our main results we fix some terminology and notation. Let R be a fixed commutative ring having a fixed multiplicatively closed countable subset S of non-zero divisors such that $0 \neq 1 \in S$ and $\bigcap_{s \in S} sR = 0$. An R-module M is S-reduced if $\bigcap_{s \in S} Ms = 0$ and M is S-torsion-free if $ms = 0$ implies $m = 0$ for $m \in M$, $s \in S$. We enumerate the non-units in S as $s_1, s_2, \ldots$ and define $q_n \in S$ by $q_n = \prod_{i \leq n} s_i$. Since S is fixed we shall normally omit the prefix S. In particular we shall always assume that R viewed as an R-module R_R is reduced and torsion-free. We remark that over a field, or more generally any Artinian ring, a non-zero R-module cannot be both reduced and torsion-free for any S. In particular our results are vacuous if R is a left perfect ring (see [10, p. 89]). We normally attribute properties to an R-algebra A if the corresponding R-module A_R possesses them. The word group shall always denote an additively written Abelian group.

In ZFC + CH the most striking and simplest to state of our results is a special case of Theorem 5.1.

Corollary 5.2. *For any positive integer r, there exists a strongly $\aleph_1$-free group G of cardinality $\aleph_1$ such that $G^{(m)} \cong G^{(n)}$ if and only if $m \equiv n \bmod r$.*

By making additional set-theoretic assumptions we can strengthen such a result and so we also derive, under the Weak Continuum Hypothesis $2^{\aleph_0} < 2^{\aleph_1}$, a result most easily stated for groups as follows:

It follows that the usual pathologies of direct decomposition, which defeat the Kaplansky Test Problems for torsion-free groups, persist in this class of strongly $\aleph_n$-free groups of power $\aleph_n$.

It seems worthwhile to make some remarks about another novel feature of our construction: the use of radicals. Similar algebraic results in V = L are obtained as a consequence of algebraic step-lemmas involving countable chains of summands and exploit the "room" between a free module and its natural (adic) completion. They are heavily dependent on the validity of the axiom $E(\kappa)$ (see [10, p. 154]), for κ not weakly compact, in the universe V = L. Since we have not made this assumption, we are forced to deal with chains of summands of uncountable length. However, the reader can easily convince herself that the topological arguments used to handle this situation will breakdown. To circumvent this, we have resorted to homological methods and in particular have exploited the so-called free radical: recall that, if H is an R-module and $\mathscr{X}$ is any class of R-modules, then the $\mathscr{X}$-radical of H is given by $R_{\mathscr{X}}(H) = \bigcap \{U \leq H \mid H/U \in \mathscr{X}\}$. In particular if $\mathscr{X}$ is the class of free R-modules, then we obtain a radical which we denote $R_R(H)$. It is easy to see that this is equivalent to $R_R(H) = \bigcap \{\operatorname{Ker} \eta \mid \eta \in \operatorname{Hom}_R(H, R)\}$ for PIDs R.

We note that such a radical is a subfunctor of the identity and if $\varphi : H \to H$ is an R-homomorphism, then $(R_R(H))\varphi \leq R_R(H)$.

Further details of such radicals may be found in [7]. Our construction of a κ-free, strongly κ-free R-module F exhibits $F = \bigcup_{\alpha < \kappa} F_\alpha$ in such a way that $F_{\alpha+1}$ is "frequently" the radical closure of F_α: $R_R(F_\beta/F_\alpha) = F_{\alpha+1}/F_\alpha$ for all $\beta > \alpha$.

We close this introduction by noting that standard algebraic terminology may be found in [11] while [17] and [10] provide the necessary set-theoretical background. Our terminology and notation are largely in accord with these references but we note that maps are written here on the right and the symbol $\sqsubset$ is used to denote a direct summand. Finally we note that appending a set-theoretical statement (e.g. $2^{\aleph_0} < 2^{\aleph_1}$) to a theorem, lemma, etc., indicates that we are assuming that this statement holds in the proof of the result.

2. Preliminaries

A subset C of an ordinal α is called a cub in α if C is closed and unbounded in α (in the order topology on α). A subset E of α is said to be stationary in α if $E \cap C \neq \emptyset$ for all cubs C in α; a subset E of α is non-reflecting in α if for all limit ordinals $\mu < \alpha$, with $\operatorname{cf} \mu > \omega$, there is a cub C in μ with $E \cap C = \emptyset$. We shall make use of the following observation of R. Solovay which is proved in [9, p. 75] (see also [10, p. 37]).

Lemma 2.1. *Let κ be a regular cardinal, $\lambda = \kappa^+$ and $\lambda^0 = \{\rho < \lambda \mid \operatorname{cf} \rho = \kappa\}$. Then λ^0 is stationary and non-reflecting in λ.* □

The following proposition is presumably well known.

Proposition 2.2. *Let A be any ring and κ a regular uncountable cardinal. If $X = \bigcup_{\alpha < \kappa} X_\alpha$ is the union of a smooth chain of A-modules X_α, each free of rank $< \kappa$,* *then there exists a free A-module resolution* $0 \to \bar{K} \to \bar{F} \to X \to 0$ *(i.e.* $\bar{K}$, $\bar{F}$ *are free A-modules).*

Proof. Let $\{x_{\alpha i} \mid i < \lambda_\alpha < \kappa\}$ be a system of generators of X_α. Set $F = \bigoplus_{\alpha < \kappa} \bigoplus_{i < \lambda_\alpha} e_{\alpha i} A$ so that F is a free A-module of rank κ. Then the mapping $\theta : (e_{\alpha i} \to x_{\alpha i})$ induces an exact sequence $F \to X \to 0$ of A-modules. Let K be the kernel of this epimorphism, so that $0 \to K \to F \to X \to 0$ is exact. For each $\beta < \kappa$, set $F_\beta = \bigoplus_{\alpha \le \beta} \bigoplus_{i < \lambda_\alpha} e_{\alpha i} A$ and set $K_\beta = F_\beta \cap K$. Then $F_\beta / K_\beta = F_\beta / (K \cap F_\beta)$ is isomorphic to the image of θ restricted to F_β and so $F_\beta / K_\beta \cong X_\beta$, which is free. Thus we conclude $K_\beta \sqsubset F_\beta$ and so K_β is projective.

