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Non-Linear Modelling of Internal Geophysical Waves

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Thesis submitted for the award of Doctor of Philosophy (PhD)

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Abstract

Geophysical waves are waves that are found naturally in the Earth's atmosphere and oceans. Internal waves, that act as an interface between fluids of different density are examples of geophysical waves. The system set-up will incorporate a model with a flat bottom, flat surface and internal wave. The system has a depth-dependent current which mimics a typical ocean set-up and, as it is based on the surface of the rotating Earth, includes Coriolis forces. Using well established fluid dynamic techniques, the total energy is calculated and used to determine the dynamics of the system using a procedure called the Hamiltonian approach. By tuning a system variable several special cases such as irrotational or current-free are easily recovered. An approximate model utilising a small amplitude, long-wave regime, the so called Intermediate Long Wave (ILW) model is then derived using perturbation expansion techniques. Solutions are obtained that model waves that move without change of form called solitary waves. These waves can be referred to as solitons when their particle-like behaviour is considered. The Coriolis effect on the internal wave propagation is also examined following the idea of "nearly" Hamiltonian approach, developed in series of papers like [14, 15, 20, 48] and generalising the Hamiltonian approach of Zakharov [88]. The presented models have applications for climatologists, meteorologists, oceanographers, marine engineers, marine biologists and applied mathematicians.

Declaration

I certify that this thesis which I now submit for examination for the award of Doctor of Philosophy (PhD), is entirely my own work and has not been taken from the work of others, save and to the extent that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for graduate study by research of the Technological University, Dublin and has not been submitted in whole or in part for another award in any other third level institution. The work reported on in this thesis conforms to the principles and requirements of the TU Dublin's guidelines for ethics in research.

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Signature: ____

(Joseph Cullen)

Date: ____________________________

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Chapter 1

INTRODUCTION

1.1 Outline of Thesis

The thesis contains six main chapters as follows:

1. **Introduction** to the subject of internal geophysical waves and ocean currents.

2. **Preliminaries** where we introduce the fundamentals of fluid mechanics, the system set up, general assumptions, velocity potential and stream function, Euler's equation and the Bernoulli condition, the kinematic boundary condition.

3. **Hamiltonian Formulation** - the Hamiltonian is defined and derived for the system of 2 layers with an internal wave at the interface. The Dirichlet-Neumann operator is introduced and using functional calculus, the variation of the Hamiltonian leads to the equations of motion.

4. **The Intermediate Long Wave Equation (ILWE)** is derived as an approximate model including the additional complications of current and vorticity, which is novel. Comparisons are also made with the regimes of other approximations, e.g. KdV and BO.

5. **Internal Waves with Coriolis Force** The Coriolis effect on internal wave propagation away from the equator is examined. A new Intermediate Long Wave (ILW) type equation is derived, extending the equation from Chapter 4, in the irrotational, current free case.

6. **Future Work and Open Questions**

The first problem examined by this thesis concerns the Intermediate Long Wave Equation (ILWE). This approximation leads to an integrable model and the significance is that the model can be solved with the methods of Soliton Theory, for example, using the inverse scattering method.

This is a new development, since the parameters characterising the current are included in the model for this propagation regime. The published article, [34], on the ILW equation in the presence of currents has been independently cited by several articles, where a range of related topics are discussed.

The long-wave KdV and BO regimes have been studied previously by Ivanov and Compelli and my work has involved the ILW regime with current and vorticity included. The following papers may be regarded as complimentary, covering all regimes of geophysical relevance: [15, 16, 34]

The second problem is about the inclusion of the Coriolis force effects for internal waves, away from the equator. Analysis and results have been published: refer to [35] for details.

The novelty in this case is in the use of a modification of the Hamiltonian approach for systems with Coriolis force. Two regimes are considered:

1. Long waves, leading to a nonintegrable equation of mKdV type, also known as the Ostrovsky equation;

2. Intermediate long waves, leading to a nonintegrable model of ILWE type, which is new.

Although non-integrable, the obtained models can be treated within the theory of soliton perturbations, or by numerical methods, developed for the mKdV and ILWE equations, which is an important advantage. The results of the second published article will be used in the forthcoming works of Prof. Rossen Ivanov and co-authors, about internal waves interacting with surface waves under the influence of the Coriolis force.

1.2 Geophysical Waves

Geophysical Fluid Dynamics (GFD) deals mainly with the dynamics of the oceans and atmosphere on a giant rotating sphere, the Earth. On small scales GFD is classical fluid dynamics. The Rossby number is commonly used in geophysical phenomena in the oceans and atmosphere, where it characterizes the importance of Coriolis accelerations arising from planetary rotation. The Rossby number is a fundamental dimensionless number in Geophysical Fluid Dynamics and is defined as [79]:

$$R_O = \frac{U}{\Omega L},$$

where Ω is the angular frequency of the Earth's rotation and U and L are the chosen velocity and length scales respectively. A small Rossby number signifies a system strongly affected by Coriolis forces, and a large Rossby number signifies a system in which inertial and centrifugal forces dominate. The term geophysical waves refers to waves that are found naturally in the Earth's atmosphere and oceans. They may also be observed in lakes and fjords. In oceanic systems the Rossby number is typically of the order of unity. Surface and internal waves, as shown in Figure 1.1, are examples of geophysical waves found in the oceans. They are wind driven with typical speeds in excess of 10 m/s. Average surface wave height is 4 m in the Equatorial Pacific, but extreme wave heights approaching nearly 20 metres (excluding rogue waves) have been observed. Internal waves, propagating beneath the surface, have characteristic speed 0.1 m/s, average wavelength of the order of several kilometers and average height of 20 m in the equatorial pacific. Internal waves are disturbances which act along an interface between fluid bodies which have different densities. This stratification of the ocean is at its most prevalent in a band of about 2° latitude from the Equator.

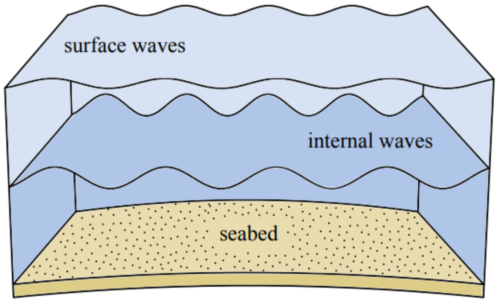

Figure 1.1: Ocean Profile

The ocean dynamics is deeply affected by waves - on the surface as well as internal waves, which represent the motion of the interface (thermocline/pycnocline). The wave motion in oceans however usually takes place in the presence of currents with shear and therefore nonzero vorticity and this important aspect needs to be included in the models. For example, the field provides evidence for a spectacular wave phenomenon, recently discovered in the South China Sea: tide-generated internal waves with amplitudes in excess of 100 m, extending over 100 km [82]. The internal energy produced by the tides is essential in generating these waves.

The Hamiltonian formulation of the problem of internal waves between two bodies of immiscible fluid with free surface and interface **without currents and vorticities** has been a focus for many studies for a long time, however a particularly convenient setting for a Hamiltonian approach based on the Dirichlet-Neumann operators has been put forward by Craig and co-authors, for example [31]. From the Hamiltonian formulation, a perturbation theory for the long wave limits has been developed. The Hamiltonian approach is central to the modelling of internal waves in the presence of current. It originates from Zakharov's paper [88] for irrotational surface waves over infinitely deep water. The Hamiltonian formalism is often utilised in the study of nonlinear waves in continuous media, see for example the review article [89].

1.3 Ocean Flow - Currents

Currents are steady mean flows of ocean water in a prevailing direction. There are three main categories of ocean currents:

- near-surface wind-driven currents (with typical speed 0.1 m/s, confined to the upper 100-200 m ocean region)
- a deeper thermohaline circulation, driven by the differences in the Ocean's density, which is related to temperature (thermo) and salinity (haline), with typical speeds of 0.01 m/s
- tidal flows, caused by gravitational attraction of the Moon (the lunar tide) and the Sun (the solar tide) and the gravitational force of the Earth

Nonlinear waves in the oceans often propagate in the presence of currents which significantly affect their dynamics. In the oceans, currents very often exist as undercurrents. The Equatorial Undercurrent (EUC) flows in a region that is roughly within 200 - 300 km (below 3° latitude) of the Equator, it is symmetric about the Equator and extends nearly across the whole length (more than 12000 km) of the Pacific Ocean basin [46]. With speeds in excess of 1 m/s, the EUC is one of the fastest permanent currents in the world.

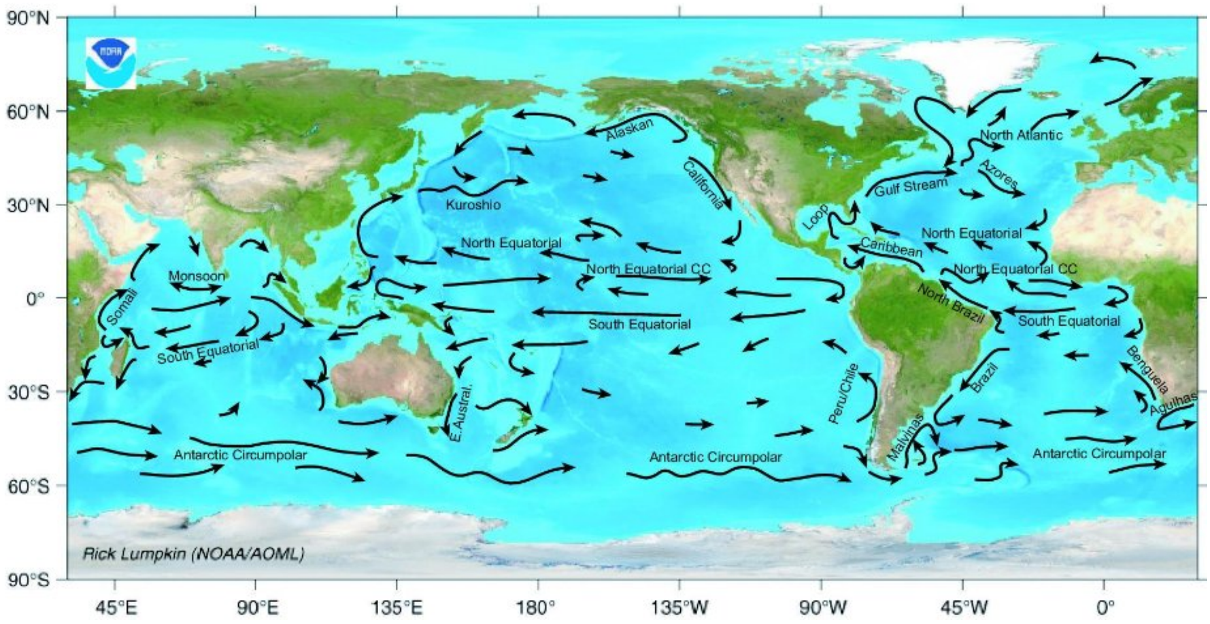

Figure 1.2: Ocean Currents (image credit: NOAA)

The equatorial region in the Pacific is characterised by a thin shallow layer of warm and less dense water over a much deeper layer of cold denser water. The two layers are separated by a sharp *thermocline* (where the temperature gradient has a maximum, it is very close to the *pycnocline*, where the pressure gradient has a maximum) at a depth, depending on the location, but usually at 100 – 200 m beneath the surface. Both layers are to a big extent homogeneous and their sharp boundary is the thermocline/pycnocline.

The flow has nearly two-dimensional character, with small meridional variations, being combinations of longitudinal non-uniform currents and waves, and presenting a significant fluid stratification that results in a pycnocline/thermocline separating two internal layers of practically constant density (see [54]). While at depths in excess of about 240 m there is, essentially, an abyssal layer of still water, the ocean dynamics near the surface is quite complex. In this region the wave motion typically comprises surface gravity waves with amplitudes of 1-2 m and oscillations with an amplitude of 10-20 m at the thermocline (of mean depth between 50 m and 150 m). These waves interact with the underlying currents. The vanishing of the Coriolis parameter at the Equator distinguishes the dynamics of the equatorial zone from the ocean dynamics at higher latitudes. The strong stratification confines the wind-driven currents to a shallow near-surface region, less than 200m deep. In the Atlantic and Pacific, the westward trade winds induce a westward surface flow at speeds of 25-75 cm/s, while a jet-like current – the Equatorial Undercurrent (EUC) – flows below it toward the East (counter to the surface current), attaining speeds of more than 1 m/s at a depth of nearly 100 m.

The choice of a piecewise linear current captures the primary structure of the equatorial current system (see Fig. 1.3). The above setting applies not only to equatorial flows, being typical for the evolution of large amplitude oscillations of an interface between two internal fluid layers, and its coupling with the motion of an overlying free surface, in the presence of wind-generated currents.

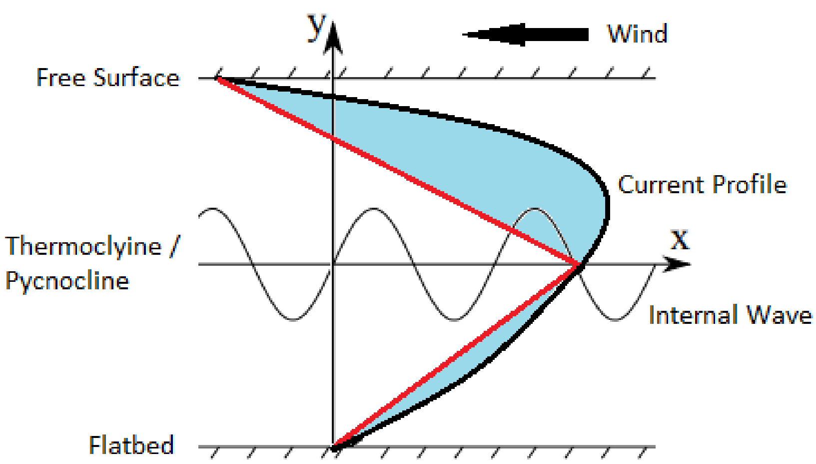

Figure 1.3: A typical wind-induced current (curve in black) in a two-layered fluid with a flat bed, and its inviscid approximation – a current with piecewise constant vorticity (red line). The wind drives a return flow, initiated at some sub-surface depth and extending beneath the pycnocline.

Complexity is added to the study of the oceans by the consideration of vorticity. Vorticity effectively means that particles in the fluid experience local rotation as they move and therefore systems where vorticity is considered are called ‘rotational’. Often vorticity is neglected, in which cases the system is said to be considered as being ‘irrotational’.

Two oceanic systems are presented in this thesis: both consisting of two layers with a flat surface and flat bottom separated by an internal wave. As the systems are present on the Earth’s rotatin surface Coriolis forces will be considered. These forces exist as the Earth is, essentially, a rotating solid spherical body. The Coriolis effect is examined for the internal wave propagation following the idea of a “nearly” Hamiltonian approach, developed in series of research papers like [14, 15, 20, 48] and generalising the Hamiltonian approach of Zakharov [88].

The dynamics of geophysical waves will be analysed using the Hamiltonian formulation. This is achieved by calculating the total energy of the system (the ‘Hamiltonian’) in terms of two variables (the ‘conjugate’ variables) and then determining how the Hamiltonian varies as they vary.

This requires using fundamental concepts from fluid dynamics such as mass and momentum conservation, determination of pressure and the analysis of boundary conditions.

Approximations can also be calculated rather than using exact representations. This technique, known as perturbation, scales the sytem variables, allowing for grouping of different orders (that is constant, linear and nonlinear terms) which can be truncated at different desired levels. Approximations are useful to many groups outside of applied mathematics, such as oceanography and marine engineering, and readily facilitate computational modelling. The first step is nondimensionalisation meaning that the variables are transformed from *physical* variables which have dimensions, such as metres and seconds, to dimensionless variables. This is achieved by scaling the variables against constants which have significance to the system, such as the system depth, the wave amplitude and the (assumed constant) acceleration due to gravity. Using these techniques numerous models can be developed in the form of nonlinear PDEs (partial differential equations). Solutions to these PDEs can be obtained by solving them analytically or numerically. For instance the ILW (Intermediate Long Wave) equation is an equation with nonlinear and dispersive (variation of wave speed with wavelength) components which, by balancing these terms, can be used to model 'solitary waves' which are waves that retain their shape as they propagate. The particle-like manner in which they behave is reflected in the use of the term 'soliton' to describe them [8, 65].

Chapter 2

PRELIMINARIES

2.1 Introduction

The Hamiltonian approach is a formulation which can play an important role in the modelling of internal waves in the presence of current. It originates from Zakharov's paper [88] for irrotational surface waves over infinitely deep water. The Hamiltonian formalism is often utilised in the study of nonlinear waves in continuous media, see for example the review article [89].

2.1.1 Wave Characteristics

When describing travelling waves some important characteristics such as wavelength: the distance between repeating features such as crests (or troughs), denoted by λ, and amplitude: the maximum deviation of the wave height from the mean water level, denoted by a, are highlighted in Figure 2.1. The wave propagates so that every period T it completes one cycle. In the figure the solid line represents the wave at some initial time $t = 0$, the dashed line represents the wave at a later time t. The frequency, f is the inverse of the period and therefore $f = \frac{1}{T}$. The wave speed is

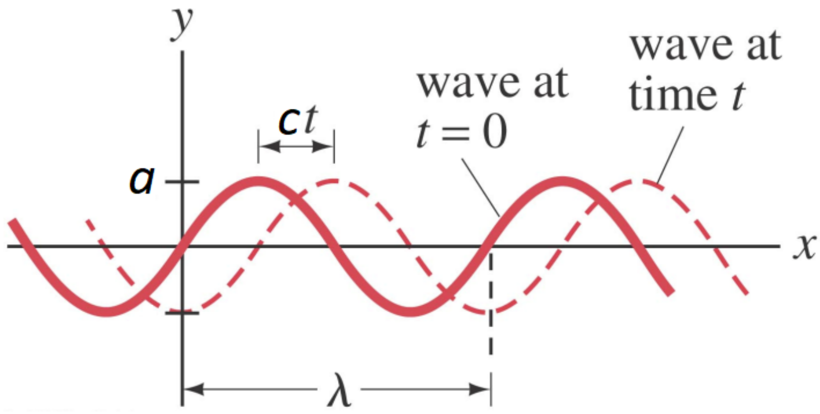

Figure 2.1: Wave Characteristics

given by

$$c = \lambda f = \frac{\lambda}{T}.$$

By introducing the wave number k and the angular frequency ω as

$$k = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi f$$

the wave speed can be written as

$$c = \frac{\omega}{k}.$$

The propagation of waves in the x-direction can be described by the function

$$\eta(x,t) = a \sin(kx - \omega t)$$

where η is the height of the wave which is often called the 'wave elevation function'. Solitary waves, unlike the periodic ones, are waves whose profiles decay rapidly to zero away from the maximum. A typical example is the sech^2 profile:

$$\eta(x,t) = a\,\text{sech}^2(x - ct). \tag{2.1.1}$$

Figure 2.2 shows a solitary wave of amplitude $a = 0.5$ at time $t = -10$.

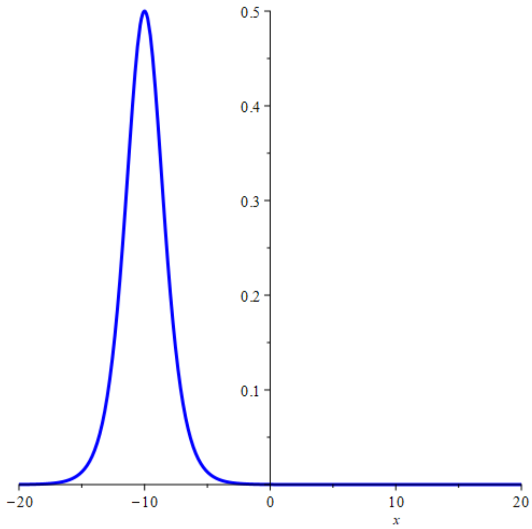

Figure 2.2: Solitary wave propagation

2.1.2 Fundamentals of Fluid Mechanics

The following important principles of fluid mechanics are now considered: conservation of mass and conservation of momentum. Conservation of mass is the fundamental assumption that matter (mass) is neither created nor destroyed anywhere in the fluid system. This requires an additional assumption that there are no energy releasing processes in the system, for example breaking-waves, that is waves that become turbulent. By examining the changes in mass for some arbitrary fixed volume the equation of mass conservation (or continuity equation) can be established as [52]

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

where $\rho(x_1, x_2, x_3, t)$ is the density, t means the partial derivative with respect to time (noting that the use of subscript notation for partial derivatives will be used throughout the thesis),

$$\nabla = (\partial_1, \partial_2, \partial_3)$$

(called 'nabla' or the 'del' operator) is a differential operator and the gradient.
The operation $\nabla\cdot$ is called the 'divergence' which is defined for some vector $\mathbf{a} = (a_1, a_2, a_3)$ as

$$\nabla \cdot \mathbf{a} := (\partial_1, \partial_2, \partial_3).(a_1, a_2, a_3) = a_{1,1} + a_{2,2} + a_{3,3}$$

and

$$\mathbf{u} = (u_1, u_2, u_3)$$

is the velocity field, with respective x_1, x_2 and x_3 components equal to u_1, u_2 and u_3.
The continuity equation is usually expanded as

$$\rho_t + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0$$

and the ‘material’ or ’convective’ derivative is introduced -

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla.$$

This differs from the ’usual’ Eulerian derivative in that it considers the flow of particular particles in the fluid as opposed to considering the changing matter that passes through a fixed region. The continuity equation can therefore be written as

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0$$

‘Incompressibility’ refers to the local resistance of a fluid to changes in density for particles travelling with the flow, that is for material volumes. For an incompressible fluid this therefore means that the material derivative of ρ is zero and it can be established that for incompressible fluids, constant density is equivalent to the velocity having zero divergence-free, that is

$$\nabla \cdot \mathbf{u} = 0$$

One of the fundamental laws of classical mechanics, Newton’s second law, is an expression of the conservation of momentum. It states that the forces acting on an inertial body (a body with mass) are proportional to the acceleration departed to the body. Analogously, for a continuum, the material derivative of the velocity is related to the sum of all the forces (internal and external) acting on the continuum by

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{g} + \nu\nabla^2\mathbf{u}$$

where P is the total pressure, $\mathbf{g}$ is the acceleration due to Earth’s gravity, that is $\mathbf{g} := (0, 0, -g)$ and ν is the kinematic viscosity. Viscosity is a quantity related to the frictional forces between fluid molecules. This equation is called the Navier-Stokes equation. For inviscid systems, that is systems with zero viscosity, $\nu = 0$ and the equation is written as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{g}$$

and is called Euler's equation.

Fluids with vorticity, that is *rotational*, means that there is local rotation of particles in the fluid. This can be present in a spinning fluid. It can also be present for shear flows where there is some linear dependency between depth and flow speed which can result in particles rotating. The vorticity $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{\gamma} := \nabla \times \mathbf{u}$$

where the operation $\nabla\times$ is called the 'curl' which is defined for some vector $\mathbf{a} = (a_1, a_2, a_3)$ as

$$\nabla \times \mathbf{a} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{bmatrix} a_{3,2} - a_{2,3} \\ a_{3,1} - a_{1,3} \\ a_{2,1} - a_{1,2} \end{bmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the orthogonal unit vectors for the x, y and z axes respectively. For two-dimensional flows, $\mathbf{u} = (u, 0, v) = (u_1, 0, u_3)$ and the vorticity is therefore

$$\nabla \times \mathbf{u} = \text{curl } \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & 0 & \partial_3 \\ u_1 & 0 & u_3 \end{vmatrix} = (0, -\mathbf{j}(u_{3,1} - u_{1,3}), 0)$$

So γ, the vorticity for 2-dim flows is defined as

$$\gamma := u_y - v_x. \tag{2.1.2}$$

2.1.3 General assumptions

The following section outlines some of the general assumptions made for the model set up.

Consideration of the inclusion of the effects of surface tension in the models is a matter of determining the domination of either gravitational effects or capillary waves (waves due to surface tension, often referred to as 'ripples' for distinction). For wavelengths with a critical wavelength λ_c given by [64]

$$\lambda_c = 2\pi\sqrt{\frac{T}{\rho g}},$$

where T is surface tension, effectively the two processes are equal.

For water $T \approx 0.074\, N\, m^{-1}$, $\rho \approx 1000\, kg\, m^{-3}$ and (for Earth) $g \approx 9.8\, m\, s^{-2}$ and so

$$\lambda_c \approx 2\pi\sqrt{\frac{0.074}{1000 \times 9.8}} = 1.74\, cm$$

and therefore, as wavelengths for oceanic waves are generally substantially greater then 1.74 centimetres surface tension will be neglected (*cf.* [20, 27]) and as such gravity waves, not capillary waves, will be considered.

As the system is present on the surface of a rotating solid body (the Earth) Coriolis forces will be considered. These forces manifest themselves as a tendency for bodies accelerating eastward along a latitude in the northern hemisphere to *tend to the right.* It is an effect of the earth's rotation and is an apparent deflection of the path of an object that moves within a rotating coordinate system. The object does not actually deviate from its path, but it appears to do so because of the motion of the coordinate system.

The fluids under study will be considered to be inviscid by the assumption [52] that all time scales and length scales attributable to viscous movement are long compared to the period of the wave and the wavelength.

All flow is considered to be two-dimensional in the $x - y$ plane as shown in Figure 2.3, meaning there is no lateral (z-direction in this setup) movement, and so the velocity is described by $\mathbf{u} = (u, 0, v)$. This is representative of many observed flows. The waves propagate in the x-direction and for large absolute values of x, they attenuate and vanish at infinity and hence are described by functions which belong to the Schwartz class: $\mathcal{S}(\mathbb{R})$. These are functions which rapidly decrease.

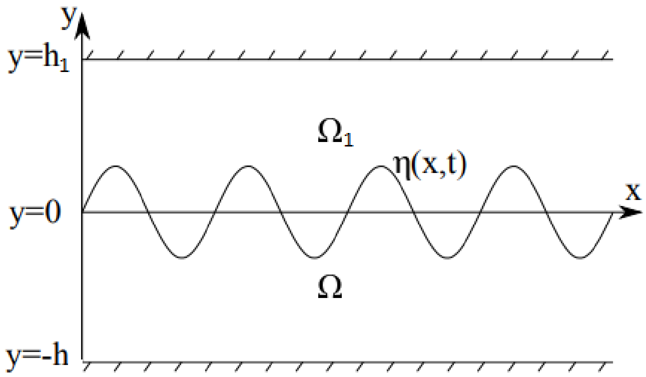

Figure 2.3: 2-Media Water Wave System

2.2 Model for Internal Equatorial Waves

2.2.1 System Set Up

The internal wave along the equator is modelled by two layers of water in two dimensions, and so meridional (latidudinal) motion is neglected. The layers are separated by a free common interface, the thermocline/pycnocline, as per Figure 2.4. The two fluid domains are

$$\begin{cases} \Omega = \{(x,y) \in \mathbb{R}^2 : -h < y < \eta(x,t)\} \\ \Omega_1 = \{(x,y) \in \mathbb{R}^2 : \eta(x,t) < y < h_1\} \end{cases} \tag{2.2.1}$$

The system is bounded at the bottom by an impermeable flatbed at a constant depth h and is considered as being bounded on the surface at a height h_1 by an assumption of absence of surface motion, that is a rigid lid approximation. Typically h_1 is of the order of hundreds of metres and h the order of kilometres.

The functions and parameters associated with the upper layer will be marked with subscript 1. Also, subscript c (implying *common interface*) will be used to denote evaluation on the internal wave. Propagation of the internal wave is assumed to be in the positive x-direction which is considered to be 'eastward'. The direction of the gravity force is in the negative y-axis.

The wave amplitude is described by a and the wave is characterised by the elevation function $\eta(x,t)$. In other words fluid particles at the interface (or on the wave) have a y-component described by

$$y = \eta(x,t). \tag{2.2.2}$$

The mean of η is assumed to be zero, that is $\int_{\mathbb{R}} \eta(x,t)dx = 0$. *Note:* It is sufficient that $\int_{\mathbb{R}} \eta(x,t)dx = \mathfrak{p} < \infty$, because the mean value involves division of the length of the corresponding interval, which in this case is $\int_{\mathbb{R}} dx = \infty$.

The fluids are incompressible with constant densities ρ and ρ_1 for the fluid domains

Ω and Ω_1 respectively. The fluids have different densities due to different salinity levels or temperatures, with stable stratification given by the immiscibility condition $\rho > \rho_1$. Figure 2.4 illustrates the system set up.

