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Propagating Degrees of Truth on an Argumentation Framework: an Abstract Account of Fuzzy Argumentation

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ABSTRACT

This paper proposes a computational framework to reason with conflicting and gradual evidence. The framework is a synthesis of Dung's seminal work in argumentation semantics with multi-valued logic. Abstract grounded semantics is used to identify the conditions under which a conclusion can be accepted, while multi-valued logic operators are used to quantify the degree of truth of such conditions. We propose a truth-compositional recursive computation based on the notion of *irrelevant* arguments, and we discuss examples using the major multi-valued logics: Godel's, Zadeh's and Lukasiewicz's logic.

CCS Concepts

Computer Methodologies → **Vagueness and fuzzy logic.** **Computer Methodologies** → **Non-monotonic, default reasoning and belief revision.**

Keywords

Argumentation semantics, multi-valued logics, approximate reasoning, fuzzy logic.

1. INTRODUCTION

The aim of this paper is to provide a sound framework for reasoning with imprecise and conflicting evidence. The core idea is to define a novel synthesis between the two main research areas relevant to the problem, namely abstract argumentation, used as a conflict-resolution strategy, and multi-valued logics, used to model gradual information.

In the argumentation setting, conclusions are reached by evaluating arguments. An argument is a construct used in discussions with a support and a claim that is derived from the support. Arguments are not proof, but rather defeasible constructs whose validity can be challenged by other arguments attacking them.

In order to analyze defeasible arguments, Dung [1] introduced the notion of abstract argumentation framework, a directed graph where nodes represent arguments and arrows represent an attack relation among arguments. Various argumentation semantics have been proposed to identify the set of acceptable arguments. Following the labelling approach proposed by [2], the effect of an abstract argumentation semantics is to assign to each argument a label *in*, *out* or *undec*, meaning that the argument is considered consistently acceptable, non-acceptable or undecided (i.e. no decision can be taken on argument acceptability).

In Dung's original work, arguments are either fully asserted or not asserted at all, and as a consequence abstract argumentation is often too strict and coarse to support decisions. In the quest for an argumentation system able to handle a generic notion of *quantifiable strength*, few approaches have been proposed. Among the non-probabilistic approaches we mention the degree of justification of arguments proposed by Pollock ([3]); the notion of gradualism derived from the topology of the argumentation graph proposed by [4], and the concept of weighted argumentation in [5].

Recent approaches [6] [7] have tried to marry abstract argumentation and probability calculus. In such frameworks, a probability distribution P is defined over the arguments of an abstract argumentation graph Ar . In a mainstream interpretation, the probability of each argument quantifies the likelihood of the argument premises to hold, and therefore the likelihood that the claim of the argument has the right to be considered in the argumentation process. Given Ar and P , a probabilistic argumentation framework (PAF) provides a way to compute the probability to which each argument is accepted by a given semantics.

Following a similar conceptual framework, here we investigate how to marry abstract argumentation and multi-valued logic. In the setting of this paper an argument has a degree of truth associated with it. A possible non-exclusive interpretation of such degree is that it could represent the degree of truth of the claim of an argument, inferred by the degree of its support. This can be illustrated by considering a rule-based model of arguments, where arguments are modelled using inference rules from a support to a claim. If arguments are built using gradual or vague pieces of evidence, the rules used to model arguments are *fuzzy rules*, which are rules

whose support or claim might be represented using fuzzy concepts. The rule “if it is sunny, then Joe is happy” is an example of argument built using a gradual fuzzy rule: the sunnier the weather, the happier is Joe. The terms *sunny* and *happy* are arguably better represented as fuzzy terms rather than Boolean; and the strength of the argument is quantified by the degree of truth of its claim *Joe is happy*, inferred from the degree to which the weather is sunny (its support).

The aim of this paper is to compute how the degree of truth of arguments changes in the presence of other attacking arguments. This first paper presents an abstract account of the framework, while non-abstract instances of the framework are presented in a forthcoming paper. In the abstract interpretation, arguments are abstracted into nodes connected by the attack relation, and the degree attached to each argument represents the degree to which the argument belongs to (=is present in) the argumentation graph. Note how this was the similar dual interpretation given by Li [x] in its first account of probabilistic argumentation framework, where the probability associated to each node was the probability of the argument to be present in the graph.

The link between the abstract account and other semi-abstract and structured models (such as the fuzzy rule-based model we mentioned) could be the following. Since we are dealing with gradual evidence, the support of a structured argument could be partially satisfied and partially not. The degree to which the support of the argument is satisfied by the state of affairs quantifies the degree to which the argument (and its claim) can be justifiably included in the dialectical process. In the abstract case, the dialectical process is represented by a graph and the degree of truth to which the support/claim of an argument is satisfied represents the degree of truth to which the corresponding abstract argument (a node) is present in the graph.

The paper is organized as follows. The next section describes the required background knowledge about abstract argumentation and multi-valued logic, while section 3 and 4 describe our computational framework, section 5 investigates the properties of the framework, section 6 provides an informal semi-abstract description of our computational framework with an illustrative example of its application, and section 6 contains related works in the area before our conclusions.