Moreover, if $\sigma > \beta$, $\sigma \in C$, then $F_\sigma / K_\beta / K_\sigma / K_\beta \cong F_\sigma / K_\sigma \cong X_\sigma$, is free and so K_σ / K_β is also projective. Thus $K_\beta \sqsubset K_\sigma$ and since $K = \bigcup_{\beta < \kappa} K_\beta$, it follows that K is a direct sum of projectives and hence projective. However, it follows from Lemma 2.3 below that there exists a free A-module L such that $K \oplus L$ is free of rank κ. Set $\bar{K} = K \oplus L$, $\bar{F} = F \oplus L$ and observe that $0 \to \bar{K} \to \bar{F} \to X \to 0$ is exact and $\bar{K}$, $\bar{F}$ are free A-modules. □

Lemma 2.3. *If K is a direct summand of a free A-module F of infinite rank* λ, *then there exists a free A-module L such that* $K \oplus L$ *is free of rank* λ.

Proof. By assumption $K \oplus N = F$. Set $L = \bigoplus_{\aleph_0} F$; clearly L is a free A-module of rank λ. Moreover, $K \oplus L = K \oplus \bigoplus_{\aleph_0} (K \oplus N) \cong L$ which is free of rank λ. □

Lemma 2.4. *Let A be any ring and* κ *be a regular uncountable cardinal. Let* $0 \to B \to H \to X \to 0$ *be a free resolution of the A-module X, where* $B \to H$ *denotes the identity map, and suppose X has a* κ*-filtration by free A-modules* X_α. *Then if*

$$H = \bigoplus_{\nu < \kappa} A_\nu, \quad A_\nu \cong A, \quad H_\alpha = \bigoplus_{\nu < \alpha} A_\nu \quad \text{and} \quad B_\alpha = B \cap H_\alpha,$$

there exists a cub C in κ *such that* H/B_α *is free and* B_α *is a free summand of B with* B/B_α *free for all* $\alpha \in C$.

Proof. $H/B_\alpha = H_\alpha/B_\alpha \oplus \bigoplus_{\nu \ge \alpha} A_\nu$ and so if we show H_α/B_α is free, then H/B_α is also free. Now $H_\alpha/B_\alpha = H_\alpha/(H_\alpha \cap B) \cong (H_\alpha + B)/B$. Then, as observed in the proof of Proposition 2.2, there is a cub C_0 in κ with $(H_\alpha + B)/B = X_\alpha$ for all $\alpha \in C_0$. Thus H_α/B_α and H/B_α are free for all $\alpha \in C_0$. However B is also free, $B = \bigoplus_{\nu < \kappa} A'_\nu$, $A'_\nu \cong A$ and so B has its own natural filtration by free modules via this decomposition. In addition $B = \bigcup B_\alpha$ is a κ-filtration of B and so there is a cub C_1 with B_α a free canonical summand for each $\alpha \in C_1$. Set $C = C_0 \cap C_1$; C is then a cub and H/B_α is free and B_α is a free canonical (i.e. B/B_α is free) summand of B, for each $\alpha \in C$. □

Lemma 2.5. *Suppose A is a unital R-algebra with* A_R *free and* $X = \bigcup_{n < \omega} X_n$ *is a countable A-module with* X_n *a free A-module for each* $n < \omega$. *Then there exists a free*

A-module resolution $0 \to B \to H \to X \to 0$ and A-submodules B_n of B such that $B = \bigcup B_n$, B_n is A-free, B_{n+1}/B_n is free for all n and H/B_n is free.

Proof. Our proof is based on Lemma 1.4(XII) in [10]. Choose a free resolution $0 \to K_0 \to F_0 \xrightarrow{\varphi_0} X_1 \to 0$; this is trivial since X_1 is free. Suppose F_{n-1}, K_{n-1}, and φ_{n-1} have been defined so that φ_i is an extension of φ_j if $j < i \leq n-1$ and we have

$$0 \to \bigoplus_{i=0}^{n-1} K_i \to \bigoplus_{i=0}^{n-1} F_i \to X_n \to 0 \quad \text{with } K_i, F_i \text{ free.}$$

Now choose F_n isomorphic to X_{n+1}; F_n is free since by assumption X_{n+1} is free. Let ψ_n be the isomorphism. Define $\varphi_n : \bigoplus_{i=0}^{n} F_i \to X_{n+1}$ by $\varphi_n |_{\bigoplus_{i=0}^{n-1} F_i} = \varphi_{n-1}$, $\varphi_n |_{F_n} = \psi_n$.

Let $\{b_i \mid i \in I\}$ be a set of free generators of $X_n \psi_n^{-1}$ in F_n and choose $\{f_i \mid i \in I\} \subseteq \bigoplus_{i=0}^{n-1} F_i$ such that $b_i \psi_n = f_i \varphi_n$. Set $K_n = \langle (b_i - f_i) \mid i \in I \rangle$, a free A-module. Clearly we have

$$0 \to \bigoplus_{i=0}^{n} K_i \to \bigoplus_{i=0}^{n} F_i \xrightarrow{\varphi_n} X_{n+1} \to 0$$

and φ_n extends φ_{n-1}.