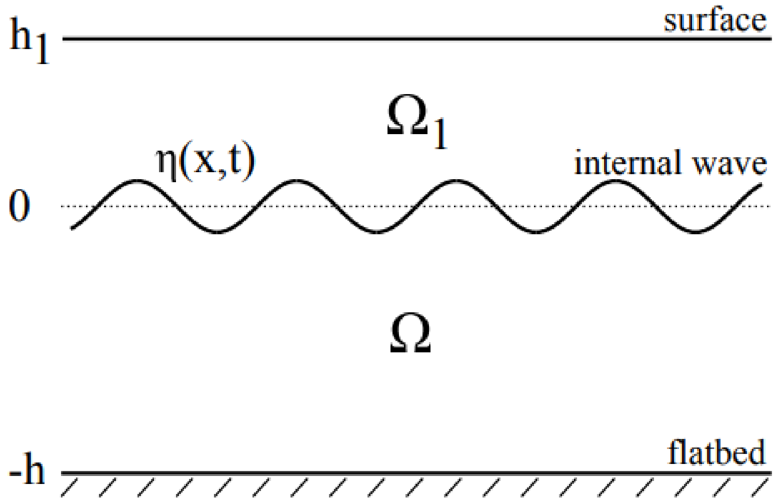

Figure 2.4: System set up. The function $\eta(x,t)$ describes the elevation of the internal wave.

The system is assumed to be on the surface of the Earth, that is on a rotating solid body and so Coriolis forces per unit mass in each fluid domain will be taken into consideration - refer to Appendix A for details of Coriolis and Centripetal force terms that will be included in the governing equations. Perturbations of the fluids are acted upon by the restorative action of gravity. The Earth's centre of gravity is in the negative y-direction with regard to the chosen co-ordinate system. Both fluids are considered to be inviscid, that is having zero viscosity.

2.2.2 The Stream Function and Velocity Potential

$\mathbf{u}(x, y, t) = (u, v)$ and $\mathbf{u}_1(x, y, t) = (u_1, v_1)$ are the velocity fields of the lower Ω and upper Ω_1 media respectively and velocity potentials φ and φ_1 are introduced such that:

$$\begin{cases} u = \varphi_x + \gamma y \\ v = \varphi_y \\ u_1 = \varphi_{1,x} + \gamma_1 y \\ v_1 = \varphi_{1,y} \end{cases} \tag{2.2.3}$$

where $\gamma = -v_x + u_y$ and $\gamma_1 = -v_{1,x} + u_{1,y}$ are the constant vorticities.

The Laplacian, Δ, of the respective velocity potentials is given by

$$\Delta\varphi = \nabla \cdot (\nabla\varphi) = \nabla \cdot \mathbf{u},$$
$$\Delta\varphi_1 = \nabla \cdot (\nabla\varphi_1) = \nabla \cdot \mathbf{u}_1.$$

However, for incompressible fluids the velocity vectors have zero divergence, that is $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}_1 = 0$ and so

$$\Delta\varphi = \Delta\varphi_1 = 0 \tag{2.2.4}$$

meaning φ and φ_1 are harmonic functions.

The stream functions, $\psi(x, y, t)$ and $\psi_1(x, y, t)$ are related to the velocity fields $\mathbf{u} = (u, v)$ and $\mathbf{u_1} = (u_1, v_1)$ as follows

$$u = \psi_y, \;\; v = -\psi_x, \;\; u_1 = \psi_{1,y} \text{ and } v_1 = -\psi_{1,x}. \tag{2.2.5}$$

A depth dependent current is considered, such as the Equatorial Undercurrent described in Section 1.2. The current as shown in Figure 2.5 is approximately linear.

This set up allows for a realistic representation of the stratification that exists in the oceans [29] adjacent to the internal wave. The current profiles for Ω and Ω_1 are given by

$$U(y) = \gamma y + \kappa \tag{2.2.6}$$

and

$$U(y) = \gamma_1 y + \kappa_1 \tag{2.2.7}$$

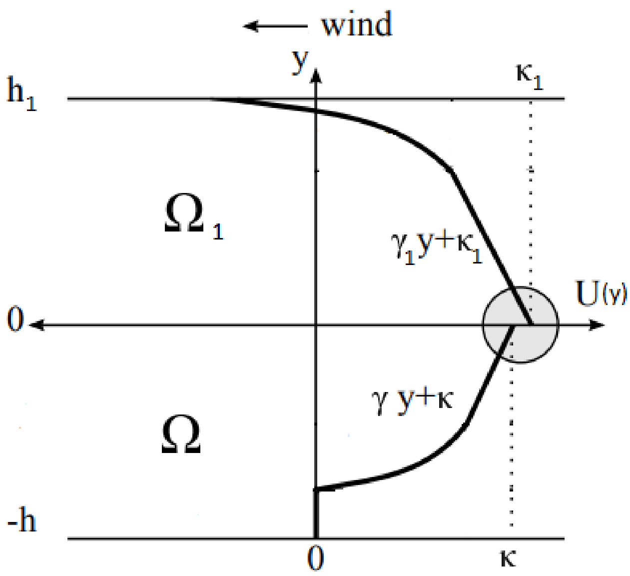

Figure 2.5: Depth Dependent Current Profile

Note: In Figure 2.5, the jump in the tangential speed is allowed since there is no viscosity. The situation without such a jump is modelled by taking $\kappa = \kappa_1$.

The velocity fields can now be written in the following form, representing the piece-wise linear current profile:

$$
\begin{cases}
u = \widetilde{\varphi}_x + \gamma y + \kappa \\
v = \widetilde{\varphi}_y \\
u_1 = \widetilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1 \\
v_1 = \widetilde{\varphi}_{1,y}
\end{cases} \tag{2.2.8}
$$

where κ and κ_1 are the time-independent currents at $y = 0$ and the 'wave-only' components have been separated out by introducing a tilde notation.

It is assumed that the functions $\eta(x,t)$, $\widetilde{\varphi}(x,y,t)$ and $\widetilde{\varphi}_1(x,y,t)$ belong to the Schwartz Class $\mathcal{S}(\mathbb{R})$ with respect to x, for any y and t. This means that they are smooth, rapidly decreasing functions and implies that for large absolute values of x, the internal wave attenuates. Therefore

$$
\lim_{|x|\to\infty} \eta(x,t) = \lim_{|x|\to\infty} \tilde{\varphi}(x,y,t) = \lim_{|x|\to\infty} \tilde{\varphi}_1(x,y,t) = 0 \tag{2.2.9}
$$

noting that an implication of this assumption is that

$$
\lim_{|x|\to\infty} \psi(x,\eta,t) = \lim_{|x|\to\infty} \psi_1(x,\eta,t) = 0. \tag{2.2.10}
$$

The following normal vectors will be used: $(\mathbf{n})_c$, $(\mathbf{n})_b$, $(\mathbf{n}_1)_c$ and $(\mathbf{n}_1)_t$, which are the outward normal vectors as shown in Figure 2.6,

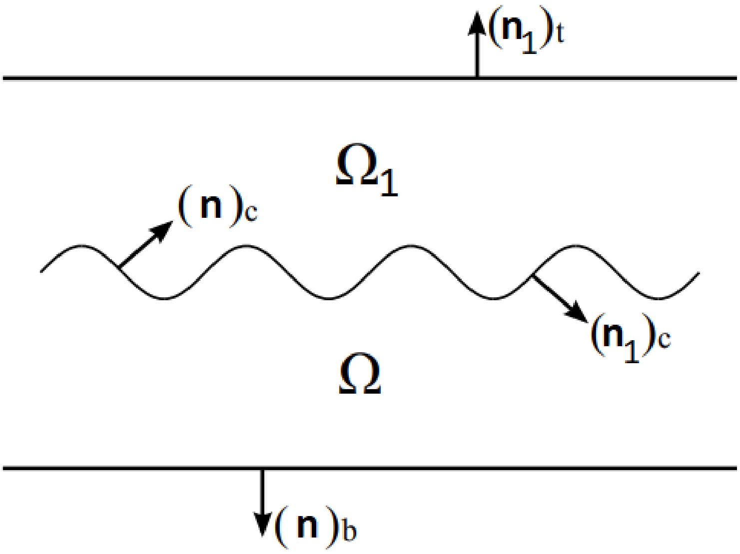

Figure 2.6: System normal vectors

where the subscript notation c, b and t refer to values at the (common) interface, bottom and top respectively. The outward normal vectors are given by

$$(\mathbf{n})_c = (-\eta_x, 1) \text{ and } (\mathbf{n})_b = (0, -1) \tag{2.2.11}$$

for Ω and

$$(\mathbf{n}_1)_c = (\eta_x, -1) \text{ and } (\mathbf{n}_1)_t = (0, 1) \tag{2.2.12}$$

for Ω_1.

2.3 Governing equations

The governing equations consist of both dynamic and kinematic boundary conditions. The dynamic condition is an interface condition and so recalling (2.2.8), the stream functions and velocity potentials adjacent to the internal wave are defined as

$$
\begin{cases}
u = \widetilde{\varphi}_x + \gamma y + \kappa = \psi_y \\
v = \widetilde{\varphi}_y = -\psi_x \\
u_1 = \widetilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1 = \psi_{1,y} \\
v_1 = \widetilde{\varphi}_{1,y} = -\psi_{1,x}
\end{cases}
\tag{2.3.1}
$$

There is an interdependency between ψ and $\widetilde{\varphi}$ and between ψ_1 and $\widetilde{\varphi}_1$ (*cf.* [52]) given by the complex potentials

$$
f(z) = \widetilde{\varphi} + i\Big(\psi - \frac{1}{2}\gamma y^2 - \kappa y\Big),
$$

and

$$
f_1(z) = \widetilde{\varphi}_1 + i\Big(\psi_1 - \frac{1}{2}\gamma_1 y^2 - \kappa_1 y\Big)
$$

where $z = x + iy$.

The functions $f(z)$ and $f_1(z)$ are analytical by construction. This can be checked by the Cauchy-Riemann relations. This reflects the fact that the fluid motion on the interface allows an analytic extension in the bulk of the fluid (in terms of $\psi(z)$ for example) and thus all physical quantities in the fluid volume can be recovered. The Laplacians of $\widetilde{\varphi}$ and $\widetilde{\varphi}_1$ are given by

$$
\Delta\widetilde{\varphi}_1 = \nabla \cdot (\nabla\widetilde{\varphi}_1) = u_{1,x} + v_{1,y},
$$

and

$$
\Delta\widetilde{\varphi} = \nabla \cdot (\nabla\widetilde{\varphi}) = u_x + v_y.
$$

Due to the assumption of incompressibility $\nabla \cdot \mathbf{u}_1 = \nabla \cdot \mathbf{u} = 0$ (see Section 2.1.2) and therefore

$$
\Delta\widetilde{\varphi}_1 = \Delta\widetilde{\varphi} = 0 \tag{2.3.2}
$$

which are expressions of Laplace's equation and so $\widetilde{\varphi}_1$ and $\widetilde{\varphi}$ are harmonic functions.

The Laplacians of ψ_1 and ψ are given by

$$\Delta\psi_1 = \nabla \cdot (\nabla\psi_1) = u_{1,y} - v_{1,x}, \tag{2.3.3}$$

and

$$\Delta\psi = \nabla \cdot (\nabla\psi) = u_y - v_x. \tag{2.3.4}$$

Recalling (2.1.2) and using (2.2.5) it follows that the Laplacians of ψ and ψ_1 are equal to the respective vorticities, that is

$$\Delta\psi_1 = \psi_{1,xx} + \psi_{1,yy} = \gamma_1, \tag{2.3.5}$$

and

$$\Delta\psi = \psi_{xx} + \psi_{yy} = \gamma. \tag{2.3.6}$$

Note that ψ_1 and ψ are not harmonic for non-zero vorticities but are for the irrotational case, that is by setting $\gamma_1 = \gamma = 0$.

2.3.1 Euler's Equation and the Bernoulli Condition

Analogous to Newton's second law for rigid solid bodies, the acceleration of a fluid element can be related to the net forces per unit mass acting on it for an observer in an inertial frame of reference. It can therefore be shown that, for the fluid domains Ω_1 and Ω, the velocity vectors $\mathbf{u}_1$ and $\mathbf{u}$ are related to the net forces per unit mass as [52]

$$\mathbf{u}_{1,t} + (\mathbf{u}_1.\nabla)\mathbf{u}_1 = -\frac{1}{\rho_1}\nabla P_1 + \mathbf{g}, \tag{2.3.7}$$

and

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{g}. \tag{2.3.8}$$

where $\mathbf{g} = (0, 0, -g)$ is the Earth's acceleration due to gravity and

$$P_1 = \rho_1 g y + p_1 + p_{\text{atm}} \tag{2.3.9}$$

and

$$P = \rho g y + p + p_{\text{atm}} \tag{2.3.10}$$

are the total pressures given as respective static, dynamic and constant atmospheric pressure terms, ρ_1 and ρ (due to the assumption of incompressibility) are the constant respective densities and g is the acceleration due to gravity. The static term is due to external 'body' forces, that is gravity, and the dynamic term is due to internal 'local' forces, that is due to the fluid motion. The system is situated on the surface of Earth, that is in a rotational frame of reference, using the procedure in [39, 74] (which is reproduced in Appendix A) the following Euler equations will be used to establish the Bernoulli condition:

$$\mathbf{u}_{1,t} + (\mathbf{u}_1.\nabla)\mathbf{u}_1 + 2\mathbf{\Omega} \times \mathbf{u}_1 + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\frac{1}{\rho_1}\nabla P_1 + \mathbf{g}, \tag{2.3.11}$$

and

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla P + \mathbf{g} \tag{2.3.12}$$

where $\mathbf{\Omega}$ is the Earth's angular velocity, $\mathbf{r}$ is a position vector and $\mathbf{g} = (0, 0, -g)$ is the Earth acceleration, that is g is the magnitude of the acceleration due to gravity. $2\mathbf{\Omega} \times \mathbf{u}_1$ and $2\mathbf{\Omega} \times \mathbf{u}$ are the Coriolis acceleration terms and $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is the centripetal acceleration. Using (2.2.8) the following substitutions can be made

$$\mathbf{u}_{1,t} = \begin{bmatrix} \widetilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1 \\ \widetilde{\varphi}_{1,y} \end{bmatrix}_t = \nabla(\widetilde{\varphi}_{1,t}) \text{ and } \mathbf{u}_t = \begin{bmatrix} \widetilde{\varphi}_x + \gamma y + \kappa \\ \widetilde{\varphi}_y \end{bmatrix}_t = \nabla(\widetilde{\varphi}_t)$$

and

$$(\mathbf{u}_1.\nabla)\mathbf{u}_1 = \begin{bmatrix} \psi_{1,y}\psi_{1,yx} - \psi_{1,x}\psi_{1,yy} \\ -\psi_{1,y}\psi_{1,xx} + \psi_{1,x}\psi_{1,xy} \end{bmatrix} \text{ and } (\mathbf{u} \cdot \nabla)\mathbf{u} = \begin{bmatrix} \psi_y\psi_{yx} - \psi_x\psi_{yy} \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} \end{bmatrix}.$$

Now using (2.3.5) and (2.3.6)

$\psi_{1,xx} = \gamma_1 - \psi_{1,yy}$, $\psi_{xx} = \gamma - \psi_{yy}$ and $\psi_{1,yy} = \gamma_1 - \psi_{1,xx}$, $\psi_{yy} = \gamma - \psi_{xx}$

giving

$$(\mathbf{u}_1.\nabla)\mathbf{u}_1 = (\nabla\psi_1.\nabla)\nabla\psi_1 - \gamma_1\nabla\psi_1.$$

and

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\psi.\nabla)\nabla\psi - \gamma\nabla\psi.$$

Using the vector identities -

$$(\nabla\psi_1.\nabla)\nabla\psi_1 = \nabla\Big(\frac{1}{2}|\nabla\psi_1|^2\Big),$$

and

$$(\nabla\psi.\nabla)\nabla\psi = \nabla\Big(\frac{1}{2}|\nabla\psi|^2\Big)$$

it follows that

$$(\mathbf{u}_1.\nabla)\mathbf{u}_1 = \nabla\Big(\frac{1}{2}|\nabla\psi_1|^2\Big) - \gamma_1\nabla\psi_1, \tag{2.3.13}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\Big(\frac{1}{2}|\nabla\psi|^2\Big) - \gamma\nabla\psi. \tag{2.3.14}$$

Recalling the definitions of total pressure in each layer from (2.3.9) and (2.3.10) gives, for the lower medium Ω,

$$P = \rho_1 g h_1 - \rho g y + p + p_{\text{atm}},$$

that is the static pressure due to the column of fluid in the upper medium plus the column of fluid above the position in the lower medium plus the dynamic pressure plus the atmospheric pressure. For the upper medium Ω_1, the total pressure is

$$P_1 = \rho_1 g(h_1 - y) + p_1 + p_{\text{atm}}$$

that is the pressure due to the column of fluid above the position in the upper medium plus the dynamic pressure plus the atmospheric pressure. This gives

$$-\frac{1}{\rho_1}\nabla P_1 = -\nabla gy - \frac{1}{\rho_1}\nabla p_1, \tag{2.3.15}$$

$$-\frac{1}{\rho}\nabla P = -\nabla gy - \frac{1}{\rho}\nabla p. \tag{2.3.16}$$

The Coriolis acceleration terms can be written for the two-dimensional flows under consideration as [74]

$$2\mathbf{\Omega} \times \mathbf{u}_1 = 2\omega \begin{bmatrix} v_1 \\ -u_1 \end{bmatrix} = -2\omega\nabla\psi_1, \tag{2.3.17}$$

and
$$2\mathbf{\Omega} \times \mathbf{u} = 2\omega \begin{bmatrix} v \\ -u \end{bmatrix} = -2\omega\nabla\psi \tag{2.3.18}$$

where ω is the Earth's rotational speed. The centripetal acceleration term $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ is directed towards the Earth's axis of rotation and is therefore conservative and can be incorporated into the potentials $\nabla(gy)$ in (2.3.15) and (2.3.16). Euler's equation can now be expressed in terms of gradients as

$$\nabla(\widetilde{\varphi}_{1,t}) + \nabla\Big(\frac{1}{2}|\nabla\psi_1|^2\Big) - \nabla(\gamma_i\psi_1) - 2\omega\nabla\psi_1 = -\nabla(gy) - \frac{1}{\rho_1}\nabla p_1,$$

and
$$\nabla(\widetilde{\varphi}_t) + \nabla\Big(\frac{1}{2}|\nabla\psi|^2\Big) - \nabla(\gamma_i\psi) - 2\omega\nabla\psi = -\nabla(gy) - \frac{1}{\rho}\nabla p$$

and so the gradient of the dynamic pressures are given as

$$\nabla p_1 = -\rho_1\nabla\Big(\widetilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - (\gamma_1 + 2\omega)\psi_1 + gy\Big), \tag{2.3.19}$$

and
$$\nabla p = -\rho\nabla\Big(\widetilde{\varphi}_t + \frac{1}{2}|\nabla\psi|^2 - (\gamma + 2\omega)\psi + gy\Big). \tag{2.3.20}$$

This means that

$$\widetilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - (\gamma_1 + 2\omega)\psi_1 + gy + \frac{p_1}{\rho_1} = \alpha(t) \tag{2.3.21}$$

and

$$\widetilde{\varphi}_t + \frac{1}{2}|\nabla\psi|^2 - (\gamma + 2\omega)\psi + gy + \frac{p}{\rho} = \beta(t) \tag{2.3.22}$$

for some arbitrary functions of time $\alpha(t)$ and $\beta(t)$. At the interface, where $y = \eta$, the dynamic pressures p and p_1 are equal, therefore using (2.3.21) and (2.3.22) this gives

$$\rho_1\alpha(t) - \rho_1\Big((\widetilde{\varphi}_{1,t})_c + \frac{1}{2}|\nabla\psi_1|_c^2 - (\gamma_1 + 2\omega)\chi_1 + g\eta\Big) = \rho\beta(t) - \rho\Big((\widetilde{\varphi}_t)_c + \frac{1}{2}|\nabla\psi|_c^2 - (\gamma + 2\omega)\chi + g\eta\Big) \tag{2.3.23}$$

noting the use of the subscript c to signify evaluation at the common interface and introducing

$$\chi_1 := \psi_1(x, \eta, t), \tag{2.3.24}$$

and

$$\chi := \psi(x, \eta, t) \tag{2.3.25}$$

as the interface stream functions. From the assumptions in Section 2.2, as the absolute value of x goes to infinity then the terms in the large brackets in (2.3.23) go to zero giving

$$\rho_1\alpha(t) = \rho\beta(t) \tag{2.3.26}$$

therefore giving the Bernoulli condition

$$\rho_1\Big((\widetilde{\varphi}_{1,t})_c + \frac{1}{2}|\nabla\psi_1|_c^2 - (\gamma_1 + 2\omega)\chi_1 + g\eta\Big) = \rho\Big((\widetilde{\varphi}_t)_c + \frac{1}{2}|\nabla\psi|_c^2 - (\gamma + 2\omega)\chi + g\eta\Big). \tag{2.3.27}$$

The interface is defined as the region having a vertical displacement equal to the surface function, that is $y - \eta = 0$. The kinematic boundary condition on the interface is equivalent to saying that a particle on the interface will stay on the interface [36]. In other words the Lagrangian (or Convective) derivative of the interface function gives

$$(y-\eta)_t + \mathbf{u}_1.\nabla(y-\eta) = (y-\eta)_t + \mathbf{u}\cdot\nabla(y-\eta) = 0.$$

Noting the independence of x, y and t under the Eulerian framework, and the independence of $\eta(x,t)$ and y, this therefore gives the kinematic boundary condition at the interface as

$$\eta_t = v_1 - u_1\eta_x = v - u\eta_x. \tag{2.3.28}$$

This can be expressed in terms of the stream functions, using (2.2.5), as

$$\eta_t = -(\psi_{1,x})_c - (\psi_{1,y})_c\eta_x = -(\psi_x)_c - (\psi_y)_c\eta_x, \tag{2.3.29}$$

and in terms of the velocity potentials, using (2.2.8), as

$$\eta_t = (\widetilde{\varphi}_{1,y})_c - \big((\widetilde{\varphi}_{1,x})_c + \gamma_1\eta + \kappa_1\big)\eta_x = (\widetilde{\varphi}_y)_c - \big((\widetilde{\varphi}_x)_c + \gamma\eta + \kappa\big)\eta_x. \tag{2.3.30}$$

The kinematic boundary condition at the bottom, requiring that there is no velocity component in the y-direction on the flatbed, is given by

$$\big(\widetilde{\varphi}(x,-h,t)\big)_y = 0 \text{ and } \big(\psi(x,-h,t)\big)_x = 0 \tag{2.3.31}$$

and, additionally, there is a kinematic boundary condition at the top, requiring that there is no velocity component in the y-direction on the surface, given by

$$\big(\widetilde{\varphi}_1(x,h_1,t)\big)_y = 0 \text{ and } \big(\psi_1(x,h_1,t)\big)_x = 0. \tag{2.3.32}$$

Chapter 3

HAMILTONIAN FORMULATION

3.1 Definition of the Hamiltonian

The Hamiltonian, H, of a single medium system with a domain defined by $\{(x, y) \in \mathbb{R}^2 : a < y < b\}$ with density ρ and velocity field $\mathbf{u}$=(u, v) is a functional representing the total energy given by

$$H = T + U \tag{3.1.1}$$

where T is the kinetic energy given by

$$T = \frac{1}{2}\rho \int\limits_{\mathbb{R}} \int_a^b (u^2 + v^2) dydx, \tag{3.1.2}$$

and U is the potential energy given by

$$U = \rho g \int\limits_{\mathbb{R}} \int_a^b y \, dydx \tag{3.1.3}$$

with g the acceleration due to gravity.

3.2 The Hamiltonian of the 2-Media System

The Hamiltonian for the 2-media system as depicted in Figure 3.1 is given as the sum of the kinetic and potential energies by the functional H as

$$\begin{aligned} H(\eta, \mathbf{u}, \mathbf{u}_1) = \frac{1}{2}\rho \int_{\mathbb{R}} \int_{-h}^{\eta} (u^2 + v^2) dy dx + \frac{1}{2}\rho_1 \int_{\mathbb{R}} \int_{\eta}^{h_1} (u_1^2 + v_1^2) dy dx \\ + \rho g \int_{\mathbb{R}} \int_{-h}^{\eta} y dy dx + \rho_1 g \int_{\mathbb{R}} \int_{\eta}^{h_1} y dy dx + \int_{\mathbb{R}} \mathfrak{h}_0 dx \end{aligned} \tag{3.2.1}$$

where $\mathfrak{h}_0$ is a constant Hamiltonian density (with zero variations), compensating for any constant terms that arise in the other integrals, so that the overall Hamiltonian density is a function from the Schwartz class $\mathcal{S}(\mathbb{R})$.

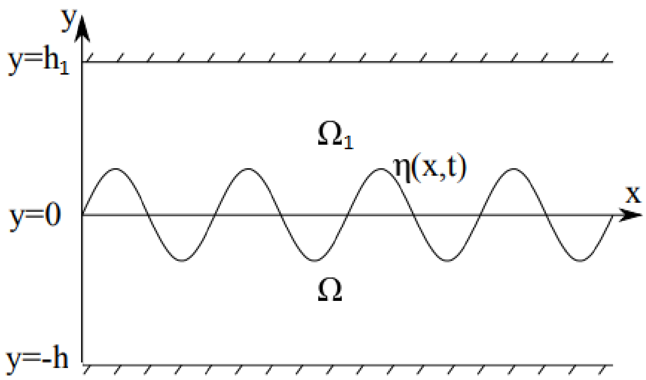

Figure 3.1: 2-Media Water Wave System

Recalling the definitions of $\widetilde{\varphi}$ and $\widetilde{\varphi}_1$ in (2.2.8), the Hamiltonian can be expanded and expressed in terms of the dependent variables $\eta(x,t)$, $\widetilde{\varphi}(x,t)$ and $\widetilde{\varphi}_1(x,t)$ as

$$H(\eta,\widetilde{\varphi},\widetilde{\varphi}_1)=\frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx+\frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx$$

$$+\rho\gamma\int_{\mathbb{R}}\int_{-h}^{\eta}y\widetilde{\varphi}_xdydx+\rho_1\gamma_1\int_{\mathbb{R}}\int_{\eta}^{h_1}y\widetilde{\varphi}_{1,x}dydx+\rho\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}\widetilde{\varphi}_xdydx+\rho_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\widetilde{\varphi}_{1,x}dydx$$

$$+\frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}(\gamma y+\kappa)^2dydx+\frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}(\gamma_1y+\kappa_1)^2dydx$$

$$+\rho g\int_{\mathbb{R}}\int_{-h}^{\eta}y\,dydx+\rho_1g\int_{\mathbb{R}}\int_{\eta}^{h_1}y\,dydx+\int_{\mathbb{R}}\mathfrak{h}_0dx.\quad(3.2.2)$$

3.3 The Hamiltonian using Dirichlet-Neumann operators

The Dirichlet-Neumann (DN) operator is introduced to enable the Hamiltonian to be expressed in terms of wave quantities only. The DN operator is a mapfrom a boundary type condition called the 'Dirichlet condition' to a derivative type condition called the 'Neumann condition' for a PDE. For a fluid domain bounded below by a flatbed, and with a free surface, the Dirichlet-Neumann operator, denoted by $G(\eta)$, is the normal derivative of the velocity potential at the surface. Adapting this to the two-media system under study, the Dirichlet-Neumann operators $G(\eta)$ and $G_1(\eta)$ are defined as [33, 62]

$$G(\eta)\phi=(\widetilde{\varphi}_{\mathbf{n}})_c\sqrt{1+\eta_x^2}\quad(3.3.1)$$

and

$$G_1(\eta)\phi_1=(\widetilde{\varphi}_{\mathbf{n}_1})_c\sqrt{1+\eta_x^2},\quad(3.3.2)$$

where $\mathbf{n}$ and $\mathbf{n}_1$ are the *unit* outward normal vectors to the corresponding domains, $\widetilde{\varphi}_{\mathbf{n}}$ and $\widetilde{\varphi}_{\mathbf{n}_1}$ are the normal derivatives in each domain, with $\mathbf{n}=-\mathbf{n}_1$ and $\sqrt{1+(\eta_x)^2}$

is a normalisation factor and

$$\phi := \widetilde{\varphi}(x, \eta(x,t), t) \tag{3.3.3}$$

and

$$\phi_1 := \widetilde{\varphi}_1(x, \eta(x,t), t) \tag{3.3.4}$$

have been introduced as the interface velocity potentials.