2. BACKGROUND

2.1 Multi-valued Logic

In the setting of multi-valued logics, the convention prescribing that a proposition is either true or false is changed. A sentence is now not true or false only, but may have a truth degree taken from an ordered scale, called truth space S , such as $[0,1]$. Multi-valued logic can model situations affected by vagueness, where a statement is satisfied to a certain extent and the concepts discussed are graded. This is usual in natural language when words are modeled by fuzzy sets, such as tall, young, fast. We identify a proposition with a fuzzy set and the degree of membership of a state of affairs to this fuzzy set evaluates the degree of fit between the proposition and the state of facts it refers to. This degree of fit is called degree of truth of a proposition ϕ . Semantically, a many-valued interpretation I maps each basic proposition ϕ, ψ into $[0,1]$ and is then extended inductively as follows:

$$I(\phi \wedge \psi) = I(\phi) \otimes I(\psi) ; I(\phi \vee \psi) = I(\phi) \oplus I(\psi)$$

$$I(\phi \rightarrow \psi) = I(\phi) \triangleright I(\psi) ; I(\bar{\phi}) = \ominus I(\psi)$$

where $\otimes, \oplus, \triangleright$ and \ominus are called triangular norms, triangular co-norms, implication functions and negation functions, which extend the classical Boolean conjunction, disjunction, implication and negation to the many-valued case. These functions have all to satisfy the following properties: tautology, contradiction, commutativity, associativity and monotonicity, but not all of them satisfy excluded middle ($x \otimes \ominus x = 0$) or double negation ($\ominus \ominus x = x$). We usually distinguish two main logics: Łukasiewicz’s and Gödel’s logic; the Zadeh’s logic is a sublogic of Łukasiewicz’s logic. Their operators are shown in table 1. For a comprehensive analysis see [8].

Table 1. Combination functions of various fuzzy logics

	Łukasiewicz’s	Gödel’s	Zadeh’s
$a \otimes b$	$\max(a+b-1, 0)$	$\min(a, b)$	$\min(a, b)$
$a \oplus b$	$\min(a+b, 1)$	$\max(a, b)$	$\max(a, b)$
$\ominus a$	$1-a$	$\begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$1-a$

2.2 Abstract Argumentation Semantics

Definition 1 An argumentation framework AF is a pair (Ar, R) , where Ar is a non-empty finite set whose elements are called arguments and $R \subseteq Ar \times Ar$ a binary relation, called the attack relation. If $(a, b) \in R$ we say that a attacks b in. Two arguments a, b are **rebuttals** iff $(a, b) \in R \wedge (b, a) \in R$.

Definition 2 (conflict-free). $Args$ is conflict-free iff $\nexists a, b \in Args \mid (a, b) \in R$.

Definition 3 (*admissible set*). $Args$ defends an argument $a \subseteq Ar$ iff $\forall b \in Ar$ so that $(b, a) \in R, \exists c \in Args$ so that $(c, b) \in R$. The set of arguments defended by $Args$ is denoted $F(Args)$. A set $Args$ is *admissible* if $Args \subseteq F(Args)$ and it is complete if $Args = F(Args)$.

An abstract argumentation semantics identifies a set of arguments that can survive the conflicts encoded by the attack relation R . We follow the labelling approach of [2], where a semantics assigns to each argument a label in, out or undec.

Definition 4 (*labelling*). Let $AF = (Ar, R)$. A *labelling* is a total function $L : Ar \rightarrow \{in, out, undec\}$. We write $in(L)$ for $\{a \in Ar | L(a) = in\}$, $out(L)$ for $\{a \in Ar | L(a) = out\}$ and $undec(L)$ for $\{a \in Ar | L(a) = undec\}$.

Definition 5 (*complete labelling*, from definition 5 in [2]). Let (Ar, R) be an argumentation framework. A *complete labelling* is a labelling that for every $a \in Ar$ holds that: 1. if a is labelled in then all attackers of a are labelled out; 2. if all attackers of a are labelled out then a is labelled in; 3. if a is labelled out then a has an attacker labelled in; 4. if a has an attacker labelled in then a is labelled out.

Theorem 1 (from [2]) Let L be a labelling of $AF = (Ar, R)$. It holds that L is a complete labelling iff for each argument $a \in Ar$ it holds that: 1. if a is labelled in then all its attackers are labelled out; 2. if a is labelled out then it has at least one attacker that is labelled in; 3. if a is labelled undec then it has at least one attacker that is labelled undec and it does not have an attacker that is labelled in.

Theorem 2 (from theorem 6 and 7 in [2]) Given $AF = (Ar, R)$, L is the grounded labelling iff L is a complete labelling where $undec(L)$ is maximal (w.r.t. set inclusion) among all complete labellings of AF .

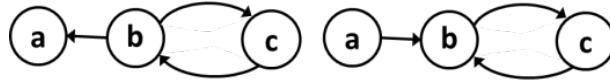


Figure 1. Two Argumentation Graphs (A) and (B).

Referring to figure 1, grounded semantics assigns the status of *undec* to all the arguments of the argumentation framework on the left, since it represents the complete labelling with the maximal *undec* set, while in the argumentation framework on the right, according to theorem 1, there is only one complete labelling (thus grounded), where argument a is *in* (no attackers), b is *out* and c is *in*. Note how a reinstates c .