Set $B_n = \bigoplus_{i=0}^{n} K_i$ so that $B = \bigoplus_{i<\omega} K_i = \bigcup B_n$ and set $H = \bigoplus_{i<\omega} F_i$. Clearly B_n, B, H and B_{n+1}/B_n are free A-modules and $H/B_n = \bigoplus_{i<\omega} F_i / \bigoplus_{i \leq n} K_i \cong X_{n+1} \oplus \bigoplus_{i>n} F_i$ is a free A-module. □

The following simple observations shall be of use in the sequel.

Observation 2.6. *Suppose A is a free R-module and $0 \neq L$ is a direct summand of a free A-module, then $L^* = \mathrm{Hom}_R(L, R) \neq 0$.*

Proof. Since $L \oplus C = \bigoplus_{i \in I} e_i A$ for some A-module C, we find a projection $\pi : L \oplus C \to e_i A$ with $\eta = \pi|_L \neq 0$. However $A = \oplus R$ is a free R-module and so η can be extended to an R-homomorphism $\eta' : L \to R$ which is non-trivial. □

Observation 2.7. *Let Y be an R-module having endomorphism algebra* $\mathrm{End}_R Y = A$, *and suppose A is free as an R-module. If*

(i) *$A = R$ and Y contains a copy of $R \oplus R$* or

(ii) *A has R-rank at least 2 and Y contains a copy of A,*

then $Y^* = \mathrm{Hom}_R(Y, R) = 0$.

Proof. In either case Y contains a free R-submodule $e_0 R \oplus e_1 R$; in case (ii) we may further assume that $e_0 R \oplus e_1 R \leq e_0 A$. Any $\phi \in \mathrm{Hom}_R(Y, R)$ can be viewed as an endomorphism $\hat{\phi}$ of Y by $y\hat{\phi} = e_0(y\phi)$ for all $y \in Y$. Thus $\hat{\phi}$ is scalar multiplication by some $a \in A$. Thus

$$e_0 a = e_0 \hat{\phi} = e_0(e_0 \phi) \in e_0 R$$

and so $a \in R$ since $\mathrm{Ann}_A e_0 = 0$ in either case. Moreover,

$$e_1 a = e_1 \hat{\phi} = e_0(e_1 \phi) \in e_1 R \cap e_0 R = 0,$$

which forces $a = 0$. But then $\phi = 0$ follows immediately. $\square$

The following definition will simplify our terminology.

Definition. A subset $D \subseteq \bigcup_{\nu < \alpha} F_\nu$ is called *unbounded* in $\bigcup_{\nu < \alpha} F_\nu$ if $D \nsubseteq F_\nu$ for all $\nu < \alpha$.

Lemma 2.8. *Suppose A is an R-algebra which is free as an R-module and $|A| < \kappa$, an infinite cardinal. If $F_\alpha = \bigoplus_{i<\alpha} e_i A$ and $F = \bigcup_{\alpha<\kappa} F_\alpha$ is a free A-module of rank κ and $h: F \to R$ is an R-homomorphism, then there exists a free A-summand D of F with $Dh = 0$, F/D A-free and D unbounded in $\bigcup_{\alpha<\kappa} F_\alpha$.*

Proof. Since A is R-free, F is a free R-module and so h vanishes on an R-summand K of corank 1; $F = xR \oplus K$, where $Kh = 0$. Then for each $a \in A$, $i < \kappa$ we have that $e_i a = x r_a^i + k_a^i$ with $r_a^i \in R$ and $k_a^i \in K$. Let J_a denote the set of all $i \in \kappa$ for which $r_a^i \neq r_a^i$; clearly $|J_a| \leq |R|$. Set $J = \bigcup_{a \in A} J_a$ and observe that $|J| \leq |A| < \kappa$. Hence, if $I = \kappa \backslash J$, then I is cofinal in κ. Moreover if $i,j \in I$, then $e_i a - e_j a \in K$ for all $a \in A$ and so $(e_i - e_j)\ A\ h = 0$ for all $i,j \in I$. Fix $i \in I$ and set $D = \bigoplus_{j \in I} (e_i - e_j)\ A$ which is the desired summand. $\square$

3. Free resolutions of modules with trivial duals

If X is an R-module, then we define the dual of X, X^* to be $\mathrm{Hom}_R(X, R)$. We say that X has trivial dual if $X^* = 0$. Suppose that A is a unital R-algebra. In the sequel we shall often need to refer to A-modules having some special properties and so we separate out the following condition:

An A-module X of cardinality κ satisfies condition $(*)$ provided:

X has trivial dual and there is a free A-module resolution.

$0 \to B \to H \to X \to 0$ where $B \to H$ is the identity map,

$B = \bigcup_{i<\kappa} B_i$ and B_{i+1}/B_i is free for all i.

We remark that condition $(*)$ is a strengthening of the condition $\mathrm{F}(\kappa)$ of Eklof and Mekler [10, p. 188].

Observation 3.0. *Observe that a κ-free A-module X of cardinality κ satisfies $(*)$ if κ is uncountable and $\mathrm{End}_R X = A$. Moreover, $(*)$ also holds for countable A-modules $X = \bigcup_{n<\omega} X_n$ where each X_n is free and $\mathrm{End}_R X = A$.*

(Apply Lemma 2.4 and Observation 2.7 if κ is uncountable; in the countable case apply Lemma 2.5 and Observation 2.7.)

The following lemma plays a role analogous to a step-lemma in V = L (see e.g. [13]) and will be a vital ingredient in our construction in Section 4. We state it in a slightly more general form than will be needed for the purposes of this paper.