The Dirichlet-Neumann operator can also be defined as [31]

$$G(\eta)\phi = (\nabla\varphi)_c.(\mathbf{n})_c\sqrt{1+\eta_x^2} \tag{3.3.5}$$

and

$$G_1(\eta)\phi_1 = (\nabla\varphi_1)_c.(\mathbf{n}_1)_c\sqrt{1+\eta_x^2}. \tag{3.3.6}$$

Recalling $(\mathbf{n})_c$ from (2.2.11) means that, for Ω,

$$G(\eta)\phi = (\nabla\varphi)_c.(-\eta_x, 1),$$

and similar for Ω_1, so

$$G(\eta)\phi = -\eta_x(\widetilde{\varphi}_x)_c + (\widetilde{\varphi}_y)_c \tag{3.3.7}$$

and

$$G_1(\eta)\phi_1 = \eta_x(\widetilde{\varphi}_{1,x})_c - (\widetilde{\varphi}_{1,y})_c. \tag{3.3.8}$$

Recalling the first and second terms of the Hamiltonian given by (3.2.2)

$$\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx + \frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx \tag{3.3.9}$$

with the following identity for some vector field $\mathbf{A}$ and some scalar field Φ

$$\nabla\cdot(\Phi\mathbf{A}) = \mathbf{A}.\nabla\Phi + \Phi\nabla\cdot\mathbf{A},$$

it can be written that

$$\nabla\cdot\big((\nabla\varphi)\varphi\big) = |\nabla\varphi|^2 + (\Delta\varphi)\varphi$$

and
$$\nabla\cdot\big((\nabla\varphi_1)\varphi_1\big) = |\nabla\varphi_1|^2 + (\Delta\varphi_1)\varphi_1.$$

However, $\Delta\varphi$ and $\Delta\varphi_1$ are zero as per (2.3.2) and so

$$|\nabla\varphi|^2 = \nabla\cdot\big((\nabla\varphi)\varphi\big) \tag{3.3.10}$$

and
$$|\nabla\varphi_1|^2 = \nabla\cdot\big((\nabla\varphi_1)\varphi_1\big). \tag{3.3.11}$$

Using the divergence theorem the following integral for the Ω domain can be expressed using outward normals as shown in Figure 2.6 as

$$\begin{aligned}&\int_{\mathbb{R}}\int_{-h}^{\eta} \nabla\cdot\big((\nabla\widetilde{\varphi})\widetilde{\varphi}\big)dydx\\ &= \int_{\mathbb{R}} \big((\nabla\widetilde{\varphi})_c\phi\big).(\mathbf{n})_c\sqrt{1+(\eta_x)^2}dx + \int_{\mathbb{R}} \big((\nabla\widetilde{\varphi})_b(\widetilde{\varphi})_b\big).(\mathbf{n})_b\sqrt{1+(\eta_x)^2}dx,\end{aligned} \tag{3.3.12}$$

recalling from (2.2.11) that

$$(\mathbf{n})_c = (-\eta_x, 1) \text{ and } (\mathbf{n})_b = (0,-1).$$

From (2.3.31) $(\widetilde{\varphi}_y)_b = 0$ and therefore

$$(\nabla\widetilde{\varphi})_b.(\mathbf{n})_b = (\widetilde{\varphi}_x, 0).(0,-1) = 0$$

giving

$$\int_{-h}^{\eta} \nabla\cdot\big((\nabla\widetilde{\varphi})\widetilde{\varphi}\big)dy = \big((\nabla\widetilde{\varphi})_c\phi\big).(\mathbf{n})_c\sqrt{1+(\eta_x)^2}.$$

Comparing this to the expression given for the Dirichlet-Neumann operator in (3.3.5)

$$G(\eta)\phi = \nabla\widetilde{\varphi}.(\mathbf{n})_c\sqrt{1+(\eta_x)^2}$$

the first term of the Hamiltonian can be represented in terms of η and ϕ as

$$\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx = \frac{1}{2}\rho\int\limits_{\mathbb{R}}\phi G(\eta)\phi dx. \tag{3.3.13}$$

and similarly for Ω_1

$$\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx = \frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\phi_1 G_1(\eta)\phi_1 dx \tag{3.3.14}$$

so the first 2 terms of (3.2.2) can be expressed in terms of Dirichlet-Neumann operators as

$$\begin{aligned}\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx + \frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx \\ = \frac{1}{2}\rho\int\limits_{\mathbb{R}}\phi G(\eta)\phi dx + \frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\phi_1 G_1(\eta)\phi_1 dx.\end{aligned} \tag{3.3.15}$$

Using the kinematic boundary conditions from (2.3.30)

$$G(\eta)\phi = -\eta_x(\widetilde{\varphi}_x)_c + (\widetilde{\varphi}_y)_c = \eta_t + (\gamma\eta+\kappa)\eta_x$$

and $$G_1(\eta)\phi_1 = \eta_x(\widetilde{\varphi}_{1,x})_c - (\widetilde{\varphi}_{1,y})_c = -\eta_t - (\gamma_1\eta+\kappa_1)\eta_x$$

giving

$$G(\eta)\phi + G_1(\eta)\phi_1 = \mu \tag{3.3.16}$$

where the function μ has been introduced as

$$\mu(x) := \big((\gamma - \gamma_1)\eta + (\kappa - \kappa_1)\big)\eta_x. \tag{3.3.17}$$

The term $\rho\phi - \rho_1\phi_1$ is defined as per [4, 5] as

$$\xi(x,t) := \rho\phi - \rho_1\phi_1 \tag{3.3.18}$$

so

$$\big(\rho G_1(\eta) + \rho_1 G(\eta)\big)\phi = G_1(\eta)\xi + \rho_1\mu$$

and

$$\big(\rho G_1(\eta) + \rho_1 G(\eta)\big)\phi_1 = -G(\eta)\xi + \rho\mu\cdot$$

Introducing the operator B [31] as

$$B := \rho G_1(\eta) + \rho_1 G(\eta) \tag{3.3.19}$$

the potentials ϕ and ϕ_1 can be written as

$$\phi = B^{-1}\big(G_1(\eta)\xi + \rho_1\mu\big) \tag{3.3.20}$$

and

$$\phi_1 = B^{-1}\big(- G(\eta)\xi + \rho\mu\big). \tag{3.3.21}$$

Recalling (3.3.15), $G_1(\eta)\phi_1$ is replaced using (3.3.16) and also using (3.3.18) means

$$\frac{1}{2}\rho\int_{\mathbb{R}} \phi G(\eta)\phi dx + \frac{1}{2}\rho_1\int_{\mathbb{R}} \phi_1 G_1(\eta)\phi_1 dx = \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)\phi dx + \frac{1}{2}\rho_1\int_{\mathbb{R}} \phi_1\mu\, dx.$$

Using the expression (3.3.20) for ϕ gives

$$\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\varphi|^2dydx+\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\varphi_1|^2dydx$$
$$=\frac{1}{2}\int\limits_{\mathbb{R}}\xi G(\eta)B^{-1}G_1(\eta)\xi\,dx+\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\xi G(\eta)B^{-1}\mu\,dx$$
$$-\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\big(B^{-1}G(\eta)\xi\big)\mu\,dx+\frac{1}{2}\rho\rho_1\int\limits_{\mathbb{R}}\mu B^{-1}\mu\,dx. \quad (3.3.22)$$

The second and third terms on the right hand side cancel because the operators G and B are self-adjoint (*cf.* [31], [32]) therefore the first two terms on the left hand side of (3.2.2) can be written as

$$\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\varphi|^2dydx+\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\varphi_1|^2dydx$$
$$=\frac{1}{2}\int\limits_{\mathbb{R}}\xi G(\eta)B^{-1}G_1(\eta)\xi\,dx+\frac{1}{2}\rho\rho_1\int\limits_{\mathbb{R}}\mu B^{-1}\mu\,dx. \quad (3.3.23)$$

For a differentiable function $F(x,y)$ [20]

$$\left(\int\limits_{a(x)}^{b(x)}F\,dy\right)_x=\int\limits_{a(x)}^{b(x)}F_xdy-F[x,a]a_x+F[x,b]b_x. \quad (3.3.24)$$

The following corollary of (3.3.24) is introduced.

Corollary 1. *Considering (3.3.24), and letting $F=y\widetilde{\varphi}$, $a=-h$ and $b(x)=\eta(x)$, the left-hand side is zero due to assumption (2.2.9) and therefore*

$$\int\limits_{-h}^{\eta(x)}y\widetilde{\varphi}_xdy=-\phi\eta\eta_x.$$

Similarly letting $F = y\widetilde{\varphi}_1$, $a(x) = \eta(x)$ *and* $b = h_1$

$$\int\limits_{\eta(x)}^{h_1} y\widetilde{\varphi}_{1,x}dy = \phi_1\eta\eta_x.$$

Terms three and four of (3.2.2) are rewritten using Corollary 1 as

$$\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta} y\widetilde{\varphi}_x dydx + \rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1} y\widetilde{\varphi}_{1,x}dydx = -\int\limits_{\mathbb{R}}(\rho\gamma\phi\eta\eta_x - \rho_1\gamma_1\phi_1\eta\eta_x)dx.$$

Now, inserting the expressions for ϕ and ϕ_1 from (3.3.20) and (3.3.21), and expanding gives

$$\begin{aligned}&\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta} y\widetilde{\varphi}_x dydx + \rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1} y\widetilde{\varphi}_{1,x}dydx \\ &= \int\limits_{\mathbb{R}}(-\rho\gamma B^{-1}G_1(\eta)\xi - \rho\rho_1\gamma B^{-1}\mu - \rho_1\gamma_1B^{-1}G(\eta)\xi + \rho\rho_1\gamma_1B^{-1}\mu)\eta\eta_x dx. \quad (3.3.25)\end{aligned}$$

Similarly terms five and six of (3.2.2) are rewritten using Corollary 1 as

$$\rho\kappa\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta} \widetilde{\varphi}_x dydx + \rho_1\kappa_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1} \widetilde{\varphi}_{1,x}dydx = -\int\limits_{\mathbb{R}}(\rho\kappa\phi\eta_x - \rho_1\kappa_1\phi_1\eta_x)dx.$$

Again using expressions for ϕ and ϕ_1 from (3.3.20) and (3.3.21) and expanding gives

$$\begin{aligned}&\rho\kappa\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta} \widetilde{\varphi}_x dydx + \rho_1\kappa_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1} \widetilde{\varphi}_{1,x}dydx \\ &= \int\limits_{\mathbb{R}}\big(-\rho\kappa B^{-1}G_1(\eta)\xi - \rho\rho_1\kappa B^{-1}\mu - \rho_1\kappa_1B^{-1}G(\eta)\xi + \rho\rho_1\kappa_1B^{-1}\mu\big)\eta_x dx. \quad (3.3.26)\end{aligned}$$

Combining terms (3.3.25) and (3.3.26), and using

$$\big((\rho_1\gamma - \rho_1\gamma)\eta + \rho_1\kappa - \rho_1\kappa\big)B^{-1}G(\eta)\xi\eta_x = 0$$

and recalling the definitions of μ (from (3.3.17)) and B (from (3.3.19)), the terms are written as

$$\int_{\mathbb{R}} \big(- \rho\rho_1 \mu B^{-1}\mu + \rho_1 \mu B^{-1} G(\eta)\xi - (\gamma\eta + \kappa)\xi\eta_x \big) dx. \tag{3.3.27}$$

The remaining terms of (3.2.2) are rewritten as

$$\begin{aligned} \frac{1}{2}\rho \int_{\mathbb{R}} \int_{-h}^{\eta} (\gamma y + \kappa)^2 dy dx + \frac{1}{2}\rho_1 \int_{\mathbb{R}} \int_{\eta}^{h_1} (\gamma_1 y + \kappa_1)^2 dy dx \\ + \rho g \int_{\mathbb{R}} \int_{-h}^{\eta} y \, dy dx + \rho_1 g \int_{\mathbb{R}} \int_{\eta}^{h_1} y \, dy dx \\ = \frac{\rho}{6\gamma} \int_{\mathbb{R}} (\gamma\eta + \kappa)^3 dx - \frac{\rho_1}{6\gamma_1} \int_{\mathbb{R}} (\gamma_1\eta + \kappa_1)^3 dx + \frac{1}{2} g(\rho - \rho_1) \int_{\mathbb{R}} \eta^2 dx. \end{aligned} \tag{3.3.28}$$

Substituting (3.3.23), (3.3.27) and (3.3.28) into the expression for the Hamiltonian given in (3.2.2) gives the Hamiltonian of the system in terms of the conjugate variables η and ξ as

$$\begin{aligned} H(\eta, \xi) = \frac{1}{2} \int_{\mathbb{R}} \xi G(\eta) B^{-1} G_1(\eta)\xi \, dx - \frac{1}{2}\rho\rho_1 \int_{\mathbb{R}} \mu B^{-1}\mu \, dx - \int_{\mathbb{R}} (\gamma\eta + \kappa)\xi\eta_x dx \\ + \rho_1 \int_{\mathbb{R}} \mu B^{-1} G(\eta)\xi \, dx + \frac{\rho}{6\gamma} \int_{\mathbb{R}} (\gamma\eta + \kappa)^3 dx - \frac{\rho_1}{6\gamma_1} \int_{\mathbb{R}} (\gamma_1\eta + \kappa_1)^3 dx + \frac{1}{2} g(\rho - \rho_1) \int_{\mathbb{R}} \eta^2 dx \\ + \int_{\mathbb{R}} \mathfrak{h}_0 dx. \end{aligned} \tag{3.3.29}$$

In the special case $\gamma_1 = \gamma$, $\kappa_1 = \kappa$ and $\mu = 0$ the Hamiltonian acquires the form

$$\begin{aligned} H(\eta, \xi) = \frac{1}{2} \int_{\mathbb{R}} \xi G(\eta) B^{-1} G_1(\eta)\xi \, dx - \int_{\mathbb{R}} (\gamma\eta + \kappa)\xi\eta_x \, dx \\ + \frac{\rho - \rho_1}{6\gamma} \int_{\mathbb{R}} (\gamma\eta + \kappa)^3 dx + \frac{1}{2} g(\rho - \rho_1) \int_{\mathbb{R}} \eta^2 dx \end{aligned} \tag{3.3.30}$$

thus recovering the Hamiltonian determined in [15]. When $\gamma_1 \neq \gamma$ and $\kappa_1 = \kappa = 0$ then $\mu = (\gamma - \gamma_1)\eta\eta_x$ and the Hamiltonian becomes

$$\begin{aligned} H(\eta,\xi) =& \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\, dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}} \eta\eta_x B^{-1}G(\eta)\xi\, dx \\ &- \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2\int_{\mathbb{R}} \eta\eta_x B^{-1}\eta\eta_x\, dx - \gamma\int_{\mathbb{R}} \xi\eta\eta_x\, dx \\ &+ \frac{1}{6}(\rho\gamma^2 - \rho_1\gamma_1^2)\int_{\mathbb{R}} \eta^3 dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx \end{aligned} \tag{3.3.31}$$

which, given the definition of B from (3.3.19), recovers the result in [14].

In the situation with $\kappa_1 = \kappa$ which is physically realistic because the unperturbed currents at the two layers have the same speed at the interface (that is absence of a vortex sheet) and $\mu = (\gamma - \gamma_1)\eta\eta_x$, giving

$$\begin{aligned} H(\eta,\xi) =& \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\, dx - \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2\int_{\mathbb{R}} \eta\eta_x B^{-1}\eta\eta_x\, dx \\ &- \int_{\mathbb{R}} (\gamma\eta+\kappa)\xi\eta_x\, dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}} \eta\eta_x B^{-1}G(\eta)\xi\, dx \\ &+ \frac{\rho\gamma^2 - \rho_1\gamma_1^2}{6}\int_{\mathbb{R}} \eta^3 dx + \frac{g(\rho-\rho_1) + (\rho\gamma - \rho_1\gamma_1)\kappa}{2}\int_{\mathbb{R}} \eta^2 dx. \end{aligned} \tag{3.3.32}$$

3.4 Equations of Motion

The dynamics of the system will be calculated using variational calculus. The term 'dynamics', refers to the calculation of the time-derivatives of the two conjugate variables in terms of functional derivatives of the functional H. These will be referred to as the 'equations of motion'.

Recalling the Hamiltonian from (3.2.2) the variation in the first two terms is

$$\delta\left[\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx+\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx\right]=\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}(\nabla\widetilde{\varphi}).\nabla\delta\widetilde{\varphi}dydx$$
$$+\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}(\nabla\widetilde{\varphi}_1).\nabla\delta\widetilde{\varphi}_1dydx+\frac{1}{2}\rho\int\limits_{\mathbb{R}}|\nabla\widetilde{\varphi}|_c^2\delta\eta\,dx-\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}|\nabla\widetilde{\varphi}_1|_c^2\delta\eta\,dx. \quad (3.4.1)$$

Looking at $\widetilde{\varphi}$ and using the product rule

$$\nabla\cdot\big((\nabla\widetilde{\varphi})\delta\widetilde{\varphi}\big)=\nabla\cdot(\nabla\widetilde{\varphi})\delta\widetilde{\varphi}+(\nabla\widetilde{\varphi}).\nabla(\delta\widetilde{\varphi})$$

but, due to the assumption of incompressibility (see (2.3.2)), $\Delta\varphi=0$ and hence

$$(\nabla\widetilde{\varphi}).\nabla\delta\widetilde{\varphi}=\nabla\cdot\big((\nabla\widetilde{\varphi})\delta\widetilde{\varphi}\big).$$

Similarly for $\widetilde{\varphi}_1$

$$\nabla\cdot\big((\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1\big)=\nabla\cdot(\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1+(\nabla\widetilde{\varphi}_1).\nabla(\delta\widetilde{\varphi}_1)$$

and due to the assumption of incompressibility (see (2.3.2)), $\Delta\widetilde{\varphi}_1=0$ and hence

$$(\nabla\widetilde{\varphi}_1).\nabla\delta\widetilde{\varphi}_1=\nabla\cdot\big((\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1\big).$$

Using the divergence theorem

$$\iint\limits_{\Omega} \nabla \cdot \big((\nabla\widetilde{\varphi})\delta\widetilde{\varphi}\big)dydx = \int\limits_{\mathbb{R}} \big((\nabla\widetilde{\varphi})\delta\widetilde{\varphi}\big).\mathbf{n}dS \tag{3.4.2}$$

and

$$\iint\limits_{\Omega_1} \nabla \cdot \big((\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1\big)dydx = \int\limits_{\mathbb{R}} \big((\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1\big).\mathbf{n}_1dS, \tag{3.4.3}$$

where $\mathbf{n}$ and $\mathbf{n}_1$ are the outward normal vectors in the corresponding domains and dS is an infinitesimal surface area.

Recalling $(\mathbf{n})_c$ and $(\mathbf{n})_b$ from (2.2.11) it follows that

$$\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta} \nabla \cdot \big((\nabla\widetilde{\varphi})\delta\widetilde{\varphi}\big)dydx$$

$$= \int\limits_{\mathbb{R}} \left[\begin{bmatrix}(\widetilde{\varphi}_x)_c \\ (\widetilde{\varphi}_y)_c\end{bmatrix}(\delta\widetilde{\varphi})_c\right] \cdot \begin{bmatrix}-\eta_x \\ 1\end{bmatrix} dx + \int\limits_{\mathbb{R}} \left[\begin{bmatrix}(\widetilde{\varphi}_x)_b \\ (\widetilde{\varphi}_y)_b\end{bmatrix}(\delta\widetilde{\varphi})_b\right] \cdot \begin{bmatrix}0 \\ -1\end{bmatrix} dx$$

and, recalling $(\mathbf{n}_1)_c$ and $(\mathbf{n}_1)_t$ from (2.2.12), it follows that

$$\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1} \nabla \cdot \big((\nabla\widetilde{\varphi}_1)\delta\widetilde{\varphi}_1\big)dydx$$

$$= \int\limits_{\mathbb{R}} \left[\begin{bmatrix}(\widetilde{\varphi}_{1,x})_c \\ (\widetilde{\varphi}_{1,y})_c\end{bmatrix}(\delta\widetilde{\varphi}_1)_c\right] \cdot \begin{bmatrix}\eta_x \\ -1\end{bmatrix} dx + \int\limits_{\mathbb{R}} \left[\begin{bmatrix}(\widetilde{\varphi}_{1,x})_t \\ (\widetilde{\varphi}_{1,y})_t\end{bmatrix}(\delta\widetilde{\varphi}_1)_t\right] \cdot \begin{bmatrix}0 \\ 1\end{bmatrix} dx.$$

Noting that the variations in the velocity potential at the flatbed and at the surface are zero, the variation in the first two terms of (3.2.2) can be written as

$$\delta\left[\frac{1}{2}\rho\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}|\nabla\widetilde{\varphi}|^2dydx+\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}|\nabla\widetilde{\varphi}_1|^2dydx\right]$$

$$=\rho\int\limits_{\mathbb{R}}\big((\widetilde{\varphi}_y)_c-(\widetilde{\varphi}_x)_c\eta_x\big)(\delta\widetilde{\varphi})_cdx-\rho_1\int\limits_{\mathbb{R}}\big((\widetilde{\varphi}_{1,y})_c-(\widetilde{\varphi}_{1,x})_c\eta_x\big)(\delta\widetilde{\varphi}_1)_cdx$$

$$+\frac{1}{2}\rho\int\limits_{\mathbb{R}}|\nabla\widetilde{\varphi}|_c^2\delta\eta\,dx-\frac{1}{2}\rho_1\int\limits_{\mathbb{R}}|\nabla\widetilde{\varphi}_1|_c^2\delta\eta\,dx. \quad (3.4.4)$$

Next, the variation in the third and fourth terms of (3.2.2) is

$$\delta\left[\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}y\widetilde{\varphi}_xdydx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}y\widetilde{\varphi}_{1,x}dydx\right]=\rho\gamma\int\limits_{\mathbb{R}}\eta(\widetilde{\varphi}_x)_c\delta\eta dx$$

$$-\rho_1\gamma_1\int\limits_{\mathbb{R}}\eta(\widetilde{\varphi}_{1,x})_c\delta\eta dx+\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}y\delta(\widetilde{\varphi}_x)dydx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}y\delta(\widetilde{\varphi}_{1,x})dydx.$$

Now, using Corollary 1 (p.38)

$$\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}y\delta(\widetilde{\varphi}_x)dydx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}y\delta(\widetilde{\varphi}_{1,x})dydx$$

$$=-\rho\gamma\int\limits_{\mathbb{R}}\eta\eta_x(\delta\widetilde{\varphi})_cdx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\eta\eta_x(\delta\widetilde{\varphi}_1)_cdx,$$

and hence

$$\delta\left[\rho\gamma\int\limits_{\mathbb{R}}\int\limits_{-h}^{\eta}y\widetilde{\varphi}_xdydx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\int\limits_{\eta}^{h_1}y\widetilde{\varphi}_{1,x}dydx\right]=\rho\gamma\int\limits_{\mathbb{R}}\eta(\widetilde{\varphi}_x)_c\delta\eta dx$$

$$-\rho_1\gamma_1\int\limits_{\mathbb{R}}\eta(\widetilde{\varphi}_{1,x})_c\delta\eta dx-\rho\gamma\int\limits_{\mathbb{R}}\eta\eta_x(\delta\widetilde{\varphi})_cdx+\rho_1\gamma_1\int\limits_{\mathbb{R}}\eta\eta_x(\delta\widetilde{\varphi}_1)_cdx. \quad (3.4.5)$$

The variation in the fifth and sixth terms of (3.2.2) is

$$\delta\left[\rho\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}\widetilde{\varphi}_x dydx+\rho_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\widetilde{\varphi}_{1,x}dydx\right]=\rho\kappa\int_{\mathbb{R}}(\widetilde{\varphi}_x)_c\delta\eta dx$$
$$-\rho_1\kappa_1\int_{\mathbb{R}}(\widetilde{\varphi}_{1,x})_c\delta\eta dx+\rho\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}\delta(\widetilde{\varphi}_x)dydx+\rho_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\delta(\widetilde{\varphi}_{1,x})dydx.$$

Using Corollary 1 (p.38)

$$\rho\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}\delta(\widetilde{\varphi}_x)dydx+\rho_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\delta(\widetilde{\varphi}_{1,x})dydx$$
$$=-\rho\kappa\int_{\mathbb{R}}\eta_x(\delta\widetilde{\varphi})_c dx+\rho_1\kappa_1\int_{\mathbb{R}}\eta_x(\delta\widetilde{\varphi}_1)_c dx,$$

and hence

$$\delta\left[\rho\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}\widetilde{\varphi}_x dydx+\rho_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\widetilde{\varphi}_{1,x}dydx\right]=\rho\kappa\int_{\mathbb{R}}(\widetilde{\varphi}_x)_c\delta\eta dx$$
$$-\rho_1\kappa_1\int_{\mathbb{R}}(\widetilde{\varphi}_{1,x})_c\delta\eta dx-\rho\kappa\int_{\mathbb{R}}\eta_x(\delta\widetilde{\varphi})_c dx+\rho_1\kappa_1\int_{\mathbb{R}}\eta_x(\delta\widetilde{\varphi}_1)_c dx \quad (3.4.6)$$

The variations in the remaining terms of (3.2.2) are

$$\delta\left[\frac{1}{2}\rho\gamma^2\int_{\mathbb{R}}\int_{-h}^{\eta}y^2dydx+\frac{1}{2}\rho_1\gamma_1^2\int_{\mathbb{R}}\int_{\eta}^{h_1}y^2dydx\right]=\frac{1}{2}\rho\gamma^2\int_{\mathbb{R}}\eta^2\delta\eta\, dx-\frac{1}{2}\rho_1\gamma_1^2\int_{\mathbb{R}}\eta^2\delta\eta dx,$$
$$\delta\left[\frac{1}{2}\rho\kappa^2\int_{\mathbb{R}}\int_{-h}^{\eta}dydx+\frac{1}{2}\rho_1\kappa_1^2\int_{\mathbb{R}}\int_{\eta}^{h_1}dydx\right]=\frac{1}{2}\rho\kappa^2\int_{\mathbb{R}}\delta\eta dx-\frac{1}{2}\rho_1\kappa_1^2\int_{\mathbb{R}}\delta\eta dx,$$
$$\delta\left[\rho\gamma\kappa\int_{\mathbb{R}}\int_{-h}^{\eta}ydx+\rho_1\gamma_1\kappa_1\int_{\mathbb{R}}\int_{\eta}^{h_1}ydx+\rho g\int_{\mathbb{R}}\int_{-h}^{\eta}ydx+\rho_1 g\int_{\mathbb{R}}\int_{\eta}^{h_1}ydx\right]$$
$$=\rho(g+\gamma\kappa)\int_{\mathbb{R}}\eta\delta\eta dx-\rho_1(g+\gamma_1\kappa_1)\int_{\mathbb{R}}\eta\delta\eta dx, \quad (3.4.7)$$

plus some constants which will not contribute to the dynamics.

