On Integrable Wave Interactions and Lax Pairs on Symmetric Spaces

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Multi-component generalizations of derivative nonlinear Schrodinger (DNLS) type of equations having quadratic bundle Lax pairs related to Z_2-graded Lie algebras and A.III symmetric spaces are studied. The Jost solutions and the minimal set of scattering data for the case of local and nonlocal reductions are constructed. The latter lead to multi-component integrable equations with CPT-symmetry. Furthermore, the fundamental analytic solutions (FAS) are constructed and the spectral properties of the associated Lax operators are briefly discussed. The Riemann-Hilbert problem (RHP) for the multi-component generalizations of DNLS equation of Kaup-Newell (KN) and Gerdjikov-Ivanov (GI) types are derived. A modification of the dressing method is presented allowing the explicit derivation of the soliton solutions for the multi-component GI equation with both local and nonlocal reductions. It is shown that for specific choices of the reduction these solutions can have regular behavior for all finite x and t. The fundamental properties of the multi-component GI equations are briefly discussed at the end.

On integrable wave interactions and Lax pairs on symmetric spaces

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Abstract

Multi-component generalizations of derivative nonlinear Schrödinger (DNLS) type of equations having quadratic bundle Lax pairs related to $\mathbb{Z}_2$-graded Lie algebras and **A.III** symmetric spaces are studied. The Jost solutions and the minimal set of scattering data for the case of local and nonlocal reductions are constructed. The latter lead to multi-component integrable equations with $\mathcal{CPT}$-symmetry. Furthermore, the fundamental analytic solutions (FAS) are constructed and the spectral properties of the associated Lax operators are briefly discussed. The Riemann-Hilbert problem (RHP) for the multi-component generalizations of DNLS equation of Kaup-Newell (KN) and Gerdjikov-Ivanov (GI) types is derived. A modification of the dressing method is presented allowing the explicit derivation of the soliton solutions for the multi-component GI equation with both local and nonlocal reductions. It is shown that for specific choices of the reduction these solutions can have regular behavior for all finite x and t. The fundamental properties of the multi-component GI equations are briefly discussed at the end.

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1 Introduction

The generalized Zakharov-Shabat systems:

$$L\psi(x,t,\lambda) \equiv i\frac{\partial \psi}{\partial x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \tag{1}$$

and the associated nonlinear evolutionary equations (NLEE) have attracted considerable attention of the mathematical and physical communities over the last four decades and have been rather well classified and analyzed [56, 23, 15, 36, 13, 35]. This class of NLEE contains such physically important equations as the nonlinear Schröodinger equation (NLS), the sine-Gordon, the 2-dimensional Toda chain and modified Kortewegde-Vriez (mKdV) equations. All these models are integrable by the Inverse Scattering Method (ISM). Here $Q(x,t)$ is a matrix-valued function belonging in general to a simple Lie algebra $\mathfrak{g}$ of rank n, J is an (constant) element of its Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $\lambda \in \mathbb{C}$ is a spectral parameter.

Another classical example of completely integrable NLEE is the derivative nonlinear Schrödinger (DNLS) equation (also known as Kaup-Newell equation) [41, 33]:

$$\mathrm{i}q_t + q_{xx} + \epsilon \mathrm{i}(|q|^2 q)_x = 0. \tag{2}$$

This equation has many physical applications, especially in plasma physics and nonlinear optics [60, 61, 52, 53]: it describes small-amplitude nonlinear Alfvén waves in a low-β plasma, propagating strictly parallel or at a small angle to the ambient magnetic field; it models also large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma propagating at an arbitrary angle to the ambient magnetic field. Here the subscripts x and t stay for partial derivatives with respect to the variables x and t respectively and $\epsilon = \pm 1$. It is related to a Lax pair quadratic in λ [41]:

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 \sigma_3; \tag{3}$$

$$M(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{3} M_k(x,t)\lambda^k - 2\lambda^4 \sigma_3 \tag{4}$$

where

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \epsilon q^*(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_3(x,t) = 2Q(x,t) \tag{5}$$

$$M_2(x,t) = \epsilon|q^2(x,t)|\sigma_3, \qquad M_1(x,t) = \frac{\mathrm{i}}{2}[\sigma_3 Q_x(x,t)] + \epsilon|q^2(x,t)|Q(x,t). \tag{6}$$

DNLS is closely related (via gauge transformations) to three other integrable NLEEs: the one studied by Chen-Lee-Liu [49]

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}|q|^2 q_x = 0, \tag{7}$$

and the equation studied by Gerdjikov-Ivanov (GI) [31, 32, 16, 71]

$$\mathrm{i}q_t + q_{xx} + \epsilon \mathrm{i} q^2 q_x^* + \frac{1}{2}|q|^4 q(x,t) = 0, \tag{8}$$

and the 2-dimensional Thirring model [48]. Due to their similarity with DNLS, (7) and (8) are sometimes termed DNLS II and DNLS III, respectively. All the three versions of the DNLS equation along with the 2-dimensional Thirring model are integrable by the ISM and are related to spectral problems for generic quadratic bundle Lax operators (related to the algebra $sl(2,\mathbb{C})$)

$$L(\lambda) = \mathrm{i}\partial_x + U_0(x,t) + \lambda U_1(x,t) - \lambda^2\sigma_3, \tag{9}$$

where $U_0(x,t)$ is a generic (2×2) traceless matrix, $U_1(x,t)$ is an off-diagonal matrix and σ_3 being the Pauli matrix. The N-th AKNS flow of the corresponding integrable hierarchy [2, 36, 56] is given by a second Lax operator $M(\lambda)$ of a polynomial form:

$$M(\lambda) = i\partial_t + \sum_{k=0}^{N} V_k(x,t)\lambda^k. \tag{10}$$

A very fruitful trend in the theory of integrable systems is the study of multi-component generalizations of integrable scalar NLEEs. The applications of the differential geometric and Lie algebraic methods to soliton type equations led to the discovery of a close relationship between the multi-component (matrix) integrable equations (of nonlinear Schrödinger type) and the symmetric and homogeneous spaces [18]. Later on, this approach was extended to other types of multi-component integrable models, like the derivative NLS, Korteweg–de Vries and modified Korteweg–de Vries, N-wave, Davey–Stewartson, Kadomtsev–Petviashvili equations [6, 17]. For example, the equation [17] (see also [68, 70])

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \frac{2m\mathrm{i}}{m+n}(\mathbf{q}\mathbf{q}^\dagger\mathbf{q})_x = 0, \tag{11}$$

is related to symmetric space $\mathbf{A.III} \simeq SU(m+n)/S(U(m)\times U(n))$ in the Cartan classification [39]. Here $\mathbf{q}(x,t)$ is a smooth $n\times m$ matrix-valued function and "†" stands for Hermitian conjugation. Similarly, Tsuchida and Wadati [67] proved the complete integrability of a class of matrix generalizations of the Chen-Lee-Liu equation [49].

Another important trend in the development of ISM was the introduction of the reduction group by A. V. Mikhailov [51], and further developed in [27, 28, 29, 30, 38]. This allows one to prove that some of the well known integrable models and also a number of new interesting NLEE are integrable by the ISM and possess special symmetry properties. As a result the potential $Q(x,t)$ has a very special form imposed by the reduction. The reduction group concept is important also because of the fact, that when one considers Lax operators on Lie algebras, the number of independent fields grows rather quickly with the rank of the algebra: the corresponding NLEEs are solvable for any rank, but their possible applications to physics do not seem realistic. However, one still may extract new integrable and physically useful NLEE by imposing reductions on $L(\lambda)$, i.e. algebraic restrictions on the potential of L, which diminish the number of independent functions in them and the number of equations. Of course, such restrictions must be compatible with the dynamics of the NLEE.

Along with the well-known local reductions, recently nonlocal integrable reductions gained a fast-growing interest: in [3] was proposed a nonlocal integrable equation of nonlinear Schrödinger type with $\mathcal{PT}$-symmetry, due to the invariance of the so-called self-induced potential $V(x,t) = \psi(x,t)\psi^*(-x,-t)$ under the combined action of parity and time reversal symmetry (14). In the same paper, the 1-soliton solution for this model is derived and it was shown that it develops singularities in finite time. Soon after this, nonlocal $\mathcal{PT}$-symmetric generalizations are found for the Ablowitz-Ladik model in [4]. All these models are integrable by the Inverse Scattering Method (ISM) [5]. To be fair, we should note that in the context of the theory of integrable systems such nonlocal reductions appear for the first time in studies of integrable boundary conditions [12, 64, 44].

$\mathcal{PT}$-symmetric systems gained a special interest in the last decade mainly due to their applications in Nonlinear Optics [76, 9, 10, 54, 55, 62].

The initial interest in such systems was motivated by quantum mechanics [9, 54]. In [9] it was shown that quantum systems with a non-Hermitian Hamiltonian admit states with real eigenvalues, i.e. the Hermiticity of the Hamiltonian is not a necessary condition to have real spectrum. Using such Hamiltonians one can build up new quantum mechanics [9, 10, 54, 55]. Starting point is the fact that in the case of a non-Hermitian Hamiltonian with real spectrum, the modulus of the wave function for the eigenstates is time-independent even in the case of complex potentials.

Historically the first pseudo-Hermitian Hamiltonian with real spectrum is the $\mathcal{PT}$-symmetric one in [9]. Pseudo-Hermiticity here means that the Hamiltonian $\mathcal{H}$ commutes with the operators of spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$: $\mathcal{PTH} = \mathcal{HPT}$. The action of these operators is defined as follows: $\mathcal{P} : x \to -x$ and $\mathcal{T} : t \to -t$.

Supposing that the wave function is a scalar, this leads to the following action of the operator of spatial reflection on the space of states:

$$\mathcal{P}\psi(x,t) = \psi(-x,t). \tag{12}$$

Applying similar arguments to the (anti-linear and anti-unitary) time reversal operator $\mathcal{T}$ shows that it should act on the space of states as follows:

$$\mathcal{T}\psi(x,t) = \psi^*(x,-t). \tag{13}$$

Therefore, the Hamiltonian and the wave function are $\mathcal{PT}$-symmetric, if

$$\mathcal{H}(x,t) = \mathcal{H}^*(-x,-t), \qquad \psi(x,t) = \psi^*(-x,-t). \tag{14}$$

In addition if one imposes charge conjugation symmetry (particle-antiparticle symmetry) $\mathcal{C}$ [57], this will lead to additional complex conjugation of the wave function and the Hamiltonian:

$$\mathcal{C}\mathcal{H}^*(x,t) = \mathcal{H}(x,t), \qquad \mathcal{C}\psi^*(x,t) = \psi(x,t). \tag{15}$$

One can show that the $\mathcal{C}$-symmetry can be realized by a unitary linear operator [57]. The Hamiltonian and the wave function are $\mathcal{CPT}$-symmetric, if

$$\mathcal{H}(x,t) = \mathcal{H}(-x,-t), \qquad \psi(x,t) = \psi(-x,-t). \tag{16}$$

Integrable systems with $\mathcal{PT}$-symmetry were studied extensively over the last two decades [19, 20, 1, 7, 8, 26].