Step-Lemma 3.1. *Let κ be a regular uncountable cardinal and X an A-module satisfying $(*)$ and having a free A-submodule G' with X and X/G' κ-free A-modules. Let $F = \bigcup_{\nu<\kappa} F_\nu$ be a free A-module of rank κ with a κ-filtration by a chain of A-summands F_ν such that F_β/F_ν is free for $\beta > \nu$. If $F = G \oplus D$ where G is isomorphic to G' and $D \not\leq F_\nu$ for any $\nu < \kappa$, then there exists a free A-module $H > F$ such that*

(i) *$H/D \cong X$, $H/F \cong X/G'$,*

(ii) *H/F_ν is free for all $\nu < \kappa$,*

(iii) *if M is a summand of H containing D then $M = H$.*

Proof. Since X satisfies $(*)$, there is a free A-module resolution $0 \to K \to H \to X \to 0$ with H, K free of rank κ.

Since $G' \leq X \cong H/K$, we can write $G' = H'/K$; moreover, $H' = K \oplus \tilde{G}$ for some $\tilde{G} \leq H$ with $\tilde{G} \cong G'$ as G' is free. Note that

(a) H/K and $H/(K \oplus \tilde{G})$ are both κ-free

since the former is isomorphic to X while the latter is isomorphic to X/G'.

Since H is free, we can write $H = \bigoplus_{i<\kappa} e_i A$ and set $H_\nu = \bigoplus_{i<\nu} e_i A$; clearly H/H_ν is free. Now set $(K \oplus \tilde{G})_\nu = (K \oplus \tilde{G}) \cap H_\nu$, $K_\nu = K \cap H_\nu$ and $\tilde{G}_\nu = \tilde{G} \cap H_\nu$. We claim there is a cub C' in κ such that

(b) $(K \oplus \tilde{G})_\nu = K_\nu \oplus \tilde{G}_\nu$ and

(c) $H/(K \oplus \tilde{G})_\nu$, H/K_ν are free for all $\nu \in C'$.

To see this consider the following filtrations: $K = \bigcup_{\nu<\kappa} K_\nu$, $\tilde{G} = \bigcup_{\nu<\kappa} \tilde{G}_\nu$, $K \oplus \tilde{G} = \bigcup_{\nu<\kappa} (K \oplus \tilde{G})_\nu$ and $K \oplus \tilde{G} = \bigcup_{\nu<\kappa} K_\nu \oplus \tilde{G}_\nu$. Thus we have two κ-filtrations of the module $K \oplus \tilde{G}$ and since κ is regular uncountable, it follows from [10, Lemma IV 1.4] that there is a cub C'' such that (b) holds. A similar argument shows that $\tilde{G}_\nu$ is free for all ν in some cub $\subseteq C''$; we continue to call this C''.

To establish (c) note firstly that it follows from condition $(*)$ that H/K_ν is free. Moreover, $H/(K_\nu \oplus \tilde{G}_\nu) = H_\nu/(K_\nu \oplus \tilde{G}_\nu) \oplus \bigoplus_{i \geq \nu} e_i A$. However $H_\nu/(K_\nu \oplus \tilde{G}_\nu) = H_\nu/(K \oplus \tilde{G}) \cap H_\nu \cong H_\nu + (K \oplus \tilde{G})/(K \oplus \tilde{G})$ and $\bigcup_{\nu<\kappa} [H_\nu + (K \oplus \tilde{G})/(K \oplus \tilde{G})]$ is a κ-filtration of the κ-free module $H/(K \oplus \tilde{G})$ ($\cong X/G'$). Since X/G' has a κ-filtration by free modules, there exists a cub C_0 for which $H_\nu + (K \oplus \tilde{G})/(K \oplus \tilde{G})$ is free for all $\nu \in C_0$. But then $H_\nu/(K_\nu \oplus \tilde{G}_\nu)$ is free for all $\nu \in C_0$ and so $H/(K_\nu \oplus \tilde{G}_\nu)$ is free also. Set $C' = C_0 \cap C''$, a cub in κ; clearly (b) and (c) hold for all $\nu \in C'$.

Now set $G_\nu = G \cap F_\nu$, $D_\nu = D \cap F_\nu$ so that $G = \bigcup G_\nu$, $D = \bigcup D_\nu$. Then by a similar argument to the one used above, there is a cub C_1 such that $F_\nu = G_\nu \oplus D_\nu$, for all $\nu \in C_1$. Set $C_0 = C' \cap C_1$ which is again a cub and $F_\nu = G_\nu \oplus D_\nu$, $K_\nu \oplus \tilde{G}_\nu = (K \oplus \tilde{G})_\nu$ for all $\nu \in C_0$. Since $|G| = |\tilde{G}|$, by passing to a suitable cub C in C_0, we may assume G_ν is free, $\operatorname{rk} G_\nu = \operatorname{rk} \tilde{G}_\nu$ and $\operatorname{rk} K_\nu = \operatorname{rk} D_\nu$ for all $\nu \in C$.

Now for $\nu \in C$ we identify $G_\nu \leftrightarrow \tilde{G}_\nu$, $K_\nu \leftrightarrow D_\nu$ which gives identifications $G \leftrightarrow \tilde{G}$, $K \leftrightarrow D$ since C is a cub. Hence $G \oplus D = F \subset H$ and we show (i)–(iii) in the lemma hold.

(i) $H/D = H/K \cong X$ and $H/F = H/(G \oplus D) = H/(\tilde{G} \oplus K) \cong H/K/(\tilde{G} \oplus K/K) \cong X/G'$.

(ii) Consider any F_γ. Since C is unbounded, there is a $\nu \in C$ with $\nu \geq \gamma$ and so F_ν/F_γ is free by hypothesis on the chain F_γ. Since $\nu \in C$, $F_\nu = G_\nu \oplus D_\nu = \tilde{G}_\nu \oplus K_\nu$ which is a summand of H by (c) above. Hence $F_\gamma \sqsubset H$ as required.

