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On Separable Torsion-Free Modules of Countable Density Character*

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The endomorphism algebras of modules of large cardinalities have been extensively studied in recent years using the combinatorial set-theoretic techniques of Shelah—the so-called black-box methods (see, e.g., [4, 5, 15]). Despite the spectacular success of these methods, they are not suitable for realization theorems at small carinalities. Of course at the level of countability (or rather more generally for cardinals $<2^{\aleph_0}$) there are in some cases the original dramatic results of A. L. S. Corner [1, 2, 3] and the more recent generalizations of Göbel and May [11]. Very recently the study of realization problems at cardinalities $<2^{\aleph_0}$ has been relooked at in [8, 13] in relation to separable torsion-free abelian groups (and some generalizations to modules). In the latter paper a new type of support argument (which has roots in a much earlier work [12]) was introduced in an effort to circumvent the lack of a "black box." It is this technique, which we exploit in the present paper to derive a basic realization result (Proposition 4), which can be readily adopted to, e.g., separable torsion-free abelian groups or modules over a complete discrete valuation ring. Among others we derive the following simple result, which appears to be new.

THEOREM 8. *If A is a torsion-free algebra over a complete discrete valuation ring R and A is generated by $<2^{\aleph_0}$ elements as an R-module, then there*

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exists a free R-module F of countable rank and a pure submodule G of $\hat{F}$ containing F such that $E(G) = A \oplus E_0(G)$.

Here, as throughout, $\hat{F}$ refers to a suitable completion (in this case p-adic) of F and $E_0(G)$ denotes the ideal of all finite rank endomorphisms of G.

Note that the module G has countable "basic rank" and that this is typical of all our results: all realizations are, in the usual terminology of topology, of "countable density character."

However, it should be pointed out that, unlike in the situation for cotorsion-free algebras [11], the module G always has rank $2^{\aleph_0}$. This, however, is no mere coincidence for modules with our "separability-like" properties; this must always hold as we now show. Suppose that $F = \bigoplus_{i<\omega} e_i R$ is a free R-module for some commutative ring R and G is a pure submodule of $\hat{F}$, where $\hat{}$ denotes completion in some suitable adic-like topology. Then we have

PROPOSITION 1. *If $F \leqslant G \leqslant \hat{F}$, where G is pure in $\hat{F}$ and* rank $G < 2^{\aleph_0}$ *then* $\operatorname{rk}(E(G)/E_0(G)) \geqslant 2^{\aleph_0}$.

Proof. Suppose that $E(G)/E_0(G)$ has rank $< 2^{\aleph_0}$. Let $\mathscr{H}$ denote a family of $2^{\aleph_0}$ almost-disjoint subsets of ω and let π_X $(X \in \mathscr{H})$ denote the projection $\hat{F} \to \bigoplus_{i \in X} (e_i R)\hat{}$. If each $\pi_X \restriction G$ is an endomorphism of G for all X in some $\mathscr{H}' \subseteq \mathscr{H}$ of size $2^{\aleph_0}$, then the family $\pi_X + E_0(G)$ $(X \in \mathscr{H}')$ is a family of $2^{\aleph_0}$ elements of $E(G)/E_0(G)$ and moreover they are independent by the almost-disjoint property of $\mathscr{H}$. Thus we conclude that there exist $2^{\aleph_0}$ subsets X such that $\pi_X \restriction G$ is not an endomorphism of G. Hence there exist elements $g_X \in G$ such that $g_X \pi_X$ is not in G. But it follows from a simple support argument (remember that the X's are almost disjoint) that the elements g_X are independent elements of G and rank $G = 2^{\aleph_0}$, a contradiction.

We remark that of course there are situations in which $E(G)/E_0(G)$ has rank $2^{\aleph_0}$, e.g., take $G = F$. Since cartesian products may be regarded as submodules of suitable completions, the above result clearly holds for separable abelian groups.

It is of course, quite standard to apply such realization theorems to obtain various pathological decomposition properties. For torsion-free separable abelian groups these have been discussed in, e.g., [6] while the situation in relation to modules over complete discrete valuation rings is discussed in [7].

We close this introduction by noting that standard algebraic terms may be found in Fuchs [9], our terminology and notations largely agree with those in [9] with the exception that we write maps on the right. The book [14] contains all the necessary references for matters relating to set theory.