The main purpose of the present paper is to study generic quadratic in λ Lax operators

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J, \tag{17}$$

related to $\mathbb{Z}_2$-graded Lie algebras and symmetric spaces of **A.III**-type and the corresponding multi-component generalizations of DNLS III or Gerdjikov-Ivanov (GI) equation. This includes: the spectral properties of the associated Lax operator, the direct scattering transform, the effect of reductions on the scattering data, the dressing method, soliton solutions and related Riemann-Hilbert problem. We also aim to study the effect of local and nonlocal reductions on the multi-component generalizations of the GI equation.

The paper is organized as follows: In Section 2 we outline the construction of Lax pairs related to symmetric spaces of **A.III**-type and the related NLEE, the direct scattering problem for the corresponding Lax operator: this includes the construction of the Jost solutions and the minimal set of scattering data. In Section 3 we describe the local and nonlocal reductions and their action on the Lax operators. The construction of the fundamental analytic solutions (FAS) is outlined in Section 4. In Section 5 we construct the Riemann-Hilbert problem for the Kaup-Newell (KN) and GI equations on **A.III**-symmetric spaces and derive explicit parametrization of the associated Lax operators. The effect of local and nonlocal reductions on the scattering data is described briefly in Section 6. The modification of the dressing method and 1-solitons for the multi-component GI equation with local and nonlocal reductions are presented in Section 7. The integrals of motion are briefly discussed in Section 8. We finish up with conclusions and a brief outlook in Section 9.

2 Preliminaries

2.1 NLEE related to graded Lie algebras and symmetric spaces

We will start with NLEE related to $\mathbb{Z}_2$-graded Lie algebras, see [39]:

$$\mathfrak{g} \simeq \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, \qquad \mathfrak{g}^{(0)} \equiv \{X \in \mathfrak{g},\ [J,X]=0\}, \qquad \mathfrak{g}^{(1)} \equiv \{Y \in \mathfrak{g},\ JY + YJ = 0\}, \tag{18}$$

where $J = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}$. More specifically in the typical representation of the algebra $\mathfrak{g} \simeq sl(n)$ with $n = p+q$ the subalgebra $\mathfrak{g}^{(0)}$ consists of block-diagonal matrices with two nontrivial blocks $p\times p$ and $q \times q$; the linear subspace $\mathfrak{g}^{(1)}$ consists of block-off-diagonal matrices with two nontrivial blocks $p \times q$ and $q \times p$.

We can also say that $\mathfrak{g}^{(1)}$ is the co-adjoint orbit passing through J which will play role of phase space for our NLEE.

Remark 1. *In fact, the grading (18) can be related to the tangent hyperplane to $SL(p+q)/(SL(p)\otimes SL(q))$. If we introduce in addition a complex structure on $\mathfrak{g}$ the grading (18) can be related to the Hermitian symmetric space* **A.III** $\simeq SU(p+q)/(S(U(p)\otimes U(q))$.

As we mentioned above we will be considering Lax operators that are given by:

$$\begin{aligned} L\psi &\equiv i\frac{\partial\psi}{\partial x} + (U_2(x,t) + \lambda Q(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \qquad Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}, \\ M\psi &\equiv i\frac{\partial\psi}{\partial t} + (V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J)\psi(x,t,\lambda) = 0. \end{aligned} \tag{19}$$

where $Q(x,t)$, $V_3(x,t) \in \mathfrak{g}^{(1)}$ and $U_2(x,t)$, $V_2(x,t)$ and $V_4(x,t) \in \mathfrak{g}^{(0)}$. Such Lax pairs give rise to multicomponent derivative NLS type equations. The Lax pair (19), as we shall see below, allows one to solve the system of NLEE:

$$\begin{aligned} i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i}{2}\boldsymbol{q}\frac{\partial \boldsymbol{p}}{\partial x}\boldsymbol{q} + \frac{1}{4}\boldsymbol{qpqpq} = 0, \\ -i\frac{\partial \boldsymbol{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{p}}{\partial x^2} + \frac{i}{2}\boldsymbol{p}\frac{\partial \boldsymbol{q}}{\partial x}\boldsymbol{p} + \frac{1}{4}\boldsymbol{pqpqp} = 0. \end{aligned} \tag{20}$$

Along with it we will consider the Lax pair:

$$\begin{aligned} &\tilde{L}\tilde{\psi} \equiv i\frac{\partial \tilde{\psi}}{\partial x} + (\lambda \tilde{Q}(x,t) - \lambda^2 J)\tilde{\psi}(x,t,\lambda) = 0, \qquad \tilde{Q}(x,t) = \begin{pmatrix} 0 & \tilde{\boldsymbol{q}} \\ \tilde{\boldsymbol{p}} & 0 \end{pmatrix}, \\ &\tilde{M}\tilde{\psi} \equiv i\frac{\partial \tilde{\psi}}{\partial t} + (\lambda \tilde{V}_3(x,t) + \lambda^2 \tilde{V}_2(x,t) + \lambda^3 \tilde{Q}(x,t) - \lambda^4 J)\tilde{\psi}(x,t,\lambda) = 0. \end{aligned} \tag{21}$$

which is gauge equivalent to (19) and which allows one to solve the system:

$$\begin{aligned} i\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{q}}}{\partial x^2} + i\frac{\partial \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}}{\partial x} = 0, \\ -i\frac{\partial \tilde{\boldsymbol{p}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{p}}}{\partial x^2} - i\frac{\partial \tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}}{\partial x} = 0. \end{aligned} \tag{22}$$

The second system also complies with the $\mathbb{Z}_2$-grading introduced above: $\tilde{Q}(x,t)$, $\tilde{V}_3(x,t) \in \mathfrak{g}^{(1)}$ and $\tilde{U}_2(x,t)$ and $\tilde{V}_2(x,t) \in \mathfrak{g}^{(0)}$.

Note that the use of $\mathbb{Z}_2$-grading is in fact additional reduction as compared to the block-matrix AKNS model [2, 36]. Indeed, the block-matrix AKNS method requires that only $Q(x,t) \in \mathfrak{g}^{(1)}$.

2.2 The scattering problem for L

Here we briefly outline the scattering problem for the system (19) for the class of potentials $Q(x,t)$ satisfying the following conditions:

C1: $Q(x,t)$ is smooth enough and falls off to zero fast enough for $x \to \pm\infty$ for all t.

C2: $Q(x,t)$ is such that L has at most finite number of simple discrete eigenvalues.

In this subsection t plays the role of an additional parameter; for the sake of brevity the t-dependence is not always shown. The condition C2 cannot be formulated as a set of explicit conditions on $Q(x,t)$; its precise meaning will become clear below.

The main tool here is the Jost solutions defined by their asymptotics at $x \to \pm\infty$:

$$\lim_{x\to\infty} \psi(x,\lambda)e^{i\lambda^2 Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda)e^{i\lambda^2 Jx} = \mathbb{1}, \tag{23}$$

Along with the Jost solutions, we introduce

$$X_+(x,\lambda) = \psi(x,\lambda)e^{i\lambda^2 Jx}, \qquad X_-(x,\lambda) = \phi(x,\lambda)e^{i\lambda^2 Jx}; \tag{24}$$

which satisfy the following linear integral equations

$$X_\pm(x,\lambda) = \mathbb{1} + i\textstyle\int_{\pm\infty}^{x} dy e^{-i\lambda^2 J(x-y)} Q(y) X_\pm(y,\lambda) e^{i\lambda^2 J(x-y)}. \tag{25}$$

These are Volterra type equations which, as is well known always have solutions provided one can ensure the convergence of the integrals on the right hand side. For real λ the exponential factors in (25) are just oscillating and the convergence is ensured by condition C1.

The Jost solutions as whole cannot be extended for $\mathrm{Im}\lambda^2 \neq 0$. Skipping the details we write down the Jost solutions $\psi(x,\lambda)$ and $\phi(x,\lambda)$ in the following block-matrix form:

$$\psi(x,\lambda) = \left(|\psi^-(x,\lambda)\rangle, |\psi^+(x,\lambda)\rangle\right), \qquad \phi(x,\lambda) = \left(|\phi^+(x,\lambda)\rangle, |\phi^-(x,\lambda)\rangle\right), \tag{26}$$

where the superscripts $+$ and (resp. $-$) show that the corresponding block-matrix allows analytic extension for $\lambda \in \Omega_1 \cup \Omega_3$ (resp. $\lambda \in \Omega_2 \cup \Omega_4$), see Figure 1. Here by $\Omega_1, \dots, \Omega_4$ we have denoted the quadrants of the complex λ-plane.

Solving the direct scattering problem means given the potential $Q(x)$ to find the scattering matrix $T(\lambda)$. By definition $T(\lambda)$ relates the two Jost solutions:

$$\phi(x,\lambda) = \psi(x,\lambda)T(\lambda), \qquad T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & -\boldsymbol{b}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & \boldsymbol{a}^-(\lambda) \end{pmatrix} \tag{27}$$

and has compatible block-matrix structure. In what follows we will need also the inverse of the scattering matrix ('hat' means:inverse matrix from now on):

$$\psi(x,\lambda) = \phi(x,\lambda)\hat{T}(\lambda), \qquad \hat{T}(\lambda) \equiv \begin{pmatrix} \boldsymbol{c}^-(\lambda) & \boldsymbol{d}^-(\lambda) \\ -\boldsymbol{d}^+(\lambda) & \boldsymbol{c}^+(\lambda) \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned}
\boldsymbol{c}^-(\lambda) &= \hat{\boldsymbol{a}}^+(\lambda)(\mathbb{1} + \rho^-\rho^+)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\hat{\boldsymbol{a}}^+(\lambda), \\
\boldsymbol{d}^-(\lambda) &= \hat{\boldsymbol{a}}^+(\lambda)\rho^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\tau^+(\lambda)\hat{\boldsymbol{a}}^-(\lambda), \\
\boldsymbol{c}^+(\lambda) &= \hat{\boldsymbol{a}}^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^-\tau^+)^{-1}\hat{\boldsymbol{a}}^-(\lambda), \\
\boldsymbol{d}^+(\lambda) &= \hat{\boldsymbol{a}}^-(\lambda)\rho^+(\lambda)(\mathbb{1} + \rho^-\rho^+)^{-1} = (\mathbb{1} + \tau^-\tau^+)^{-1}\tau^-(\lambda)\hat{\boldsymbol{a}}^+(\lambda).
\end{aligned} \tag{29}$$

The diagonal blocks of both $T(\lambda)$ and $\hat{T}(\lambda)$ allow analytic continuation off the real axis, namely $\boldsymbol{a}^+(\lambda)$, $\boldsymbol{c}^+(\lambda)$ are analytic functions of λ for $\lambda \in \Omega_1 \cup \Omega_3$, while $\boldsymbol{a}^-(\lambda)$, $\boldsymbol{c}^-(\lambda)$ are analytic functions of λ for $\lambda \in \Omega_2 \cup \Omega_4$.