Using (3.4.4), (3.4.5), (3.4.6) and (3.4.7) the variation is

$$\delta H = \int_{\mathbb{R}} \Big(\rho\big((\widetilde{\varphi}_y)_c - (\widetilde{\varphi}_x)_c\eta_x\big)(\delta\widetilde{\varphi})_c - \rho_1\big((\widetilde{\varphi}_{1,y})_c - (\widetilde{\varphi}_{1,x})_c\eta_x\big)(\delta\widetilde{\varphi}_1)_c$$
$$+ \frac{1}{2}\rho|\nabla\widetilde{\varphi}|_c^2\delta\eta - \frac{1}{2}\rho_1|\nabla\widetilde{\varphi}_1|_c^2\delta\eta + \rho\gamma\eta(\widetilde{\varphi}_x)_c\delta\eta - \rho_1\gamma_1\eta(\widetilde{\varphi}_{1,x})_c\delta\eta - \rho\gamma\eta\eta_x(\delta\widetilde{\varphi})_c$$
$$+ \rho_1\gamma_1\eta\eta_x(\delta\widetilde{\varphi}_1)_c + \rho\kappa(\widetilde{\varphi}_x)_c\delta\eta - \rho_1\kappa_1(\widetilde{\varphi}_{1,x})_c\delta\eta - \rho\kappa\eta_x(\delta\widetilde{\varphi})_c + \rho_1\kappa_1\eta_x(\delta\widetilde{\varphi}_1)_c$$
$$+\frac{1}{2}\rho\gamma^2\eta^2\delta\eta - \frac{1}{2}\rho_1\gamma_1^2\eta^2\delta\eta + \frac{1}{2}\rho\kappa^2\delta\eta - \frac{1}{2}\rho_1\kappa_1^2\delta\eta + \rho(g+\gamma\kappa)\eta\delta\eta - \rho_1(g+\gamma_1\kappa_1)\eta\delta\eta\Big)\,dx.$$

From the kinematic boundary conditions (2.3.30)

$$\rho\eta_t = \rho\big((\widetilde{\varphi}_y)_c - (\widetilde{\varphi}_x)_c\eta_x\big) - \rho\gamma\eta\eta_x - \rho\kappa\eta_x$$

and

$$\rho_1\eta_t = \rho_1\big((\widetilde{\varphi}_{1,y})_c - (\widetilde{\varphi}_{1,x})_c\eta_x\big) - \rho_1\gamma_1\eta\eta_x - \rho_1\kappa_1\eta_x$$

resulting in

$$\delta H = \int_{\mathbb{R}} \Big(\rho\eta_t(\delta\widetilde{\varphi})_c - \rho_1\eta_t(\delta\widetilde{\varphi}_1)_c + \frac{1}{2}\rho|\nabla\widetilde{\varphi}|_c^2\delta\eta - \frac{1}{2}\rho_1|\nabla\widetilde{\varphi}_1|_c^2\delta\eta$$
$$+ \rho\gamma\eta(\widetilde{\varphi}_x)_c\delta\eta - \rho_1\gamma_1\eta(\widetilde{\varphi}_{1,x})_c\delta\eta + \rho\kappa(\widetilde{\varphi}_x)_c\delta\eta - \rho_1\kappa_1(\widetilde{\varphi}_{1,x})_c\delta\eta + \frac{1}{2}\rho\gamma^2\eta^2\delta\eta$$
$$- \frac{1}{2}\rho_1\gamma_1^2\eta^2\delta\eta + \frac{1}{2}\rho\kappa^2\delta\eta - \frac{1}{2}\rho_1\kappa_1^2\delta\eta + \rho(g+\gamma\kappa)\eta\delta\eta - \rho_1(g+\gamma_1\kappa_1)\eta\delta\eta\Big)\,dx.$$

The variation of the velocity potential on the wave is given as [20]

$$(\delta\widetilde{\varphi})_c = \delta\phi - (\widetilde{\varphi}_y)_c\delta\eta \tag{3.4.8}$$

and

$$(\delta\widetilde{\varphi}_1)_c = \delta\phi_1 - (\widetilde{\varphi}_{1,y})_c\delta\eta \tag{3.4.9}$$

and hence

$$\delta H = \int\limits_{\mathbb{R}} \Big(- \rho\eta_t(\widetilde{\varphi}_y)_c + \rho_1\eta_t(\widetilde{\varphi}_{1,y})_c + \frac{1}{2}\rho|\nabla\widetilde{\varphi}|_c^2 - \frac{1}{2}\rho_1|\nabla\widetilde{\varphi}_1|_c^2$$
$$+ \rho\gamma\eta(\widetilde{\varphi}_x)_c - \rho_1\gamma_1\eta(\widetilde{\varphi}_{1,x})_c + \rho\kappa(\widetilde{\varphi}_x)_c - \rho_1\kappa_1(\widetilde{\varphi}_{1,x})_c + \frac{1}{2}\rho\gamma^2\eta^2$$
$$- \frac{1}{2}\rho_1\gamma_1^2\eta^2 + \frac{1}{2}\rho\kappa^2 - \frac{1}{2}\rho_1\kappa_1^2 + \rho(g+\gamma\kappa)\eta - \rho_1(g+\gamma_1\kappa_1)\eta\Big)\delta\eta dx$$
$$+ \rho\int_{\mathbb{R}} \eta_t\delta\phi dx - \rho_1\int_{\mathbb{R}}\eta_t\delta\phi_1 dx. \quad (3.4.10)$$

Fixing ϕ and ϕ_1 therefore gives the variation with respect to η as

$$\frac{\delta H}{\delta\eta} = -\rho\eta_t(\widetilde{\varphi}_y)_c + \rho_1\eta_t(\widetilde{\varphi}_{1,y})_c + \frac{1}{2}\rho|\nabla\widetilde{\varphi}|_c^2 - \frac{1}{2}\rho_1|\nabla\widetilde{\varphi}_1|_c^2$$
$$+ \rho\gamma\eta(\widetilde{\varphi}_x)_c - \rho_1\gamma_1\eta(\widetilde{\varphi}_{1,x})_c + \rho\kappa(\widetilde{\varphi}_x)_c - \rho_1\kappa_1(\widetilde{\varphi}_{1,x})_c + \frac{1}{2}\rho\gamma^2\eta^2$$
$$- \frac{1}{2}\rho_1\gamma_1^2\eta^2 + \frac{1}{2}\rho\kappa^2 - \frac{1}{2}\rho_1\kappa_1^2 + \rho(g+\gamma\kappa)\eta - \rho_1(g+\gamma_1\kappa_1)\eta. \quad (3.4.11)$$

Recalling the definitions of ψ, $\widetilde{\varphi}$, ψ_1 and $\widetilde{\varphi}_1$ from (2.2.8) it follows that

$$|\nabla\psi|_c^2 = |\nabla\widetilde{\varphi}|_c^2 + \gamma^2\eta^2 + \kappa^2 + 2\gamma\kappa\eta + 2\gamma\eta(\widetilde{\varphi}_x)_c + 2\kappa(\widetilde{\varphi}_x)_c$$

and $$|\nabla\psi_1|_c^2 = |\nabla\widetilde{\varphi}_1|_c^2 + \gamma_1^2\eta^2 + \kappa_1^2 + 2\gamma_1\kappa_1\eta + 2\gamma_1\eta(\widetilde{\varphi}_{1,x})_c + 2\kappa_1(\widetilde{\varphi}_{1,x})_c$$

hence

$$\frac{1}{2}|\nabla\widetilde{\varphi}|_c^2 + \frac{1}{2}\gamma^2\eta^2 + \gamma\eta(\widetilde{\varphi}_x)_c = \frac{1}{2}|\nabla\psi|_c^2 - \frac{1}{2}\kappa^2 - \gamma\kappa\eta - \kappa(\widetilde{\varphi}_x)_c$$

and $$\frac{1}{2}|\nabla\widetilde{\varphi}_1|_c^2 + \frac{1}{2}\gamma_1^2\eta^2 + \gamma_1\eta(\widetilde{\varphi}_{1,x})_c = \frac{1}{2}|\nabla\psi_1|_c^2 - \frac{1}{2}\kappa_1^2 - \gamma_1\kappa_1\eta - \kappa_1(\widetilde{\varphi}_{1,x})_c.$$

The variation with respect to η is therefore

$$\frac{\delta H}{\delta\eta} = -\rho\eta_t(\widetilde{\varphi}_y)_c + \rho_1\eta_t(\widetilde{\varphi}_{1,y})_c + \frac{1}{2}\rho|\nabla\psi|_c^2 - \frac{1}{2}\rho_1|\nabla\psi_1|_c^2 + (\rho-\rho_1)g\eta. \quad (3.4.12)$$

From the Bernoulli condition (2.3.27)

$$\frac{1}{2}\rho|\nabla\psi|_c^2 - \frac{1}{2}\rho_1|\nabla\psi_1|_c^2 + (\rho-\rho_1)g\eta$$
$$= -\rho(\widetilde{\varphi}_t)_c + \rho_1(\widetilde{\varphi}_{1,t})_c + \rho(\gamma+2\omega)\chi - \rho_1(\gamma_1+2\omega)\chi_1 \quad (3.4.13)$$

and the variation with respect to η is

$$\frac{\delta H}{\delta\eta} = -\rho\eta_t(\widetilde{\varphi}_y)_c + \rho_1\eta_t(\widetilde{\varphi}_{1,y})_c - \rho(\widetilde{\varphi}_t)_c + \rho_1(\widetilde{\varphi}_{1,t})_c + \rho(\gamma+2\omega)\chi - \rho_1(\gamma_1+2\omega)\chi_1.$$

Using (3.4.8), (3.4.9) and (3.3.18)

$$\phi_t = (\widetilde{\varphi}_t)_c + (\widetilde{\varphi}_y)_c\eta_t$$

and

$$\phi_{1,t} = (\widetilde{\varphi}_{1,t})_c + (\widetilde{\varphi}_{1,y})_c\eta_t$$

and the variation with respect to η becomes

$$\frac{\delta H}{\delta\eta} = -\xi_t + \rho(\gamma+2\omega)\chi - \rho_1(\gamma_1+2\omega)\chi_1. \quad (3.4.14)$$

At the interface, using (2.2.5), the velocity components can respectively be defined in terms of the stream functions for Ω and Ω_1 as

$$(u)_c = (\psi_y)_c \text{ and } (v)_c = -(\psi_x)_c$$

and

$$(u_1)_c = (\psi_{1,y})_c \text{ and } (v_1)_c = -(\psi_{1,x})_c.$$

However, at any moment in time any arbitrary point (x,η) at the interface will be moving at a distinct velocity which can be measured independent of knowing the vorticities or velocity potentials, that is $(u)_c = (u_1)_c$ and $(v)_c = (v_1)_c$, therefore

$$(\psi_y)_c = (\psi_{1,y})_c \text{ and } (\psi_x)_c = (\psi_{1,x})_c.$$

This means that

$$(\nabla\psi)_c = (\nabla\psi_1)_c$$

and so $(\psi)_c$ and $(\psi_1)_c$ differ only by a constant. As potentials are modulo an additive constant, using assumption (2.2.10), as the absolute value of x goes to infinity then χ, χ_1 go to zero. Therefore $(\psi)_c$ and $(\psi_1)_c$ are equal, that is

$$\chi = \chi_1,$$

and so it is a natural physical fact that there is no flow through the common interface. The function

$$\chi' := \chi = \chi_1 \tag{3.4.15}$$

is therefore defined. The constant terms $\rho(\gamma + 2\omega) - \rho_1(\gamma_1 + 2\omega)$ are defined via a new constant Γ where

$$\Gamma := \rho\gamma - \rho_1\gamma_1 + 2\omega(\rho - \rho_1) \tag{3.4.16}$$

which will be referred to as the 'system constant'. The variation with respect to η, given by (3.4.14), is therefore

$$\frac{\delta H}{\delta\eta} = -\xi_t + \Gamma\chi'. \tag{3.4.17}$$

Now recalling (3.4.10) and fixing η,

$$\delta H|_{\delta\eta=0} = \rho\int_{\mathbb{R}} \eta_t\delta\phi dx - \rho_1\int_{\mathbb{R}} \eta_t\delta\phi_1 dx.$$

Recalling the definition of ξ from (3.3.18), and noting that δ is additive,

$$\delta\xi = \rho\delta\phi - \rho_1\delta\phi_1$$

and hence

$$\frac{\delta H}{\delta \xi} = \eta_t. \tag{3.4.18}$$

Equations (3.4.17) and (3.4.18) therefore give the non-canonical system

$$\eta_t = \frac{\delta H}{\delta \xi} \text{ and } \xi_t = -\frac{\delta H}{\delta \eta} + \Gamma \chi' \tag{3.4.19}$$

referred to as a *nearly* Hamiltonian system, due to the linearity of the χ' term [20]. By introducing $\mathfrak{u} = \xi_x$, the equations of motion (3.4.19) become

$$\eta_t = -\left(\frac{\delta H}{\delta \mathfrak{u}}\right)_x \text{ and } \mathfrak{u}_t = -\left(\frac{\delta H}{\delta \eta}\right)_x + \Gamma \eta_t. \tag{3.4.20}$$

The χ' term may be written in terms of the conjugate variables via the following lemma and corollary.

Lemma 1. *The evaluation of the stream function at the interface, χ', may be written in terms of the wave elevation function, η, as:*

$$\chi'(x,t) = -\int_{-\infty}^{x} \eta_t(x',t)dx'.$$

Proof. On one hand, from (2.3.29),

$$\eta_t = -(\psi_x)_c - (\psi_y)_c \eta_x = -(\psi_{1,x})_c - (\psi_{1,y})_c \eta_x \tag{3.4.21}$$

but if t is considered as being a parameter

$$\begin{aligned} \frac{d}{dx}\chi'(x,t) &= \frac{d}{dx}\psi(x,\eta(x,t),t) = \frac{d}{dx}\psi_1(x,\eta(x,t),t) \\ &= (\psi_x)_c + (\psi_y)_c \eta_x = (\psi_{1,x})_c + (\psi_{1,y})_c \eta_x. \end{aligned} \tag{3.4.22}$$

By comparing (3.4.21) and (3.4.22), and using the fundamental theorem of calculus for $x \in [-\infty, +\infty]$, it follows that

$$\frac{d}{dx}\chi'(x,t) = -\eta_t \tag{3.4.23}$$

and hence

$$\chi'(x,t) = -\int_{-\infty}^{x} \eta_t(x',t)dx' \tag{3.4.24}$$

thus proving the lemma. □

Corollary 2. *Using Lemma 1 and the equation for η_t in (3.4.18) the evaluation of the stream function at the interface, χ', may be written in terms of the variational derivative of the Hamiltonian, H, with respect to the evaluation of the velocity potential at the interface, ξ, as:*

$$\chi'(x,t) = -\int_{-\infty}^{x} \frac{\delta H}{\delta \xi(x')} dx'.$$

The equations of motion in (3.4.19) may therefore be written, using Lemma 1, in terms of η and ξ *only* (as opposed to η, ξ and χ') as the nearly Hamiltonian system

$$\eta_t = \frac{\delta H}{\delta \xi} \text{ and } \xi_t = -\frac{\delta H}{\delta \eta} - \Gamma \int_{-\infty}^{x} \eta_t(x',t)dx', \tag{3.4.25}$$

or, using Corollary 2 as

$$\eta_t = \frac{\delta H}{\delta \xi} \text{ and } \xi_t = -\frac{\delta H}{\delta \eta} - \Gamma \int_{-\infty}^{x} \frac{\delta H}{\delta \xi(x')} dx'. \tag{3.4.26}$$

From the beginning, we assume

$$\int_{\mathbb{R}} \eta(x,t)dx = 0,$$

which is to secure the assumption of zero average value of η. Otherwise, the unperturbed interface is not $y = 0$. This is consistent with the fact that χ' decays to xero at infinity.

Lemma 2.

$$\int_{\mathbb{R}} \eta_t(x,t)dx = 0.$$

Proof. From the assumption in (2.2.10)

$$\lim_{|x|\to\infty} \chi'(x,t) = 0.$$

Using Lemma 1 this means that

$$\int_{\mathbb{R}} \eta_t(x,t)dx = 0 \tag{3.4.27}$$

therefore proving Lemma 2. □

Corollary 3. *Using Lemma 2*

$$\int_{\mathbb{R}} \eta_t(x,t)dx = 0 \quad \Rightarrow \quad \int_{\mathbb{R}} \eta(x,t)dx \text{ is constant.}$$

Note: As $\int_{\mathbb{R}} \eta\, dx$ is a conserved quantity it can be taken, to be zero.

Canonical equations of motion can be achieved using a variable transformation as follows. The velocity potential at the interface, ξ, is transformed to a new variable, ζ, via the transformation (*cf.* [86])

$$\xi \quad \to \quad \zeta = \xi + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x',t)dx'. \tag{3.4.28}$$

The Hamiltonian structure of the equations of motion is proven by the following theorem:

Theorem 1. *The system under study is a Hamiltonian system described by the phase space variables η and ζ.*

Proof. Equation (3.4.19) can be written via (3.4.17) and (3.4.18) as

$$\delta H = \int_{\mathbb{R}} (-\xi_t + \Gamma\chi')\delta\eta \, dx + \int_{\mathbb{R}} \eta_t \delta\xi \, dx.$$

Applying the variable transformation given by (3.4.28) the variation can be written as

$$\delta H = \int_{\mathbb{R}} \Big(-\zeta_t + \frac{\Gamma}{2} \int_{-\infty}^{x} \eta_t(x')dx' + \Gamma\chi' \Big)\delta\eta(x)dx + \int_{\mathbb{R}} \Big(\eta_t\delta\zeta - \frac{\Gamma}{2}\eta_t \int_{-\infty}^{x} \delta\eta(x')dx' \Big) dx.$$

Using Lemma 1 (p.50) the second and third terms can be combined giving

$$\delta H = \int_{\mathbb{R}} \Big(-\zeta_t + \frac{\Gamma}{2}\chi' \Big)\delta\eta(x)dx + \int_{\mathbb{R}} \Big(\eta_t\delta\zeta - \frac{\Gamma}{2}\eta_t \int_{-\infty}^{x} \delta\eta(x')dx' \Big) dx. \tag{3.4.29}$$

Using integration by parts:

$$\begin{aligned} &\int_{\mathbb{R}} \eta_t \Big(\int_{-\infty}^{x} \delta\eta(x')dx' \Big) dx \\ &\quad = \Big[\int_{-\infty}^{x} \delta\eta(x')dx' \int_{-\infty}^{x} \eta_t(x'')dx'' \Big]_{-\infty}^{+\infty} - \int_{\mathbb{R}} \Big(\int_{-\infty}^{x} \eta_t(x'')dx'' \Big)\delta\eta(x)dx. \end{aligned} \tag{3.4.30}$$

The first term on the right-hand side is zero due to Lemma 2 and applying Lemma 1 to the other term means (3.4.29) can be rewritten as

$$\delta H = \int_{\mathbb{R}} \Big(-\zeta_t + \frac{\Gamma}{2}\chi' \Big)\delta\eta(x)dx + \int_{\mathbb{R}} \Big(\eta_t\delta\zeta - \frac{\Gamma}{2}\chi'\delta\eta(x) \Big) dx.$$

Noting the cancellation of the χ' terms this gives the Hamiltonian system

$$\zeta_t = -\frac{\delta H}{\delta\eta} \text{ and } \eta_t = \frac{\delta H}{\delta\zeta}. \tag{3.4.31}$$

□

The Hamiltonian (3.3.29) can now be expressed in terms of canonical variables η and ζ as

$$\begin{aligned} H(\eta,\zeta) = \frac{1}{2}\int_{\mathbb{R}} \left(\zeta + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x',t)dx'\right) G(\eta)B^{-1}G_1(\eta)\left(\zeta + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x',t)dx'\right)dx \\ - \frac{1}{2}\rho\rho_1 \int_{\mathbb{R}} \mu B^{-1}\mu\, dx - \int_{\mathbb{R}} (\gamma\eta+\kappa)\left(\zeta + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x',t)dx'\right)\eta_x dx \\ + \rho_1 \int_{\mathbb{R}} \mu B^{-1}G(\eta)\left(\zeta + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x',t)dx'\right)dx + \frac{\rho}{6\gamma}\int_{\mathbb{R}} (\gamma\eta+\kappa)^3 dx \\ - \frac{\rho_1}{6\gamma_1}\int_{\mathbb{R}} (\gamma_1\eta+\kappa_1)^3 dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx + \int_{\mathbb{R}} \mathfrak{h}_0 dx. \quad (3.4.32) \end{aligned}$$

3.5 Discussion and conclusions

A two-media fluid system separated by a two-dimensional internal wave, bounded above by a flat surface and below by a flat bottom, was examined. The system was considered to be inviscid and incompressible. A realistic representation of a depth-dependent current was presented and the system included a vortex sheet. As the system was considered to be present on the surface of the Earth, Coriolis forces were included.

The Hamiltonian approach was taken. The introduction of an interface velocity potential, interface stream function and Dirichlet-Neumann operators resulted in the Hamiltonian being expressed in terms of interface quantities only. The Hamiltonian was then rewrittem in terms of conjugate variables and the equations of motion were found to be 'nearly' Hamiltonian. Canonical equations were established by a transformation of the overall interface velocity potential. The importance of the introduction of the interface quantities means that it is not necessary to know what is happening in the bulk of the fluid at all times but only what is happening at the interface. The introduction of the complex potential allows an analytic continuation of these quantities to the body of the fluid.

Chapter 4

AN APPROXIMATE MODEL - THE INTERMEDIATE LONG WAVE EQUATION (ILWE)

4.1 Introduction

The main features of the ocean dynamics in a band about 300 km wide, centred on the Equator, stretching over 12,000 km in the Pacific are:

- there is a pronounced large scale stratification, greater than anywhere else in the ocean, with lower-density fluid atop an abyssal, practically motionless, layer
- the underlying currents, while confined to the near-surface (within the upper 200-300 m of the 4 km deep ocean), are non-uniform and present flow-reversal (beneath the westward wind-driven surface current is a strong eastward jet, the EUC, one of the strongest currents in the ocean, discovered in 1952 by T. Cromwell, at depths between 100 m and 200 m)

- 3-4 m high gravity waves are common on the surface of the ocean, while large internal waves (with heights in excess of 30 m) propagate as oscillations of the thermocline – the interface separating the two adjacent layers of different constant density

This chapter considers the Hamiltonian framework already established for the 2-media system and adopts an approach using perturbation techniques, whereby small parameters are used to separate the order of the terms of the model. This facilitates truncation at different orders and the production of both linear and non-linear approximations.

4.2 Nondimensionalisation

The system under study in the previous chapter has been expressed in terms of dimensional physical quantities, such as metres and seconds. For the approximate models the physical quantities are transformed to nondimensional quantities using constants that have some meaning in the context of the systems, specifically the domain depths, the wave amplitude and the Earth's acceleration due to gravity. Note, the introduction of bar notation to identify a transformed variable.
In the case of *long*-waves the phase velocity v_p is given by [52]

$$v_p \approx \sqrt{\frac{g}{k}\tanh\left(kh_1\right)}$$

where k is the wave number and g is the acceleration due to gravity. For shallow water models the wave number is small and therefore $\tanh\left(kh_1\right)$ is approximately kh_1 meaning

$$v_p \approx \sqrt{gh_1}$$

is a velocity that is intrinsic to the system and therefore the velocity components are nondimensionalised by the transformation

$$
\begin{aligned}
(\bar{u}, \bar{v}) &\rightarrow \sqrt{gh_1}(u, v) \\
\text{and} \qquad (\bar{u}_1, \bar{v}_1) &\rightarrow \sqrt{gh_1}(u_1, v_1).
\end{aligned}
$$

By scaling the spatial coordinates using h_1 they are nondimensionalised by the transformation

$$
(\bar{x}, \bar{y}) \rightarrow h_1(x, y)
$$

and then also

$$
\bar{t} \rightarrow \frac{h_1}{\sqrt{gh_1}} t.
$$

As the κ and κ_1 terms are velocities

$$
\begin{aligned}
\bar{\kappa} &\rightarrow \sqrt{gh_1}\kappa \\
\text{and} \qquad \bar{\kappa_1} &\rightarrow \sqrt{gh_1}\kappa_1
\end{aligned}
$$

and as the γ and γ_1 terms are spatial-derivatives of velocities

$$
\begin{aligned}
\bar{\gamma} &\rightarrow \sqrt{\frac{g}{h_1}}\gamma \\
\text{and} \qquad \bar{\gamma_1} &\rightarrow \sqrt{\frac{g}{h_1}}\gamma_1.
\end{aligned}
$$

Finally, the wave elevation function is nondimensionalised using the wave amplitude by the transformation

$$
\bar{\eta} \rightarrow a\eta.
$$

4.3 Linearisation using the small amplitude regime

Linearised approximations can be used for computational modelling in oceanography for example. The small amplitude regime will be studied. The Hamiltonian under study can be expanded in terms of orders of the dependent variables η and ξ as [87]

$$H(\eta, \xi) = \sum_{j=0}^{\infty} H^{(j)}(\eta, \xi) \tag{4.3.1}$$

where j is the order and therefore $H^{(0)}$ is a constant term, $H^{(1)}(\eta, \xi)$ is a linear term, $H^{(2)}(\eta, \xi)$ is a quadratic term, etc. The operators in the Hamiltonian will also need to be expanded. The Dirichlet-Neumann operators can be expanded in terms of orders of η as

$$G_i(\eta) = \sum_{j=0}^{\infty} G_i^{(j)}(\eta).$$

The operator B, as defined in (3.3.19), which is a function of Dirichlet-Neumann operators, can therefore be expressed as

$$B = \rho \sum_{j=0}^{\infty} G_1^{(j)}(\eta) + \rho_1 \sum_{j=0}^{\infty} G^{(j)}(\eta).$$

A scale parameter is now introduced

$$\varepsilon = \frac{a}{h_1} \tag{4.3.2}$$

where the constant a represents the average amplitude of the waves $\eta(x, t)$ under consideration and $\varepsilon \ll 1$ is a small dimensionless parameter which will be used to separate the order of the terms in the model.

The Dirichlet-Neumann (DN) operators have the following structure

$$G = G^{(0)} + G^{(1)} + G^{(2)} + \dots \tag{4.3.3}$$

where $G^{(n)}(\eta)$ is an operator, such that $G^{(n)}(\nu\eta) = \nu^n G^{(n)}(\eta)$ for any constant ν that is $G^{(n)} \sim \varepsilon^n \sim \eta^n$, since $\eta \sim h_1\varepsilon$ and similarly for G_1. The corresponding expansions are [31, 32]

$$G(\eta) = D\tanh(hD) + D\eta D - D\tanh(hD)\eta D\tanh(hD) + \mathcal{O}(\eta^2) \qquad (4.3.4)$$

$$\text{and } G_1(\eta) = D\tanh(h_1 D) - D\eta D + D\tanh(h_1 D)\eta D\tanh(h_1 D) + \mathcal{O}(\eta^2) \qquad (4.3.5)$$

where

$$D := -i\partial/\partial x.$$

To derive the equations of motion we will also make the additional approximation that the wavelengths L are much bigger than h_1, that is

$$\delta = \frac{h_1}{L} \ll 1.$$

Noting that the wave number $k = 2\pi/L$ is an eigenvalue or a Fourier multiplier for the operator D (when acting on waves of the form e^{ikx}). Ref. Appendix B.

Further assumptions that we will make about the scales are detailed as follows:

1. $\delta = \mathcal{O}(\varepsilon)$;

2. $hk = \mathcal{O}(1)$ and $h_1 k = \mathcal{O}(\delta)$ that is $h_1/h \sim \delta \ll 1$. This corresponds to a deep lower layer;

3. $\xi = \mathcal{O}(1)$.

4. The physical constants h_1, ρ, ρ_1, γ, γ_1 are $\mathcal{O}(1)$.

Since the operator D has an eigenvalue k, it follows that $hD = \mathcal{O}(1)$ and $h_1 D = \mathcal{O}(\delta)$.

The DN operators can now be written as

$$G(\eta) = \frac{1}{h_1}\Big[(h_1 D)\tanh(hD) + (h_1 D)\frac{\eta}{h_1}(h_1 D) \\ -(h_1 D)\tanh(hD)\frac{\eta}{h_1}(h_1 D)\tanh(hD) + \ldots\Big] \tag{4.3.6}$$

and

$$G_1(\eta) = \frac{1}{h_1}\Big[(h_1 D)\tanh(h_1 D) - (h_1 D)\frac{\eta}{h_1}(h_1 D) \\ +(h_1 D)\tanh(h_1 D)\frac{\eta}{h_1}(h_1 D)\tanh(h_1 D) + \ldots\Big] \tag{4.3.7}$$

with the scale factors determined explicitly as

$$G(\eta) = \frac{1}{h_1}\Big[\delta(h_1\bar{D})\tanh(h\bar{D}) + \delta^3(h_1\bar{D})\frac{\bar{\eta}}{h_1}(h_1\bar{D}) \\ -\delta^3(h_1\bar{D})\tanh(h\bar{D})\frac{\bar{\eta}}{h_1}(h_1\bar{D})\tanh(h\bar{D})\Big] + \mathcal{O}(\delta^4) \tag{4.3.8}$$

and

$$G_1(\eta) = \frac{1}{h_1}\Big[\delta(h_1\bar{D})\left(\delta(h_1\bar{D}) - \delta^3\frac{(h_1\bar{D})^3}{3}\right) - \delta^3(h_1\bar{D})\frac{\bar{\eta}}{h_1}(h_1\bar{D}) \\ +\delta^5(h_1\bar{D})(h_1\bar{D})\frac{\bar{\eta}}{h_1}(h_1\bar{D})(h_1\bar{D})\Big] + \mathcal{O}(\delta^4) \tag{4.3.9}$$

where the barred (dimensional) quantities and operators together with h and h_1 are assumed to be of order 1.