3. SETTING THE PROBLEM

A multi-valued argumentation framework is an extension of Dung's abstract framework where a degree of truth is associated to each argument.

Definition 7 (multi-valued argumentation frameworks) A *multi-valued argumentation framework (MV-AF)* is a tuple $((Ar, R), \mu)$ where (Ar, R) is an abstract argumentation framework and $\mu : Ar \rightarrow [0,1]$ assigns a degree of truth to each argument in Ar .

Given $a \in Ar$, we write μ_A as a shortcut for $\mu(a)$. Given a MV-AF, the focus of this work is to identify the degree of truth to which argument a is accepted, called μ_{AIN} . We stress the crucial difference between μ_A and μ_{AIN} . μ_A refers to the degree of the *isolated* argument, inferred from its support only; while μ_{AIN} is the resulting degree of truth of a after having accounted for the effect of other attacking arguments.

In the abstract account presented here the reader has to intend the degree of truth *associated* to each argument as the degree of membership of the argument to the argumentation graph. To a degree μ an argument belongs to the graph, while to a degree $\ominus \mu$ it does not. Note how we follow the same abstract interpretation that Li proposed for probabilistic argumentation framework in [6].

Space limitation prevents the description of a semi-abstract account of MV-AF. However, for clarity and to show the potential applications of a MV-AF we briefly discuss a possible exemplar instance. In a rule-based model, arguments are represented by rules from a support to a conclusion. If the concepts used to build rules are gradual, we use fuzzy rules. In a fuzzy rule the support and/or the conclusion could contain fuzzy terms with corresponding membership functions and degrees of truth. In general (but not necessarily) the degree of truth of the conclusion depends on the degree of truth of the support of the rule and the relation between support and conclusion. This could be modelled by a Cartesian product between the fuzzy set of the support and the fuzzy set of the conclusion. Alternatively, fuzzy rules can be also modelled as multi-valued implications or ad-hoc functions [9].

The rule “if it is sunny, Joe is happy” is an example of *gradual* fuzzy rule. The higher the degree of sunshine, the happier is Joe. Rules of this kind can be abstracted into an MV-AF; the degree of truth of the claim is used as the degree associated with the argument in an MV-AF, therefore quantifying the strength of the conclusion of the argument.

What about the meaning of attacks in a MV-AF?

In general, in a rule-based model the defeasible nature of arguments is modeled using defeasible rules, rules whose validity can be questioned by other attacking rules. Defeasible rules could be represented with rules containing assumptions (often implicit) in their supports, assumptions that other rules could disprove [10].

An undercutting (asymmetrical) attack refers to the situation where a conclusion of a rule disproves the assumption of another. If we deal with gradual truths, the effect of the attack is also gradual: the stronger the degree of truth μ_{att} of the attacker, the stronger the effect of invalidating the attacked rule.

The extreme case is when $\mu_{att} = 1$ coincides with the rule fully defeated, as in the Boolean case. If $\mu_{att} < 1$ the attack might leave the attacked rule still valid but with a diminished degree of truth of its claim. Going back to our example, let us presume that *sunshine makes Joe happy* assuming *Joe has no homework*. Homework undercuts the rule: the more homework Joe gets, the less the sunshine has an effect on Joe's happiness. In general, his happiness is diminished (even with full sunshine) when we prove he has homework. Of course, other arguments can also make Joe happy.

Rebuttals attacks are a clash of conflicting statements. In general the two arguments will have claims with different degrees, and it is reasonable to expect that the two degrees will be diminished and the greater of the two will reasonably remain greater than zero.

3.1 Subgraph Notation and Labelling of Subgraphs

Given an argumentation framework $AF = (Ar, R)$ with $|Ar| = n$, and the graph \mathcal{g} identified by Ar and R , $a \in Ar$, we consider the set \mathcal{S} of all the subgraphs of \mathcal{g} . We focus on particular sets of subgraphs, i.e. elements of $2^{\mathcal{S}}$. We call A and \bar{A} respectively the set of subgraphs where argument a is present and the complementary set of subgraphs where a is absent. If $Ar = \{a_1, \dots, a_n\}$, a single subgraph \mathcal{s} can be expressed by an intersection of n sets A_i or \bar{A}_i ($i \leq n$) depending on whether the i^{th} argument a_i is or is not contained in \mathcal{s} .

A set of subgraphs can be expressed by combining some of the sets $A_1, \dots, A_n, \bar{A}_1, \dots, \bar{A}_n$. with the connectives $\{\cup, \cap\}$. We write AB to denote $A \cap B$ and $A + B$ for $A \cup B$. For instance, in figure 1 left the single subgraph with only b and c present is denoted with $\bar{A}BC$, while the expression AB denotes a set of two subgraphs (ABC and $AB\bar{C}$) where arguments a and b are present and the status of c (not in the expression AB) is not specified.