Let R be a commutative Noetherian ring and we always assume $1 \neq 0$. Under this assumption, the set $E_0(G)$ becomes a two-sided ideal of $E(G)$ for any R-module G and hence $E(G)/E_0(G)$ is an R-algebra as desired. Let $S \leqslant R$ be a countable multiplicatively closed subset consisting of non-zero divisors (and $1 \in S$) such that $\bigcap sR = 0$. The S-topology on an R-module M has the submodules $\{sM : s \in S\}$ as a basis of neighbourhoods of zero. Such a topology is of course Hausdorff precisely if $\bigcap_{s \in S} sM = 0$ or, in a more algebraic spirit, if and only if M is S-reduced. We denote the completion of such an S-reduced module M by $\hat{M}$. In a similar fashion we have the notions of S-pure, S-dense, S-divisible, and S-torsion free (see, e.g., [3, 4, 5, 10, 11], where such notions have been extensively used previously). Since there is no possibility of ambiguity we drop the prefix S when referring to the above notions.

Finally we note that there is no loss in generality in labelling the elements of S as $s_1, ..., s_n, ...$, where we require that $s_n \mid s_{n+1}$, which means $s_{n+1} \in s_n R$ and then it follows that if $s \in S$ then $s \mid s_n$ for almost all n.

Let $F = \bigoplus_{i \in \omega} e_i R$ be a free R-module of countable rank. Set $B = \bigoplus_{n < \omega} B_n$, where each B_n is isomorphic to F. We make B into an $E(F)$-module as follows: If $x = \sum b_n$, $b_n \in B_n$, is an element of B, then we define, for θ in $E(F)$, $x\theta^* = \sum_n (b_n \theta)$ (cf. [8]). We normally identify θ^* with θ. In a similar fashion we can define an action of $E(\hat{F})$ on $\hat{B}$.

We always work within the module $(\hat{B} \cap \prod_{n \in \omega} B_n)$ and accordingly we define supports of such elements in the normal way:

> if $x \in (\hat{B} \cap \prod B_n)$ then $[x] = \{n \in \omega \mid b_n \neq 0$, where $x = \sum_{n \in \omega} b_n$ is in $(\hat{B} \cap \prod B_n)\}$.

In an attempt to mirror some constructions normally carried out in larger cardinalities via black-box combinatorics, we introduce an additional notion of $*$-support: Choose any family $\mathscr{H} \subseteq \mathscr{P}(\omega)$ which satisfies

(a) all $X \in \mathscr{H}$ are infinite

(b) if X, $Y \in \mathscr{H}$ and $|X \cap Y|$ is infinite, then $X = Y$

(c) $|\mathscr{H}| = 2^{\aleph_0}$

(d) $\mathscr{H}$ is maximal with respect to (a)–(c).

It is well known that such "almost-disjoint" families exist. They have been used already in [8, 12, 13]. We now choose one such $\mathscr{H}$ and keep it fixed throughout the remainder of the paper. By analogy with the tree ${}^{\omega >}\omega$, we call the elements of $\mathscr{H}$ *branches*. Let $\bar{B}$ denote the submodule of $(\hat{B} \cap \prod B_n)$ defined by

$$\bar{B} = B + \sum_{X \in \mathscr{H}} \left(\left(\bigoplus_{n \in X} B_n \right)^{\wedge} \cap \prod_{n \in X} B_n \right).$$

If $E \subseteq \mathscr{H}$, then we write $\bar{B}_E$ to denote the submodule obtained by replacing $\mathscr{H}$ with E above. If $u \in \hat{B}$, define the *-support $[u]^*$ of u to be

$$[u]^* = \{ Y \in \mathscr{H} : u \notin \bar{B}_{\mathscr{H} \setminus \{Y\}} \}.$$

The following observation follows immediately and is similar to [4, p. 456] (cf. [13, Lemma 4.3]).