(iii) Suppose $M \geq D$ and $M \oplus L = H$. Then $X \cong H/D = (M \oplus L)/D = M/D \oplus (L \oplus D)/D \cong X' \oplus L$ for some $X' \leq X$. If $L \neq 0$, then by Observation 2.6, $L^* \neq 0$ which contradicts $X^* = 0$. This forces $M = H$ as claimed. □

Corollary 3.2. *Suppose X and $G \oplus D = F = \bigcup_{\gamma < \kappa} F_\nu \subset H$ are as in Lemma 3.1 and $\operatorname{End}_R X = A$. If φ is an R-endomorphism of F which leaves D invariant and extends to an endomorphism φ of H, then there is an $a \in A$ such that $x(\varphi - a) \in D$ for all $x \in H$.*

Proof. If $\varphi \in \operatorname{End} F$ is as above and φ extends to an endomorphism of H then it induces a homomorphism $\bar{\varphi}: H/D \to H/D$. Since $H/D \cong X$ and $\operatorname{End}_R X = A$, there exists $a \in A$ such that $\bar{\varphi} - a = 0$ and so $\varphi - a: H \to D$ as required. □

4. The main theorem and radicals

Throughout this section κ is a regular cardinal and X is a fixed κ-free A-module of cardinality κ with $\operatorname{End}_R X = A$ and $X^* = 0$.

Let $\lambda = \kappa^+$ and recall from Lemma 2.1 that $\lambda^0 = \{\alpha \in \lambda \mid \operatorname{cf} \alpha = \kappa\}$ is a stationary, non-reflecting subset of λ. We remark that for the rest of this section, it would be possible to work with any stationary subset of λ^0; in particular, by suitable choices of such subsets, it would be possible to construct 2^λ non-isomorphic modules in our main theorem. It would be possible also to achieve a similar result by varying the module X over a rigid family. However, in the interest of clarity of presentation we restrict our attention to the set λ^0.

Our principal interest here is in the construction of almost-free counter-examples to Kaplansky's Test Problems without assuming V = L. It is well known that, for this purpose, it suffices to realize suitable algebras as endomorphism algebras modulo a two-sided ideal (cf. [5]). With this in mind we now define a suitable ideal in $\mathrm{End}_R F$, where F is a strongly λ-free A-module having a filtration $F = \bigcup_{\nu < \lambda} F_\nu$ by free A-modules F_ν with $F_{\nu+1}/F_\nu = X_\nu \cong X$ for all $\nu \in \lambda^0$. We shall call such a filtration, the X-filtration of F.

While we are able to replace this ideal by 0 under $\diamondsuit$ in this paper, we suspect that it will be needed for a result in ZFC. Because of results under $\neg$CH we know that some ideal $\neq 0$ is definitely necessary.

Before we give the appropriate definition, we derive a simple result which, nonetheless, plays a crucial role in our determination of endomorphism algebras. Recall that the free radical of an R-module Y is $R_R(Y) = \bigcap \{U \leq Y \mid Y/U \text{ is a free } R\text{-module}\}$.

Lemma 4.1. *If F is an A-module with a given X-filtration and $\nu \in \lambda^0$, then if A is free as an R-module, $R_B(F_\beta/F_\nu) = F_{\nu+1}/F_\nu$ for all $\beta > \nu$ and $R_R(F_\beta/F_\nu)$ is fully invariant under R-homomorphisms of F_β/F_ν.*

Proof. Since $F_\beta/F_\nu/F_{\nu+1}/F_\nu \cong F_\beta/F_{\nu+1}$ and $\nu+1$ is a successor, it follows that $F_\beta/F_{\nu+1}$ is a free A-module and hence a free R-module since A is free as an R-module. It follows easily that $R_R(F_\beta/F_\nu) \leq F_{\nu+1}/F_\nu$. Moreover we may write $F_\beta = F_{\nu+1} \oplus C$, where C is a free R-module and so, if $U/F_\nu \leq F_{\nu+1}/F_\nu$, then $F_\beta/U \cong F_{\nu+1}/U \oplus C$. However, $F_{\nu+1}/U$ is an epimorphic image of $X_\nu = F_{\nu+1}/F_\nu$ and so has trivial dual. But now if $V/F_\nu = R_R(F_\beta/F_\nu)$, it follows from the definition of $R_R(F_\beta/F_\nu)$ as an intersection that F_β/V is isomorphic to a cartesian product of free modules. Hence F_β/V will have non-trivial dual. Moreover by a simple modification of Observation 2.6 the same is true of any non-trivial summand of it. However, if $V \leq F_{\nu+1}$ then as shown above taking $U = V$, we have that $F_{\nu+1}/V$ is a direct summand of F_β/V having trivial dual. This can happen only if $F_{\nu+1}/V = 0$ as required. Finally the invariance of $R_R(F_\beta/F_\nu)$ follows from the fact that such a radical is a subfunctor of the identity. $\square$

Definition 4.2. Suppose F is a λ-free, strongly λ-free A-module with an X-filtration $F = \bigcup_{\nu < \lambda} F_\nu$. An R-endomorphism σ of F is said to be *inessential* if the set

$$S_\sigma = \{\alpha \in \lambda^0 \mid F_\alpha \sigma \leq F_\alpha \text{ and the induced map } \sigma^\alpha : F/F_\alpha \to F/F_\alpha \text{ is zero on } X_\alpha\}$$

is a cub in λ^0.

We denote the set of inessential endomorphisms of F by Ines F. The second half of the following proposition is crucial.

Proposition 4.3. *Let F be as in Definition* 4.2. *Then the following hold:*

(i) Ines F *is filtration independent,*

(ii) Ines F *is an ideal in* $\mathrm{End}_R F$.

Proof. (i) Suppose S_σ as above is a cub in λ^0 and $F = \bigcup_{\nu<\lambda} F'_\nu$ is another X-filtration, then $F'_\nu = F_\nu$ for all $\nu \in C'$ for some cub C' in λ. Let $S'_\sigma = \{\beta \in \lambda^0 \mid F'_\beta\sigma \le F'_\beta$ and $\sigma^\beta|_{X'_\beta} = 0\}$ as in Definition 4.2, replacing α by β.