Given a subgraph $\mathcal{s} \in \mathcal{S}$, the labelling of \mathcal{s} follows the rules of the chosen semantics. An issue is how to label an argument a in Ar that is absent in the subgraph \mathcal{s} . Instead of using extra labels such as *on*, *off*, we decided to label absent arguments with the label *out*, de facto extending its meaning. Our decision is justified by the fact that under grounded semantics an *out*-labelled argument is equivalent to a removed argument w.r.t the computation of the semantics. In both situations the claim arguments can be neglected in the decision-making process. Therefore an argument in a subgraph \mathcal{s} is labelled *out* in two cases: it is labelled *out* by the chosen semantics in the subgraph \mathcal{s} , representing the effect on a of the other arguments, or because a is absent in the subgraph \mathcal{s} (meaning that a cannot be claimed even *on its own*).

Finally, we define as A_{IN}, A_{OUT}, A_U the set of subgraphs where argument a is labelled *in*, *out*, *undec*.

Example 1. In the graph of figure 1 left, there are 3 arguments and 2^3 subgraphs; argument a is labelled *in* in all the subgraphs where a is present and b is not present (and c becomes irrelevant), i.e. $A_{IN} = A\bar{B}$. It is *undec* when all the arguments are present (the single subgraph $A_U = ABC$) while a is *out* when it is not present or when b is present and c is not present, i.e. $A_{OUT} = \bar{A} + AB\bar{C}$.

3.2 Computing A_{IN}

A subgraph-based brute force algorithm to find A_{IN} simply computes the grounded semantics in all the subgraphs of Ar and selects the subgraphs where a is labelled *in*. The set A_{IN} will therefore be expressed as a conjunction of subgraphs. Here we propose an alternative recursive algorithm to compute A_{IN} that extends what we proposed in [11].

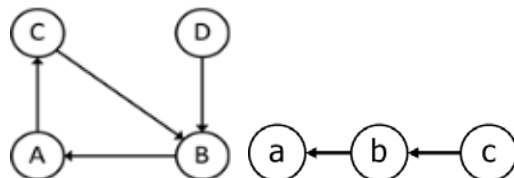


Figure 2. Two argumentation graphs

Inputs: an argument a , l a label (*in* or *out*), P is an auxiliary variable holding the list of nodes visited before a .

Outputs: an expression representing all the cases where a has the label l .

FindSet(a, l, P):

```

1 if a in P:
2   return  $\emptyset$            //cycle found,  $A_{IN} = \emptyset$ 
3 if l = IN:
4   if a terminal:
5     return A           //terminal condition,  $A_{IN} = A$ 
6   else:
7     add a to P
8      $A_{IN} = A$ 
9     for each child c of a
10       $A_{IN} = A_{IN} \cap \text{FindSet}(c, OUT, P)$  //condition c1
11    return  $A_{IN}$ 
12 if l = OUT:
13  if a terminal:
14    return  $\bar{A}$  //terminal condition,  $A_{OUT} = \bar{A}$ 
15  else
16    add a to P
17     $A_{OUT} = \bar{A}$ 
18    for each child c of a
19       $A_{OUT} = A_{OUT} \cup \text{FindSet}(c, IN, P)$ 
20    return  $A_{OUT}$            // condition c2

```

Given a starting argument a and a label $l \in \{in, out\}$, the algorithm traverses the transpose graph (a graph with reversed arrows) from a down to its attackers, propagating the constraints of the grounded labelling. The constraints needed are listed in definition 5 and theorem 1. If argument a – attacked by n arguments x_n – is required to be labelled *in*, we impose A_{IN} to be:

$$A_{IN} = A X_{1_{OUT}} X_{2_{OUT}} \dots X_{n_{OUT}} \text{ (condition c1, line 10)}$$

meaning that argument a is labelled *in* in the cases where:

1. a is present (i.e. the case A) and
2. all the attacking arguments x_i are *out* (sets $X_{i_{OUT}}$).

If a is required to be labelled *out*, the set of cases is:

$$A_{OUT} = \bar{A} + X_{1_{IN}} + X_{2_{IN}} + \dots + X_{n_{IN}} \text{ (condition c2, line 20)}$$

i.e. a is labelled *out* in all the cases where it is absent or at least one of the attackers is labelled *in*. Thus we recursively traverse the graph, finding the cases that are compatible with the starting label of a . The sets $X_{n_{OUT}}, X_{n_{IN}}$ are found when terminal nodes are reached. When a terminal node x_T is reached the following conditions are applied:

1. if x_T is required to be *in* then $X_{T_{IN}} = X_T$.
2. if node x_T is required to be *out* then $X_{T_{OUT}} = \bar{X}_T$.

The way algorithm 1 treats cycles guarantees that only grounded labellings are identified. If a cycle is detected, the recursion path terminates, returning an empty set for that recursive branch.

Example 2. Referring to figure 2 left, a is labelled *in* when:

$$A_{IN} = AB_{OUT} = A(\bar{B} + D_{IN} + C_{IN}) = A(\bar{B} + D + CA_{OUT}) = A(\bar{B} + D).$$

Note how CA_{OUT} identifies a cycle and returns the empty set.

4. COMPUTING $\mu_{A_{IN}}$

A starting idea simply translates the approach used in probabilistic argumentation to the multi-valued case. In a PAF, the nodes of the graph have a probability assigned, to be intended as the probability that the node belongs to the graph. The goal is to compute the probability of each argument to be accepted by the chosen semantics.