RECOGNITION LEMMA. (a) *If $u \in \bar{B}$, then $[u]^*$ is finite.*

(b) *$u \in \bar{B}_E$ if and only if $[u]^* \subseteq E$ and $[u]^*$ is finite.*

Given a branch $X \in \mathscr{H}$, we define an *X-element* b as follows: Take any bijection $\tau: X \to \omega$; then using the definition of F as $\bigoplus_{i \in \omega} e_i R$, $e_{\tau(n)} \in F$ for each $n \in X$. Then the X-element b, which we also denote v_X, is given by $v_X = \sum_{n \in X} b_n$ with $b_n = e_{\tau(n)} s_n$ if $n \in X$ and 0 otherwise. Clearly $[v_X] = X$ and $[v_X]^* = \{X\}$.

If X is not specified, we call v_X an $\mathscr{H}$-element.

We also have a simple but useful finite recognition lemma. Observe that $\bar{B}$ is an $E(F)$-module under the identification $\theta = \theta^*$ from above.

LEMMA 2. *If X is a branch and $\theta \in E(F)$ and $[v_X \theta]$ is finite then $\theta \in E_0(\hat{F})$.*

Proof. Observe that $F = \bigoplus_{i < \omega} e_i R$ and hence it suffices to show that $e_n \theta = 0$ for almost all $n \in \omega$. By definition of the action of $E(\hat{F})$ we have that $v_X \theta = v_X \theta^* = \sum_{n \in X} (b_n \theta)$, where $v_X = \sum b_n$ with $b_n = e_{\tau(n)} s_n$. But then $[v_X \theta]$ finite implies $b_n \theta = 0$ for almost all n. Hence $e_{\tau(n)} \theta = 0$ for almost all n and so $\theta \in E_0(\hat{F})$.

For technical reasons we need the following property (P) of a ring R related to homomorphisms from $E_0(\bar{B})$. The concept has clear connections with the notion of slenderness and $\aleph_0$-cotorsion freedom (See, e.g., [4].)

(P) If G is any torsion-free, reduced R-module which is generated by $< 2^{\aleph_0}$ elements then every homomorphism $\tau: \bar{B} \to G$ has finite rank.

Property (P) is established in Lemma 7 for a complete discrete valuation ring and is a familiar fact for $R = \mathbb{Z}$; this has been extended recently for the more general rings discussed in Theorem 5 in [13].

The following simple lemma collects together some useful consequences of the action of $E(\hat{F})$ on $\bar{B}$.

LEMMA 3. (a) $E(\hat{F}) \cap E_0(\bar{B}) = \{0\}$.

(b) *If A is a pure submodule of $E(\hat{F})$ and $\bar{B}A \leqslant \bar{B}$ then A is pure in $E(\bar{B})$ and $A \oplus E_0(\bar{B})$ is pure in $E(\bar{B})$.*

Proof. (a) Suppose $\delta \in E(\hat{F})$ and $\delta \neq 0$. Then there exists $e_n \in F$ such that $e_n\delta$ is a non-zero element, $e_n\delta = x$, say. Choose any branch X in $\mathscr{H}$ and without loss identify X with ω. Set $b_n = x \in B_n$ for each $n \in X$. Then the elements $y_k = \sum_{n \leqslant k, n \in X} b_n$ are linearly independent elements of $\bar{B}$ and $y_k = t_k\delta$ for a suitable $t_k \in B$. This contradicts $\delta \in E_0(B)$ and so we conclude that $\delta = 0$.

(b) If $\tau \in A \cap sE(\bar{B})$ then since any component $\hat{B}_n$ of $\bar{B}$ is left invariant by the action of A, $\tau \restriction \hat{B}_n$ is a multiple of s and so purity in $E(\hat{F})$ gives the first result immediately. Clearly from (a), $A \cap E_0(\bar{B}) = \{0\}$ and so $A \oplus E_0(\bar{B}) \leqslant E(\bar{B})$. However, if $\theta \in E(\bar{B})$ and $s\theta = a + \eta$, where $a \in A$, $\eta \in E_0(\bar{B})$, then if $x \in \bar{B} \setminus \bar{B}\eta$, then $xs\theta = xa + a\eta$, and so $s \mid a$, say, $a = sa'$. But then $s(\theta - a') \in E_0(\bar{B})$ and since this is certainly pure in $E(\bar{B})$ the result follows.

Notation. Let $\Delta^* = \{(s, a) \mid s \in S \cup \{0\},\ a \in A\}$ and let Δ denote the subset of Δ^* obtained by restricting the first component to S.