Introducing

$$\mathfrak{t}_h := \tanh(hD) \tag{4.3.10}$$

and

$$\mathcal{D}_1 := h_1 D \tag{4.3.11}$$

and omitting the bars for convenience, the DN expansions are truncated as follows:

$$G(\eta) = \frac{1}{h_1}\left[\delta\mathcal{D}_1\mathfrak{t}_h + \delta^3\mathcal{D}_1\frac{\eta}{h_1}\mathcal{D}_1 - \delta^3\mathcal{D}_1\mathfrak{t}_h\frac{\eta}{h_1}\mathcal{D}_1\mathfrak{t}_h\right] + \mathcal{O}(\delta^4) \tag{4.3.12}$$

$$G_1(\eta) = \frac{\delta^2}{h_1}\mathcal{D}_1\left[1 - \delta\frac{\eta}{h_1} - \delta^2\frac{\mathcal{D}_1^2}{3}\right]\mathcal{D}_1 + \mathcal{O}(\delta^5). \tag{4.3.13}$$

Note that h appears only in the definition of the operator $\mathfrak{t}_h$, which is of order 1. Since h_1 is assumed of order 1, then formally the order of the differentiation ∂_x is δ. Hence the order of the integration measure dx is $1/\delta$.

The leading order terms in G are $\mathcal{O}(\delta)$ and the leading order terms in G_1 are δ^2 hence $G_1 G^{-1} \sim \delta \ll 1$ and so it follows:

$$\begin{aligned} GB^{-1}G_1 &= G\frac{1}{\rho_1 G + \rho G_1}G_1 = G\frac{1}{\rho_1(1+\frac{\rho}{\rho_1}G_1G^{-1})G}G_1 \\ &= \frac{1}{\rho_1}GG^{-1}\left[1 - \frac{\rho}{\rho_1}G_1G^{-1} + \frac{\rho^2}{\rho_1^2}(G_1G^{-1})^2 - \ldots\right]G_1 \\ &= \frac{1}{\rho_1}\left[G_1 - \frac{\rho}{\rho_1}G_1G^{-1}G_1 + \frac{\rho^2}{\rho_1^2}G_1G^{-1}G_1G^{-1}G_1 - \ldots\right]. \end{aligned} \tag{4.3.14}$$

Since both G and G_1 are self-adjoint, it is now evident that $GB^{-1}G_1$ is self-adjoint too at this order of approximation. The substitution of (4.3.12) and (4.3.13) in (4.3.14) gives

$$\begin{aligned} GB^{-1}G_1 &= \delta^2\frac{h_1}{\rho_1}D^2 - \delta^3\frac{1}{\rho_1}\left(D\eta D + i\frac{\rho h_1^2}{\rho_1}D^3\mathcal{T}_h\right) \\ &\quad + \delta^4\frac{\rho h_1}{\rho_1^2}\left(iD\eta D^2\mathcal{T}_h + i\mathcal{T}_h D^2\eta D\right) - \delta^4\frac{h_1^3}{\rho_1}\left(\frac{1}{3} + \frac{\rho^2}{\rho_1^2}\mathcal{T}_h^2\right)D^4 + \mathcal{O}(\delta^5) \end{aligned} \tag{4.3.15}$$

where

$$\mathcal{T}_h := -i\coth(hD) = (i\mathfrak{t}_h)^{-1}$$

is introduced. Details of this operatror are given in Appendix C.

Next

$$\begin{aligned} B^{-1} &= \frac{1}{\rho_1}\left[G^{-1} - \frac{\rho}{\rho_1}G^{-1}G_1G^{-1} + \frac{\rho^2}{\rho_1^2}G^{-1}G_1G^{-1}G_1G^{-1} - \ldots\right] \\ &= \delta^{-1}\frac{1}{\rho_1}D^{-1}\mathfrak{t}_h^{-1} + \mathcal{O}(1) \end{aligned} \tag{4.3.16}$$

and

$$\begin{aligned}B^{-1}G &= \frac{1}{\rho_1}\left[1 - \frac{\rho}{\rho_1}G^{-1}G_1 + \frac{\rho^2}{\rho_1^2}G^{-1}G_1G^{-1}G_1 - \dots\right] \\ &= \frac{1}{\rho_1}\left(1 - \delta\frac{\rho_1}{\rho_1}i\mathcal{T}_h D\right) + \mathcal{O}(\delta^2). \end{aligned} \tag{4.3.17}$$

Recalling the Hamiltonian (3.3.32)

$$\begin{aligned}H(\eta,\xi) =& \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\,dx - \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2\int_{\mathbb{R}}\eta\eta_x B^{-1}\eta\eta_x\,dx \\ & - \int_{\mathbb{R}}(\gamma\eta+\kappa)\xi\eta_x\,dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}}\eta\eta_x B^{-1}G(\eta)\xi\,dx \\ & + \frac{\rho\gamma^2-\rho_1\gamma_1^2}{6}\int_{\mathbb{R}}\eta^3 dx + \frac{g(\rho-\rho_1)+(\rho\gamma-\rho_1\gamma_1)\kappa}{2}\int_{\mathbb{R}}\eta^2 dx.\end{aligned}$$

The quantity $\eta\eta_x = ih_1^3(\eta/h_1)\mathcal{D}_1(\eta/h_1) \sim \delta^3$. The contribution of the integral density $\eta\eta_x B^{-1}\eta\eta_x$ in the Hamiltonian is therefore of order δ^5. Recall that $dx \sim 1/\delta$. Hence, by keeping terms up to δ^3 in (3.3.32) and recalling

$$\mathfrak{u} := \xi_x$$

the scaled Hamiltonian can now be written as

$$\begin{aligned}H(\eta,\mathfrak{u}) =& \delta\frac{h_1}{2\rho_1}\int_{\mathbb{R}}\mathfrak{u}^2\,dx + \delta\frac{A}{2}\int_{\mathbb{R}}\eta^2\,dx + \delta\kappa\int_{\mathbb{R}}\eta\mathfrak{u}\,dx \\ & - \delta^2\frac{1}{2\rho_1}\int_{\mathbb{R}}\eta\mathfrak{u}^2\,dx - \delta^2\frac{h_1^2\rho}{2\rho_1^2}\int_{\mathbb{R}}\mathfrak{u}\mathcal{T}_h\mathfrak{u}_x\,dx + \delta^2\frac{\gamma_1}{2}\int_{\mathbb{R}}\eta^2\mathfrak{u}\,dx \\ & + \delta^2\frac{\rho\gamma^2-\rho_1\gamma_1^2}{6}\int_{\mathbb{R}}\eta^3\,dx - \delta^3\frac{h_1^3}{2\rho_1}\int_{\mathbb{R}}\mathfrak{u}_x\left(\frac{1}{3}+\frac{\rho^2}{\rho_1^2}\mathcal{T}_h^2\right)\mathfrak{u}_x\,dx \\ & + \delta^3\frac{h_1\rho}{\rho_1^2}\int_{\mathbb{R}}\eta\mathfrak{u}\mathcal{T}_h\mathfrak{u}_x\,dx + \delta^3\frac{(\gamma-\gamma_1)h_1\rho}{2\rho_1}\int_{\mathbb{R}}\eta^2\mathcal{T}_h\mathfrak{u}_x\,dx,\end{aligned} \tag{4.3.18}$$

where the constant $A = g(\rho-\rho_1) + \kappa(\rho\gamma-\rho_1\gamma_1)$.

H is of order δ. This gives the proper scaling of ∂_t which should also be of order δ, same as the order of ∂_x. The variation $\delta\mathfrak{u}$ bears a scale factor δ as well. The equations of motion (3.4.19) with scaling written explicitly are therefore

$$\eta_t + (\delta)^{-1}\Big(\frac{\delta H}{\delta \mathfrak{u}}\Big)_x = 0 \tag{4.3.19}$$

$$\mathfrak{u}_t + \Gamma\eta_t + (\delta)^{-1}\Big(\frac{\delta H}{\delta \eta}\Big)_x = 0 \tag{4.3.20}$$

producing the coupled system

$$\begin{aligned}\eta_t + \kappa\eta_x + \frac{h_1}{\rho_1}\mathfrak{u}_x - \delta\frac{1}{\rho_1}(\eta\mathfrak{u})_x - \delta\frac{\rho h_1^2}{\rho_1^2}\mathcal{T}_h\mathfrak{u}_{xx} + \delta\gamma_1\eta\eta_x + \delta^2\frac{h_1^3}{\rho_1}\Big(\frac{1}{3} + \frac{\rho^2}{\rho_1^2}\mathcal{T}_h^2\Big)\mathfrak{u}_{xxx}\\ + \delta^2\frac{\rho h_1}{\rho_1^2}\left((\eta\mathcal{T}_h\mathfrak{u}_x)_x + \mathcal{T}_h(\eta\mathfrak{u})_{xx}\right) + \delta^2\frac{\rho h_1(\gamma-\gamma_1)}{2\rho_1}\mathcal{T}_h(\eta^2)_{xx} = 0\end{aligned} \tag{4.3.21}$$

$$\begin{aligned}\mathfrak{u}_t + \kappa\mathfrak{u}_x + \Gamma\eta_t + A\eta_x - \delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_x + \delta\gamma_1(\eta\mathfrak{u})_x + \delta(\rho\gamma^2 - \rho_1\gamma_1^2)\eta\eta_x\\ + \delta^2\frac{\rho h_1}{\rho_1^2}(\mathfrak{u}\mathcal{T}_h\mathfrak{u}_x)_x + \delta^2\frac{\rho h_1(\gamma-\gamma_1)}{\rho_1}(\eta\mathcal{T}_h\mathfrak{u}_x)_x = 0.\end{aligned} \tag{4.3.22}$$

These equations can be viewed as a generalisation of the irrotational case ($\Gamma = \gamma_1 = \gamma = 0$, $\kappa = 0$) derived in [31].

4.4 The Intermediate Long Wave Equation (ILWE)

Keeping terms of up to order δ in (4.3.21) and (4.3.22), results in

$$\eta_t + \kappa\eta_x + \frac{h_1}{\rho_1}\mathfrak{u}_x - \delta\frac{1}{\rho_1}(\eta\mathfrak{u})_x - \delta\frac{\rho h_1^2}{\rho_1^2}\mathcal{T}_h\mathfrak{u}_{xx} + \delta\gamma_1\eta\eta_x = 0 \tag{4.4.1}$$

$$\mathfrak{u}_t + \kappa\mathfrak{u}_x + \Gamma\eta_t + A\eta_x - \delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_x + \delta\gamma_1(\eta\mathfrak{u})_x + \delta(\rho\gamma^2 - \rho_1\gamma_1^2)\eta\eta_x = 0. \tag{4.4.2}$$

Next, a Galilean transformation of coordinates is performed where

$$X = x - \kappa t, \quad T = t, \quad \partial_X = \partial_x, \quad D \to -i\partial_X \quad \text{and} \quad \partial_T = \partial_t + \kappa\partial_x \tag{4.4.3}$$

and taking into account that for the typical values of κ of several m/s, $g \gg 2\omega\kappa$ the equations of motion can be written as

$$\eta_T + \frac{h_1}{\rho_1}\mathfrak{u}_X - \delta\frac{1}{\rho_1}(\eta\mathfrak{u})_X - \delta\frac{\rho h_1^2}{\rho_1^2}\mathcal{T}_h\mathfrak{u}_{XX} + \delta\gamma_1\eta\eta_X = 0 \tag{4.4.4}$$

$$\mathfrak{u}_T + \Gamma\eta_T + (A - \Gamma\kappa)\eta_X - \delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_X + \delta\gamma_1(\eta\mathfrak{u})_X + \delta(\rho\gamma^2 - \rho_1\gamma_1^2)\eta\eta_X = 0, \tag{4.4.5}$$

where $A - \Gamma\kappa = g(\rho - \rho_1) + \kappa(\rho\gamma - \rho_1\gamma_1) - \kappa(\rho\gamma - \rho_1\gamma_1) + 2\omega\kappa\big(\rho - \rho_1\big) = (g - 2\omega\kappa)(\rho - \rho_1) \approx g(\rho - \rho_1)$ giving

$$\eta_T + \frac{h_1}{\rho_1}\mathfrak{u}_X - \delta\frac{1}{\rho_1}(\eta\mathfrak{u})_X - \delta\frac{\rho h_1^2}{\rho_1^2}\mathcal{T}_h\mathfrak{u}_{XX} + \delta\gamma_1\eta\eta_X = 0 \tag{4.4.6}$$

$$\mathfrak{u}_T + \Gamma\eta_T + g(\rho - \rho_1)\eta_X - \delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_X + \delta\gamma_1(\eta\mathfrak{u})_X + \delta(\rho\gamma^2 - \rho_1\gamma_1^2)\eta\eta_X = 0. \tag{4.4.7}$$

The leading order terms (that is neglecting the terms with δ above) produce a system of linear equations with constant coefficients from where the speed(s) of the travelling waves (in the leading order) can be determined:

$$c = -\frac{h_1}{2\rho_1}\Gamma \pm \sqrt{\frac{h_1^2}{4\rho_1^2}\Gamma^2 + \frac{h_1}{\rho_1}g(\rho - \rho_1)}. \tag{4.4.8}$$

The plus sign is for the right-running waves and the minus sign is for the left-running waves. These speeds coincide with the speeds in the case of infinitely deep lower layer [16], noting that the h-dependence comes only from the term $\mathcal{T}_h$ which is of order δ.

For the travelling wave, which depends on the characteristic variable $X - cT$,

$$\mathfrak{u} = \frac{\rho_1}{h_1}c\eta.$$

In order to obtain a single nonlinear equation for η, a relation which involves terms of order δ is also needed, that is an expansion of the form

$$\mathfrak{u} = \frac{\rho_1}{h_1} c\eta + \delta\alpha\eta^2 + \delta\beta\mathcal{T}_h\eta_X, \tag{4.4.9}$$

for some yet undetermined constants α and β. This type of relation is also known as the Johnson transformation. The substitution of $\mathfrak{u}$ from (5.4.8) in (4.4.4) and (4.4.5) when keeping only the terms up to order δ leads to two equations for η and therefore these two equations must coincide. This leads to equality of the coefficients in front of the terms of the same type, which further allows to determine the previously unknown

$$\alpha = \frac{\rho_1(\rho_1 c^2 + 2h_1\Gamma c - \gamma_1 h_1^2\Gamma + \rho_1\gamma_1 h_1 c + h_1^2)}{2h_1^2\big(2\rho_1 c + h_1\Gamma\big)} \tag{4.4.10}$$

and

$$\beta = \frac{\rho(\rho_1 c^2 + h_1\Gamma c)}{2\rho_1 c + h_1\Gamma}. \tag{4.4.11}$$

The equation for η is

$$\eta_T + c\eta_X - \delta\frac{\rho h_1 c^2}{2\rho_1 c + h_1\Gamma}\mathcal{T}_h\eta_{XX} + \delta\frac{-3\rho_1 c^2 + 3\rho_1\gamma_1 h_1 c + h_1^2(\rho\gamma^2 - \rho_1\gamma_1^2)}{h_1(2\rho_1 c + h_1\Gamma)}\eta\eta_X = 0. \tag{4.4.12}$$

The obtained equation is known as the Intermediate Long Wave Equation (ILWE) introduced in [50, 61]. It is an integrable equation. The soliton theory for ILWE has been developed in a number of works, for example [?, 55, 59, 67, 81].

The ILWE in the irrotational case ($\gamma = \gamma_1 = \omega = 0$, $\kappa = \Gamma = 0$) becomes[1]

$$\eta_t + c\eta_x - \delta\frac{\rho h_1 c}{2\rho_1}\mathcal{T}_h\eta_{xx} - \delta\frac{3c}{2h_1}\eta\eta_x = 0, \tag{4.4.13}$$

where, from (5.6.4) the wave speed(s) are

$$c = \pm\sqrt{\frac{h_1}{\rho_1}g(\rho - \rho_1)}.$$

[1]Note that in this case $\kappa = 0$ and hence $(X, T) \equiv (x, t)$.

The ILWE can be written in the following form

$$\eta_T + c\eta_X + \delta\mathcal{A}\eta\eta_X - \delta\mathcal{B}\mathcal{T}_h\eta_{XX} = 0 \tag{4.4.14}$$

where

$$\mathcal{A} := \frac{-3\rho_1 c^2 + 3\rho_1\gamma_1 h_1 c + h_1^2(\rho\gamma^2 - \rho_1\gamma_1^2)}{h_1(2\rho_1 c + h_1\Gamma)}, \qquad \mathcal{B} := \frac{\rho h_1 c^2}{2\rho_1 c + h_1\Gamma}. \tag{4.4.15}$$

The one-soliton solution of (4.4.14) has the form

$$\eta(X,T) = \frac{2\mathcal{B}}{\mathcal{A}} \cdot \frac{k_0 \sin(k_0 h)}{\cos(k_0 h) + \cosh[k_0(X - X_0 - (c - \delta\mathcal{B}k_0\cot(k_0 h))T)]}, \qquad 0 < k_0 < \frac{2\pi}{h}. \tag{4.4.16}$$

In the above formula X_0 and k_0 are the soliton parameters, that is arbitrary constants within their range of allowed values. X_0 is the initial position of the crest of the soliton and k_0 is related to its amplitude. The wavespeed of the soliton is $c - \delta\mathcal{B}k_0\cot(k_0 h)$ and the correction of order δ depends on the coefficient $\mathcal{B}$ and the dispersion law related to the dispersive term and also on the parameter k_0. Both the amplitude of the soliton and its speed are related through k_0. Another feature of this solution is the fact that the function cot is unbounded. The physical relevance of the solution however requires that the choice of k_0 should be such that the quantity $k_0\cot(k_0 h)$ is of order 1. As will be shown in the next section, there is no such anomaly for the related Benjamin-Ono equation, for which the limiting procedure requires a special choice of k_0.

4.5 Connection to the Benjamin-Ono equation

The BO model of waves in the presence of uniformly-sheared currents has been derived previously in [16]. In this section, it will be shown that the BO equation can be obtained as a special kind of a long-wave limit from the ILWE (4.4.12).
In the limit $h \to \infty$ which corresponds to an infinitely deep lower layer

$$\mathcal{T}_h = -i \coth(hD) \to -i\,\mathrm{sign}(D), \qquad \mathcal{T}_h \partial_X \to |D|,$$

and the equation (4.4.12) becomes the well known Benjamin-Ono (BO) equation [3, 16, 70]:

$$\eta_T + c\eta_X + \delta \mathcal{A} \eta \eta_X - \delta \mathcal{B} |D| \eta_X = 0. \tag{4.5.1}$$

Similar to the ILWE, the BO is an integrable equation whose solutions can be obtained by the Inverse Scattering method [38, 58, 66].
The one soliton solution of the BO equation (from (4.4.16) with $h \to \infty$ but $k_0 h$ finite, $k_0 h = \pi - k_0/q$ where q is a constant) and can be written in the form

$$\eta(X,T) = \frac{\eta_0}{1 + \left(\frac{\eta_0 \mathcal{A}}{4\mathcal{B}}\right)^2 [X - X_0 - (c + \frac{1}{4}\delta \mathcal{A} \eta_0) T]^2} \tag{4.5.2}$$

where the constants are the initial position X_0 of the soliton and its amplitude η_0. The relation to the constant q (and hence k_0) is

$$\eta_0 = 4\mathcal{B}q/\mathcal{A} = 4\mathcal{B}k_0/[\mathcal{A}(\pi - k_0 h)].$$

4.6 Connection to the KdV equation

The KdV model of internal waves in the presence of uniformly-sheared currents has been derived previously in [15]. The special situation of KdV with $h_1/h \sim \delta$ has been provided in Appendix D for convenience. The provided analysis shows that it can be obtained as a special kind of a long-wave limit from the ILWE.

The KdV limit can be obtained assuming $hk \ll 1$ or $|hD| \ll 1$. The operator

$$\mathcal{T}_h = -i\coth(hD) \approx -i\left(\frac{1}{hD} + \frac{1}{3}hD\right) = \frac{1}{h}\partial_X^{-1} - \frac{1}{3}h\partial_X$$

giving from (4.4.14)

$$\eta_T + \left(c - \delta\frac{\rho c^2}{2\rho_1 c + h_1\Gamma}\frac{h_1}{h}\right)\eta_X + \delta\mathcal{A}\eta\eta_X + \delta\frac{h\mathcal{B}}{3}\eta_{XXX} = 0. \tag{4.6.1}$$

Noting that $h_1/h \simeq \delta \ll 1$ the correction to c in the second term is of order δ^2 and can be neglected. Thus the KdV equation in the following form is obtained

$$\eta_T + c\eta_X + \delta\mathcal{A}\eta\eta_X + \delta\frac{h\mathcal{B}}{3}\eta_{XXX} = 0 \tag{4.6.2}$$

which coincides with (D5), Appendix D since in the ILWE setup $\delta \simeq \varepsilon$ and

$$\mathcal{B}_1 = \frac{h\mathcal{B}}{3} = \frac{c^2\rho h h_1}{3(2c\rho_1 + \Gamma h_1)}.$$

The one-soliton solution of (4.6.2) can also be obtained from (4.4.16). For $k_0 h \ll 1$, $\sin(k_0 h) \approx k_0 h$, $\cos(k_0 h) \approx 1$, and using the identity

$$\begin{aligned}
1 + \cosh Z &= 2\cosh^2(Z/2), \\
\mathcal{B}k_0\cot(k_0 h) &\approx \frac{3\mathcal{B}_1}{h}k_0\left(\frac{1}{k_0 h} - \frac{k_0 h}{3}\right) = \frac{3\mathcal{B}_1}{h^2} - \mathcal{B}_1 k_0^2 \\
&= \frac{\rho c^2}{2c\rho_1 + \Gamma h_1}\cdot\frac{h_1}{h} - \mathcal{B}_1 k_0^2.
\end{aligned}$$

The first term

$$\frac{c^2\rho}{2c\rho_1 + \Gamma h_1}\cdot\frac{h_1}{h} = \frac{c^2\rho}{2c\rho_1 + \Gamma h_1}\cdot\delta \ll 1$$

does not depend on k_0 and represents the small $\mathcal{O}(\delta^2)$ correction to the constant wave speed c. The second term is proportional to k_0^2, and the approximation leads to

$$\eta(X,T) = \frac{6\mathcal{B}_1}{\mathcal{A}} \cdot \frac{k_0^2}{1+\cosh[k_0(X - X_0 - (c+\delta\mathcal{B}_1 k_0^2)T))]}$$
$$= \frac{3\mathcal{B}_1}{\mathcal{A}} \cdot \frac{k_0^2}{\cosh^2[\frac{k_0}{2}(X - X_0 - (c+\delta\mathcal{B}_1 k_0^2)T))]}. \quad (4.6.3)$$

Introducing a new constant $K = k_0/2$, gives

$$\eta(X,T) = \frac{12\mathcal{B}_1}{\mathcal{A}} \cdot \frac{K^2}{\cosh^2[K(X - X_0 - (c+4\delta K^2\mathcal{B}_1)T))]} \quad (4.6.4)$$

which coincides with the KdV one-soliton solution (E3), Appendix E.

4.7 Discussion and conclusions

The integrable ILWE for the case of solitary waves on the interface of two fluids with constant vorticities was derived, modeling equatorial internal waves interacting with the undercurrent. The surface waves which are usually of much smaller amplitude were neglected, so a "rigid lid" approximation for the upper fluid was assumed. The ILWE is an integrable model and the inverse scattering method or other methods like Darboux transforms and Hirota's method allow for the derivation of explicit multisoliton solutions. The one-soliton solution for example is provided in (4.4.16). Two important integrable limits were applied to the ILWE one-soliton solution, and the BO and KdV one-soliton solutions were recovered. The limits of course exist for the multisoliton ILWE solutions as well and in principle for all types of solutions. It has to be noted that the ILWE, BO and KdV correspond to different scales of the physical quantities, as can be seen from Table 4.1.

	ILWE	BO	KdV (D1)
$\mathcal{O}(h_1/h)$	δ	0	1
$\mathcal{O}(\eta/h_1)$	δ	δ	δ^2
$\mathcal{O}(h_1 k)$	δ	δ	δ
$\mathcal{O}(hk)$	1	∞	δ

Table 4.1: The scales of the three approximation models.

The BO limit corresponds to a limit to an infinitely deep lower layer. The limit from ILWE to KdV gives a KdV equation in its particular form (4.6.2). Schematically the relations between the ILWE and the KdV equations involved could be represented as follows:

$$\mathrm{KdV}\,(D1) \xrightarrow{(h_1/h)=\varepsilon\ll 1} \mathrm{KdV}\,(D5) \xrightarrow{\delta:=\varepsilon} \boxed{\mathrm{KdV}\ \ (4.6.2)} \xleftarrow{hk\ll 1} \mathrm{ILWE}$$

This shows that the ILWE should be used as a "master" equation with some caution. An interesting aspect for further studies is the development of theoretical models for internal waves with currents over variable bottom, extending the results from [75, 76].

Chapter 5

INTERNAL WAVES WITH CORIOLIS FORCE

5.1 Introduction

As previously discussed, internal wave propagation is affected by various factors, such as currents and interactions with other waves, since the ocean dynamics near the surface is quite complex. However at great depths in the ocean (depths in excess of about 240 m) there is, essentially, an abyssal layer of still water. In what follows, the influences of current, vorticity and interactions with other waves will not be taken into account. It turns out that the highly idealised theoretical two-layer model describes accurately the reported observations [44, 71].

In this chapter, the Coriolis effect on the internal wave propagation is examined following the idea of the "nearly" Hamiltonian approach and generalising the Hamiltonian approach of Zakharov [88]. This nearly-Hamiltonian approach however, unlike the one in the previous chapter, is about the inclusion of the Coriolis effect, not an underlying current. The aim is to illustrate the mathematical usefulness of the Hamiltonian approach in a systematic study of the internal wave propagation, rather than to present new equations on internal waves. The approach could be used in further studies, including detailed analysis with higher order approximations or effects, not included here. The details, analysis and results of this chapter have been published: refer to article [35].

The Coriolis effect is present both in the ocean and in the atmosphere. A mass of moving air or water subject only to the Coriolis force travels in a circular trajectory called an "inertial circle". The Coriolis effect has been intensively studied as well, see for example [44, 63, 77].

For ocean waves of large magnitude, the viscosity does not play an essential role and can be neglected, so effectively the fluid dynamics are governed by Euler's equation.

5.2 The System Setup

The Euler equation with included Coriolis force as per (2.3.12) is

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{g} \tag{5.2.1}$$

where the velocity vector field $\mathbf{u} = (u, v, w)$ is presented through its components in a local coordinate system. Similar to the previous setup (Figure 2.3), the more traditional oceanographic coordinate system is now used where the geophysical axis x is oriented to the East, the y axis is pointing to the North and the z axis is vertical to the Earth surface, as shown in Figure 5.1.

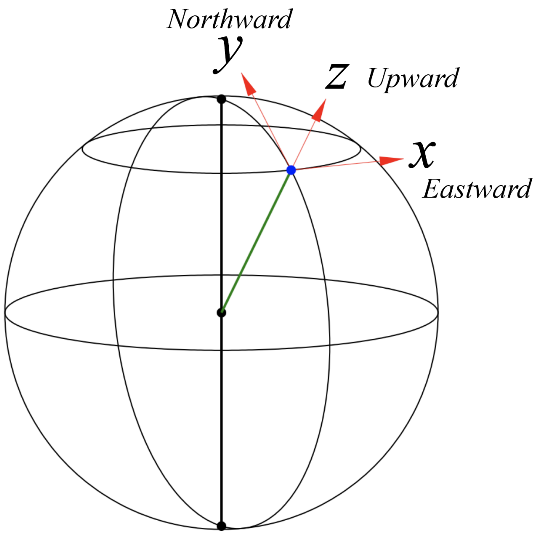

Figure 5.1: Local Cartesian coordinates for a point on the surface of the Earth.

In addition incompressibility is given by div $\mathbf{u} = 0$, P is the pressure in the fluid and $\mathbf{g}$ is the acceleration due to the Earth's gravity.