By $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ above we have denoted the multicomponent generalizations of the reflection coefficients (for the scalar case, see [2, 41]):

$$\rho^\pm(\lambda) = \boldsymbol{b}^\pm\hat{\boldsymbol{a}}^\pm(\lambda) = \hat{\boldsymbol{c}}^\pm\boldsymbol{d}^\pm(\lambda), \qquad \tau^\pm(\lambda) = \hat{\boldsymbol{a}}^\pm\boldsymbol{b}^\mp(\lambda) = \boldsymbol{d}^\mp\hat{\boldsymbol{c}}^\pm(\lambda), \tag{30}$$

We will need also the asymptotics for $\lambda \to \infty$:

$$\lim_{\lambda\to-\infty} \phi(x,\lambda)e^{i\lambda Jx} = \lim_{\lambda\to\infty} \psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{\lambda\to\infty} T(\lambda) = \mathbb{1}, \tag{31}$$

i.e. $\lim_{\lambda\to\infty} \boldsymbol{a}^\pm(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^\pm(\lambda) = \mathbb{1}$.

The inverse to the Jost solutions $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ are solutions to:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,\lambda)(U_2(x,t) + \lambda Q(x,t) - \lambda^2 J) = 0, \tag{32}$$

satisfying the conditions:

$$\lim_{x\to\infty} e^{-i\lambda Jx}\hat{\psi}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} e^{-i\lambda Jx}\hat{\phi}(x,\lambda) = \mathbb{1}. \tag{33}$$

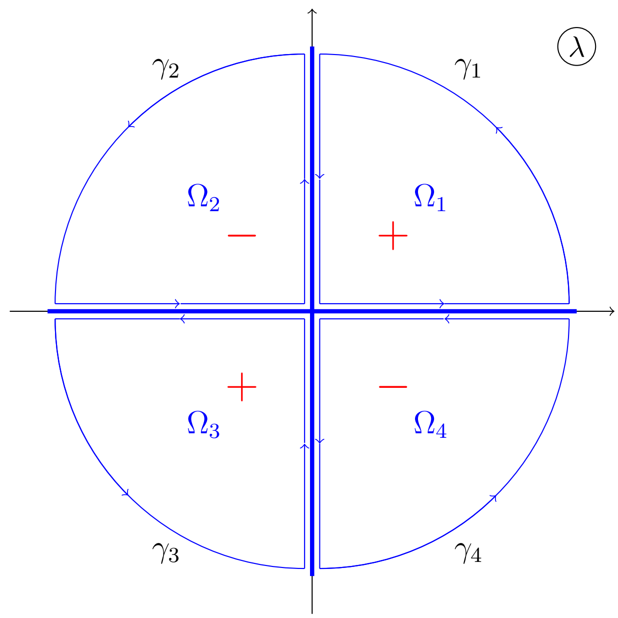

Figure 1: The continuous spectrum of a $L(\lambda)$.

Now it is the collections of rows of $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ that possess analytic properties in λ:

$$\hat{\psi}(x,\lambda) = \begin{pmatrix} \langle \hat{\psi}^+(x,\lambda)| \\ \langle \hat{\psi}^-(x,\lambda)| \end{pmatrix}, \qquad \hat{\phi}(x,\lambda) = \begin{pmatrix} \langle \hat{\phi}^-(x,\lambda)| \\ \langle \hat{\phi}^+(x,\lambda)| \end{pmatrix}, \tag{34}$$

Just like the Jost solutions, their inverse (34) are solutions to linear equations (32) with regular boundary conditions (33); therefore they can have no singularities in their regions of analyticity. The same holds true also for the scattering matrix $T(\lambda) = \hat{\psi}(x,\lambda)\phi(x,\lambda)$ and its inverse $\hat{T}(\lambda) = \hat{\phi}(x,\lambda)\psi(x,\lambda)$, i.e.

$$\boldsymbol{a}^+(\lambda) = \langle \hat{\psi}^+(x,\lambda)|\phi^+(x,\lambda)\rangle, \qquad \boldsymbol{a}^-(\lambda) = \langle \hat{\psi}^-(x,\lambda)|\phi^-(x,\lambda)\rangle, \tag{35}$$

as well as

$$\boldsymbol{c}^+(\lambda) = \langle \hat{\phi}^+(x,\lambda)|\psi^+(x,\lambda)\rangle, \qquad \boldsymbol{c}^-(\lambda) = \langle \hat{\phi}^-(x,\lambda)|\psi^-(x,\lambda)\rangle, \tag{36}$$

are analytic for $\lambda \in \mathbb{C}_\pm$ and have no singularities in their regions of analyticity. However they may become degenerate (i.e., their determinants may vanish) for some values $\lambda_j^\pm \in \mathbb{C}_\pm$ of λ. Below we analyze the structure of these degeneracies.

3 Local and Non-local Reductions

3.1 Local Reductions

An important and systematic tool to construct new integrable NLEE is the so-called reduction group [51]. It will be instructive to start with the local reductions:

$$\begin{aligned}
&1) \qquad A_1 U^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = U(x,t,\lambda), \qquad A_1 V^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = V(x,t,\lambda),\\
&2) \qquad A_2 U^T(x,t,\kappa_2\lambda)A_2^{-1} = -U(x,t,\lambda), \qquad A_2 V^T(x,t,\kappa_2\lambda)A_2^{-1} = -V(x,t,\lambda),\\
&3) \qquad A_3 U^*(x,t,\kappa_1\lambda^*)A_3^{-1} = -U(x,t,\lambda), \qquad A_3 V^*(x,t,\kappa_1\lambda^*)A_3^{-1} = -V(x,t,\lambda),\\
&4) \qquad A_4 U(x,t,\kappa_2\lambda)A_4^{-1} = U(x,t,\lambda), \qquad A_4 V(x,t,\kappa_2\lambda)A_4^{-1} = V(x,t,\lambda).
\end{aligned} \tag{37}$$

The consequences of these reductions and the constraints they impose on the FAS and the Gauss factors of the scattering matrix are well known, see [51, 56, 23, 28, 27].

Let us detail the consequences of the reductions 1) and 3) in (37) on the NLEE (20). It is easy to see that they restrict $U_0(x,t)$ and $Q(x,t)$ by:

$$\begin{aligned} &1) \quad A_1 J A_1^{-1} = J, \qquad \kappa_1 A_1 Q^\dagger A_1^{-1} = Q(x,t), \qquad A_1 U_2^\dagger(x,t) A_1^{-1} = U_2(x,t), \\ &3) \quad A_3 J A_3^{-1} = -J, \qquad \kappa_3 A_3 Q^* A_3^{-1} = -Q(x,t), \qquad A_3 U_2^*(x,t) A_3^{-1} = -U_2(x,t), \end{aligned} \tag{38}$$

where $\kappa_1^2 = \kappa_3^2 = 1$ and $A_1^2 = A_3^2 = \mathbb{1}$. From $A_1 J A_1^{-1} = J$ (resp. $A_3 J A_3^{-1} = -J$) we find that A_1 is block-diagonal (resp. A_3 is block-off-diagonal) matrix. If we introduce

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}, \tag{39}$$

we obtain

$$\begin{aligned} &1) \quad \kappa_1 a_1 \boldsymbol{p}^\dagger \hat{a}_2 = \boldsymbol{q}, \qquad \kappa_1 a_2 \boldsymbol{q}^\dagger \hat{a}_1 = \boldsymbol{p}, \qquad A_1 U_2^\dagger A_1^{-1} = U_2 \\ &3) \quad \kappa_3 b_1 \boldsymbol{p}^* \hat{b}_2 = -\boldsymbol{q}, \qquad \kappa_3 b_2 \boldsymbol{q}^* \hat{b}_1 = -\boldsymbol{p}, \qquad A_3 U_2^* A_3^{-1} = -U_2. \end{aligned} \tag{40}$$

As a result the eq. (20) reduces to a multicomponent GI equation:

$$i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i\kappa_1}{2}\boldsymbol{q} a_2 \frac{\partial \boldsymbol{q}^\dagger}{\partial x}\hat{a}_1 \boldsymbol{q} + \frac{1}{4}\boldsymbol{q} a_2 \boldsymbol{q}^\dagger \hat{a}_1 \boldsymbol{q} a_2 \boldsymbol{q}^\dagger \hat{a}_1 \boldsymbol{q} = 0. \tag{41}$$

while the equation (22) goes into a multicomponent KN equation:

$$i\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{q}}}{\partial x^2} + i\kappa_1 \frac{\partial}{\partial x}\left(\tilde{\boldsymbol{q}} a_2 \tilde{\boldsymbol{q}}^\dagger \hat{a}_1 \tilde{\boldsymbol{q}}\right) = 0. \tag{42}$$

3.2 Non-Local Reductions

It is important to note, that for the derivative NLS equations *there are no* reductions compatible with either $\mathcal{P}$- or $\mathcal{T}$-symmetry separately. However the $\mathbb{Z}_2$ reductions

$$\begin{aligned} &1) \quad C_1 U^\dagger(-x,-t,\kappa_1\lambda^*) C_1^{-1} = -U(x,t,\lambda), \qquad C_1 V^\dagger(-x,-t,\kappa_1\lambda^*) C_1^{-1} = -V(x,t,\lambda), \\ &2) \quad C_2 U^T(-x,-t,\kappa_2\lambda) C_2^{-1} = U(x,t,\lambda), \qquad C_2 V^T(-x,-t,\kappa_2\lambda) C_2^{-1} = V(x,t,\lambda), \\ &3) \quad C_3 U^*(-x,-t,\kappa_1\lambda^*) C_3^{-1} = U(x,t,\lambda), \qquad C_3 V^*(-x,-t,\kappa_1\lambda^*) C_3^{-1} = V(x,t,\lambda), \\ &4) \quad C_4 U(-x,-t,\kappa_2\lambda) C_4^{-1} = -U(x,t,\lambda), \qquad C_4 V(-x,-t,\kappa_2\lambda) C_4^{-1} = -V(x,t,\lambda), \end{aligned} \tag{43}$$

are obviously $\mathcal{PT}$-symmetric [69]. Here $\kappa_i^2 = 1$ and A_i and C_i, $i = 1, \dots, 4$ are involutive automorphisms of the relevant Lie algebra.