PROPOSITION 4. *Let A be a subalgebra of $E(\hat{F})$ satisfying*:

(a) *A is generated by $< 2^{\aleph_0}$ elements as an R-module*

(b) *A is pure in $E(\hat{F})$*;

then provided the module $\bar{B}$ satisfies property (P), *there exists a pure submodule G of $\bar{B}$ containing B such that*

$$E(G) = A \oplus E_0(G).$$

Proof. We construct the desired R-module G by induction, eliminating unwanted homomorphisms one at a time. Since $\mathrm{Hom}(B, \bar{B}) \leqslant \mathrm{Hom}(B, \hat{B}) \cong (\hat{B})^\omega$ and $|\hat{B}| = 2^{\aleph_0}$ we conclude that $\mathrm{Hom}(B, \hat{B})$ has cardinality at most $2^{\aleph_0}$. Enumerate the elements of $\mathrm{Hom}(B, \hat{B}) \setminus A + E_0(\bar{B})$ as $\{\phi_\alpha : \alpha \in 2^{\aleph_0}\}$. Now construct inductively a sequence of pure submodules G_α of $\bar{B}$ and elements g_α as follows: Set $G_0 = \langle BA \rangle_*$ and if G_γ $(\gamma \leqslant \alpha)$ has been constructed then we set $G_{\alpha+1} = \langle G_\alpha, g_\alpha A \rangle_* \leqslant \bar{B}$ or $G_\lambda = \bigcup_{\alpha < \lambda} G_\alpha$ if λ is a limit.

The elements g_α are required to satisfy

$$\text{(I)}\quad g_\alpha\phi_\alpha \notin \langle G_\alpha, g_\alpha A \rangle_* \qquad \text{and} \qquad (\mathrm{II}_\gamma)\quad g_\beta\phi_\beta \notin G_\gamma \qquad \text{for all} \quad \beta < \gamma.$$

Accepting for the moment the existence of such elements $\{g_\alpha : \alpha \in 2^{\aleph_0}\}$, set $G = \bigcup G_\alpha$. It follows immediately (making the usual identification of functions and their extensions) that $E(G) \leqslant A + E_0(\bar{B})$. However, G is clearly A-invariant and so it follows from the modular law that $E(G) = A + \{E(G) \cap E_0(\bar{B})\}$. Moreover since $\{E(G) \cap E_0(\bar{B})\} = E_0(G)$ it follows from Lemma 3(b) that $E(G) = A \oplus E_0(G)$. Thus it only remains to establish the existence of suitable elements $\{g_\alpha\}$ satisfying (I) and (II_γ) for $\gamma \leqslant \alpha$.

So assume that the submodules G_γ ($\gamma \leqslant \alpha$) and elements g_β ($\beta < \alpha$) have been constructed and satisfy (I) and (II_γ) for $\gamma \leqslant \alpha$. It clearly suffices to find $g_\alpha \in \bar{B}$ such that (I) $g_\alpha \phi_\alpha \notin \langle G_\alpha, g_\alpha A\rangle_*$ and ($\mathrm{II}_{\alpha+1}$) $g_\beta \phi_\beta \notin \langle G_\alpha, g_\alpha A\rangle_*$ for $\beta < \alpha$. Observe that the inductive construction shows that each G_α is generated by $< 2^{\aleph_0}$ elements as an R-module.

Since $\phi_\alpha \notin A + E_0(\bar{B})$, it follows from (b) and Lemma 3 that $s\phi_\alpha - a \notin E_0(\bar{B})$ for all $(s, a) \in \Delta$. Then since G_α is generated by $< 2^{\aleph_0}$ elements and $\bar{B}$ satisfies property (P) by assumption (b), we conclude that there exists an element $v_{sa} \in \bar{B}$ such that $v_{sa}(s\phi_\alpha - a) \notin G_\alpha$ for all $(s, a) \in \Delta$. Now consider the submodule

$$T_\alpha = G_\alpha + \bar{B} \cap \langle v_{sa}(t\phi_\alpha - b), g_\beta(s\phi_\alpha - a), g_\beta \phi_\beta \mid \beta < \alpha, \\ (s, a), (t, b) \in \Delta^* \rangle.$$