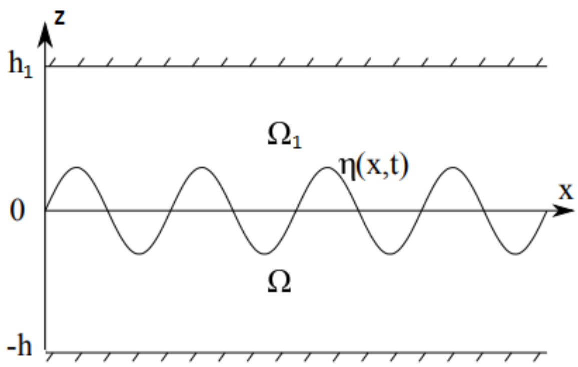

Figure 5.2: System Set up. The function $\eta(x,t)$ describes the elevation of the internal wave.

The fluid domain Ω is the fluid of higher density. The pycnocline/thermocline is the interface that separates the two fluid domains Ω and Ω_1. The Earth's angular velocity at latitude θ in this system is

$$\mathbf{\Omega} = \omega(0, \cos\theta, \sin\theta),$$

where $\omega = 7.3 \times 10^{-5}$ rad/s.

Introducing the parameters $f = 2\omega \sin\theta$ and $r = 2\omega\cos\theta$ gives

$$2\mathbf{\Omega} \times \mathbf{u} = (rw - fv, fu, -ru).$$

Note: the form (5.2.1) arises as the so-called *traditional approximation* of the Euler equations in spherical coordinates.

For Equatorial motion $\theta = 0$ and $f = 0$ so there are no forces acting in the y-direction. The Coriolis forces support the fluid to move along the Equator (in the x-direction), so that its motion remains two-dimensional. The system will be considered for $\theta > 0$. It will be assumed that the fluid motion is irrotational (that is absence of currents and vorticity), apart from the global rotation caused by the Coriolis forces.

In this approximation the velocity field is also potential, that is $\mathbf{u} = \nabla\varphi(x, y, z, t)$ and therefore $\Delta\varphi = 0$. The Coriolis effect will be presented as a perturbation to the potential motion.

The governing equations (5.2.1) acquire the form:

$$\begin{aligned}
&\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_x + r\varphi_z - f\varphi_y = 0,\\
&\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_y + f\varphi_x = 0,\\
&\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_z - r\varphi_x = 0,
\end{aligned} \tag{5.2.2}$$

where g is the Earth acceleration. The internal waves are illustrated in Fig. 2.4. For fixed y the system is bounded at the bottom by an impermeable flatbed and is considered as being bounded on the top by a flat horizontal surface, reflecting the assumption of absence of surface motion. The domains

$$\Omega = \{(x,z) \in \mathbb{R}^2 : -h < z < \eta(x,t)\},$$
$$\Omega_1 = \{(x,z) \in \mathbb{R}^2 : \eta(x,t) < z < h_1\}$$

correspond to the two fluid layers and the quantities associated with each domain are using the respective subscript notation. Also, subscript c (implying *common interface*) will be used to denote evaluation on the internal wave $z = \eta(x,t)$. Propagation of the internal wave is assumed to be in the positive x-direction, which is oriented eastward. The function $\eta(x,t)$ describes the elevation of the internal wave with the mean of η assumed to be zero,

$$\int_{\mathbb{R}} \eta(x,t)dx = 0. \tag{5.2.3}$$

The system is considered incompressible with ρ and ρ_1 being the respective constant densities of the lower and upper media and stability is given by the immiscibility condition $\rho > \rho_1$. For long internal waves with wavelength L the parameter $\delta = h/L \ll 1$ is a small non-dimensional parameter and φ is a small quantity of order δ. The effect of the terms proportional to r on the propagation in the x-direction could be estimated from the correction of the wave propagation speed in the x-direction due to the Coriolis force. The exact expression for this wave-propagation speed for equatorial waves when $\cos\theta = 1$ is [15]

$$c_0 = -\alpha_1(\rho-\rho_1)\omega \pm \sqrt{\alpha_1^2(\rho-\rho_1)^2\omega^2 + \alpha_1(\rho-\rho_1)g}, \quad \alpha_1 = \frac{hh_1}{\rho_1 h + \rho h_1}, \tag{5.2.4}$$

The terms with ω (which are proportional to r) are much smaller, thus the approximate value of c_0 is

$$c_0 = \pm\sqrt{\frac{hh_1(\rho-\rho_1)g}{(\rho_1 h + \rho h_1)}}$$

and the relative change due to the Coriolis force can be estimated to be of a magnitude

$$\left|\frac{\Delta c_0}{c_0}\right| = \sqrt{\frac{hh_1(\rho-\rho_1)}{(\rho_1 h+\rho h_1)g}}(2\omega\cos\theta) \sim 10^{-5} \text{ to } 10^{-4}$$

for the feasible values of the parameters. Thus the dependence of the motion in the (x, z)-plane due to the terms containing r can be neglected.

The motion in the y direction is very slow in comparison to the wave propagation in the x-direction, therefore in the leading order, $p = p(x, z)$ and the second equation in (5.2.2) can be used in the linear approximation to exclude the y dependence. This means

$$\varphi_{ty} + f\varphi_x = 0 \tag{5.2.5}$$

giving formally

$$\varphi_y = -f\partial_t^{-1}\varphi_x.$$

In what follows it will be assumed that f is of order $\delta^{3/2}$ or $\delta^2 \ll 1$. Noting that the ∂_x operator with an eigenvalue $k = 2\pi/\lambda$ is also of order δ (since $k = 2\pi\delta/h$), for compatible time-scales $\partial_t \sim \delta$ thus the y-derivative $\varphi_y \ll \varphi_x$. The first equation from (5.2.2) gives the following generalisation of the Bernoulli equation:

$$\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz + f^2\partial_t^{-1}\varphi = 0.$$

The nonlinear contribution

$$|\nabla\varphi|^2 = \varphi_x^2 + \varphi_y^2 + \varphi_z^2 \approx \varphi_x^2 + \varphi_z^2.$$

Therefore, the quantities in the leading order do not depend on the y variable and in the following sections this dependence will be supressed and essentially only two-dimensional wave motion will be considered (the y-dependence will be discussed again and reintroduced in Section 5.7). Note that the two-dimensional motion is common at the Equator ($\theta = 0$), but not necessarily in most other parts of the ocean where meandering is quite frequent. Nevertheless it is also physically realistic when $\theta > 0$: one example of such two-dimensional flow propagation in terms of sperical coordinates is the Antarctic Circumpolar Current - see for example [24]. It follows that the Bernoulli equation for this case is

$$\varphi_t + \frac{\varphi_x^2 + \varphi_z^2}{2} + \frac{p(x,z)}{\rho} + gz + f^2 \partial_t^{-1} \varphi = 0. \tag{5.2.6}$$

This is effectively a (2+1)-dimensional equation for the x, z dependent variables (considering y fixed) and it will provide one of the governing equations for the arising models.

It is again assumed that the functions $\eta(x,t)$, $\varphi(x,z,t)$ and $\varphi_1(x,z,t)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ with respect to the x variable (for any z and t). This reflects the localised nature of the wave disturbances, including disturbances in the form of solitary waves. The assumption of course implies that for large absolute values of x the internal wave attenuates

$$\lim_{|x|\to\infty} \eta(x,t) = 0, \quad \lim_{|x|\to\infty} \varphi(x,z,t) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \varphi_1(x,z,t) = 0. \tag{5.2.7}$$

The action of the operators ∂_x^{-1}, is not uniquely defined, however for the special subclass of functions satisfying the condition (5.2.3) the action is unique,

$$\partial_x^{-1} \eta(x,t) = \int_{\pm\infty}^{x} \eta(x',t) dx', \tag{5.2.8}$$

and moreover $\partial_x^{-1} \eta(x,t)$ belongs to $\mathcal{S}(\mathbb{R})$ with respect to the x variable as well. Therefore, the differentiation operator is invertible on the functional subclass of $\mathcal{S}(\mathbb{R})$ satisfying (5.2.3).

5.3 (Nearly) Hamiltonian representation of the internal wave dynamics

Recall that subindex 1 is used for the quantities associated to the upper layer. Also, subscript c (implying *common interface*) will also be used to denote evaluation on the internal wave $z = \eta(x, t)$. In addition, $\eta(x, t)$ satisfies the boundary kinematic condition on the interface

$$\eta_t = (w - u\eta_x)_c = (w_1 - u_1\eta_x)_c \tag{5.3.1}$$

or

$$\eta_t = (\varphi_z - \varphi_x\eta_x)_c = (\varphi_{1,z} - \varphi_{1,x}\eta_x)_c \tag{5.3.2}$$

representing the fact that the particles from the surface $z = \eta(x, t)$ (which is actually the interface, also the thermocline and the pycnocline) do not move in the direction, perpendicular to the surface [52].

Propagation of the internal wave is assumed to be in the positive x-direction which is considered to be "eastward". The main result is formulated in the following theorem:

Theorem 2. *The time evolution of the internal wave motion in the x-direction is given by the following system in quasi-Hamiltonian form*

$$\xi_t = -\frac{\delta H_0}{\delta \eta} - f^2 \left(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1)\right)_c \tag{5.3.3}$$

$$\eta_t = \frac{\delta H_0}{\delta \xi} \tag{5.3.4}$$

where

$$\xi(x, t) := (\rho\varphi - \rho_1\varphi_1)_c.$$

and H_0 is the Hamiltonian that corresponds to the irrotational motion ($f = 0$)

Proof: At $z = \eta(x,t)$, $p(x,\eta,t) = p_1(x,\eta,t)$ and therefore the Bernoulli condition is

$$\rho\Big((\varphi_t)_c + \frac{1}{2}|\nabla\varphi|_c^2 + g\eta + f^2(\partial_t^{-1}\varphi)_c\Big) = \rho_1\Big((\varphi_{1,t})_c + \frac{1}{2}|\nabla\psi_1|_c^2 + g\eta + f^2(\partial_t^{-1}\varphi_1)_c\Big) \tag{5.3.5}$$

or

$$\left(\rho\varphi_t - \rho_1\varphi_{1,t}\right)_c + \frac{\rho}{2}|\nabla\varphi|_c^2 - \frac{\rho_1}{2}|\nabla\varphi_1|_c^2 + g(\rho-\rho_1)\eta + f^2\left(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1)\right)_c = 0. \tag{5.3.6}$$

The functional H_0, which describes the total energy of the system, can be written as the sum of the kinetic, $\mathcal{K}$, and potential energy, (Π) contributions. The potential part, must be

$$\mathcal{V}(\eta) = \rho g \int_{\mathbb{R}}\int_{h}^{\eta} z\,dzdx + \rho_1 g \int_{\mathbb{R}}\int_{\eta}^{h_1} z\,dzdx. \tag{5.3.7}$$

However, the potential energy is always measured from some reference value, for example $\mathcal{V}(\eta = 0)$ which is the potential energy of the undisturbed interface, without wave motion. Therefore, the relevant part of the potential energy, contributing to the wave motion is

$$\Pi(\eta) = \mathcal{V}(\eta) - \mathcal{V}(0) = \rho g \int_{\mathbb{R}}\int_{0}^{\eta} z\,dzdx + \rho_1 g \int_{\mathbb{R}}\int_{\eta}^{0} z\,dzdx = \frac{1}{2}(\rho-\rho_1)g\int_{\mathbb{R}} \eta^2 dx. \tag{5.3.8}$$

The kinetic energy of the wave motion is

$$\mathcal{K} = \frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}(u^2 + w^2)dzdx + \frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}(u_1^2 + w_1^2)dzdx \tag{5.3.9}$$

and in terms of the velocity potentials

$$\begin{aligned} H_0 = \mathcal{K} + \Pi = \frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}|\nabla\varphi|^2 dzdx + \frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}|\nabla\varphi_1|^2 dzdx \\ + \frac{1}{2}(\rho-\rho_1)g\int_{\mathbb{R}}\eta^2 dx. \end{aligned} \tag{5.3.10}$$

The next step is to compute the variation of H_0, that is δH_0 :

$$\begin{aligned}\delta H_0 = \rho \int_{\mathbb{R}} \int_{-h}^{\eta} \nabla\varphi \cdot \nabla(\delta\varphi)\, dzdx + \rho_1 \int_{\mathbb{R}} \int_{\eta}^{h_1} \nabla\varphi_1 \cdot \nabla(\delta\varphi_1)\, dzdx \\ + \frac{1}{2}\rho \int_{\mathbb{R}} |\nabla\varphi|_c^2 \delta\eta\, dx - \frac{1}{2}\rho_1 \int_{\mathbb{R}} |\nabla\varphi_1|_c^2 \delta\eta\, dx + (\rho-\rho_1) g \int_{\mathbb{R}} \eta\delta\eta\, dx.\end{aligned} \tag{5.3.11}$$

Taking into account $\Delta\varphi = 0$ and $\Delta\varphi_1 = 0$

$$\begin{aligned}\delta H_0 = \rho \int_{\mathbb{R}} \int_{-h}^{\eta} \mathrm{div}[(\nabla\varphi)\delta\varphi]\, dzdx + \rho_1 \int_{\mathbb{R}} \int_{\eta}^{h_1} \mathrm{div}[(\nabla\varphi_1)\delta\varphi_1]\, dzdx \\ + \int_{\mathbb{R}} \Big(\frac{\rho}{2}|\nabla\varphi|_c^2 - \frac{\rho_1}{2}|\nabla\varphi_1|_c^2 + (\rho-\rho_1) g\eta\Big)\, \delta\eta\, dx.\end{aligned} \tag{5.3.12}$$

By applying the Divergence Theorem for the domains Ω and Ω_1 and noting that the outward normal to the domain Ω is $\mathbf{N} = (-\eta_x, 1)$, the outward normal to Ω_1 is $-\mathbf{N}$, $|\mathbf{N}| = \sqrt{1+\eta_x^2}$, with the line element $dl = \sqrt{1+\eta_x^2}dx$ and that there is no contribution from the flat bed or top surface, then

$$\begin{aligned}\delta H_0 = \rho \int_{\mathbb{R}} (\varphi_z - \eta_x \varphi_x)_c (\delta\varphi)_c\, dx - \rho_1 \int_{\mathbb{R}} (\varphi_{1,z} - \eta_x \varphi_{1,x})_c (\delta\varphi_1)_c\, dx \\ + \int_{\mathbb{R}} \Big(\frac{\rho}{2}|\nabla\varphi|_c^2 - \frac{\rho_1}{2}|\nabla\varphi_1|_c^2 + (\rho-\rho_1) g\eta\Big)\, \delta\eta\, dx.\end{aligned} \tag{5.3.13}$$

and using (5.3.1)

$$\begin{aligned}\delta H_0 = \int_{\mathbb{R}} \eta_t [\rho(\delta\varphi)_c - \rho_1 (\delta\varphi_1)_c]\, dx \\ + \int_{\mathbb{R}} \Big(\frac{\rho}{2}|\nabla\varphi|_c^2 - \frac{\rho_1}{2}|\nabla\varphi_1|_c^2 + (\rho-\rho_1) g\eta\Big)\, \delta\eta\, dx.\end{aligned} \tag{5.3.14}$$

Now recalling the relations

$$(\delta\varphi)_c = \delta\varphi_c - (\varphi_z)_c \delta\eta$$

and

$$(\delta\varphi_1)_c = \delta(\varphi_1)_c - (\varphi_{1,z})_c \delta\eta$$

this leads to

$$\delta H_0 = \int_{\mathbb{R}} \eta_t \delta\xi \, dx + \int_{\mathbb{R}} \Big(\frac{\rho}{2} |\nabla\varphi|_c^2 - \frac{\rho_1}{2} |\nabla\varphi_1|_c^2 + (\rho - \rho_1) g\eta - (\rho(\varphi_z)_c - \rho_1(\varphi_{1,z})_c)\eta_t \Big) \delta\eta \, dx. \tag{5.3.15}$$

From (5.3.15), (5.3.4) follows and also

$$\frac{\delta H_0}{\delta \eta} = \frac{\rho}{2} |\nabla\varphi|_c^2 - \frac{\rho_1}{2} |\nabla\varphi_1|_c^2 + (\rho - \rho_1) g\eta - (\rho\varphi_z - \rho_1\varphi_{1,z})_c \eta_t. \tag{5.3.16}$$

Since $(\varphi_c)_t = (\varphi_t)_c + (\varphi_z)_c \eta_t$ and $(\varphi_{1,c})_t = (\varphi_{1,t})_c + (\varphi_{1,z})_c \eta_t$, then

$$\xi_t = (\rho\varphi_t - \rho_1\varphi_{1,t})_c + (\rho\varphi_z - \rho_1\varphi_{1,z})_c \eta_t. \tag{5.3.17}$$

From (5.3.16) and (5.3.17) it follows that

$$\frac{\delta H_0}{\delta \eta} = \frac{\rho}{2} |\nabla\varphi|_c^2 - \frac{\rho_1}{2} |\nabla\varphi_1|_c^2 + (\rho - \rho_1) g\eta + (\rho\varphi_t - \rho_1\varphi_{1,t})_c - \xi_t. \tag{5.3.18}$$

Now (5.3.3) follows from (5.3.18) and (5.3.6). Hence (5.3.3) – (5.3.4) represent the nearly Hamiltonian formulation of the internal wave dynamics. □

In the following sections the Theorem 2 will be applied in the derivation of several important approximate model equations under various assumptions for the smallness of the physical quantities.

5.4 Long wave and small amplitude approximation

In this approximation we assume that $\delta \ll 1$, $\eta \sim \delta^2$, $D = -i\partial_x \sim \delta$, $\xi \sim \delta$, and $f \sim \delta^2$. The expansion with respect to δ of the Hamiltonian is [15]

$$H_0[\xi,\eta] = \frac{\delta^4}{2}\int_{\mathbb{R}} \xi D\left(\alpha_1 + \delta^2(\alpha_3\eta - \alpha_2 D^2)\right) D\xi\, dx + \delta^4 g(\rho-\rho_1)\int_{\mathbb{R}} \frac{\eta^2}{2} dx \quad (5.4.1)$$

where

$$\alpha_1 = \frac{hh_1}{\rho_1 h + \rho h_1}, \qquad \alpha_2 = \frac{h^2 h_1^2(\rho h + \rho_1 h_1)}{3(\rho_1 h + \rho h_1)^2}, \qquad \alpha_3 = \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2} \quad (5.4.2)$$

are constant parameters.

The Hamiltonian equations that follow from (5.3.3)-(5.3.4) expressed in terms of η and $\tilde{u} = \xi_x$ are

$$\begin{aligned} &\eta_t + \alpha_1\tilde{u}_x + \delta^2\alpha_2\tilde{u}_{xxx} + \delta^2\alpha_3(\eta\tilde{u})_x = 0, \\ &\tilde{u}_t + g(\rho-\rho_1)\eta_x + \delta^2\alpha_3\tilde{u}\tilde{u}_x + \delta^2 f^2\left[(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_{z=\eta(x,t)}\right]_x = 0. \end{aligned} \quad (5.4.3)$$

Taking the term:

$$\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_{z=\eta(x,t)} = \int^t [\rho\varphi(x,\eta(x,t),t') - \rho_1\varphi_1(x,\eta(x,t),t')]dt'$$

and evaluating

$$\partial_t[\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_{z=\eta(x,t)}] =$$

$$[\rho\varphi(x,\eta(x,t),t) - \rho_1\varphi_1(x,\eta(x,t),t)] + \eta_t\int^t [\rho\varphi_z(x,\eta(x,t),t') - \rho_1\varphi_{1,z}(x,\eta(x,t),t')]dt'$$

$$= \xi(x,t) + \text{smaller order terms} \quad (5.4.4)$$

since $\xi \sim \delta$ and $\eta_t \sim \delta^3$ etc. Therefore

$$\left[(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_{z=\eta(x,t)}\right]_x = \partial_t^{-1}\xi_x + \ldots = \partial_t^{-1}\tilde{u} + \ldots,$$

which leads to the system of coupled equations

$$\begin{aligned} &\eta_t + \alpha_1\tilde{u}_x + \delta^2\alpha_2\tilde{u}_{xxx} + \delta^2\alpha_3(\eta\tilde{u})_x = 0, \\ &\tilde{u}_t + g(\rho - \rho_1)\eta_x + \delta^2\alpha_3\tilde{u}\tilde{u}_x + \delta^2 f^2(\partial_t^{-1}\tilde{u}) = 0. \end{aligned} \tag{5.4.5}$$

In the leading order

$$\eta_t + \alpha_1\tilde{u}_x = 0, \qquad \tilde{u}_t + g(\rho - \rho_1)\eta_x = 0$$

or

$$\eta_{tt} = -\alpha_1\tilde{u}_{xt} = g\alpha_1(\rho - \rho_1)\eta_{xx}$$

so

$$\eta_{tt} - g\alpha_1(\rho - \rho_1)\eta_{xx} = 0, \tag{5.4.6}$$

which is the wave equation for η giving the wave speed

$$c_0 = \pm\sqrt{\alpha_1(\rho - \rho_1)g}. \tag{5.4.7}$$

For an observer, moving with the flow, that is there are left- ($-$ sign) and right-running ($+$ sign) waves. In the leading approximation, for linear waves, the functions depend on the characteristic variable $x - c_0 t$, therefore $\tilde{u} = \frac{c_0}{\alpha_1}\eta$. In the next order approximations with respect to δ so that

$$\tilde{u} = \frac{c_0}{\alpha_1}\eta + \delta^2(\ldots). \tag{5.4.8}$$

Differentiating the first equation in (5.4.5) with respect to t

$$\eta_{tt} + \alpha_1 \tilde{u}_{tx} + \delta^2 \alpha_2 \tilde{u}_{txxx} + \delta^2 \alpha_3 (\eta_t \tilde{u} + \eta \tilde{u}_t)_x = 0$$

and substituting in it $\tilde{u}_t$ from the second equation in (5.4.5), $\tilde{u}$ from (5.4.8), and

$$\eta_t = -\alpha_1 \tilde{u}_x + \delta^2(\dots)$$

where necessary, neglecting δ^4 terms, the following generalised Boussinesq equation for η is obtained:

$$\eta_{tt} - c_0^2 \eta_{xx} - \delta^2 \frac{3\alpha_3 c_0^2}{2\alpha_1} (\eta^2)_{xx} - \delta^2 \frac{\alpha_2 c_0^2}{\alpha_1} \eta_{xxxx} + \delta^2 f^2 \eta = 0. \qquad (5.4.9)$$

The dispersion law of this equation is

$$\tilde{\omega}^2(k) = c_0^2 k^2 - \delta^2 \frac{\alpha_2 c_0^2}{\alpha_1} k^4 + \delta^2 f^2, \qquad \tilde{\omega}(k) \approx c_0 k - \delta^2 \frac{\alpha_2 c_0}{2\alpha_1} k^3 + \delta^2 \frac{f^2}{2kc_0}. \quad (5.4.10)$$

A generalised Korteweg - de Vries (or KdV, [57]) type equation of the form

$$\eta_t + c_0 \eta_x + \delta^2 a_1 \eta_{xxx} + \delta^2 a_2 (\eta^2)_x + \delta^2 a_3 f^2 \partial_x^{-1} \eta = 0 \qquad (5.4.11)$$

for some constants a_1, a_2, a_3 (yet unknown) can be obtained from (5.4.9). Differentiating the above equation with respect to t gives

$$\eta_{tt} + c_0 \eta_{xt} + \delta^2 a_1 \eta_{txxx} + \delta^2 a_2 (\eta^2)_{xt} + \delta^2 a_3 f^2 \partial_x^{-1} \eta_t = 0$$

in which η_t is substituted from (5.4.11) to obtain (neglecting δ^4 terms)

$$\eta_{tt} - c_0^2 \eta_{xx} - \delta^2 2a_1 c_0 \eta_{xxxx} - \delta^2 2a_2 c_0 (\eta^2)_{xx} - \delta^2 2a_3 c_0 f^2 \eta = 0.$$

The comparison with (5.4.9) gives

$$a_1 = \frac{\alpha_2 c_0}{2\alpha_1}, \qquad a_2 = \frac{3\alpha_3 c_0}{4\alpha_1}, \qquad a_3 = -\frac{1}{2c_0}.$$

Then finally the KdV-type equation acquires the form

$$\eta_t + c_0\eta_x + \delta^2\frac{\alpha_2 c_0}{2\alpha_1}\eta_{xxx} + \delta^2\frac{3c_0\alpha_3}{4\alpha_1}(\eta^2)_x = \delta^2\frac{f^2}{2c_0}\partial_x^{-1}\eta. \tag{5.4.12}$$

The leading order term $\eta_t + c_0\eta_x$ an be transformed to just a $T-$derivative via an appropriate Galilean transformation $X \to (x - c_0 t)$, $T \to \delta^2 t$ which introduces the characteristic variable X and the slow time T such that $\eta_t = \delta^2\eta_T - c_0\eta_X$, $\eta_x = \eta_X$ and $\eta_t + c_0\eta_x = \delta^2\eta_T$ and therefore

$$\eta_T + \frac{\alpha_2 c_0}{2\alpha_1}\eta_{XXX} + \frac{3c_0\alpha_3}{4\alpha_1}(\eta^2)_X = \frac{f^2}{2c_0}\partial_X^{-1}\eta. \tag{5.4.13}$$

In the chosen scaling all quantities are of the same order and the scale variable δ has disappeared. The equation (5.4.13) is also known as Ostrovsky's equation [77]. Note that the dispersion law of the Ostrovsky equation (5.4.12) is like in (5.4.10):

$$\tilde{\omega}(k) = c_0 k - \delta^2\frac{\alpha_2 c_0}{2\alpha_1}k^3 + \delta^2\frac{f^2}{2kc_0}.$$

The condition (5.2.3) could be relaxed to $\int_{\mathbb{R}}\eta(x,t)dx =$ constant, which is typical for the soliton-like solutions. Such a condition does not change the average value of η over the whole real line $\mathbb{R}$, since the interval in this case has an infinite length, and the average value is still zero. The inverse operators of ∂_x and ∂_X however then are not unique and we can use (5.4.13) only in the form

$$\left[\eta_T + \frac{\alpha_2 c_0}{2\alpha_1}\eta_{XXX} + \frac{3c_0\alpha_3}{4\alpha_1}(\eta^2)_X\right]_X = \frac{f^2}{2c_0}\eta. \tag{5.4.14}$$

A derivation directly from Euler's equations is presented in Leonov's paper [63].

The Ostrovsky equation itself is Hamiltonian and possesses three conservation laws, however it is not bi-Hamiltonian and it is not integrable by the Inverse Scattering Method [9]. Solutions from perturbations of the KdV solitons [52, 65] can be derived in principle, although this is technically difficult, see for example [40] and the references therein. Some special solutions like travelling waves for the Ostrovsky equation have been well known (see for example [41]) and can be used in the analysis of the Earth's rotation in processes like energy transfer [41, 43] by ocean waves. Various other aspects of the equation have been studied extensively in numerous works, see for example [41, 85] and the references therein.