Now the consequences of the reductions 1) and 3) in (43) on the NLEE (20). It is easy to see that they restrict $U_0(x,t)$ and $Q(x,t)$ by:

$$\begin{aligned} &1) \quad C_1 J C_1^{-1} = -J, \qquad \kappa_1 C_1 Q^\dagger(-x,-t) C_1^{-1} = -Q(x,t), \qquad C_1 U_2^\dagger(-x,-t) C_1^{-1} = -U_2(x,t), \\ &3) \quad C_3 J C_3^{-1} = J, \qquad \kappa_3 C_3 Q^*(-x,-t) C_3^{-1} = Q(x,t), \qquad C_3 U_2^*(-x,-t) C_3^{-1} = U_2(x,t), \end{aligned} \tag{44}$$

where $\kappa_1^2 = \kappa_3^2 = 1$ and $C_1^2 = C_3^2 = \mathbb{1}$. From $C_1 J C_1^{-1} = -J$ (resp. $C_3 J C_3^{-1} = J$) we find that C_3 is block-diagonal (resp. C_1 is block-off-diagonal) matrix. If we introduce

$$C_1 = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \tag{45}$$

we obtain

$$
\begin{aligned}
&1) \quad \kappa_1 c_1 \boldsymbol{q}^\dagger(-x,-t)\hat{c}_2 = -\boldsymbol{q}(x,t), \qquad \kappa_1 c_2 \boldsymbol{p}^\dagger(-x,-t)\hat{c}_1 = -\boldsymbol{p}(x,t), \\
&\qquad C_1 U_2^\dagger(-x,-t) C_1^{-1} = -U_2(x,t) \\
&3) \quad \kappa_3 d_1 \boldsymbol{q}^*(-x,-t)\hat{d}_2 = \boldsymbol{q}(x,t), \qquad \kappa_3 d_2 \boldsymbol{p}^*(-x,-t)\hat{d}_1 = \boldsymbol{p}(x,t), \\
&\qquad C_3 U_2^*(-x,-t) C_3^{-1} = U_2(x,t).
\end{aligned}
\tag{46}
$$

As a result the equations (20) and (22) retain their form, the only difference being that $\boldsymbol{q}$ and $\boldsymbol{p}$ are now restricted by (46).

On the Jost solutions we have

$$
\phi^\dagger(x,t,\lambda^*) = \psi^{-1}(-x,t,-\lambda), \qquad \psi^\dagger(x,t,\lambda^*) = \phi^{-1}(x,t,-\lambda),
$$

so for the scattering matrix we have

$$
T^\dagger(t,-\lambda^*) = T(t,\lambda),
$$

As a consequence for the Gauss factors we get:

$$
T^{-\dagger}(-\lambda^*) = \hat{S}^+(\lambda), \qquad T^{+\dagger}(-\lambda^*) = \hat{S}^-(\lambda), \qquad D^{\pm\dagger}(\lambda^*) \;= \hat{D}^\pm(-\lambda).
$$

In analogy with the local reductions, the kernel of the resolvent has poles at the points $\lambda_2^\pm$ at which $D^\pm(\lambda)$ has poles or zeroes. In particular, if λ_2^+ is an eigenvalue, then $-\lambda_2^+$ is also an eigenvalue. For the reflection coefficients we obtain the constraints:

$$
\tau^+(-\lambda) = -\rho^{+,*}(\lambda), \qquad \tau^-(-\lambda) = -\rho^{-,*}(\lambda),
$$

Remark 2. *In what follows for the sake of simplicity we specify $A_1 = C_3 = J$ and $A_3 = C_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. In the latter case we restrict ourselves to the special case when $\boldsymbol{p}$ and $\boldsymbol{q}$ are square matrices, i.e. our symmetric space is $SU(2q)/S(U(q)\otimes U(q))$.*

4 The fundamental analytic solutions and the RHP

The next step is to construct the fundamental analytic solutions of (19). In our case this is done simply by combining the blocks of Jost solutions with the same analytic properties:

$$
\begin{aligned}
\chi^+(x,\lambda) &\equiv \left(|\phi^+\rangle, |\psi^+\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^+(\lambda) = \psi(x,\lambda)\boldsymbol{T}^-(\lambda), \\
\chi^-(x,\lambda) &\equiv \left(|\psi^-\rangle, |\phi^-\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^-(\lambda) = \psi(x,\lambda)\boldsymbol{T}^+(\lambda),
\end{aligned}
\tag{47}
$$

where the block-triangular functions $\boldsymbol{S}^\pm(\lambda)$ and $\boldsymbol{T}^\pm(\lambda)$ are given by:

$$
\begin{aligned}
\boldsymbol{S}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & \boldsymbol{d}^-(\lambda) \\ 0 & \boldsymbol{c}^+(\lambda) \end{pmatrix}, \qquad & \boldsymbol{T}^-(\lambda) &= \begin{pmatrix} \boldsymbol{a}^+(\lambda) & 0 \\ \boldsymbol{b}^+(\lambda) & \mathbb{1} \end{pmatrix}, \\
\boldsymbol{S}^-(\lambda) &= \begin{pmatrix} \boldsymbol{c}^-(\lambda) & 0 \\ -\boldsymbol{d}^+(\lambda) & \mathbb{1} \end{pmatrix}, \qquad & \boldsymbol{T}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & -\boldsymbol{b}^-(\lambda) \\ 0 & \boldsymbol{a}^-(\lambda) \end{pmatrix},
\end{aligned}
\tag{48}
$$

These triangular factors can be viewed also as generalized Gauss decompositions (see [39]) of $T(\lambda)$ and its inverse:

$$
T(\lambda) = \boldsymbol{T}^-(\lambda)\hat{\boldsymbol{S}}^+(\lambda) = \boldsymbol{T}^+(\lambda)\hat{\boldsymbol{S}}^-(\lambda), \qquad \hat{T}(\lambda) = \boldsymbol{S}^+(\lambda)\hat{\boldsymbol{T}}^-(\lambda) = \boldsymbol{S}^-(\lambda)\hat{\boldsymbol{T}}^+(\lambda).
\tag{49}
$$

The relations between $\boldsymbol{c}^{\pm}(\lambda)$, $\boldsymbol{d}^{\pm}(\lambda)$ and $\boldsymbol{a}^{\pm}(\lambda)$, $\boldsymbol{b}^{\pm}(\lambda)$ in eq. (29) ensure that equations (49) become identities. From eqs. (47), (48) we derive:

$$\begin{aligned} \chi^{+}(x,\lambda) &= \chi^{-}(x,\lambda)G_0(\lambda), \qquad & G_0(\lambda) &= \hat{D}^{-}(\lambda)(\mathbb{1}+K^{-}(\lambda)),\\ \chi^{-}(x,\lambda) &= \chi^{+}(x,\lambda)\hat{G}_0(\lambda), \qquad & \hat{G}_0(\lambda) &= \hat{D}^{+}(\lambda)(\mathbb{1}-K^{+}(\lambda)), \end{aligned} \tag{50}$$

valid for $\lambda \in \mathbb{R}$, where

$$\begin{aligned} D^{-}(\lambda) &= \begin{pmatrix} \boldsymbol{c}^{-}(\lambda) & 0 \\ 0 & \boldsymbol{a}^{-}(\lambda) \end{pmatrix}, \qquad & K^{-}(\lambda) &= \begin{pmatrix} 0 & \boldsymbol{d}^{-}(\lambda) \\ \boldsymbol{b}^{+}(\lambda) & 0 \end{pmatrix},\\ D^{+}(\lambda) &= \begin{pmatrix} \boldsymbol{a}^{+}(\lambda) & 0 \\ 0 & \boldsymbol{c}^{+}(\lambda) \end{pmatrix}, \qquad & K^{+}(\lambda) &= \begin{pmatrix} 0 & \boldsymbol{b}^{-}(\lambda) \\ \boldsymbol{d}^{+}(\lambda) & 0 \end{pmatrix}, \end{aligned} \tag{51}$$

Obviously the block-diagonal factors $D^{+}(\lambda)$ and $D^{-}(\lambda)$ are matrix-valued analytic functions for $\lambda \in \Omega_1 \cup \Omega_3$ and $\lambda \in \Omega_2 \cup \Omega_4$ respectively. Another well known fact about the FAS $\chi^{\pm}(x,\lambda)$ concerns their asymptotic behavior for $\lambda \to \pm\infty$, namely:

$$\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)e^{i\lambda^2 Jx}, \qquad \lim_{\lambda\to\infty} \xi^{\pm}(x,\lambda) = \mathbb{1}. \tag{52}$$

On the real and imaginary axis $\xi^{+}(x,\lambda)$ and $\xi^{-}(x,\lambda)$ are related by

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda)G(x,\lambda), \qquad G(x,\lambda) = e^{-i\lambda^2 Jx}G_0(\lambda)e^{i\lambda^2 Jx}, \qquad G_0(\lambda) = S^{+}(\lambda)\hat{S}^{-}(\lambda). \tag{53}$$

The function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (19) [63, 22].

Thus eq. (53) combined with eq. (52) can be understood as a Riemann-Hilbert problem with canonical normalization: given the sewing function $G_0(x,\lambda)$ construct $\xi^{\pm}(x,\lambda)$.