Since T_α is generated by $< 2^{\aleph_0}$ elements as an R-module, it follows that the family $T_\alpha^* = \{(\bigcup [u]^* : u \in T_\alpha)\}$ has cardinality less than $2^{\aleph_0}$. So there exists a branch $X \in \mathscr{H}$ such that $X \notin T_\alpha$. Now consider the X-element $b_\alpha = v_\alpha$. If $b_\alpha \phi_\alpha \notin \langle G_\alpha, b_\alpha A\rangle_*$ choose $g_\alpha = b_\alpha$. Otherwise we can find $(s, a) \in \Delta$ with $b_\alpha(s\phi_\alpha - a) \in G_\alpha$. In this case set $g_\alpha = b_\alpha + v_{sa}$. Claim that (I) holds for either choice of g_α. We need only to check the second type of choice of g_α. If (I) does not hold then $(b_\alpha + v_{sa})(t\phi_\alpha - b) \in G_\alpha$ for some (t, b) in Δ. As noted earlier we may assume, without loss, that $s \mid t$ and write $t = qs$. Then on subtracting we get

$$x = b_\alpha(b - qa) - v_{sa} t\phi_\alpha + v_{sa} b \in G_\alpha \leqslant T_\alpha.$$

But $b_\alpha(b - qa)$ is thus an element of T_α since $v_{sa}(t\phi_\alpha - b) \in T_\alpha$. But this implies $[b_\alpha(b - qa)]^* \in T_\alpha^*$ and since b was an X-element for some $X \notin T^*$, this can only happen if $[b_\alpha(b - qa)]$ is finite. Since b_α is an $\mathscr{H}$-element, this implies, by Lemma 2, that $b - qa \in E_0(\hat{F}) \cap A = 0$. But then the element x above reduces to $qv_{sa}(s\phi_\alpha - a)$ and $x \in G_\alpha$. Purity now forces $v_{sa}(s\phi_\alpha - a) \in G_\alpha$, a contradiction. So we conclude that (I) holds for g_α.

Finally suppose $g_\beta \phi_\beta \in \langle G_\alpha, g_\alpha A\rangle_*$ for some $\beta < \alpha$. Then $tg_\beta \phi_\beta - g_\alpha b \in G_\alpha \leqslant T_\alpha$ for some $(t, b) \in \Delta$. If g_α was an $\mathscr{H}$-element b_α then clearly $b_\alpha b \in T_\alpha$; if, however, g_α was given by $g_\alpha = b_\alpha + v_{sa}$ then $b_\alpha b + v_{sa} b + tg_\beta \phi_\beta \in T_\alpha$ and again $b_\alpha b \in T_\alpha$ since both other terms belong to T_α by definition. So in either case $[b_\alpha b]^* \in T_\alpha^*$ and as above this forces $b = 0$. But then $g_\beta \phi_\beta \in G_\alpha$, contrary to the induction hypothesis. So ($\mathrm{II}_{\alpha+1}$) holds and the proof is completed by the observation that G is generated by $\leqslant 2^{\aleph_0}$ elements since the chain G_α has length $2^{\aleph_0}$.

With the aid of some results established in [13] we can easily deduce the principal result in that work:

THEOREM 5. *Let R be a commutative Noetherian ring of cardinality $\leqslant 2^{\aleph_0}$ (in the case $|R| = 2^{\aleph_0}$ we demand that R be slender) which is torsion-free and reduced as an R-module. Then if A is a subalgebra of $E(P)$, where $P = R^\omega$, satisfying:*

(a) *A is generated by $< 2^{\aleph_0}$ elements as an R-module*

(b) *$DA \leqslant D$ for the S-adic closure D of $C = R^{(\omega)}$ in P*

(c) *A is a pure R-submodule of P*

then there exists a pure submodule G of P^ω ($\cong P$) containing $B = C^{(\omega)}$ such that $E(G) = A \oplus E_0(G)$.

Remark. We require the restriction that R be Noetherian in order to conclude that $E(G)/E_0(G)$ is an R-algebra (cf. introductory information).

Proof. We deduce this result from Proposition 4: Identify $P \leqslant \hat{B}_n$ for each n and then $P^\omega \leqslant \prod B_n$. Note also that this identification gives $\bar{B} \leqslant P^\omega$ and the inclusion is pure. Since $B \leqslant G$ and G is pure in $\bar{B}$, we deduce that G is a pure submodule of P^ω containing the corresponding direct sum B. It only remains to show that $\bar{B}$ satisfies the property (P). This, however, follows from (b), Lemma 3.4, and Corollary 4.5 of [13].