Since $C' \cap S_\sigma$ is a cub and $\sigma^\alpha|_{X'_\alpha} = 0$ for all $\alpha \in C' \cap S_\sigma$, we have a cub in λ^0 contained in S'_σ. Hence S'_σ is a cub and σ is inessential with respect to $\{F'_\nu\}$.

(ii) Since the sum of two mappings which act as zero on cubs again acts as zero on a cub, it is immediate that Ines F is closed under addition. Suppose $\varphi \in \mathrm{End}_R F$ and $\sigma \in$ Ines F. Then $F_\alpha\varphi = F_\alpha$ for all α in some cub C' in λ and so $(\varphi\sigma)^\alpha = \varphi^\alpha\sigma^\alpha$ for all $\alpha \in C' \cap S_\sigma$. Since λ is regular, we can find $\beta > \alpha$ such that $(F_\beta/F_\alpha)\varphi^\alpha \le F_\beta/F_\alpha$ and $\varphi^\alpha|_{F_\beta/F_\alpha}$ is then an endomorphism of F_β/F_α. Since X_α is the radical of F_β/F_α by Proposition 4.1, $X_\alpha\varphi^\alpha \le X_\alpha$. Hence $X_\alpha(\varphi\sigma)^\alpha = (X_\alpha\varphi^\alpha)\sigma^\alpha = 0$ for all $\alpha \in C' \cap S_\sigma$. But $C' \cap S_\sigma$ is a cub in λ^0, so $S_{\varphi\sigma}$ is a cub in λ. Thus $\varphi\sigma \in$ Ines F. Moreover, if $\alpha \in C' \cap C \cap \lambda^0$, then $X_\alpha(\sigma\varphi)^\alpha = X_\alpha\sigma^\alpha\varphi^\alpha = (0)\varphi^\alpha = 0$ and $\sigma\varphi$ is also inessential. □

Remark. Since we usually will show that our modules F have $\mathrm{End}_R F = A \oplus$ Ines F *as modules*, it would have sufficed to show that Ines F is a right ideal which does not require the above radical argument. However, the radical argument used above is essential for showing End $F = A \oplus$ Ines F as a ring-split extension.

Let $\diamondsuit(\lambda^0)$ denote the diamond principle [10, p. 139] for λ^0 and recall that $V = L$ implies $\diamondsuit(\lambda^0)$.

Main Theorem 4.4. ($\diamondsuit(\lambda^0)$) *Let $\lambda = \kappa^+$, where κ is either countable or a regular uncountable cardial and let X be a κ-free A-module of cardinality κ satisfying* (*) *in Section* 3 *and having endomorphism algebra* $\mathrm{End}_R X = A$, *which is a free R-module with $|A| \le \kappa$. Then there exists a λ-free, strongly λ-free A-module* (*and hence λ-free, strongly λ-free R-module*) *F such that*

(i) $|F| = \lambda$,
(ii) $\mathrm{End}_R F = A$,
(iii) $\Gamma F = \tilde{\lambda}^0$.

Remark. The invariant ΓF is defined in [10, pp. 85–86].

The construction.

Case 1: κ is uncountable. The module F is constructed by induction as the union of a λ-filtration $\{F_\alpha \mid \alpha < \lambda\}$ subject to the following conditions:

(a) F_α is a free A-module for all $\alpha < \lambda$,
(b) if $\alpha < \beta$, $\alpha \notin \lambda^0$, then F_β/F_α is a free A-module,
(c) if β is a limit ordinal $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$,
(d) if $\beta \in \lambda^0$, $F_{\beta+1}/F_\beta = X_\beta \cong X$, provided β is not in case (e) or (f).

Suppose we have constructed F_α for $\alpha < \beta$.

If cf $\beta = \omega$, then we can find an ascending sequence $\beta_n \notin \lambda^0$, with sup $\beta_n = \beta$. Since $F_{\beta_{n+1}} = F_{\beta_n} \oplus C_n$, where C_n is a free A-module, it is easy to see that $F_\beta = \bigcup_{n<\omega} F_{\beta_n} = F_{\beta_0} \oplus \bigoplus_{n<\omega} C_n$, which is free so that (a) holds.

If $\operatorname{cf}\beta > \omega$, then there is a cub C in β such that $C \cap \lambda^0 = \emptyset$, since λ^0 is non-reflecting by Lemma 2.1. So there is an ascending sequence $\alpha^* \in \beta$ ($\alpha < \operatorname{cf}\beta$) such that $\alpha^* \notin \lambda^0$, $\sup \alpha^* = \beta$. A similar argument for the case $\operatorname{cf}\beta = \omega$ shows that F_β is a free A-module.

If $\beta = \alpha + 1$ and $\alpha \notin \lambda^0$, then we choose $F_\beta = F_\alpha \oplus C_\alpha$ where C_α is a free A-module of rank κ. Observe that conditions (a)–(d) have been satisfied in all of these cases. The core of the construction is the remaining case where $\beta = \alpha + 1$ and $\alpha \in \lambda^0$.