5 Parametrization of Lax pairs

Here we will outline a natural parametrization of $U(x,t,\lambda)$ and $V(x,t,\lambda)$ in terms of the local coordinate $Q_1(x,t)$ on the co-adjoint orbit $\mathfrak{g}^{(1)}$. Below we will choose it in the form:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix}, \tag{54}$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are generic $p \times q$ and $q \times p$ matrices. Following [14, 24] we also introduce the solution $\xi(x,t,\lambda)$ of a RHP with canonical normalization. Since $\xi(x,t,\lambda)$ must be an element of the corresponding Lie group we define it by

$$\xi(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t), \tag{55}$$

where $\mathcal{Q}(x,t,\lambda)$ is a formal series over the negative powers of λ whose coefficients Q_s take values in $\mathfrak{g}^{(0)}$ if s is even and in $\mathfrak{g}^{(1)}$ if s is odd. Therefore the first few of these coefficients take the form:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix}, \quad Q_2(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{r} & 0 \\ 0 & \boldsymbol{s} \end{pmatrix}, \quad Q_3(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{v} \\ -\boldsymbol{w} & 0 \end{pmatrix}. \tag{56}$$

With such choice for $\xi(x,t,\lambda)$ we obviously have

$$\lim_{\lambda\to\infty} \xi(x,t,\lambda) = \mathbb{1} \tag{57}$$

which provides the canonical normalization of the RHP. Besides we have requested that $\mathcal{Q}(x,t,\lambda)$ takes values in the Kac-Moody algebra determined by the grading (18); in other words $\mathcal{Q}(x,t,\lambda)$ satisfies

$$\mathcal{Q}(x,t,\lambda) = C_0 \mathcal{Q}(x,t,-\lambda) C_0^{-1}, \qquad C_0 = \exp(\pi i J). \tag{58}$$

Then we can introduce $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as the non-negative parts of [21, 14]:

$$U(x,t,\lambda) = -\left(\lambda^a \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+, \qquad V(x,t,\lambda) = -\left(\lambda^b \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+, \tag{59}$$

where a and b can be any integers. For simplicity and definiteness we will fix up $a=2$ and $b=4$. The explicit calculation of $U(x,t,\lambda)$ and $V(x,t,\lambda)$ in terms of $Q_s(x,t)$ can be done using the well known formula

$$\xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda) = J + \sum_{s=1}^{\infty} \frac{1}{s!} \mathrm{ad}_{\mathcal{Q}}^{\,s} J, \qquad \mathrm{ad}_{\mathcal{Q}} J = [\mathcal{Q}, J], \qquad \mathrm{ad}_{\mathcal{Q}}^{\,2} J = [\mathcal{Q},[\mathcal{Q},J]], \quad \dots \tag{60}$$

Since in (59) we need only the non-negative powers of λ for any a and b we will need only finite number of terms. In particular for $a=2$ and $b=4$ we have:

$$\begin{aligned}
U(x,t,\lambda) &= -\left(\lambda^2 \xi J \hat{\xi}\right)_+ = -\lambda^2 J + \lambda Q(x,t) + U_2(x,t), \\
Q(x,t) &= -[Q_1, J] = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}, \\
U_2(x,t) &= -\frac{1}{2}[Q_1,[Q_1,J]] - [Q_2(x,t), J] = \frac{1}{2} \begin{pmatrix} \boldsymbol{qp} & 0 \\ 0 & -\boldsymbol{pq} \end{pmatrix}.
\end{aligned} \tag{61}$$

Note that since $Q_2(x,t) \in \mathfrak{g}^{(0)}$ then $[Q_2(x,t), J] = 0$. Similarly

$$\begin{aligned}
V(x,t,\lambda) &= -\left(\lambda^4 \xi^{\pm} J \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J, \\
V_2(x,t) &= U_2(x,t), \qquad V_3(x,t) = -\frac{1}{2} \mathrm{ad}_{Q_2} \mathrm{ad}_{Q_1} J - \frac{1}{6} \mathrm{ad}_{Q_1}^{\,3} J, \\
V_4(x,t) &= -\frac{1}{2}\left(\mathrm{ad}_{Q_3} \mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_1} \mathrm{ad}_{Q_3} J\right) - \frac{1}{6}\left(\mathrm{ad}_{Q_1} \mathrm{ad}_{Q_2} \mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_2} \mathrm{ad}_{Q_1}^{\,2} J\right) - \frac{1}{24} \mathrm{ad}_{Q_1}^{\,4} J.
\end{aligned} \tag{62}$$

Here we used again $[Q_2(x,t), J] = 0$ and $[Q_4(x,t), J] = 0$. Below we will pay special attention to the particular case $p=1$ which corresponds to the vector GI equation.

5.1 RHP and multi-component GI equations

Here we assume that the FAS of L and M satisfy a canonical RHP with special reduction:

$$\xi^{\pm}(x,t,-\lambda) = \xi^{\pm,-1}(x,t,\lambda), \tag{63}$$

i.e., $\mathcal{Q}(x,t,\lambda) = -\mathcal{Q}(x,t,-\lambda)$ and therefore $Q_{2s}(x,t) = 0$. As a result the expression for the Lax pair simplifies to

$$
\begin{aligned}
&L\psi \equiv i\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi = 0, && M\psi \equiv i\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi = 0,\\
&U(x,t,\lambda) = U_2(x,t) + \lambda Q(x,t) - \lambda^2 J, && Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}, \qquad U_2(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{qp} & 0 \\ 0 & -\boldsymbol{pq} \end{pmatrix},\\
&V_2(x,t) = U_2(x,t), && V_3(x,t) = \begin{pmatrix} 0 & \boldsymbol{v} - \frac{1}{6}\boldsymbol{qpq} \\ \boldsymbol{w} - \frac{1}{6}\boldsymbol{pqp} & \end{pmatrix},\\
&V_4(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{qw} + \boldsymbol{vp} - \frac{1}{12}\boldsymbol{qpqp} & 0 \\ 0 & -\boldsymbol{wq} - \boldsymbol{pv} + \frac{1}{12}\boldsymbol{pqpq} \end{pmatrix}.
\end{aligned}
\tag{64}
$$

The commutation $[L, M]$ must vanish identically with respect to λ. It is polynomial in λ with the following coefficients:

$$
\begin{aligned}
&\lambda^5: && -[J, V_1] - [Q, J] = 0, && \Rightarrow \quad V_1 = Q,\\
&\lambda^4: && -[J, V_2] + [Q, V_1] - [U_2, J] = 0, && \Rightarrow \quad \text{identity}\\
&\lambda^3: && i\frac{\partial V_1}{\partial x} + [U_2, V_1] + [Q, V_2] = [J, V_3],
\end{aligned}
\tag{65}
$$

The last of these equations is fulfilled iff

$$
\boldsymbol{v} = \frac{i}{2}\frac{\partial \boldsymbol{q}}{\partial x} + \frac{1}{6}\boldsymbol{qpq}, \qquad \boldsymbol{w} = -\frac{i}{2}\frac{\partial \boldsymbol{p}}{\partial x} + \frac{1}{6}\boldsymbol{pqp}.
\tag{66}
$$

The next equations are:

$$
\begin{aligned}
&\lambda^2: \quad i\frac{\partial V_2}{\partial x} + [U_2, V_2] + [Q, V_3] = [J, V_4] \equiv 0,\\
&\lambda^1: \quad i\frac{\partial V_3}{\partial x} - i\frac{\partial Q}{\partial t} + [U_2, V_3] + [Q, V_4] = 0, \qquad \lambda^0: \quad i\frac{\partial V_4}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_4] = 0.
\end{aligned}
\tag{67}
$$

The first of the above equations is satisfied identically. The second one written in block-components gives the following NLEE which can be viewed as multicomponent GI equations related to the **D.III** symmetric space:

$$
\begin{aligned}
i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i}{2}\boldsymbol{q}\frac{\partial \boldsymbol{p}}{\partial x}\boldsymbol{q} + \frac{1}{4}\boldsymbol{qpqpq} = 0,\\
-i\frac{\partial \boldsymbol{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{p}}{\partial x^2} + \frac{i}{2}\boldsymbol{p}\frac{\partial \boldsymbol{q}}{\partial x}\boldsymbol{p} + \frac{1}{4}\boldsymbol{pqpqp} = 0.
\end{aligned}
\tag{68}
$$

The last equation in (67) is a consequence of the expressions for Q_0 and V_4 and the eqs. (68) and (66).

5.2 RHP and multi-component Kaup-Newel equations

The Lax pair for the KN-system is obtained from (64) by a gauge transformation:

$$
\begin{aligned}
\tilde{L}\tilde{\chi}^{\pm}(x,t,\lambda) &\equiv i\frac{\partial \tilde{\chi}^{\pm}}{\partial x} + (\lambda\tilde{Q}(x,t) - \lambda^2 J)\tilde{\chi}^{\pm}(x,t,\lambda) = 0,\\
\tilde{M}\tilde{\chi}^{\pm}(x,t,\lambda) &\equiv i\frac{\partial \tilde{\chi}^{\pm}}{\partial t} + (\lambda\tilde{V}_3(x,t) + \lambda^2\tilde{V}_2(x,t) + \lambda^3\tilde{Q}(x,t) - \lambda^4 J)\tilde{\chi}^{\pm}(x,t,\lambda) = 0,
\end{aligned}
\tag{69}
$$

where

$$\tilde{V}_3(x,t) = g_0^{-1} V_3(x,t) g_0(x,t), \qquad \tilde{V}_2(x,t) = g_0^{-1} V_2(x,t) g_0(x,t), \qquad \tilde{Q}(x,t) = g_0^{-1} Q(x,t) g_0(x,t) \tag{70}$$

and the gauge $g_0(x,t)$ is defined uniquely by the equations:

$$i\frac{\partial g_0}{\partial x} + U_2(x,t) g_0(x,t) = 0, \qquad i\frac{\partial g_0}{\partial t} + V_4(x,t) g_0(x,t) = 0. \tag{71}$$

Note that $g_0(x,t)$ must be block-diagonal, so similarity transformations with it preserve the grading (the block-matrix structure) of the coefficients in $U(x,t,\lambda)$ and $V(x,t,\lambda)$; in particular, $g_0^{-1} J g_0(x,t) = J$. Therefore we introduce

$$\tilde{Q} = \begin{pmatrix} 0 & \tilde{\boldsymbol{q}} \\ \tilde{\boldsymbol{p}} & 0 \end{pmatrix} \tag{72}$$

Applying the gauge transformation to $V_2(x,t)$ we easily obtain:

$$\tilde{V}_2(x,t) = -\frac{1}{2} g_0^{-1} [Q_1, [Q_1(x,t), J]] = \frac{1}{2} \begin{pmatrix} \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}} & 0 \\ 0 & -\tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}} \end{pmatrix}. \tag{73}$$

Next we apply the compatibility condition of $\tilde{L}$ and $\tilde{M}$ and obtain

$$V_3 = \mathrm{ad}_J^{-1} \left(i\frac{\partial \tilde{Q}}{\partial x} + [\tilde{Q}, V_2] \right) = \frac{1}{2} \begin{pmatrix} 0 & i\tilde{\boldsymbol{q}}_x - \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}} \\ -i\tilde{\boldsymbol{p}}_x + \tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}} & \end{pmatrix}. \tag{74}$$

Finally the multi-component Kaup-Newell equation takes the form:

$$\begin{aligned} i\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{q}}}{\partial x^2} + i\frac{\partial \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}}{\partial x} &= 0, \\ -i\frac{\partial \tilde{\boldsymbol{p}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{p}}}{\partial x^2} - i\frac{\partial \tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}}{\partial x} &= 0. \end{aligned} \tag{75}$$

6 Effects of reductions on the scattering matrix

Let us briefly outline the effects of the reductions on the scattering matrix. We will start with the properties of the Jost solutions and then with determining the reduction of the scattering data. In particular we will find the symmetries on the discrete eigenvalues of L.