5.5 Special case of small or vanishing α_3

There is one special configuration when the coefficient α_3 from (5.4.2) is small (or vanishing), for example $\mathcal{O}(\alpha_3) \leq \delta$, or $\rho h_1^2 \approx \rho_1 h^2$. Then the nonlinearities like $\eta^2\eta_x$ can contribute significantly in balancing the dispersive terms. It can be shown that there is a scaling which corresponds to this situation and it is $\eta \sim \delta$, $f \sim \delta^2$ and $\tilde{u} \sim \delta$. Because the scaling differs from the one in the previous section this case is presented separately. The Hamiltonian acquires some extra terms. Looking at the Hamiltonian (5.9) from the article [15] and expanding up to terms of order δ^4 gives

$$H_0[\eta, \tilde{u}] = \frac{1}{2}\int_{\mathbb{R}} \left(\delta^2\alpha_1 + \delta^3\alpha_3\eta\right) \tilde{u}^2 dx + \delta^2 \int_{\mathbb{R}} \alpha_4 \frac{\eta^2}{2} dx - \delta^4 \frac{1}{2}\int_{\mathbb{R}} \beta_1 \eta^2 \tilde{u}^2 dx + \delta^4 \frac{1}{2}\int_{\mathbb{R}} \alpha_2 \tilde{u}\tilde{u}_{xx} dx, \quad (5.5.1)$$

where $\alpha_4 = g(\rho - \rho_1)$ and

$$\beta_1 = \frac{\rho\rho_1(h+h_1)^2}{(\rho_1 h + \rho h_1)^3}.$$

The term with α_3 is small of order δ^4 or even smaller, but because α_3 is a constant coefficient and for some other choice of the parameters it could be significant, the coefficient α_3 is not scaled explicitly. Taking into account the scaling, the "nearly"

Hamiltonian formulation can be written in the form

$$\begin{aligned} \eta_t &= -[\delta^{-2}]\left(\frac{\delta H_0}{\delta \tilde{u}}\right)_x \\ \tilde{u}_t &= -[\delta^{-2}]\left(\frac{\delta H_0}{\delta \eta}\right)_x - \delta^2 f^2 \partial_t^{-1}\tilde{u} \end{aligned} \tag{5.5.2}$$

where $[\delta^{-2}]$ is a scale factor and not a variation. This results in the system of equations

$$\eta_t + \alpha_1 \tilde{u}_x + \left(\delta\alpha_3\eta\tilde{u} - \delta^2\beta_1\eta^2\tilde{u} + \delta^2\alpha_2\tilde{u}_{xx}\right)_x = 0 \tag{5.5.3}$$

$$\tilde{u}_t + \alpha_4\eta_x + \left(\delta\frac{\alpha_3\tilde{u}^2}{2} - \delta^2\beta_1\eta\tilde{u}^2\right)_x + \delta^2 f^2\partial_t^{-1}\tilde{u} = 0. \tag{5.5.4}$$

Again, the leading order gives $c_0^2 = \alpha_1\alpha_4 = \alpha_1 g(\rho - \rho_1)$. From the coupled system of equations $\tilde{u}$ can be eliminated by using an expansion like (5.4.8) however taking into account all possible terms

$$\tilde{u} = \frac{\alpha_4}{c_0}\eta + \delta^2 b_1\eta_{xx} + \delta b_2\frac{\eta^2}{2} + \delta^2 b_3\eta^3 + \delta^2 b_4\partial_x^{-2}\eta \tag{5.5.5}$$

for some yet unknown coefficients $b_1, \ldots, b_4$. The substitution of (5.5.5) in (5.5.3) leads to an equation for η, keeping only terms up to order δ^2:

$$\begin{aligned} \eta_t + c_0\eta_x + \Bigg[\delta^2\left(b_1\alpha_1 + \frac{\alpha_2\alpha_4}{c_0}\right)\eta_{xx} + \delta\left(\frac{\alpha_1 b_2}{2} + \frac{\alpha_3\alpha_4}{c_0}\right)\eta^2 \\ +\delta^2\left(\alpha_1 b_3 + \frac{\alpha_3 b_2}{2} - \frac{\beta_1\alpha_4}{c_0}\right)\eta^3\Bigg]_x + \delta^2\alpha_1 b_4\partial_x^{-1}\eta = 0. \end{aligned} \tag{5.5.6}$$

The next step is in a similar fashion to eliminate $\tilde{u}$ from (5.5.4). A substitution of (5.5.5) in (5.5.4) is required, however there will appear $t-$ derivatives in various terms. In the non-leading order terms η_t can be eliminated by using the expression (5.5.6) and taking into account that in the leading order

$$\partial_t^{-1} = c_0^{-1}\partial_x^{-1}$$

results in

$$\eta_t + c_0\eta_x - \delta^2 \frac{c_0^2 b_1}{\alpha_4}\eta_{xxx} + \delta\left(\frac{\alpha_3\alpha_4}{c_0} - \frac{b_2 c_0^2}{\alpha_4}\right)\eta\eta_x$$
$$+ \delta^2\left(\frac{\alpha_3 b_2}{2} - \frac{\beta_1\alpha_4}{c_0} - \frac{c_0 b_2}{3\alpha_4}\left(b_2\alpha_1 + \frac{2\alpha_3\alpha_4}{c_0}\right)\right)(\eta^3)_x - \delta^2\left(\frac{f^2}{c_0} + \frac{c_0^2 b_4}{\alpha_4}\right)\partial_x^{-1}\eta = 0. \tag{5.5.7}$$

The two equations (5.5.6) and (5.5.7) need to coinside identically, and therefore their coefficients are equal. This leads to the following identities (recall $c_0^2 = \alpha_1\alpha_4$)

$$\begin{aligned}
-\frac{c_0^2 b_1}{\alpha_4} &= b_1\alpha_1 + \frac{\alpha_2\alpha_4}{c_0} \quad \text{giving} \quad b_1 = -\frac{c_0\alpha_2}{2\alpha_1^2},\\
\frac{\alpha_3\alpha_4}{c_0} - \frac{b_2 c_0^2}{\alpha_4} &= b_2\alpha_1 + \frac{2\alpha_3\alpha_4}{c_0} \quad \text{giving} \quad b_2 = -\frac{c_0\alpha_3}{2\alpha_1^2},\\
\alpha_1 b_3 + \frac{\alpha_3 b_2}{2} - \frac{\beta_1\alpha_4}{c_0} &\\
= \frac{\alpha_3 b_2}{2} - \frac{\beta_1\alpha_4}{c_0} - \frac{c_0 b_2}{3\alpha_4}&\left(b_2\alpha_1 + \frac{2\alpha_3\alpha_4}{c_0}\right) \quad \text{giving} \quad b_3 = \frac{c_0\alpha_3^2}{8\alpha_1^3},\\
\alpha_1 b_4 &= -\frac{f^2}{c_0} - \frac{c_0^2 b_4}{\alpha_4} \quad \text{giving} \quad b_4 = -\frac{f^2}{2c_0\alpha_1}.
\end{aligned} \tag{5.5.8}$$

These coefficients completely determine the relation (5.5.5) giving $\tilde{u}$ in terms of η. Moreover, the substitution of the coefficients in any of the equations (5.5.6) or (5.5.7) leads to the following equation for η :

$$\eta_t + c_0\eta_x + \delta^2 \frac{c_0\alpha_2}{2\alpha_1}\eta_{xxx} + \delta\frac{3c_0\alpha_3}{2\alpha_1}\eta\eta_x - \delta^2\frac{c_0}{8\alpha_1^2}\left(\alpha_3^2 + 8\alpha_1\beta_1\right)(\eta^3)_x$$
$$- \delta^2\frac{f^2}{2c_0}\partial_x^{-1}\eta = 0. \tag{5.5.9}$$

The coefficient in front of $(\eta^3)_x$ in terms of the physical parameters is

$$-\frac{c_0}{8\alpha_1^2}\left(\alpha_3^2 + 8\alpha_1\beta_1\right) = -\frac{c_0(\rho^2 h_1^4 + 8\rho\rho_1 h h_1(h^2 + h_1^2) + 14\rho\rho_1 h^2 h_1^2 + \rho_1 h^4)}{8h^2 h_1^2(\rho_1 h + \rho h_1)^2}. \tag{5.5.10}$$

Note that if α_3 is of the order of δ or smaller, then all non-leading order terms are of order δ^2. In terms of the (X, T) variables introduced in the previous section, the leading order term $\eta_t + c_0\eta_x = \delta^2\eta_T$ is of the same order, δ^2.

Finally, the very special case without quadratic nonlinearities is realised when $\alpha_3 = 0$. Then the equation (5.5.9) when $f = 0$ is of mKdV-type for the real variable η. Since the two densities are very similar, it is the difference $\Delta\rho = \rho - \rho_1$ that matters for the evaluation of α_1. Elsewhere $\rho = \rho_1$, and hence $\alpha_3 = 0$ meaning $h = h_1$. Then (5.5.9) becomes

$$\eta_t + c_0\eta_x + \delta^2\frac{c_0h^2}{6}\eta_{xxx} - \delta^2\frac{c_0}{h^2}(\eta^3)_x - \delta^2\frac{f^2}{2c_0}\partial_x^{-1}\eta = 0 \qquad (5.5.11)$$

with

$$c_0^2 = \frac{gh\Delta\rho}{2\rho},$$

or

$$\eta_T + \frac{c_0h^2}{6}\eta_{XXX} - \frac{c_0}{h^2}(\eta^3)_X - \frac{f^2}{2c_0}\partial_X^{-1}\eta = 0. \qquad (5.5.12)$$

The mKdV equation (when $f = 0$) is an integrable system, while the equation with a nonzero f is not.

5.6 Intermediate Long Wave approximation

In this section, the equations of motion are examined under the additional approximation that the wavelengths L are much bigger than h_1, that is the shallowness parameter will be taken as

$$\delta = \frac{h_1}{L} \ll 1.$$

Noting that the wave number $k = 2\pi/L$ is an eigenvalue or a Fourier multiplier for the operator D (when acting on monochromatic waves of the form e^{ikx}). The following further assumptions about the scales are made:

1. η and $\tilde{u}$ are both of order δ;

2. $hk = \mathcal{O}(1)$ and $h_1 k = \mathcal{O}(\delta)$ that is $h_1/h \sim \delta \ll 1$. This corresponds to a deep lower layer;

3. The physical constants h_1, ρ, ρ_1, are $\mathcal{O}(1)$, and $f \sim \delta^{3/2}$ (so that the f^2-term in the equation is of order δ).

Since the operator D has an eigenvalue k,it follows that $hD = \mathcal{O}(1)$ and $h_1 D = \mathcal{O}(\delta)$. The Hamiltonian, expanded over δ is derived in section (3.3) and could be obtained again with an asymptotic expansion of the general Hamiltonian from [15]. The Hamiltonian with terms up to order δ^3 is given by (4.3.18)

$$H_0[\eta, \mathfrak{u}] = \delta^2 \frac{h_1}{2\rho_1} \int_{\mathbb{R}} \tilde{u}^2 \, dx + \delta^2 \frac{\alpha_4}{2} \int_{\mathbb{R}} \eta^2 \, dx - \delta^3 \frac{1}{2\rho_1} \int_{\mathbb{R}} \eta \tilde{u}^2 \, dx \\ - \delta^3 \frac{h_1^2 \rho}{2\rho_1^2} \int_{\mathbb{R}} \tilde{u} \mathcal{T}_h \tilde{u}_x \, dx \quad (5.6.1)$$

where again the constant $\alpha_4 = g(\rho - \rho_1)$. Note that H_0 is of order δ^2. The equations follow from the nearly-Hamiltonian formulation in the case of Coriolis force (5.5.2), with the only difference that the Coriolis term with f^2 is now of order δ:

$$\eta_t + \frac{h_1}{\rho_1} \tilde{u}_x - \delta \frac{1}{\rho_1} (\eta \tilde{u})_x - \delta \frac{\rho h_1^2}{\rho_1^2} \mathcal{T}_h \tilde{u}_{xx} = 0 \quad (5.6.2)$$

$$\tilde{u}_t + \alpha_4 \eta_x - \delta \frac{1}{\rho_1} \tilde{u} \tilde{u}_x + \delta f^2 \partial_t^{-1} \tilde{u} = 0. \quad (5.6.3)$$

The leading order terms (that is neglecting the terms with δ above) produce a system of linear equations with constant coefficients from where the speed(s) of the travelling waves (in the leading order) is

$$c_0^2 = \frac{h_1}{\rho_1} g(\rho - \rho_1) \quad (5.6.4)$$

and clearly corresponds to the limit of large h in (5.4.7).

With the Johnson transformation

$$\tilde{u} = \frac{c_0\rho_1}{h_1}\eta + \delta b_1 \mathcal{T}_h \eta_x + \delta b_2 \frac{\eta^2}{2} + \delta b_3 \partial_x^{-2}\eta \tag{5.6.5}$$

and as before, the $\tilde{u}$ variable is excluded in order to obtain the two equations for η :

$$\eta_t + c_0\eta_x + \delta\left(\frac{b_1 h_1}{\rho_1} - \frac{c_0\rho h_1}{\rho_1}\right)\mathcal{T}_h\eta_{xx} + \delta\left(\frac{h_1 b_2}{\rho_1} - \frac{2c_0}{h_1}\right)\eta\eta_x + \delta\frac{b_3 h_1}{\rho_1}\partial_x^{-1}\eta = 0,$$
$$\eta_t + c_0\eta_x - \delta\frac{b_1 h_1}{\rho_1}\mathcal{T}_h\eta_{xx} - \delta\left(\frac{h_1 b_2}{\rho_1} + \frac{c_0}{h_1}\right)\eta\eta_x - \delta\left(\frac{b_3 h_1}{\rho_1} + \frac{f^2}{c_0}\right)\partial_x^{-1}\eta = 0.$$

From the comparison of the corresponding coefficients the following quantities are obtained

$$b_1 = \frac{c_0\rho}{2}, \quad b_2 = \frac{c_0\rho_1}{2h_1^2}, \quad b_3 = -\frac{f^2\rho_1}{2c_0 h_1} \tag{5.6.6}$$

which leads to the following equation for η :

$$\eta_t + c_0\eta_x - \delta\frac{c_0\rho h_1}{2\rho_1}\mathcal{T}_h\eta_{xx} - \delta\frac{3c_0}{2h_1}\eta\eta_x - \delta\frac{f^2}{2c_0}\partial_x^{-1}\eta = 0. \tag{5.6.7}$$

With an appropriate Galilean transformation $X \to (x - c_0 t)$, $T \to \delta t$ the equation transforms into

$$\eta_T - \frac{c_0\rho h_1}{2\rho_1}\mathcal{T}_h\eta_{XX} - \frac{3c_0}{2h_1}\eta\eta_X - \frac{f^2}{2c_0}\partial_X^{-1}\eta = 0. \tag{5.6.8}$$

Note that the speed c_0 depends only on the depth of the upper layer h_1. The h-dependence comes only from the term $\mathcal{T}_h$. For c_0 there are two possible choices, corresponding to the left or the right-running waves. For both choices, the equations (5.6.8) are different. The obtained equation (5.6.8) extends the integrable Intermediate Long Wave (ILW) equation [8, 50, 61].

In the limit $h \to \infty$ the operator $\mathcal{T}_h = -i\coth(hD)$ becomes the Hilbert transform operator $\mathcal{H} = -i\,\mathrm{sign}(D)$ or $\mathcal{T}_h\partial_x \to |D| = |\partial_X|$. Then (5.6.8) becomes an extension of the Benjamin-Ono (BO) equation:

$$\eta_T - \frac{c_0 \rho h_1}{2\rho_1}|\partial_X|\eta_X - \frac{3c_0}{2h_1}\eta\eta_X - \frac{f^2}{2c_0}\partial_X^{-1}\eta = 0. \tag{5.6.9}$$

The BO equation [1, 70] like KdV and ILWE is a classical example of an integrable equation solvable by the inverse scattering method [58].

The models with Coriolis term (5.6.8),(5.6.9) are not integrable, however the solitary wave solutions could be treated in the framework of the theory of soliton perturbations, as well as with other methods.

5.7 Including the y-dependence

Up to this point, the y-dependence of the physical quantities has been ignored, and there are physical situations where this is justified. However, from (5.2.5) it is clear that the y derivative, or, rather the operator $-i\partial_y$ and its eigenvalue k_y are of the same order as f, that is $\delta^{n/2}$ where $n = 4$ for the KdV models and $n = 3$ for the ILWE and BO models. The y-dependence is now reintroduced through the following Lemma:

Lemma 3. *The approximate model equations for small amplitude and long or intermediate long wave regimes have the following general form*

$$\eta_t + c_0\eta_x + \delta^{n-2}\left[\frac{R(D)}{2c_0}\eta + \left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta\right] + nonlin.\ terms = 0, \tag{5.7.1}$$

where $R(k)$ is an appropriate odd function.

Proof: Since $\varphi_y \ll \varphi_x$ from the symmetry of the original equations it is clear that when extending the gradient there will be no ∂_y contributions in the nonlinear part of the equation, since ∂_y would not contribute to the leading order of the nonlinear part. The contribution of ∂_y in the dispersive part of the equation could be determined from the extension of the dispersion law. The dispersion law in principle is given by some model-related odd function $R(k)$ - like in (5.4.10) where $R(k) = -\frac{c_0^2\alpha_2}{\alpha_1}k^3$, and in addition - a Coriolis contribution, given by the f^2 term,

$$\tilde{\omega}^2(k) = c_0^2k^2 + \delta^{n-2}\left[kR(k) + f^2\right]. \tag{5.7.2}$$

The leading order term could be extended in the two-dimensional case noting that $k^2 \to \vec{k}^2 = k^2 + k_y^2$, or taking into account the scaling

$$\delta^2k^2 \to \delta^2k^2 + \delta^n k_y^2 = \delta^2(k^2 + \delta^{n-2}k_y^2)$$

hence

$$\tilde{\omega}^2(k, k_y) = c_0^2k^2 + \delta^{n-2}\left[c_0^2k_y^2 + kR(k) + f^2\right]. \tag{5.7.3}$$

The evaluation of the square root up to δ^{n-2} gives

$$\tilde{\omega}(k, k_y) = c_0k + \delta^{n-2}\left[\frac{R(k)}{2c_0} + \frac{c_0k_y^2}{2k} + \frac{f^2}{2c_0k}\right]. \tag{5.7.4}$$

The equation for η is of the form $\eta_t = i\tilde{\omega}(D, -i\partial_y)\eta+$ nonlinear terms, that is (5.7.1).

□

Therefore the addition to the model equations in each case is

$$\delta^{n-2}\left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta.$$

Note that the y-derivatives term and the Coriolis term are both of the same order. The main results, following from the Main Theorem and (5.3.3)-(5.3.4) with the y-dependence extension from Lemma 3 therefore can be summarised into the following corollaries about the approximate models describing the time-evolution of the internal wave in two dimensions:

Corollary 4. *(a) In the case of small amplitude (of order δ^2) and long-wave regime,* *the approximation gives the KP - Ostrovsky type equation*

$$\eta_t + c_0\eta_x + \delta^2 \frac{\alpha_2 c_0}{2\alpha_1}\eta_{xxx} + \delta^2 \frac{3c_0\alpha_3}{4\alpha_1}(\eta^2)_x + \delta^2 \left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta = 0; \qquad (5.7.5)$$

(b) In the case of small amplitude (of order δ) with parameter α_3 of order δ or smaller, and long-wave regime, the approximation gives the mKdV - KP - Ostrovsky type equation

$$\eta_t + c_0\eta_x + \delta^2 \frac{c_0\alpha_2}{2\alpha_1}\eta_{xxx} + \delta \frac{3c_0\alpha_3}{2\alpha_1}\eta\eta_x - \delta^2 \frac{c_0}{8\alpha_1^2}\left(\alpha_3^2 + 8\alpha_1\beta_1\right)(\eta^3)_x + \delta^2 \left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta = 0. \quad (5.7.6)$$

By taking formally $f = 0$ in (5.7.5) one can recover the corresponding Kadomtsev-Petviashvili (KP)-type equations, [52, 60, 65].

Corollary 5. *(a) In the case of small amplitude (of order δ) and intermediate long wave regime, the approximation gives the ILW - Ostrovsky type equation*

$$\eta_t + c_0\eta_x - \delta \frac{c_0\rho h_1}{2\rho_1}\mathcal{T}_h\eta_{xx} - \delta \frac{3c_0}{2h_1}\eta\eta_x + \delta \left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta = 0, \qquad (5.7.7)$$

and,

(b) In the limit $h \to \infty$, the previous model becomes the BO-Ostrovsky type equation:

$$\eta_t + c_0\eta_x - \delta \frac{c_0\rho h_1}{2\rho_1}|\partial_x|\eta_x - \delta \frac{3c_0}{2h_1}\eta\eta_x + \delta \left(\frac{c_0}{2}\partial_y^2 - \frac{f^2}{2c_0}\right)\partial_x^{-1}\eta = 0. \qquad (5.7.8)$$

5.8 Discussion and conclusions

A consistent approach has been presented for the derivation of the simplified long-wave and intermediate long-wave models (5.7.5)-(5.7.8) based on the extension of the Hamiltonian approach in the irrotational case [30, 31]. A similar Hamiltonian approach has been successful for treating internal waves with shear currents [15, 16, 17, 18, 19, 47]. For surface waves the derivation is analogous, only the Hamiltonian H_0 is the corresponding Hamiltonian for the propagation of surface waves.

The obtained approximate models are integrable only in the special case of $f = 0$, and they are not integrable in the situation that includes the Coriolis forces. Nevertheless the structure of the equations usually allows several conservation laws (like "mass" and "energy") and also allows the application of the soliton perturbation theory to the soliton solutions of the corresponding integrable equations [40]. In addition, the structure of perturbed integrable equations has certain advantages in the development of numerical methods for these equations.

Chapter 6

FUTURE WORK AND OPEN QUESTIONS

This thesis presents a description of internal geophysical wave models using the Hamiltonian formulation. Two main incompressible and inviscid water wave systems were considered. The first system was comprised of two discrete rotational fluid domains with a depth-dependent current separated by an internal wave bounded below by a flat bottom and on top by a lid. The second model was also comprised of two discrete fluid domains, but in this case the domains were considered to be irrotational and current free. Approximate models were derived using linearisation and perturbation techniques. The governing equations are Euler's equations.

For the intermediate long wave propogation for internal waves in the presence of currents, a realistic oceanic representation was considered. Vorticity and Coriolis forces were also taken into account. Using the Hamiltonian approach, and introducing Dirichlet-Neumann operators and interface quantities, the Hamiltonian was derived in terms of interface quantities only. Equations of motion were found to be 'nearly' Hamiltonian. Canonical equations of motion were established by a variable transformation. The importance of the introduction of the interface quantities means that the dynamics in the bulk can be obtained analytically from the dynamics of the wave.

The ILWE approximate model equation was then derived. In order to achieve this, the system quantities were nondimensionalised and the introduction of a small arbitrary perturbation parameter, ε, established a small amplitude approximation with linear equations of motion. The introduction of a second perturbation parameter, δ, established a long-wave model. A solitary wave solution was then identified which depends on the model parameters.

For the second system, the Coriolis effect on the internal wave propagation following the idea of the "nearly" Hamiltonian approach was examined, suggested in [20] but in a different setting, generalising the Hamiltonian approach of Zakharov [88]. The Coriolis effect is present both in the ocean and in the atmosphere. A derivation of the model equations for the internal wave propagation taking into account the Coriolis effect was performed. Two propagation regimes were examined, the long-wave and the intermediate long-wave propagation with a small amplitude approximation for certain geophysical scales of the physical variables. The obtained models are of the type of the well-known Ostrovsky equation and describe the wave propagation over the two spatial horizontal dimensions of the ocean surface. The main aim of this work was to illustrate the mathematical usefulness of the Hamiltonian approach in a systematic study of the internal wave propagation, rather than to present new equations on internal waves. The approach could be used in further studies, including detailed analysis with higher order approximations or effects, not included in this thesis.

An interesting area of active research is the development of theoretical models for internal waves with currents over a variable bottom. For example, the article written by Ivanov, Martin and Todorov, [49], presents a derivation of a model equation (with variable coefficients) of the KdV type which describes the propagation of interfacial internal waves in two-layer domains in a flow with a variable depth and a flat surface in the presence of currents, vorticity (wave–current interactions) and density stratification.

The general KdV type equation derived in [49] takes the following form

$$c^2\Big(2(c-\kappa)+\alpha_1\Gamma\Big)\eta_X+\Big(c^2c_X-\kappa c(c-\kappa)\frac{\alpha_{1,X}}{\alpha_1}\Big)\eta+\frac{\alpha_2(c-\kappa)^2}{\alpha_1 c^2}\eta_{\theta\theta\theta}$$
$$+\Big(3\frac{\alpha_3}{\alpha_1}(c-\kappa)^2+3\alpha_4(c-\kappa)+\alpha_1\alpha_6\Big)\eta\eta_\theta=0$$

where, given the small amplitude parameter $\epsilon=\frac{|\eta_{max}|}{h}$ and long wave parameter $\delta=\frac{h}{\lambda}$,

$$\theta=\frac{1}{\epsilon}R(X)-t,$$

and

$$X=\delta^2 x.$$

Note: $R(X)$ is a function such that $R'(X)=\frac{1}{c}(X)$.

This general equation in various limits leads to several known simplified cases with variable bottom, like the irrotational case, the single layer case with and without background current, which goes back to the well-known work of Johnson [51].

The equatorial waves and currents in the equatorial Pacific Ocean, where the so-called Equatorial Undercurrent (EUC) resides and where the abyssal hills are the most abundant seabed structures near the equator, provide an example for a potentially realistic situation.

It is worth noting that Ostrovsky and Helfrich's paper [78] provides a discussion on some interesting properties of soliton dynamics in a two layer, rotating fluid over a variable bottom, with Coriolis force included. They discuss the cumulative effect of Coriolis force and variable bottom for internal waves.

The published article [34], (co-authored by the thesis author) has been independently cited by several articles, where a range of topics are discussed. Some of these articles are noted here for reference:

- Martin, C.I., 2021. *Azimuthal equatorial flows in spherical coordinates with discontinuous stratification.* Physics of Fluids, 33(2), p.026602.

- Fan, L., Liu, R. and Gao, H., 2023. *Hamiltonian model for coupled surface and internal waves over currents and uneven bottom.* Physica D: Nonlinear Phenomena, 443, p.133558.

- Geyer, A. and Quirchmayr, R., 2022. *Weakly nonlinear waves in stratified shear flows.* Communications on Pure and Applied Analysis, 21(7), pp.2309-2325.

- Li, G., 2023. *Deep-water and shallow-water limits of the intermediate long wave equation: from deterministic and statistical viewpoints.*

The results of a second published article, [35], also co-authored by the thesis author, will be used in the forthcoming works of Prof. Rossen Ivanov and co-authors, about internal waves interacting with surface waves under the influence of the Coriolis force.

Further areas for future work include -

- Model equations including meridional varations.

- 3-dimensional models.

- Models with both vorticity and Coriolis Force.

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Appendices

Appendix A

Euler's equation in a non-inertial frame of reference

Considering an inviscid fluid, with two-dimensional flow in the x–y plane and with a free surface at $y = 0$, situated on the surface of Earth. For the non-inertial 'spinning-Earth' frame of reference, as shown in Figure A.1 (*cf.* [26]), it is noted that x_1 is considered *eastward*, x_2 is considered *northward* and x_3 is considered *perpendicular to the surface.*

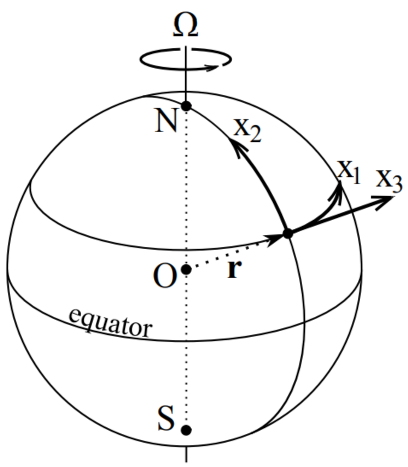


Figure A.1: The rotational frame of reference

For an observer in an inertial frame of reference the measurements determined by the observer in the rotational frame of reference must be adjusted as follows [39, 74]

(where the subscript *rot* means in the rotational frame of reference)

$$\left(\frac{D\mathbf{u}}{Dt}\right)_{in} = \left(\frac{D\mathbf{u}}{Dt}\right)_{rot} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \left(\frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}\right) \tag{A1}$$

where $\boldsymbol{\Omega}$ is the Earth's angular velocity and $\mathbf{r}$ is a position vector, as shown in Figure A.1. The term $2\boldsymbol{\Omega} \times \mathbf{u}$ is the Coriolis acceleration, $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u})$ is the centripetal acceleration and the final term, which is related to the rate of change of $\boldsymbol{\Omega}$, can be ignored due to the assumption that time-scales involved are short compared to the time-scale over which $\boldsymbol{\Omega}$ changes. By dropping the *rot* subscript notation (A1) can therefore be written as

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla P + \mathbf{g}$$

where $\mathbf{g} = (0, 0, -g)$ is the Earth's acceleration due to gravity and

$$P = \rho g y + p + p_{\text{atm}} \tag{A2}$$

is the total pressure given as hydrostatic (due to gravity), dynamic and constant atmospheric pressure terms respectively. This is Euler's Equation for an incompressible fluid (that is the inviscid form of the Navier-Stokes equation for an incompressible fluid). Alternatively, by using the Eulerian derivative, the equation can be written as

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla P + \mathbf{g}.$$

For the two-media system under study the following Euler equations will be used to establish the Bernoulli condition:

$$\mathbf{u}_{1,t} + (\mathbf{u}_1.\nabla)\mathbf{u}_1 + 2\boldsymbol{\Omega} \times \mathbf{u}_1 + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho_1}\nabla P_1 + \mathbf{g} \tag{A3}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla P + \mathbf{g}. \tag{A4}$$

Appendix B

Expansion of the Dirichlet-Neumann Operator

Using the approach in [31] the velocity potential $\varphi(x, y)$ for a single medium system can be represented by

$$\varphi_k(x, y) = a(k)e^{ky}e^{ikx} + b(k)e^{-ky}e^{ikx} \tag{B1}$$

where

$$a(k) = \frac{e^{kh}}{e^{kh} + e^{-kh}} \text{ and } b(k) = \frac{e^{-kh}}{e^{kh} + e^{-kh}}.$$

Using the definition of $\tanh(z)$ in Appendix **??** it is noted that

$$a(k) - b(k) = \tanh(hk) \text{ and } a(k) + b(k) = 1. \tag{B2}$$

Consider that

$$-i\big(\varphi_k(x, y)\big)_x = -i\big(ika(k)e^{ky}e^{ikx} + ikb(k)e^{-ky}e^{ikx}\big) = k\big(\varphi_k(x, y)\big).$$

Remark. As a result of the previous equation the operation $D = -i\partial_x$ is equivalent to multiplication by the wavenumber k, that is, the differential operator D has an eigenvalue equivalent to the wave number.