If $\alpha \in \lambda^0$ then $\operatorname{cf}\alpha = \kappa$ and we can choose an ascending, continuous sequence $\{\beta^* \in \alpha \mid \beta < \kappa\}$ with $\beta^* \notin \lambda^0$ and $\sup_{\beta<\kappa}\beta^* = \alpha$. From conditions (a)–(c) we note that the free module F_α is the union of an ascending chain of free summands F_{β^*} with F_{β^*}, F_{ν^*}/F_{β^*} free of rank κ for all $\beta < \nu \le \alpha$. Now apply Step-Lemma 3.1 taking identifying F_{β^*} with F_β ($\beta < \kappa$) so that $F_\alpha = F$ in the Step-Lemma as follows. From $\diamondsuit(\lambda^0)$ we have Jensen functions $\{h_\alpha : F_\alpha \to F_\alpha, \alpha \in \lambda^0\}$ such that for any map $h : F \to F$ the set $\{\alpha \in \lambda^0, h|_{F_\alpha} = h_\alpha\}$ is stationary in λ. Suppose we can find $F_\alpha = G_\alpha \oplus D_\alpha$ with D_α unbounded in $\bigcup_{\nu<\alpha} F_\nu$, then we say $\alpha \in I$, if $D_\alpha h_\alpha \subseteq D_\alpha$ and $\alpha \in II$ if $0 \ne h_\alpha : F_\alpha \to R \subset F_\alpha$ and $D_\alpha h_\alpha = 0$. Decompose λ^0 into two stationary sets E, E'.

(e) If $\alpha \in E \cap I$ and h_α is a homomorphism which does not induce scalar multiplication by an $a \in A$ on F_α/D_α, then choose $F_{\alpha+1} = H$ by (3.1) (identifying $F = F_\alpha$, $D = D_\alpha$ and $G = G_\alpha$).

(f) If $\alpha \in E' \cap II$ apply the same construction for the appropriate D, F, G and note that induction proceeds as desired. Thus there exists a free A-module $H > F_\alpha$ such that

(b′) H/F_{β^*} is a free A-module for all $\beta < \kappa$ and $H/F_\alpha = X_\alpha \cong X/G_\alpha$.

Set $F_{\alpha+1} = H$. Clearly conditions (a) and (d) remain satisfied since H is free and $H/F_\alpha \cong X$. In order to show (b) remains satisfied (which is all that is left to establish since (c) is vacuous here), take any $\gamma < \alpha$ with $\gamma \notin \lambda^0$ and observe that there is a $\beta < \kappa$ with $\gamma < \beta^* < \alpha$; hence F_{β^*}/F_γ is free by induction and $F_{\alpha+1}/F_{\beta^*}$ is free by (b′). Thus (b) holds again.

This completes the construction in Case 1.

Case 2: $\kappa = \aleph_0$. The construction in the case $\kappa = \aleph_0$ is a simplification of the preceding argument. The construction is identical except that in the critical case, $\alpha \in \lambda^0$, $\operatorname{cf}\alpha = \kappa$ we identify the summands F_{β^*} with B_n in the simpler Lemma 2.5. No further modifications are required.

Proof of the Main Theorem. For the moment suppose that we know

$$(+) \qquad F^* = \operatorname{Hom}(F, R) = 0.$$

The module F is a λ-free, strongly λ-free A-module by construction, so it remains to show that (ii) and (iii) hold. Condition (iii) follows immediately from the definition of the Γ-invariant since X is not even projective.

Since each F_α is an A-module, A acts faithfully on F by scalar multiplication and so we have a natural identification of $A \leq \mathrm{End}_R F$.

Conversely consider any $\varphi \in \mathrm{End}_R F \setminus A$. We find a cub C'' such that $F_\alpha \varphi \leq F_\alpha$ for all $\alpha \in C''$. Since $|X_\alpha| = \kappa < \lambda$, regular, we can find $\beta' > \alpha$ such that $F_{\alpha+1}\varphi < F_{\beta'}$. Enlarging $F_{\beta'}$ if necessary, we can find $\beta \geq \beta'$ such that $F_\beta \varphi \leq F_\beta$. Now it follows that φ induces an endomorphism φ^α of F/F_α which leaves F_β/F_α invariant. However it follows from Lemma 4.1 that X_α is a fully invariant R-submodule of F_β/F_α and so $\varphi^\alpha|_{X_\alpha} \in \mathrm{End}_R X_\alpha$. Thus $\varphi^\alpha|_{X_\alpha} = a^\alpha$ for some $a^\alpha \in A \leq \mathrm{End}_R X$ and this holds for all $\alpha \in C'' \cap \lambda^0$, a set of cardinality $\lambda > \kappa \geq |A|$.

We now distinguish two cases:

Case (i): There is a $\alpha \in C'' \cap \lambda^0$ such that the induced map φ^α is scalar multiplication by some $a \in A$. It follows that $\varphi - a: F \to F_\alpha$ and by hypothesis on φ, this is a non-trivial R-homomorphism. Since F_α is a free A-module and A is free qua R-module, this gives rise to a non-trivial R-homomorphism from F to R, contrary to $F^* = 0$. So this case does not arise.