6.1 The local involution case

Each of the reductions in (37) has a natural action on the Jost solutions and, as a consequence on the FAS and the scattering matrix. In particular the reduction 1) of eq. (37) requires that:

$$\begin{aligned} A_1 \psi^\dagger(x,t,\lambda^*) A_1^{-1} = \psi^{-1}(x,t,\lambda), \qquad A_1 \phi^\dagger(x,t,\lambda^*) A_1^{-1} &= \phi^{-1}(x,t,\lambda), \\ A_1 T^\dagger(t,\lambda^*) A_1^{-1} &= T^{-1}(t,\lambda), \end{aligned} \tag{76}$$

where A_1 was introduced in eq. (39). So for the matrix elements of the scattering matrix we have

$$\begin{aligned} a_1 (a^+(\kappa_1 \lambda^*))^\dagger a_1^{-1} &= c^-(\lambda), \qquad & a_2 (a^-(\kappa_1 \lambda^*))^\dagger a_2^{-1} &= c^+(\lambda), \\ a_1 (b^+(\kappa_1 \lambda^*))^\dagger a_2^{-1} &= d^-(\lambda), \qquad & a_2 (b^-(\kappa_1 \lambda^*))^\dagger a_1^{-1} &= d^+(\lambda), \end{aligned} \tag{77}$$

If we specify $a_1 = \mathbb{1}$ and $a_2 = \epsilon_2 \mathbb{1}$, $\epsilon_2 = \pm 1$ we get:

$$\begin{aligned} &(a^{\pm}(\kappa_1\lambda^*))^\dagger = c^{\mp}(\lambda), \qquad &&\epsilon_2(b^{\pm}(\kappa_1\lambda^*))^\dagger = d^{\mp}(\lambda), \\ &(\rho^{+}(\kappa_1\lambda^*))^\dagger = \epsilon_2\rho^{-}(\lambda), \qquad &&(\tau^{+}(\kappa_1\lambda^*))^\dagger = \epsilon_2\tau^{-}(\lambda). \end{aligned} \tag{78}$$

It is well known that the zeros of $a^{\pm}(\lambda)$ and $c^{\pm}(\lambda)$ are the discrete eigenvalues of L. From (78) one finds that if $\lambda_1^+ \in \Omega_1 \cup \Omega_3$ is a zero of $a^+(\lambda)$ then $\kappa_1(\lambda_1^+)^* \in \Omega_2 \cup \Omega_4$ is a zero of $c^-(\lambda)$.

The other local reduction 3) in (37) is treated similarly. Here for simplicity we assume $p = q$, i.e. the matrices $\boldsymbol{p}(x,t)$ and $\boldsymbol{q}(x,t)$ are square. Then

$$A_3\psi^*(x,t,\lambda^*)A_3^{-1} = \psi(x,t,\lambda), \qquad A_3\phi^\dagger(x,t,\lambda^*)A_3^{-1} = \phi(x,t,\lambda), \qquad A_3T^*(t,\lambda^*)A_3^{-1} = T(t,\lambda), \tag{79}$$

with A_3 defined by (39), which means that

$$\begin{aligned} &b_2(a^{+}(\kappa_1\lambda^*)^* b_2^{-1} = a^{-}(\lambda), \qquad &&b_1(a^{-}(\kappa_1\lambda^*)^* b_1^{-1} = a^{+}(\lambda), \\ &b_1(b^{+}(\kappa_1\lambda^*)^* b_1^{-1} = -b^{-}(\lambda), \qquad &&b_2(b^{-}(\kappa_1\lambda^*)^* b_1^{-1} = -b^{+}(\lambda), \end{aligned} \tag{80}$$

If we assume $b_1 = \mathbb{1}$ and $b_2 = \epsilon_2 \mathbb{1}$, $\epsilon_2 = \pm 1$ we get:

$$\begin{aligned} &(a^{\pm}(\kappa_1\lambda^*))^* = a^{\mp}(\lambda), \qquad &&\epsilon_2(b^{\pm}(\kappa_1\lambda^*))^* = -b^{\mp}(\lambda), \\ &(\rho^{+}(\kappa_1\lambda^*))^* = -\epsilon_2\rho^{-}(\lambda), \qquad &&(\tau^{+}(\kappa_1\lambda^*))^* = -\epsilon_2\tau^{-}(\lambda). \end{aligned} \tag{81}$$

6.2 The nonlocal involution case

These reductions also have a natural (but different from the above) effect on the Jost solutions and the scattering matrix. The reduction 1) from (43) leads to:

$$\begin{aligned} &C_1\phi^\dagger(x,t,\kappa_1\lambda^*)C_1^{-1} = \psi^{-1}(-x,-t,-\lambda), \qquad C_1\psi^\dagger(x,t,\kappa_1\lambda^*)C_1^{-1} = \phi^{-1}(-x,-t,-\lambda), \\ &C_1T^\dagger(t,\kappa_1\lambda^*)C_1^{-1} = T(-t,\lambda), \end{aligned}$$

where C_1 is given by (45). As a consequence for the matrix elements of $T(t,\lambda)$ we get:

$$c_1(a^{-}(\kappa_1\lambda^*))^\dagger c_1^{-1} = a^{+}(\lambda), \qquad c_1(b^{-}(\kappa_1\lambda^*))^\dagger c_2^{-1} = b^{-}(\lambda), \qquad c_2(b^{+}(\kappa_1\lambda^*))^\dagger c_1^{-1} = b^{+}(\lambda), \tag{82}$$

The reduction 3) from (43) leads to:

$$\begin{aligned} &C_3\phi^*(x,t,\kappa_1\lambda^*)C_3^{-1} = \psi^{-1}(-x,-t,\lambda), \qquad C_3\psi^*(x,t,\kappa_1\lambda^*)C_3^{-1} = \phi^{-1}(-x,-t,\lambda), \\ &C_3T^*(t,\kappa_1\lambda^*)C_3^{-1} = T^{-1}(-t,\lambda), \end{aligned} \tag{83}$$

with C_3 defined by (45). Thus for the matrix elements of $T(t,\lambda)$ we find:

$$\begin{aligned} &d_1(a^{+}(\kappa_1\lambda^*))^* d_1^{-1} = c^{-}(\lambda), \qquad &&d_2(a^{-}(\kappa_1\lambda^*))^* d_2^{-1} = c^{+}(\lambda), \\ &d_2(b^{+}(\kappa_1\lambda^*))^* d_1^{-1} = -d^{+}(\lambda), \qquad &&d_1(b^{-}(\kappa_1\lambda^*))^* d_2^{-1} = -d^{-}(\lambda), \end{aligned} \tag{84}$$

Particularly if we put $d_1 = \mathbb{1}$ and $d_2 = \epsilon_2\mathbb{1}$ we find:

$$(a^{\pm}(\kappa_1\lambda^*))^* = c^{\mp}(\lambda), \qquad (b^{\pm}(-t,\kappa_1\lambda^*))^* = -\epsilon_2 d^{\pm}(\lambda), \tag{85}$$

and

$$\rho^{\pm}(-t,\kappa_1\lambda^*) = -\epsilon_2\tau^{\mp}(\lambda). \tag{86}$$

From (85) we see that if $\lambda_1^+ \in \Omega_1 \cup \Omega_3$ is a zero of $a^+(\lambda)$ then $\kappa_1(\lambda_1^+)^* \in \Omega_2 \cup \Omega_4$ is a zero of $c^-(\lambda)$.

7 Soliton solutions

Here we adapt Zakharov-Shabat's dressing method [74, 75] to the above Lax pairs.

7.1 Dressing method

One of the most convenient approaches to the derivation of the soliton solutions is the so-called dressing method [74, 75] (see also [23, 36, 73, 40, 70, 26]). The rationale of the method is the construction of a nontrivial (dressed) FAS, $\chi^{\pm}(x,t,\lambda)$ from the known (bare) FAS, $\chi_0^{\pm}(x,t,\lambda)$ by the means of the so-called dressing factor $u(x,t,\lambda)$:

$$\chi^{\pm}(x,t,\lambda) = u(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda). \tag{87}$$

The dressing factor is analytic in the entire complex λ-plane, with the exception of the newly added simple pole singularities at $\lambda = \lambda_k^{\pm}$, $k = 1, 2, \ldots, N$:. It is known that these singularities are in fact discrete eigenvalues of the 'dressed' Lax operator L:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_k(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_k(x,t) \right). \tag{88}$$

As far as the FAS satisfy the Lax pair equations (64), the dressing factor must be a solution of the equation

$$iu_x + U_2 u - uU_2^{(0)} + \lambda(Qu - uQ^{(0)}) + \lambda^2[u, J] = 0, \tag{89}$$

where the upper index (0) indicates the quantities, associated to the bare solution. The equation (89) must hold identically with respect to λ. Since u has poles at finitely many points of the discrete spectrum, it will be enough to request that (89) holds for $\lambda \to \infty$ and $\lambda \to \lambda_k^{\pm}$. For $\lambda \to \infty$, $u \to \mathbb{1}$, so the derivative term in (89) disappears. The λ^2- terms are proportional to $[J, \mathbb{1}]$ that also identically vanishes. Thus, we are left with two terms, which are easily evaluated to be

$$\begin{aligned} \lambda^1 : \quad & Q - Q^{(0)} = \sum_{k=1}^{N} (\lambda_k^+ - \lambda_k^-)[J, B_k - \tilde{B}_k], \\ \lambda^0 : \quad & U_2 - U_2^{(0)} = \sum_{k=1}^{N} (\lambda_k^+ - \lambda_k^-) \left([J, \lambda_k^+ B_k - \lambda_k^- \tilde{B}_k] - Q(B_k - \tilde{B}_k) + (B_k - \tilde{B}_k)Q^{(0)} \right). \end{aligned} \tag{90}$$

Thus, if we know the residues $B_k, \tilde{B}_k$ we are able to reconstruct $Q(x,t)$ and $U_2(x,t)$. The condition that (89) holds for $\lambda \to \lambda_k^{\pm}$ leads to the following:

$$i\partial_x B_k + (U_2 + \lambda_k^+ Q)B_k - B_k(U_2^{(0)} + \lambda_k^+ Q^{(0)}) + (\lambda_k^+)^2[B_k, J] = 0. \tag{91}$$

In the simplest possible nontrivial case, B_k are rank 1 matrices of the form

$$B_k = |n_k\rangle\langle m_k| \tag{92}$$

satisfying the matrix equation (91), ($|n\rangle$ is a vector-column, $\langle m|$ is a vector-row as usual). It is straightforward to verify that B_k in the form (92) will satisfy (91), if and only if

$$\begin{aligned} i\partial_x |n_k\rangle + \left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J \right) |n_k\rangle = 0, \\ i\partial_x \langle m_k| - \langle m_k| \left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J \right) = 0, \end{aligned} \tag{93}$$

i.e.

$$|n_k\rangle = \chi^+(x,t,\lambda_k^+)|n_{k,0}\rangle, \qquad \langle m_k| = \langle m_{k,0}|\hat{\chi}_0^+(x,t,\lambda_k^+), \tag{94}$$

where $|n_{k,0}\rangle$ and $\langle m_{k,0}|$ are some constant vectors. One can start with the trivial bare solutions $Q^{(0)} = 0$, $U_2^{(0)} = 0$, so that $\chi_0^+(x,t,\lambda) = \exp i(\lambda^2 Jx + \lambda^4 Jt)$ is known explicitly.