At the surface $y = \eta$ therefore the surface potential

$$\xi_k(x) = \varphi_k(x, \eta(x)) = a(k)e^{k\eta}e^{ikx} + b(k)e^{-k\eta}e^{ikx}. \tag{B3}$$

At the shear surface $y = 0$ therefore

$$\varphi_k(x, 0) = e^{ikx}. \tag{B4}$$

The surface potential ξ_k given by (B3) is Taylor expanded about $\eta = 0$ giving

$$\xi_k(x) = \varphi_k(x, 0) + \frac{1}{1!}\varphi_k'(x, 0)\eta + \frac{1}{2!}\varphi_k''(x, 0)\eta^2 + ...$$

The following are calculated

$$\varphi_k(x, \eta) = k^0\Big(a(k)e^{k\eta} + b(k)e^{-k\eta}\Big)e^{ikx},$$

$$\varphi_k'(x, \eta) = k^1\Big(a(k)e^{k\eta} - b(k)e^{-k\eta}\Big)e^{ikx},$$

and

$$\varphi_k''(x, \eta) = k^2\Big(a(k)e^{k\eta} + b(k)e^{-k\eta}\Big)e^{ikx},$$

therefore

$$\varphi_k(x, 0) = k^0\big(a(k) + b(k)\big)e^{ikx},$$

$$\varphi_k'(x, 0) = k^1\Big(a(k) - b(k)\Big)e^{ikx},$$

and

$$\varphi_k''(x, 0) = k^2\big(a(k) + b(k)\big)e^{ikx}.$$

Note the sign of $b(k)$ is positive for even order derivatives and negative for odd order derivatives the expansion is given as

$$\xi_k(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\eta^j k^j \big(a(k) + (-1)^j b(k)\big)e^{ikx}. \tag{B5}$$

The Dirichlet-Neumann operator (3.3.7) is written as

$$G(\eta)\xi_k = -\eta_x(\varphi_{k,x})_s + (\varphi_{k,y})_s. \tag{B6}$$

The following partial derivatives $(\varphi_k)_x$ and $(\varphi_k)_y$ are calculated

$$(\varphi_k)_x = ik\varphi_k(x, y)$$

and

$$(\varphi_k)_y = k\Big(a(k)e^{ky}e^{ikx} + (-1)b(k)e^{-ky}e^{ikx}\Big).$$

Applying these to (B6) and using the expansion of $\xi_k(x)$ from (B5) gives

$$G(\eta)\xi_k = \sum_{j\geq 0} -ik\eta_x \frac{1}{j!}\eta^j k^j \big(a(k) + (-1)^j b(k)\big)e^{ikx} \\ + \sum_{j\geq 0} k\frac{1}{j!}\eta^j k^j \big(a(k) + (-1)^1(-1)^j b(k)\big)e^{ikx}$$

which can be written as

$$G(\eta)\xi_k = \sum_{j\geq 0} -i\eta_x \frac{1}{j!}\eta^j k^{j+1} \big(a(k) + (-1)^j b(k)\big)e^{ikx} \\ + \sum_{j\geq 0} \frac{1}{j!}\eta^j k^{j+1} \big(a(k) + (-1)^{j+1} b(k)\big)e^{ikx}. \tag{B7}$$

The left hand side of this equation can be expanded in terms of orders of η, that is

$$G(\eta) = \sum_{j=0}^{\infty} G^{(j)}(\eta) \tag{B8}$$

which means

$$G(\eta)\xi_k = \left[\sum_{j=0}^{\infty} G^{(j)}(\eta)\right]\left[\sum_{j=0}^{\infty} \frac{1}{j!}\eta^j k^j \big(a(k) + (-1)^j b(k)\big)e^{ikx}\right]. \tag{B9}$$

Combining (B7) and (B9) therefore gives

$$\left[\sum_{j=0}^{\infty} G^{(j)}(\eta)\right]\left[\sum_{j=0}^{\infty} \frac{1}{j!}\eta^j k^j \big(a(k) + (-1)^j b(k)\big)e^{ikx}\right]$$
$$= \sum_{j=0}^{\infty} -i\eta_x \frac{1}{j!}\eta^j k^{j+1}\big(a(k) + (-1)^j b(k)\big)e^{ikx}$$
$$+ \sum_{j=0}^{\infty} \frac{1}{j!}\eta^j k^{j+1}\big(a(k) + (-1)^{j+1} b(k)\big)e^{ikx}. \quad \text{(B10)}$$

By examining this equation it can be seen that the order of η is equal to j. Also, j gives the term of the expansion of G, that is $G^{(j)}$. By choosing different values of j the terms in the expansion of $G(\eta)$, in terms of different orders of η, can therefore be determined.

Letting $j = 0$ (B10) becomes

$$\left[G^{(0)}\right]\left[\big(a(k) + b(k)\big)e^{ikx}\right] = -i\eta_x k\big(a(k) + b(k)\big)e^{ikx} + k\big(a(k) - b(k)\big)e^{ikx}$$

and using the identities in (B2) gives

$$G^{(0)}e^{ikx} = -i\eta_x k e^{ikx} + k\tanh(hk)e^{ikx}.$$

Looking only at terms that are constant (noting that $G^{(0)} \neq G^{(0)}(\eta)$, that is it is a constant) gives

$$G^{(0)}e^{ikx} = k\tanh(hk)e^{ikx}.$$

In terms of the operator D therefore

$$G^{(0)} = D\tanh(hD). \quad \text{(B11)}$$

Letting $j = 1$ (B10) becomes

$$\Big[G^{(0)} + G^{(1)}(\eta)\Big]\Big[\big(a(k)+b(k)\big)e^{ikx} + \eta k\big(a(k)-b(k)\big)e^{ikx}\Big]$$
$$= \Big[-i\eta_x k\big(a(k)+b(k)\big) + k\big(a(k)-b(k)\big)$$
$$- i\eta_x\eta k^2\big(a(k)-b(k)\big) + \eta k^2\big(a(k)+b(k)\big)\Big]e^{ikx}$$

and using the identities in (B2) gives

$$\Big[G^{(0)} + G^{(1)}(\eta) + G^{(0)}\eta k\tanh(hk) + G^{(1)}(\eta)\eta k\tanh(hk)\Big]e^{ikx}$$
$$= \Big[-i\eta_x k + k\tanh(hk) - i\eta_x\eta k^2\tanh(hk) + \eta(x)k^2\Big]e^{ikx}.$$

Looking only at terms that are linear in η

$$G^{(1)}(\eta)e^{ikx} + G^{(0)}\eta k\tanh(hk)e^{ikx} = -i\eta_x k e^{ikx} + \eta k^2 e^{ikx}.$$

Now

$$-i(\eta k e^{ikx})_x = -i\eta_x k e^{ikx} + \eta k^2 e^{ikx}$$

therefore

$$G^{(1)}(\eta)e^{ikx} + G^{(0)}\eta k\tanh(hk)e^{ikx} = -i\big(\eta k e^{ikx}\big)_x.$$

In terms of the operator D therefore

$$G^{(1)}(\eta) = D\eta D - G^{(0)}\eta G^{(0)}. \tag{B12}$$

Letting $j = 2$ (B10) becomes

$$\Big[G^{(0)}+G^{(1)}(\eta)+G^{(2)}(\eta)\Big]\Big[\big(a(k)+b(k)\big)+\eta k\big(a(k)-b(k)\big)+\frac{1}{2}\eta^2k^2\big(a(k)+b(k)\big)\Big]e^{ikx}$$
$$= \Big[-i\eta_x k\big(a(k)+b(k)\big)+k\big(a(k)-b(k)\big)-i\eta_x\eta k^2\big(a(k)-b(k)\big)$$
$$+\eta k^2\big(a(k)+b(k)\big)-i\eta_x\frac{1}{2}\eta^2k^3\big(a(k)+b(k)\big)+\frac{1}{2}\eta^2k^3\big(a(k)-b(k)\big)\Big]e^{ikx}$$

and using the identities in (B2) gives

$$\Big[G^{(0)}+G^{(0)}\eta k\tanh(hk)+G^{(0)}\frac{1}{2}\eta^2k^2+G^{(1)}(\eta)+G^{(1)}(\eta)\eta k\tanh(hk)$$
$$+G^{(1)}(\eta)\frac{1}{2}\eta^2k^2+G^{(2)}(\eta)+G^{(2)}(\eta)\eta k\tanh(hk)+G^{(2)}(\eta)\frac{1}{2}\eta^2k^2\Big]e^{ikx}$$
$$=\Big[-i\eta_x k+k\tanh(hk)-i\eta_x\eta k^2\tanh(hk)+\eta k^2$$
$$-i\eta_x\frac{1}{2}\eta^2k^3+\frac{1}{2}\eta^2k^3\tanh(hk)\Big]e^{ikx}.$$

Looking only at terms that are quadratic in η gives

$$\frac{1}{2}k^2\eta^2G^{(0)}+G^{(1)}(\eta)\eta k\tanh(hk)+G^{(2)}(\eta)=-i\eta_x\eta k^2\tanh(hk)+\frac{1}{2}\eta^2k^3\tanh(hk)$$

therefore

$$G^{(2)}(\eta)=-\frac{1}{2}k^2\eta^2G^{(0)}-G^{(1)}(\eta)\eta k\tanh(hk)-i\eta_x\eta k^2\tanh(hk)+\frac{1}{2}\eta^2k^3\tanh(hk).$$

In terms of the operator D, and by inserting the definition of the term $G^{(1)}(\eta)$ from Equation (B12), this means that

$$G^{(2)}(\eta)=-\frac{1}{2}D^2\eta^2G^{(0)}-D\eta D\eta G^{(0)}+G^{(0)}\eta G^{(0)}\eta G^{(0)}+D\eta D\eta G^{(0)}-\frac{1}{2}G^{(0)}\eta^2D^2.$$

Cancelling terms appropriately gives

$$G^{(2)}(\eta) = -\frac{1}{2}\big(D^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)} + G^{(0)}\eta^2 D^2\big).$$

For a system with two layers the operators are therefore

$$\begin{aligned}
G^{(0)} &= D\tanh(hD), \\
G_1^{(0)} &= D\tanh(h_1 D), \\
G^{(1)}(\eta) &= D\eta D - G^{(0)}\eta G^{(0)}, \\
G_1^{(1)}(\eta) &= -D\eta D + G_1^{(0)}\eta G_1^{(0)}, \\
G^{(2)}(\eta) &= -\frac{1}{2}\big(D^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)} + G^{(0)}\eta^2 D^2\big),
\end{aligned}$$

and $$G_1^{(2)}(\eta) = -\frac{1}{2}\big(D^2\eta^2 G_1^{(0)} - 2G_1^{(0)}\eta G_1^{(0)}\eta G_1^{(0)} + G_1^{(0)}\eta^2 D^2\big)$$

noting that the two linear terms have different signs due to the fact that the wave is at the bottom of the domain from the perspective of Ω_1.

Appendix C

The operator $\mathcal{T}_h$

The action of operators like $\mathcal{T}_h = -i\coth(hD)$ is defined with the help of Fourier transforms $\mathcal{F}$,

$$\hat{v}(k) := \mathcal{F}^{-1}\{v(x)\}(k), \qquad v(x) = \mathcal{F}\{\hat{v}(k)\}(x).$$

Then

$$\mathcal{T}_h v(x) := -i\mathcal{F}\{\coth(hk)\hat{v}(k)\}(x)$$

and furthermore

$$\begin{aligned}\mathcal{T}_h v(x) &= -i\frac{1}{\sqrt{2\pi}}\int e^{ikx}\coth(hk)\hat{v}(k)dk \\ &= \frac{-i}{\sqrt{2\pi}}\int e^{ikx}\coth(hk)\left(\frac{1}{\sqrt{2\pi}}\int e^{-ikx'}v(x')dx'\right)dk.\end{aligned}$$

By changing the order of integration and using some appropriate integrals from [42] for the integration over dk

$$\mathcal{T}_h v(x) = -\frac{1}{2h}\text{P.V.}\int_{-\infty}^{\infty}\coth\frac{\pi(x-x')}{2h}v(x')dx'.$$

When $h\to\infty$,

$$\frac{1}{2h}\coth\frac{\pi(x-x')}{2h} \to \frac{1}{2h}\cdot\frac{2h}{\pi(x-x')} = \frac{1}{\pi(x-x')}$$

and $\mathcal{T}_h$ becomes the Hilbert transform, $\mathcal{H}$

$$\mathcal{T}_h v(x) \to \mathcal{H}\{v\}(x) := \text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(x')dx'}{x - x'}.$$

Appendix D

The KdV equation for internal waves with currents

The propagation of internal waves with currents in the KdV regime $\varepsilon \simeq \delta^2 \ll 1$ has been derived in [15]. Without the assumption that h_1/h is small, the equation for the elevation $\eta(X,T)$ is

$$\eta_T + c\eta_X + \varepsilon \frac{c^2\alpha_2}{\alpha_1(2c+\Gamma\alpha_1)}\eta_{XXX} + \varepsilon \frac{3c^2\alpha_3 + 3c\alpha_1\alpha_4 + \alpha_1^2\alpha_5}{\alpha_1(2c+\Gamma\alpha_1)}\eta\eta_X = 0. \tag{D1}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{hh_1}{\rho_1 h + \rho h_1}, \qquad \alpha_2 = \frac{h^2 h_1^2(\rho h + \rho_1 h_1)}{3(\rho_1 h + \rho h_1)^2}, \qquad \alpha_3 = \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2},\\
\alpha_4 &= \frac{\gamma_1\rho_1 h + \gamma\rho h_1}{\rho_1 h + \rho h_1}, \qquad \alpha_5 = \rho\gamma^2 - \rho_1\gamma_1^2,\\
c &= -\frac{\alpha_1\Gamma}{2} \pm \sqrt{\frac{\alpha_1^2\Gamma^2}{4} + \alpha_1(\rho-\rho_1)g}.
\end{aligned} \tag{D2}$$

With the further assumption $\frac{h_1}{h} = \varepsilon \ll 1$, the values of the constants become:

$$\begin{aligned}
&\alpha_1 \approx \frac{h_1}{\rho_1}, \qquad \alpha_2 \approx \frac{\rho h h_1^2}{3\rho_1^2}, \qquad \alpha_3 \approx -\frac{1}{\rho_1},\\
&\alpha_4 \approx \gamma_1, \quad \alpha_5 = \rho\gamma^2 - \rho_1\gamma_1^2, \quad c = -\frac{h_1\Gamma}{2\rho_1} \pm \sqrt{\frac{h_1^2\Gamma^2}{4\rho_1^2} + \frac{(\rho-\rho_1)gh_1}{\rho_1}}.
\end{aligned} \tag{D3}$$

Note: c coincides with (5.6.4), the wave speed of the ILW equation. The KdV equation acquires the form

$$\eta_T + c\eta_X + \varepsilon \frac{c^2 \rho h h_1}{3(2c\rho_1 + \Gamma h_1)} \eta_{XXX} + \varepsilon \frac{-3\rho_1 c^2 + 3\rho_1 h_1 \gamma_1 c + h_1^2(\rho\gamma^2 - \rho_1\gamma_1^2)}{h_1(2c\rho_1 + \Gamma h_1)} \eta\eta_X = 0, \tag{D4}$$

or

$$\eta_T + c\eta_X + \varepsilon \mathcal{B}_1 \eta_{XXX} + \varepsilon \mathcal{A} \eta\eta_X = 0, \tag{D5}$$

with

$$\mathcal{B}_1 = \frac{c^2 \rho h h_1}{3(2c\rho_1 + \Gamma h_1)}, \qquad \mathcal{A} = \frac{-3\rho_1 c^2 + 3\rho_1 h_1 \gamma_1 c + h_1^2(\rho\gamma^2 - \rho_1\gamma_1^2)}{h_1(2c\rho_1 + \Gamma h_1)},$$

moreover $\mathcal{A}$ coincides with the expression from (4.4.15) and $\mathcal{B}_1 = h\mathcal{B}/3$.

Appendix E

The KdV solitary wave solution

The KdV equation (D5) from the previous Appendix D

$$\eta_T + c\eta_X + \varepsilon\mathcal{B}_1\eta_{XXX} + \varepsilon\mathcal{A}\eta\eta_X = 0$$

describes a balance between the non-linearity term $\eta\eta_X$, which tends to steepen the wave profile, and a dispersion term η_{XXX} which essentially counteracts this steepening. The solitary wave is one of permanent form for which this balance is maintained. That is, for the KdV regime $\varepsilon \simeq \delta^2 \ll 1$ the interplay between nonlinearity and dispersion produces smooth and stable in time soliton solutions.

The solution to the *standard/canonical* type of KdV equation

$$\eta_T + \eta_{XXX} + 6\eta\eta_X = 0 \tag{E1}$$

is well known and takes the form

$$\eta(X,T) = 2k^2 \operatorname{sech}^2[k(X - 4k^2T)]. \tag{E2}$$

This represents a sech^2 type solitary wave with amplitude $2k^2$ travelling at a speed of $4k^2$. Using a technique described in [52], the KdV equation (D5) can be solved. The appropriate Galilean Transformation and scaling procedure gives the one-soliton solution as

$$\eta(X,T) = \frac{12\mathcal{B}_1 K^2}{\mathcal{A}} \operatorname{sech}^2[K(X - X_0 - (c + \varepsilon 4K^2\mathcal{B}_1)T)]. \tag{E3}$$

This describes a solitary crest of amplitude

$$\frac{12\mathcal{B}_1 K^2}{\mathcal{A}}$$

moving with speed

$$c + \varepsilon 4K^2\mathcal{B}_1.$$

The correction to the speed c is

$$\varepsilon 4K^2\mathcal{B}_1$$

which is related to the amplitude through K.

A visualisation of the one-soliton solution of the form (E2) can be seen in Figure E.1 where $k = 0.5$ and $t = -10$.

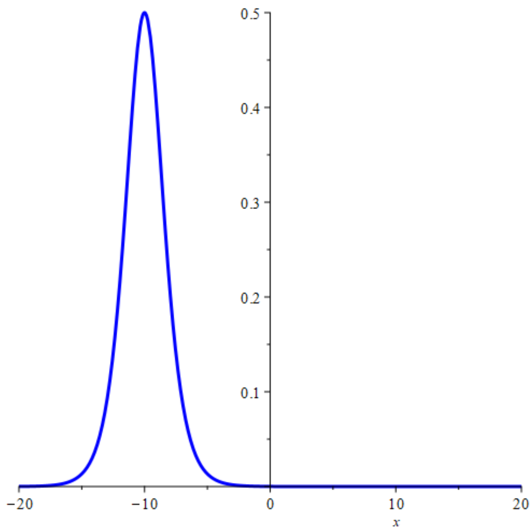
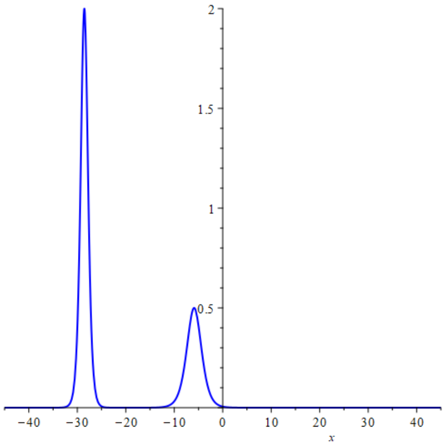


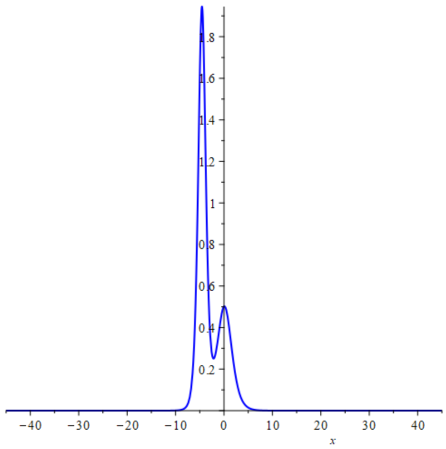
Figure E.1: Graph of 1-soliton solution

Using *Maple*, an example of a 2-soliton solution of the KdV equation is shown in Figure E.2, as described in [56]. Travelling from left to right, the taller, faster soliton catches up with the shorter one and they collide at $t = 0$. Following the collision, the solitons re-emerge retaining the shape and speed of the initial two waves but the collision causes a position or phase shift such that the faster soliton gets shifted forward and the slower one backward, indicating a type of nonlinear interaction occurring in the 2-soliton solution. Figures E.3 and E.4 depict the same 2-soliton solution, but plotted as a surface over the x,t plane.

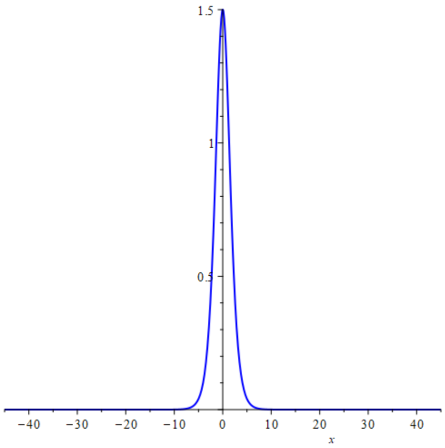
These solutions exist not only theoretically but also as physical phenomena which can be observed as surface or internal waves.



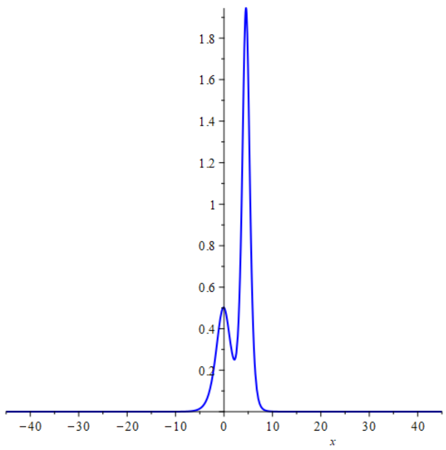
(a) t=-7



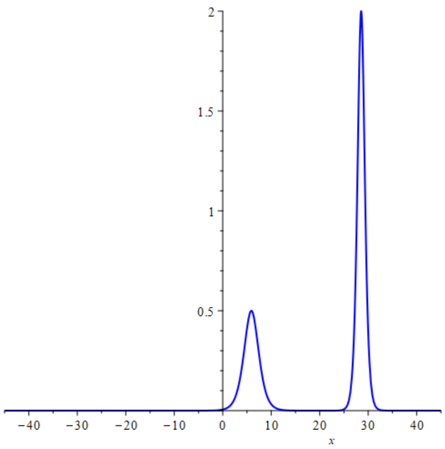
(b) t=-1.5



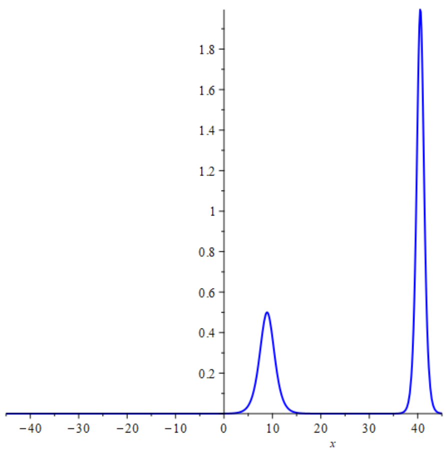
(c) t=0



(d) t=1.5



(e) t=7



(f) t=10

Figure E.2: A 2-soliton collision

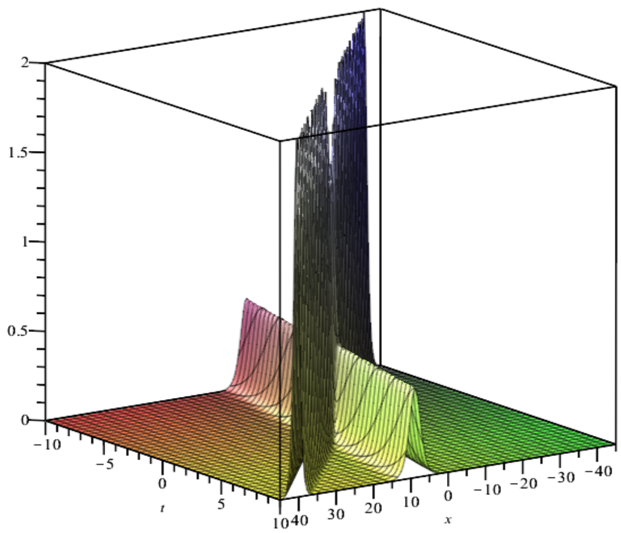


Figure E.3: The 2-soliton solution depicted as a surface over the x, t plane

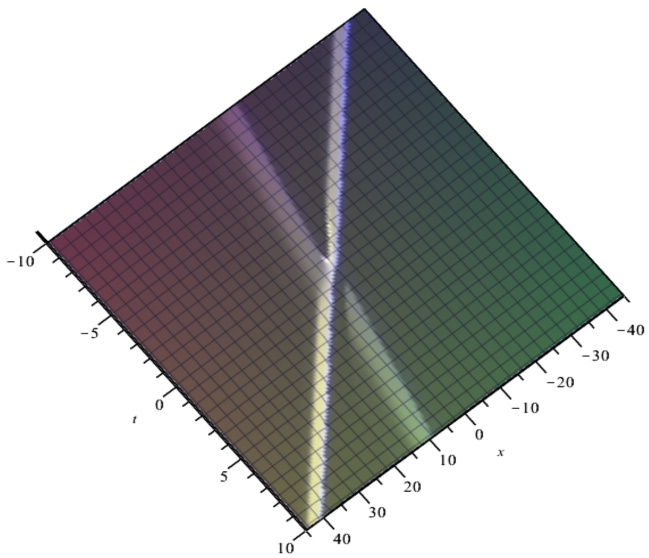


Figure E.4: The 2-soliton solution with x, t plane as viewed from above. Moving from top to bottom gives the passage of time. The solitons collide at $t = 0$.

There are infinitely many different solutions to the KdV equation (E1). For Soliton theory, and particularly the KdV equation, these are known as *pure n-soliton solutions* and can be produced from an algorithm as shown in [56]. Given any n-soliton solution $\eta(X,T)$ to the KdV equation, a function $\tau(X,T)$ can be found in polynomial exponential form

$$\tau(X,T) = \sum_{i=1}^{n} c_i e^{a_i X + b_i T} \tag{E4}$$

such that

$$\eta(X,T) = 2\partial_X{}^2 \log_e(\tau) = \frac{2\tau\tau_{XX} - 2\tau_X{}^2}{\tau^2}. \tag{E5}$$

Details regarding the soliton theory and multi-soliton solutions of the ILW and BO equations can be found in a number of works of which the following are mentioned [**?**, 55, 59, 67, 81].

List of related publications

This thesis is based on the following peer-reviewed articles (full details available in bibliography):

1. J. Cullen and R. Ivanov, On the intermediate long wave propagation for internal waves in the presence of currents, European Journal of Mechanics / B Fluids **84** (2020) 325–333.

2. Joseph D. Cullen, Rossen I. Ivanov, *Hamiltonian description of internal ocean waves with Coriolis force*, Communications on Pure and Applied Analysis **21(7)** (2022) 2291–2307.