Case (ii): For all $\alpha \in C'' \cap \lambda^0$ the induced map φ^α is not scalar multiplication by some $\alpha \in A$. If $\alpha \in C'' \cap \lambda^0$, choose $k_\alpha \in F_{\alpha+1}$ such that $k_\alpha A \sqsubset F_{\alpha+1}$ and $k_\alpha\, A \cap F_\alpha = 0$; this is possible since $F_{\alpha+1}$ is a free A-module and $F_{\alpha+1}/F_\alpha$ contains an A-cyclic submodule. As seen above, for each $x \in k_\alpha A$ we find $a^\alpha \in A$ and $f_\alpha^x \in F_\alpha$ such that $x\varphi = xa^\alpha + f_\alpha^x$. However $|k_\alpha A| < \kappa = \mathrm{cf}(\alpha)$ and since κ is a regular cardinal, we can find $\alpha^* < \alpha$ such that $\{f_\alpha^x\colon x \in k_\alpha A\} \subset F_{\alpha^*}$. But the map $*: \lambda^0 \cap C'' \to \lambda$ is a regressive function on a stationary set and it follows from Fodor's Lemma [12, p. 59] that there is a stationary set S in $\lambda^0 \cap C''$ on which $*$ is constant. There is a $\beta < \lambda$ such that $f_\alpha^x \in F_\beta$ for all $x \in k_\alpha\, A$ and $\alpha \in S$. Let $D = \bigoplus_{\alpha \in S, \alpha > \beta} k_\alpha A \oplus F_\beta$ and set $D_\alpha = D \cap F_\alpha$. The set $C = \{\alpha \in \lambda\colon D_\alpha \text{ is unbounded in } \bigcup_{\nu<\alpha} F_\nu\}$ is a cub in λ and it follows that $D\varphi \subseteq D$, $D_\alpha \varphi \subseteq D_\alpha$ for all $\alpha \in C$. If the induced map on F/D is scalar multiplication by $a \in A$, then $\varphi - a$ maps F into D and $\varphi = a$ follows exactly as in Case (i). If the induced map is not scalar multiplication, then there exists $a \in E' \cap C'' \cap C$ such that $\varphi|_{F_\alpha} = h_\alpha$ and φ does not induce scalar multiplication of F_α/D_α. However the module D_α is unbounded in $\bigcup_{\nu<\alpha} F_\nu$ since $\alpha \in C$. Moreover, if $\beta < \gamma < \alpha$, then $k_\gamma A \sqsubset F_{\gamma+1} F_\alpha$ and so $F_\alpha = D_\alpha \oplus G_\alpha$ can be established, thus $\alpha \in I \cap E'$ and by (e) and Corollary 3.2 h_α cannot extend to $F_{\alpha+1}$, a contradiction.

It remains to show $(+)$: Suppose $0 \neq \varphi \in F^*$ and we may assume $F^* \subset \mathrm{End}\, F$. Then the set $\{\alpha \in E'\colon 0 \neq \varphi|_{F_\alpha} = h_\alpha\}$ is stationary in λ. Choose a fixed α in this set. Since $\mathrm{cf}(\alpha) = \kappa$ we can choose an unbounded, strictly increasing continuous sequence of ordinals $\{\nu^* \in \alpha \setminus \lambda^0\colon \nu < \kappa\}$ and $f_\nu \in F$ with $F_{\nu^*} \oplus f_\nu A \sqsubset F_{(\nu+1)^*}$. Set $F'_\alpha = \bigoplus_{\nu<\kappa} f_\nu A$; then F'_α is a free A-module and the quotient F_α/F'_α is free as well. Now apply Lemma 2.8 to F'_α and $h_\alpha|_{F'_\alpha}$: we obtain an unbounded free A-module $D_\alpha \subseteq \ker(h_\alpha|_{F'_\alpha})$ in F_α and F'_α/D_α is A-free. Since $\alpha \in E' \cap II$, the construction followed (f), which gives a contradiction. $\square$

5. Applications

Our first application of the Main Theorem in Section 4 can be used to show that the Kaplansky Test Problems have a negative answer for strongly $\aleph_1$-free groups of cardinality $\aleph_1$, in ZFC + CH set theory.

Theorem 5.1. *If A is any countable, S-reduced, S-torsion-free R-algebra with A_R free, then there exists an $\aleph_1$-free, strongly $\aleph_1$-free R-module G of cardinality $\aleph_1$, such that* $\mathrm{End}_R G = A$.

Proof. The result will follow immediately from the case $\kappa = \aleph_0$ of our Main Theorem (Theorem 4.4), if we can exhibit an R-module X satisfying condition $(*)$ of Section 3 with $\mathrm{End}_R X = A$. Now it follows from a well-known result of Corner [1] (which has been extended to our present "(R, S) context" by Corner in an unpublished paper [4]) that any such algebra A is the full endomorphism algebra of a countable, S-reduced, S-torsion-free R-module X where X is a pure submodule of $\hat{A}$ containing A. Corner's result appears as a special case in [14], where R is not necessarily countable. Moreover it follows from these constructions that X contains a free A-module Y with $A \leq Y \leq \hat{A}$ such that $X = \langle Y \rangle_* \leq \hat{A}$. Suppose $Y = \bigoplus_{i \in I} e_i A$, see Section 1 for our terminology.

Choose independent elements $e_i^2 \in \hat{A} \backslash A$ such that $q_2 e_i^2 = e_i$; this is possible since A is dense in $\hat{A}$. Set $X_1 = Y$, $X_2 = \bigoplus_{i \in I} e_i^2 A$. Continue in this fashion defining independent elements $e_i^{n+1} \in \hat{A} \backslash A$ such that $q_{n+1} e_i^{n+1} = e_i^n$ and let $X_{n+1} = \bigoplus_i e_i^{n+1} A \oplus X_n$.

Then each X_n is a free A-module and clearly $X = \bigcup_{n < \omega} X_n$. However, as noted in Observation 3.0, such a module X automatically satisfies condition $(*)$.

There are, of course, many consequences of such a theorem describing pathological behaviour of module theory; see [5] for details. We restrict here to one example, phrased for Abelian groups but which obviously holds in a more general setting, which simultaneously defeats the Kaplansky Test Problems. Note that, as observed in the Introduction, strongly κ-free Abelian groups are necessarily κ-free.

Corollary 5.2. *For any positive integer r, there exists a strongly $\aleph_1$-free Abelian group G of cardinality $\aleph_1$ such that $G^{(m)} \cong G^{(n)}$ if and only if $m \equiv n \bmod r$.*

Proof. Take A to be the ring $\mathbb{Z}\Lambda$ constructed by Corner in [2]; see [11, Theorem 9.16]. Then $A_{\mathbb{Z}}$ is free and the result follows from Theorem 5.1 by an argument similar to that of Corner [3]. □

A curious consequence of the above examples is that whereas a single cardinal number (the rank) suffices to classify free modules, no additive cardinal invariants can hope to classify strongly $\aleph_1$-free modules.

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