7.2 Example - One soliton solution with local reduction

In the first example the dressing factor $u(x,\lambda;t)$ satisfies the reduction conditions from the first reduction of (37):

$$\text{A)} \qquad A_1 u^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = u^{-1}(x,t,\lambda), \qquad \text{B)} \qquad u(x,t,-\lambda) = u^{-1}(x,t,\lambda). \tag{95}$$

We consider the case $p=1$, i.e. $\boldsymbol{q}$ is a vector-row and $\boldsymbol{p}$ is a vector-column, J is diagonal with $J_{11} = 1$ and $J_{ii} = -1$ for $i = 2,\dots n$. $(A_1)_{ij} = \epsilon_i\delta_{ij}$ is diagonal, with $\epsilon_i = \pm 1$. Introducing the notation

$$A_1 = \mathrm{diag}(a_1, a_2)$$

for the block-diagonal matrix A_1 and noting that $A_1 = A_1^{-1}$, we have the following relations between $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{q} = \kappa_1 a_1 \boldsymbol{p}^\dagger a_2, \qquad \boldsymbol{p} = \kappa_1 a_2 \boldsymbol{q}^\dagger a_1. \tag{96}$$

A dressing factor with simple poles at $\lambda = \lambda_1^\pm$ has the form

$$u(x,t,\lambda) = \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+}B_1(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-}\tilde{B}_1(x,t) \tag{97}$$

Moreover, the reductions A and B are simultaneously satisfied if

$$\lambda_1^+ = -\kappa_1(\lambda_1^-)^*, \qquad \tilde{B}_1 = A_1 B_1^\dagger A_1^{-1}. \tag{98}$$

Let us introduce the notation $\mu \equiv \lambda_1^+$ and in polar form $\mu = \rho e^{i\varphi}$. Both reductions A, B must hold identically with respect to λ which necessitates (e.g. when $\lambda \to \mu$)

$$B_1\left(\mathbb{1} - \frac{\mu + \kappa_1\mu^*}{2\mu}B_1(x,t) + \frac{\mu + \kappa_1\mu^*}{\mu - \kappa_1\mu^*}A_1 B_1^\dagger(x,t)A_1^{-1}\right) = 0 \tag{99}$$

Looking for a rank one solution $B_1 = |n\rangle\langle m|$ of the matrix equation (99), ($|n\rangle$ is a vector-column, $\langle m|$ is a vector-row as usual) we find that

$$B_1 = z\frac{A_1|m^*\rangle\langle m|}{\langle m|A_1|m^*\rangle} \tag{100}$$

where the complex constant z satisfies the linear equation

$$1 - \frac{\mu + \kappa_1\mu^*}{2\mu}z + \frac{\mu + \kappa_1\mu^*}{\mu - \kappa_1\mu^*}z^* = 0. \tag{101}$$

In addition, from (90)– (91) it follows that

$$\begin{aligned} &i\partial_x B_1 + (U_2 + \mu Q)B_1 - B_1(U_2^{(0)} + \mu Q^{(0)}) + \mu^2[B_1, J] = 0, \\ &Q = Q^{(0)} + (\mu + \kappa_1\mu^*)[J, B_1 - C_1 B_1^\dagger(x,t)C_1^{-1}]. \end{aligned} \tag{102}$$

and together with the assumption $B_1 = |n\rangle\langle m|$ one can find out that $\langle m|$ satisfies the bare equation

$$i\partial_x \langle m| - \langle m|(U_2^{(0)} + \mu Q^{(0)} - \mu^2 J) = 0. \qquad (103)$$

Therefore, starting from the trivial solution $U_2^{(0)} = Q^{(0)} = 0$ we find

$$\langle m| = \langle m_0| e^{i(\mu^2 x + \mu^4 t)J}, \qquad (104)$$

where $\langle m_0|$ is a constant vector with components m_{0j}. Now we can write the one-soliton solution,

$$\boldsymbol{q}_{j-1}(x,t) = Q_{1j} = 4\rho r(\kappa_1) \frac{m_{0j} e^{\xi_0} e^{-i\phi(x,t)}}{m_{01}\cosh(\theta(x,t) - \xi_0)}, \qquad j = 2, \dots, n, \qquad (105)$$

where $r(1) = i\sin\varphi$, and $r(-1) = \cos\varphi$ and when $A_1 = \mathbb{1}$,

$$e^{-2\xi_0} \equiv \frac{\sum_{j=2}^{n} |m_{0j}|^2}{|m_{01}|^2}$$

is real and positive,

$$\theta(x,t) = 2\rho^2(\sin 2\varphi)x + 2\rho^4(\sin 4\varphi)t, \qquad \phi(x,t) = 2\rho^2(\cos 2\varphi)x + 2\rho^4(\cos 4\varphi)t, \qquad (106)$$

7.3 Example - One soliton solution with nonlocal reduction

In the second example the dressing factor $u(x,t,\lambda;t)$ satisfies the reduction conditions from the first reduction of (43):

$$\text{A)} \quad C_1 u^\dagger(-x,-t,\kappa_1\lambda^*)C_1^{-1} = u^{-1}(x,\lambda), \qquad \text{B)} \quad u(x,t,-\lambda) = u^{-1}(x,\lambda). \qquad (107)$$

Let us take for simplicity $p = 1$, $n = 2$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are scalar functions. The automorphism C_1 cannot be represented by a diagonal matrix, since now it must change the sign of $J \equiv \sigma_3$. Hence, we take

$$C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (108)$$

The reduction gives now the following connections between $\boldsymbol{p}$ and $\boldsymbol{q}$, under which the equations are $\mathcal{CPT}$-invariant:

$$\boldsymbol{q}(x,t) = -\kappa_1 \boldsymbol{q}^*(-x,-t), \qquad \boldsymbol{p}(x,t) = -\kappa_1 \boldsymbol{p}^*(-x,-t). \qquad (109)$$

The dressing factor satisfies the equation (89). Again it is taken to have simple poles at $\lambda = \lambda_1^\pm$:

$$u(x,t,\lambda) = \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_1(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_1(x,t) \qquad (110)$$

This time the reductions A and B are simultaneously satisfied if

$$\lambda_1^+ = -\kappa_1(\lambda_1^-)^*, \qquad \tilde{B}_1(x,t) = C_1 B_1^\dagger(-x,-t)C_1^{-1}. \qquad (111)$$

With the short notations $\mu \equiv \lambda_1^+ = \rho e^{i\varphi}$ we obtain the equation for $B_1(x,t)$

$$B_1(x,t)\left(\mathbb{1} - \frac{\mu + \kappa_1\mu^*}{2\mu} B_1(x,t) + \frac{\mu + \kappa_1\mu^*}{\mu - \kappa_1\mu^*} C_1 B_1^\dagger(-x,-t)C_1^{-1}\right) = 0 \qquad (112)$$

with a rank one solution

$$B_1(x,t) = z\frac{C_1|m^*(-x,-t)\rangle\langle m(x,t)|}{\langle m(x,t)|C_1|m^*(-x,-t)\rangle} \tag{113}$$

where the complex constant z and the components of $\langle m(x,t)|$, i.e. $m_j(x,t)$ are as before. The solution is

$$Q(x,t) = Q^{(0)}(x,t) + (\mu+\kappa_1\mu^*)[J, B_1(x,t) - C_1B_1^\dagger(-x,-t)C_1^{-1}]. \tag{114}$$

Starting with $Q^{(0)}(x,t)\equiv 0$ and real m_{0j} we obtain

$$\begin{aligned}
\boldsymbol{q}(x,t) = Q_{12} &= (\mu+\kappa_1\mu^*)(z-z^*)\frac{m_{02}e^{-i\phi(x,t)}}{m_{01}\cosh(\theta(x,t))},\\
\boldsymbol{p}(x,t) = Q_{21} &= -(\mu+\kappa_1\mu^*)(z-z^*)\frac{m_{01}e^{i\phi(x,t)}}{m_{02}\cosh(\theta(x,t))},
\end{aligned} \tag{115}$$

with $\phi(x,t)$ and $\theta(x,t)$ defined as before,

$$(z-z^*)_{\kappa_1=1} = 2i\tan\varphi, \qquad (z-z^*)_{\kappa_1=-1} = -2i\cot\varphi.$$

Note that for the solutions (115) the property (109) is evident. It is worth noting that in both cases the action of the reduction on λ is $\lambda\to\epsilon\lambda^*$. In both cases the action on λ is very nice. Indeed, the analyticity regions are $A_+ = \mathrm{Im}\lambda^2 > 0$ and $A_- = \mathrm{Im}\lambda^2 < 0$. The action on λ always maps $A_+\to A_-$.

8 Integrals of motion of the multi-component DNLS equations

From eq. (25) we conclude that block-diagonal Gauss factors $D_J^\pm(\lambda)$ are generating functionals of the integrals of motion. The principal series of integrals is generated by $m_1^\pm(\lambda)$:

$$\pm\ln m_1^\pm = \frac{1}{i}\sum_{s=1}^{\infty} I_s\lambda^{-s}. \tag{116}$$

Let us first outline a way to calculate their densities as functionals of $Q(x,t)$. To do this we make use of the third type of Wronskian identities involving $\dot\chi^\pm(x,\lambda)$. They have the form:

$$\left(i\hat\xi^\pm\dot\xi^\pm(x,\lambda) + 2\lambda Jx\right)\Big|_{x=-\infty}^{\infty} = -\int_{-\infty}^{\infty} dx\,\left(\hat\xi(Q(x)-2\lambda J)\xi(x,\lambda) + \lambda^2[J,\hat\xi^\pm\dot\xi^\pm(x,\lambda)]\right), \tag{117}$$

If we multiply both sides of (117) with J and take the Killing form we get:

$$\left\langle (i\hat\xi^\pm\dot\xi^\pm(x,\lambda) + 2\lambda Jx, J\right\rangle\Big|_{x=-\infty}^{\infty} = \pm 2i\frac{d}{d\lambda}\ln m_1^\pm(\lambda), \tag{118}$$

which means that

$$\pm i\frac{d}{d\lambda}\ln m_1^\pm(\lambda) = \frac{i}{2}\int_{-\infty}^{\infty} dx\,\left(\langle (Q(x)-2\lambda J), \xi^\pm(x,\lambda)J\hat\xi^\pm(x,\lambda)\rangle + 2\lambda\langle J,J\rangle\right). \tag{119}$$