N probabilistic arguments generate a sample space of 2^N events, each of them represented by a subgraph of the original argumentation framework and each of them with an associated probability, computed from the probability distribution defined over the arguments. The mainstream computation of *PAFs*, referred to as the *constellations approach*, requires analysing the behaviour of a semantics over all the subgraphs of the starting argumentation framework and the computation has an above-polynomial complexity. Given an argument a , the probability of the acceptance of argument a is the sum of the probabilities associated with all the subgraphs where a is labelled *in* by the chosen semantics.

Back to MV–AF: if μ_A is the degree of truth associated to argument a that, in the abstract interpretation, is the degree to which the argument belongs to the graph (i.e. a is present in the graph), the degree to which arguments a and b are present in the graph is the degree of $\mu_A \otimes \mu_B$, where \otimes is the multi-valued norm. Similarly, the degree to which a is present and b is not is $\mu_A \otimes (\ominus \mu_B)$ where \ominus is the negation operator.

Therefore a generic subgraph s also has a degree of associated truth, since a subgraph s is specified by stating which arguments are present and which are absent from the starting complete graph. As degrees of truth and probabilities measure different concepts, a subgraph of a MV–AF does not have the same interpretation as a subgraph of a PAF. In the probabilistic case, each subgraph represents a distinct and mutually exclusive event and there is uncertainty about which event will happen, while in a MV–AF a subgraph represents one of the partially true co-existing interpretations of the same perfectly known situation.

We are therefore back to a subgraph-based analysis and the temptation is to simply translate the computation of the *constellations approach* to the multi-valued logic case. If we want to compute to which degree an argument is accepted, the constellations approach for multi-valued logics first finds all the subgraphs s_i in which the argument is accepted (i.e. is labelled *in*). Then, it associates to each subgraph s_i its degree of truth μ_{s_i} and the final value of $\mu_{A_{IN}}$ is the disjunction of the degrees μ_{s_i} , computed with the co-norm \oplus . However, this proposal is flawed, as shown in the following discussion.

Let us consider the simple argumentation graph depicted in figure 2 right where argument a is attacked by b , and b is attacked by c . The constellations approach expresses A_{IN} as the disjunction of the following three subgraphs: $A_{IN} = ABC + A\bar{B}C + A\bar{B}\bar{C}$. The recursive algorithm 1 returns the following expression: $A_{IN} = AB_{OUT} = A(\bar{B} + C_{IN}) = A(\bar{B} + C)$. We can also further simplify it into $A\bar{B} + AC$, or express the set A_{IN} as $A\bar{B} + ABC$.

We list the four different expressions:

- (a) $A_{IN} = ABC + A\bar{B}C + A\bar{B}\bar{C}$, *constellations approach*;
- (b) $A_{IN} = A(\bar{B} + C)$, the output of algorithm 1;
- (c) $A_{IN} = A\bar{B} + AC$, algorithm 1 further simplified;
- (d) $A_{IN} = A\bar{B} + ABC$, an expression using *disjoint sets*.

The above expressions, even if all logically equivalent in the Boolean sense, do not evaluate the same in the multi-valued case. For instance, if $\mu_A = 0.8, \mu_B = 0.3, \mu_C = 0.9$, using Zadeh's *max* and *min* operators the *constellations approach* gives a value for $\mu_{A_{IN}}$ of 0.4¹, the recursive algorithm 0.8 and the disjoint set notation (iv) a value of 0.5. Which computation should be favoured?

Note how the above expression would evaluate the same in the probabilistic case. A first observation is linked to the fact that multi-valued operators do not satisfy the properties of classical set theory, underpinning probability calculus. For instance, $A\bar{B}C + A\bar{B}\bar{C}$ has not the same degree of truth of $A\bar{B}$, since $x \oplus \ominus x \neq 1$, and therefore the constellations-approach expression (1) is not equal to the expression (4). Moreover, the recursive expression (2) $A(\bar{B} + C)$ is not equal to the expression (3) $A\bar{B} + AC$, since the MV operators do not satisfy the distributive properties of *AND* over *OR*. However, even if these observations explain why the expressions differ and they suggest to avoid violating MV properties, they do not provide reason to favour one expression over the other.

We observe that some of the above expressions contain redundant information. Let us consider the *constellations approach* expression $A_{IN} = ABC + A\bar{B}C + A\bar{B}\bar{C}$. In the last two terms ($A\bar{B}C$ and $A\bar{B}\bar{C}$), b is absent, c becomes disconnected from a and therefore irrelevant for the grounded labelling of a . Therefore, c 's degree of truth should not alter the degree of truth of a . The same happens with expression 4. In the term ABC , why consider b ? b is labelled *out* and therefore irrelevant (under grounded semantics) for the labelling of a .

If in the probabilistic case the above observations have no relevance (since all the expressions have the same value), it has relevance when dealing with multi-valued logics.

In general, given a generic argument $b \in Ar$, each time a term B or \bar{B} appears in A_{IN} , the term identifies a constraint over argument b , requiring it to be present or absent from the set of subgraphs considered. The next definition helps to identify constraints representing redundant information.