In the case $R = \mathbb{Z}$ and $|A| = \aleph_0$ (i.e., for Abelian groups), the above is a realization theorem on separable torsion-free abelian groups which extends results in [8].

Rather surprisingly for modules over a complete discrete valuation ring R (taking $S = \{p^k : p \text{ prime in } R, k < \omega\}$ as usual) we easily derive a result which appears to be new:

THEOREM 6. *Let R be a complete discrete valuation ring and suppose A is any R-algebra which satisfies:*

(i) *A is a pure subalgebra of $E(\hat{F})$, where F is a free R-module of countable rank.*

(ii) *A is generated by $< 2^{\aleph_0}$ elements as an R-module.*

Then there exists a free R-module C of countable rank and an R-module G with $C \leqslant_* G \leqslant_* \hat{C}$ such that $E(G) = A \oplus E_0(G)$.

Proof. We apply Proposition 4 to this situation as follows: $\bar{B} = B + \sum_{X \in \mathscr{X}} ((\bigoplus_{n \in X} B_n)\hat{} \cap \prod_{n \in X} B_n)$ is clearly pure in $\hat{B}$ and so if we set $C = B$ then it oly remains to check that $\bar{B}$ satisfies property (P). Since this follows from Lemma 7 below, the result follows immediately from Proposition 4.

LEMMA 7. *Let R be a complete discrete valuation ring and let G be an R-module which is generated by $< 2^{\aleph_0}$ elements. Then every homomorphism $\phi: \bar{B} \to G$ has finite rank.*

Proof. Since R is a complete discrete valuation ring, the module $\bar{B}$ we are using is given by $\bar{B} = B + \sum_{X \in \mathscr{H}} (\bigoplus_{n \in X} B_b)\hat{}$, where $B = \bigoplus_{n \in \omega} B_n$ and $B_n = \hat{F}$ with F a free R-module of countable rank.

Let $E = \{n \in \omega : B_n\phi \neq 0\}$. If $X \in \mathscr{H}$ is a branch then $\phi \restriction (\bigoplus_{n \in X} B_n)\hat{}$ maps into G. However, since G is generated by $< 2^{\aleph_0}$ elements it must be $\aleph_0$-cotorsion free (since a completion of a countable rank module must have rank $\aleph_0^{\aleph_0} = 2^{\aleph_0}$) and so $B_n\phi = 0$ for almost all n in X. Thus $X \cap E$ is finite for all branches $X \in \mathscr{H}$. Since $\mathscr{H}$ was a maximal almost disjoint family we conclude E is finite and then ϕ has finite rank as required.

It is possible to derive abstract realization theorems from Theorems 5 and 6 by imposing suitable restrictions on the algebra A. In the case of separable torsion-free Abelian groups such conditions are stated in [6]: The algebra A has free additive group $A^+ = \bigoplus_{i \in I} e\mathbb{Z}_i$ and if $\hat{A}$ is the $\mathbb{Z}$-adic completion of A^+ then $\bar{A} = \hat{A} \cap \prod_{i \in I} e_i\mathbb{Z}$ is an A-submodule of $\hat{A}$.

For modules over a complete discrete valuation ring R we have the following:

THEOREM 8. *If A is an algebra over a complete discrete valuation ring R and A is generated by $< 2^{\aleph_0}$ elements as an R-module then there exists a free R-module F of countable rank and a pure submodule G of $\hat{F}$ containing F such that $E(G) = A \oplus E_0(G)$.*

Proof. The proof follows from the proof of Theorem 6, and it will suffice to exhibit a free R-module F of countable rank such that $A \leqslant_* E(\hat{F})$ and $A \cap E_0(\hat{F}) = 0$. This can be easily achieved by setting $H = \bigoplus_{n \in \omega} e_n A$ and choosing a basic submodule C of A; the restriction on A ensures that C can be chosen to be of at most countable rank. Then $F = \bigoplus e_n C \leqslant H \leqslant \hat{F}$ and F is free of countable rank. Let A act by scalar mutliplication on H and extend this to $\hat{F}$. Then if follows immediately that $A \cap E_0(\hat{F}) = 0$ and A is pure in $E(\hat{F})$.

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