If we introduce the notations:

$$\xi^{\pm} J \hat{\xi}^{\pm}(x,\lambda) = J + \sum_{s=1}^{\infty} \lambda^{-s} X_s, \tag{120}$$

then from eq. (60) one can calculate recursively X_s. Of course their complexity grows rather quickly with s. Knowing X_s we find the following recursive formula for I_s:

$$I_{2s} = \frac{1}{4s} \int_{-\infty}^{\infty} dx \left(\langle Q(x), X_{2s+1} \rangle - 2 \langle J, X_{2s+2} \rangle \right). \tag{121}$$

Since in our case $Q_2 = Q_4 = \cdots = 0$ we find that $X_{2s} \in \mathfrak{g}^{(0)}$ and $X_{2s+1} \in \mathfrak{g}^{(1)}$ and therefore $I_1 = I_3 = \cdots = 0$. In calculating the Lax pair for DNLS we in fact calculated the first four coefficients $X_s = V_s$ for $s = 1, \dots, 4$. Using this we get (see the appendix):

$$\begin{aligned} I_1 &= 0, \qquad I_2 = \frac{1}{4} \int_{-\infty}^{\infty} dx \left(i\langle \boldsymbol{q}_x, \boldsymbol{p} \rangle - i \langle \boldsymbol{q}, \boldsymbol{p}_x \rangle + \langle \boldsymbol{qp}, \boldsymbol{qp} \rangle \right), \\ I_3 &= 0, \qquad I_4 = \frac{1}{4} \int_{-\infty}^{\infty} dx \left(\langle \boldsymbol{q}_x, \boldsymbol{p}_x \rangle + \frac{i}{2} (\langle \boldsymbol{q}_x, \boldsymbol{pqp} \rangle - \langle \boldsymbol{qpq}, \boldsymbol{p}_x \rangle) + \frac{1}{4} \langle \boldsymbol{qpq}, \boldsymbol{pqp} \rangle \right). \end{aligned} \tag{122}$$

9 Conclusions

We have studied quadratic bundle Lax pairs on $\mathbb{Z}_2$-graded Lie algebras and on **A.III** symmetric spaces. This includes: the construction of Lax pairs and the related NLEE of Kaup-Newell and GI type. We also constructed the Jost solutions and the minimal set of scattering data for local and nonlocal reductions. The later lead to equations having $\mathcal{CPT}$-symmetry.

We have also constructed the fundamental analytic solutions (FAS) and discussed briefly the spectral properties of the associated Lax operators. It turns out that the spectral properties of the Lax operator depend crucially on the choice of representation of the underlying Lie algebra or symmetric space while the minimal set of scattering data is provided by the same set of functions [25].

We have also formulated the Riemann-Hilbert problem for the Kaup-Newell (KN) and GI equations on **A.III**-symmetric spaces and derived explicit parametrization of the associated Lax operators. This can serve as a starting point in obtaining the Lax pair and the corresponding NLEEs [24, 37].

Finally, we have presented a modification of the dressing method and obtained 1-solitons for the multi-component GI equation with local and nonlocal reductions. We have shown that for specific choices of the polarization vector these solutions can develop singularities in finite time and that there are also cases of soliton solutions with a regular behavior.

The results of this paper can be extended in several directions:

- To construct gauge covariant formulation of the multi-component KN and GI hierarchies on symmetric spaces, including the generating (recursion) operator and it spectral decomposition, the description of the infinite set of integrals of motion, the hierarchy of Hamiltonian structures.

- To study the gauge equivalent systems to the multi-component KN and GI equations on symmetric spaces.

- To study the associated Darboux transformations and their generalizations for DNLS equations over Hermitian symmetric spaces and to obtain multi-soliton solutions via such generalizations. This includes also rational solutions [50, 13, 66].

- To extend our results for the case of non-vanishing boundary conditions (a non-trivial background). Such solutions were obtained for the case of the scalar DNLS by using a slightly different version of dressing method we have employed here [42, 43, 65, 72, 73]. In the scalar case, such solutions are of interest in nonlinear optics: they arise in the theory of ultrashort femto-second nonlinear pulses in optical fibers, when the spectral width of the pulses becomes comparable with the carrier frequency and the effect of self-steepening of the pulse should be taken into account [11]. The considerations required in this case are more complicated and will be discussed it elsewhere.

- To study quadratic bundles associated with other types of Hermitian symmetric spaces both for Kaup-Newell and for GI equations [31, 33, 36, 58, 59].

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A Derivation of the integrals of motion

Let us introduce the notations:

$$\begin{aligned} i\frac{\partial \xi}{\partial x}\xi^{-1}(x,t,\lambda) &\equiv i\frac{\partial \mathcal{Q}}{\partial x} + \sum_{s=1}^{\infty} \frac{1}{(s+1)!}\mathrm{ad}\,_{\mathcal{Q}}^{s}\frac{\partial \mathcal{Q}}{\partial x} = \sum_{s=1}^{\infty}\lambda^{-s}\mathcal{X}_s, \\ \xi J\xi^{-1}(x,t,\lambda) &\equiv J + \sum_{s=1}^{\infty}\frac{1}{s!}\mathrm{ad}\,_{\mathcal{Q}}^{s} J = J + \sum_{s=1}^{\infty}\lambda^{-s}X_s. \end{aligned} \tag{123}$$

Then the fact that $\xi(x,t,\lambda)$ provide the FAS of the operator

$$i\frac{\partial \xi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t))\xi(x,t,\lambda) - \lambda^2[J,\xi(x,t,\lambda)] = 0, \tag{124}$$

leads to $U_1(x,t) \equiv Q(x,t) = [J,Q_1(x,t)]$, $U_2(x,t) = [Q_1(x,t),Q(x,t)]$ and to the recurrent relations

$$i\mathcal{X}_s + X_{s+2} = 0, \qquad s = 1,2,\ldots. \tag{125}$$

Obviously equations (123) provide a recurrence to evaluate $\mathcal{X}_s$ and X_s in terms of $Q_1(x,t)$. Below we list the explicit formulae for the first few of them.

$$\begin{aligned} X_1 &= \mathrm{ad}\,_{Q_1} J = -U_1(x,t) = -\begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}, & X_2 &= \frac{1}{2}\mathrm{ad}\,_{Q_1}^{2} J = -U_2(x,t) = -\frac{1}{2}\begin{pmatrix} \boldsymbol{q}\boldsymbol{p} & 0 \\ 0 & -\boldsymbol{p}\boldsymbol{q} \end{pmatrix}. \\ \mathcal{X}_1 &= \frac{\partial Q_1}{\partial x} = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{q}_x \\ -\boldsymbol{p}_x & 0 \end{pmatrix}, & \mathcal{X}_2 &= \frac{1}{2}\mathrm{ad}\,_{Q_1}\frac{\partial Q_1}{\partial x} = \frac{1}{8}\begin{pmatrix} \boldsymbol{q}_x\boldsymbol{p} - \boldsymbol{q}\boldsymbol{p}_x & 0 \\ 0 & \boldsymbol{p}_x\boldsymbol{q} - \boldsymbol{p}\boldsymbol{q}_x \end{pmatrix}, \end{aligned} \tag{126}$$

$$X_3 = \mathrm{ad}_{Q_3} J + \frac{1}{6}\mathrm{ad}^3_{Q_1} J = -i\mathcal{X}_1, \qquad Q_3 = \frac{i}{4}\begin{pmatrix} 0 & i\boldsymbol{q}_x + \frac{1}{3}\boldsymbol{qpq} \\ i\boldsymbol{p}_x - \frac{1}{3}\boldsymbol{pqp} & 0 \end{pmatrix},$$

$$\mathcal{X}_3 = \frac{\partial Q_3}{\partial x} + \frac{1}{6}\mathrm{ad}_{Q_1}\frac{\partial Q_1}{\partial x} = \frac{i}{4}\begin{pmatrix} 0 & \boldsymbol{q}_{xx} \\ \boldsymbol{p}_{xx} & 0 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 0 & (\boldsymbol{qpq})_x + \boldsymbol{qp}_x\boldsymbol{q} \\ -(\boldsymbol{pqp})_x - \boldsymbol{pq}_x\boldsymbol{p} & 0 \end{pmatrix}. \tag{127}$$

$$X_4 = \frac{1}{2}\left(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_3} + \mathrm{ad}_{Q_3}\mathrm{ad}_{Q_1}\right) J + \frac{1}{24}\mathrm{ad}^4_{Q_1} J$$

$$= -\frac{1}{4}\begin{pmatrix} i(\boldsymbol{q}_x\boldsymbol{p} - \boldsymbol{qp}_x) + \frac{1}{2}\boldsymbol{qpqp} & 0 \\ 0 & i(\boldsymbol{p}_x\boldsymbol{q} - \boldsymbol{pq}_x) - \frac{1}{2}\boldsymbol{pqpq} \end{pmatrix},$$

$$X_5 = \mathrm{ad}_{Q_5} J + \frac{1}{6}\left(\mathrm{ad}^2_{Q_1}\mathrm{ad}_{Q_3} + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_3}\mathrm{ad}_{Q_1} + \mathrm{ad}_{Q_3}\mathrm{ad}^2_{Q_1}\right) J + \frac{1}{120}\mathrm{ad}^5_{Q_1} J$$

$$= \frac{1}{4}\begin{pmatrix} 0 & \boldsymbol{q}_{xx} - \frac{i}{4}(\boldsymbol{qpq})_x - \frac{i}{4}\boldsymbol{qp}_x\boldsymbol{q} \\ \boldsymbol{p}_{xx} + \frac{i}{4}(\boldsymbol{pqp})_x + \frac{i}{4}\boldsymbol{pq}_x\boldsymbol{p} & 0 \end{pmatrix},$$

$$Q_5 = \frac{1}{8}\begin{pmatrix} 0 & -\boldsymbol{q}_{xx} + \frac{7i}{12}(\boldsymbol{qpq})_x - \frac{5i}{12}\boldsymbol{qp}_x\boldsymbol{q} + \frac{3}{10}\boldsymbol{qpqpq} \\ \boldsymbol{p}_{xx} + \frac{7i}{12}(\boldsymbol{pqp})_x - \frac{5i}{12}\boldsymbol{pq}_x\boldsymbol{p} + \frac{3}{10}\boldsymbol{pqpqp} & 0 \end{pmatrix}. \tag{128}$$

The above expressions readily lead to the results for the conserved quantities (122) along with the fact that

$$\langle J, \xi J \xi^{-1}(x,t,\lambda)\rangle = \langle J, J\rangle + \sum_{s=1}^{\infty}\frac{(-1)^s}{(2s)!}\langle \mathrm{ad}^s_{\mathcal{Q}} J, \mathrm{ad}^s_{\mathcal{Q}} J\rangle\,. \tag{129}$$

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