Definition of constraint and irrelevant constraint. Given an expression of A_{IN} and an argument b , we call *constraint* a single occurrence of the term B or \bar{B} in A_{IN} . We say that a constraint $c_B = B$ is *irrelevant* to the computation of A_{IN} iff by replacing B with \bar{B} then the new expression $A_{IN}^{\bar{B}}$ is logically equivalent (in the Boolean sense) to A_{IN} , i.e. they represent the same set of subgraphs. A

¹ Note how, using Łukasiewicz's operators, the value of $a \otimes b \otimes c$ is equals to $\max(\max(a + b - 1, 0) + c - 1, 0)$ and not $\max(a + b + c - 1, 0)$.

constraint $c_B = \bar{B}$ is irrelevant to the computation of A_{IN} iff by replacing \bar{B} with B then the new expression A_{IN}^B is logically equivalent to A_{IN} .

Our principle is that *irrelevant constraints are redundant information that should not be present in the expression of the set A_{IN}* . Algorithm 1 satisfies this principle, as it can be proven that the expression of A_{IN} generated by algorithm 1 does not contain *irrelevant constraints*.

However, both the expressions (2) $A_{IN} = A(\bar{B} + C)$ and (3) $A_{IN} = A\bar{B} + AC$ do not contain irrelevant constraints but they do not evaluate the same using MV operators. Why should algorithm 1 expression (2) be preferred over expression (3)?

The decisive observation is related to the properties that each computation exhibits. A reasonable principle (in the author's opinion, a necessary axiom in any argumentation framework with attack relation only) is that the strength of an argument a cannot increase if a is attacked, but it can only remain the same or be reduced. Probabilistic argumentation satisfies this property. In our context, this means that $\mu_{A_{IN}} \leq \mu_A$.

While expressions generated by algorithm 1 have $\mu_{A_{IN}} \leq \mu_A$, expressions like (3), where A_{IN} is expressed as a disjunction of conjunctive forms, do not always guarantee it. For instance, if $\mu_A = 0.8, \mu_B = 0.3, \mu_C = 0.9$, it can be verified that using Łukasiewicz's logic the degree of the expression $A\bar{B} + AC$ has a value of 1. Therefore, the argumentation process has increased the degree of truth of a !

A minor but interesting further consideration is given by the order in which arguments should be considered. We believe a computation of the degree of truth of a should move from a down to its attackers recursively, as algorithm 1 does and as suggested by Pollock in [3]. The *constellations approach*, on the other hand, fragments the computation in a collection of single subgraphs, losing the topology of the graph and consequently producing redundant information.

Moreover, algorithm 1 directly maps the definition of complete grounded labelling as found in [2]: its output is correct both for the probabilistic case and the multi-valued one. In conclusion, algorithm 1 expression should be preferred for the following reasons:

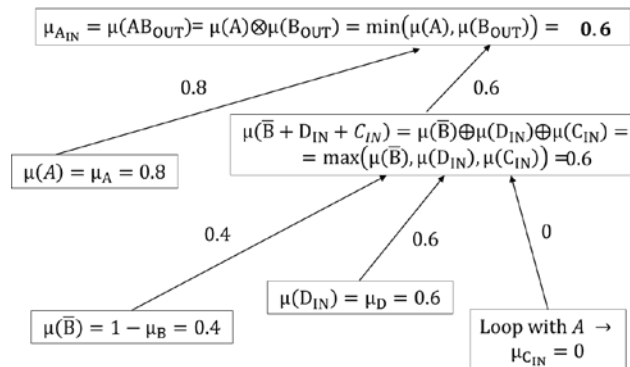
- The (graph-based) recursive order in which arguments are considered;
- The property $\mu_{A_{IN}} \leq \mu_A$ of the resulting expression;
- The absence of irrelevant constraints in the expression of A_{IN} .

As an extra condition, algorithm 1 output should be correctly evaluated without violating the properties of multi-valued operators. The next section describes a convenient way to perform such computation.

4.1 Exploiting the truth compositional operators

Unlike probability or possibility calculus the three multi-valued logics proposed have truth-functional operators, i.e. the degree of truth of an expression is fully determined by the degree of truth of its components. As stressed by Dubois [27], we are allowed to use truth-functional operators as long as we are dealing with gradual properties with no uncertainty involved, otherwise possibility theory has to be applied and the truth-compositional property is lost. Using truth-composition, degrees of truth can be computed during the recursive visit of algorithm 1. Degrees of truth of arguments are found when terminal conditions are reached and these values are propagated back through the recursive chain and combined with the truth-compositional multi-valued operators \oplus, \otimes, \ominus . By doing so, arguments are gradually replaced by their degrees of truth. The truth-compositional property makes the computation of degrees of truth under grounded semantics having the same complexity class as a recursive tree traversal, i.e. a linear complexity proportional to the number of nodes and links, while the constellations approach is of above-polynomial complexity.

Example 3. Let us continue example 2. If $\mu_A = \mu_C = \mu_D = 0.8, \mu_B = 0.6$ then $\mu_{A_{IN}}$ is given by the recursive tree below.



Note how degrees of truth are computed and propagated during the recursive steps exploiting the truth-compositional property of multi-valued operators. The computation consistently employs both grounded argumentation semantics and multi-valued logic.

5. ATTACK, REINSTATEMENT, ACCRUAL AND REBUTTALS

The following examples illustrate the behavior of MV-AF w.r.t. essential situations that any argumentation framework has to handle, namely attack, reinstatement, accrual of arguments and rebuttals.

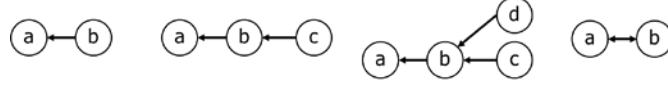


Figure 3. Argumentation graphs for the examples 4, 5, 6, 7

Example 4. Attack. If argument a is attacked by b , how is the degree of a modified?

It is $A_{IN} = A\bar{B}$. Using Zadeh's operators, $\mu_{A_{IN}} = \min(\mu_A, 1 - \mu_B)$. Therefore it is $\mu_{A_{IN}} \leq \mu_A$ (degree of truth is diminished) and $\mu_A = \mu_{A_{IN}}$ when $\mu_A < 1 - \mu_B$. The degree of truth of a is unchanged and the attack from b neglected if $\mu_A + \mu_B < 1$. This imposes a minimum degree of truth on the attacker to have an impact on μ_A . Note how this finding seems to justify the notion of a threshold for *attack activation* present in [4], where the authors suggest that an attack that is too weak (weaker than the attacked arguments) fails and has to be neglected.

Using Łukasiewicz's logic it is:

$$\mu_{A_{IN}} = \min(\mu_A + 1 - \mu_B - 1, 0) = \begin{cases} \mu_A - \mu_B & \text{if } \mu_A > \mu_B \\ 0 & \text{if } \mu_A \leq \mu_B \end{cases}$$

Therefore a is always diminished (unlike Zadeh's case) and totally defeated if the degree of the attacker is greater than μ_A . Interestingly, this is the exact behaviour proposed by Pollock [3], whose proposal was not grounded in any multi-valued logic system.

An important difference between Zadeh and Łukasiewicz's logic is the following: using Zadeh's *min* operator, an argument can be totally defeated only if $\mu_B = 1$, while using Łukasiewicz's logic it is totally defeated every time $\mu_A \leq \mu_B$.

Finally, Godel's logic negation operator always assigns a null degree of truth to $\ominus \mu_A$ if $\mu_A > 0$. In practical terms, this implies removing all the terms containing negated constraints (i.e. of the kind $c_x = \bar{X}, x \in Ar$) from the output of algorithm 1, or similarly only considering the complete starting argumentation graph and ignoring the subgraph analysis. This means that, using grounded semantics, only one out of the three quantities $\mu_{A_{IN}}, \mu_{A_{OUT}}, \mu_{A_U}$ has a not null value. In b is attacking a , it is obviously $\mu_{A_{IN}} = 0$.

Example 5. Reinstatement Chain. A chain of three arguments illustrate the reinstatement property. It is $A_{IN} = A(\bar{B} + C)$.

Under Godel's logic, only AC has a not null degree of truth and $\mu_{A_{IN}} = \min(\mu_A, \mu_C)$. Thus the argument is fully reinstated if $\mu_C > \mu_A$ or it is otherwise reinstated to the degree equal to its defender c .

Using Zadeh's logic, $\mu_{A_{IN}}$ is given by the expression $\min(\mu_A, \max(1 - \mu_B, \mu_C))$. We note that, if $1 - \mu_B > \mu_C$, nothing changes from example 4 and no reinstatement happens, while, when $1 - \mu_B < \mu_C$, $\mu_{A_{IN}}$ could be increased w.r.t. example 4. Both Zadeh's and Godel's operators fully reinstate a if $\mu_C > \mu_A$. Arguably, when $\mu_C > \mu_A$ the two logic systems neglect the degree of truth of the attacker b when computing μ_A .

Using Łukasiewicz's logic, $\mu_{A_{IN}} = \max(\mu_A + \min(1 - \mu_B + \mu_C, 1) - 1, 0)$. Argument a is fully reinstated if $1 - \mu_B + \mu_C > 1$, i.e. $\mu_C > \mu_B$, which seems a reasonable result; again, it is the same behaviour as Pollock [3].

The reinstatement example provides further evidence in favour of the usage of the recursive algorithm 1. *Constellations approach* expressions such as $A_{IN} = A\bar{B}C + A\bar{B}\bar{C} + ABC$, even if $\mu_{A_{IN}} \leq \mu_A$, exhibit a counterintuitive behaviour due to the fact that the longer conjunctive terms are harder to satisfy and the resulting degree of truth decreases rapidly. For instance, if $\mu_A = 0.5, \mu_B = 0.5, \mu_C = 1$ we have $\mu_{A_{IN}} = 0$ (even if a is defended by an argument with the maximum degree of truth, there is no reinstatement).

Example 6. Accrual of attacks. The example illustrates the accrual of attacks. It is $A_{IN} = A(\bar{B} + C + D)$. Both Godel's and Zadeh's operators do not accrue arguments, since it is the *max* of the three terms inside the parenthesis that is considered, as in Pollock [3]. Arguments accrue with Łukasiewicz's logic, since its disjunction operator does, and the effect of multiple attackers generally is not equal to the strongest one. Note how, in probabilistic argumentation, probability accrues since $P(b \cup c) \geq P(c)$ and $P(b \cup c) \geq P(b)$.

Example 7. Rebuttal. In case of two rebuttal arguments, grounded semantics gives $A_U = B_U = AB, A_{IN} = A\bar{B}, B_{IN} = B\bar{A}$. Figure 4 shows the behaviour of the three multi-valued logics discussed. Godel and Zadeh always assign a not null value to the *undec* situation equal to $\mu_{A_U} = \mu_{B_U} = \min(\mu_A, \mu_B)$. Using Łukasiewicz's operators it is $\mu_{A_U} = \max(\mu_A + \mu_B - 1, 0)$, and $\mu_{A_U} > 0$ only when $\mu_A + \mu_B > 1$. Intuitively, using Łukasiewicz, two conflicting arguments do not create an undecided situation if their degrees of truth are small enough to avoid overlapping.

Regarding $\mu_{A_{IN}}$ and $\mu_{B_{IN}}$, Godel's system assigns a null degree of truth to both; while Zadeh's logic always assigns a not null degree that has an upper bound in the degree to which the other conflicting argument is negated. Łukasiewicz's logic assigns a not null

degree equal to $|\mu_A - \mu_B|$ to the argument with the highest degree, and a null degree to the other. Each of this behaviour seems to fit some but not all the situations where gradual arguments conflict.

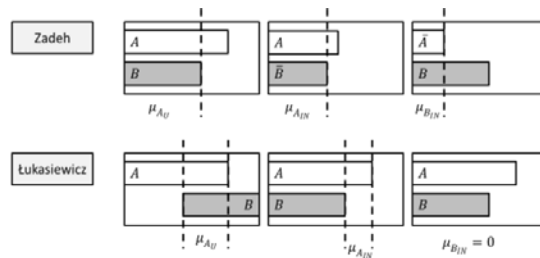


Figure 4. Rebuttals with different multi-valued logic

5.1 Properties

We report some of the properties of MV-*AF* and check if some of the properties identified for the probabilistic case hold in this setting. Table 2 summarizes the properties investigated. The first column refers to the case of probabilistic argumentation while columns *Z*, *G*, *L* refer to the multi-valued systems considered here. The basic property $\mu_{AIN} \leq \mu_A$ is verified by all the semantics, guaranteeing that the degree of a conclusion can only decrease or remain the same due to the presence of attacking arguments. Properties 3, 4 and 5 link the three quantities μ_{AU} , μ_{AIN} , μ_{AOUT} . Property 3 is not valid any more in the multi-valued case, but a weaker form (property 4 and 5) is valid respectively from *L*, *G* and for *Z*, *G* and *L*. Property 8 (accrual of attacks) has been illustrated in example 6.

Property 6 and 7 are the *rationality* and *coherency* properties of *PAF* introduced in [7]. Both of them are valid in MV-*AF*, confirming the validity of our framework w.r.t. accepted state-of-the-art principles.

Table 2. Properties of MV-*AF* ($0 < \mu \leq 1$). Column *P* refers to the same property for probabilistic argumentation. The other columns refer to MVAF using Zadeh's (*Z*), Godel's (*G*) and Łukasiewicz's (*L*) systems.

	Property	<i>P</i>	<i>Z</i>	<i>G</i>	<i>L</i>
1	$\mu_{AIN} \leq \mu_A$	+	+	+	+
2	Is $\mu_{AIN} = \mu_A$ possible if argument <i>a</i> is attacked?	-	+	+	+
3	$\mu_{AU} + \mu_{AIN} + \mu_{AOUT} = 1$	+	-	-	-
4	$\mu_{AU} + \mu_{AIN} + \mu_{AOUT} \leq 1$	+	-	+	-
5	$\mu_{AIN} + \mu_{AOUT} \leq 1$	+	+	+	+
6	Rationality: If one attacker <i>b</i> of <i>a</i> has $\mu_{BIN} > 0.5$ then $\mu_{AIN} < 0.5$	+	+	+	+
7	Coherency: $\mu_{AIN} \leq 1 - \mu_{BIN}$	+	-	+	+
8	Accrual of attacks?	+	-	-	+

6. RELATED WORKS

Conceptually, our framework is closer to the work done in the context of probabilistic argumentation frameworks. The idea of merging probabilities and abstract argumentation was first presented by Dung [12], and a more detailed formalization was provided by Li [6], along with the works by Hunter [7] and Thrimm [13]. [6] introduces the notion of constellations approach. An alternative research direction is the epistemic approach ([7] [13]). Here authors assume that there is already an uncertainty measure on the admissibility set of each argument and study which properties this uncertainty measure should satisfy in order to be rational.

Regarding multi-valued argumentation, the paper progresses our preliminary work in [14] and [15], mainly by correcting its flaws and justifying its foundations. Regarding works that explicitly define fuzzy argumentation systems, we should mention the framework by Janssen [16], where fuzzy labels may be interpreted as fuzzy membership to an extension. However, Janssen's approach differs significantly from ours due to the fact that the attack relation that defines the framework is taken to be fuzzy and the conflict-free and admissibility definitions are changed accordingly. In [17] a certitude factor is added to the labels *in*, *out* and *undec*, as we do. The work proposes an *equational* approach to abstract /argumentation, where arguments degrees have to satisfy a set of properties modelled as equations, properties that might not have any link to a multi-valued logic system. On the contrary, our computation of degrees of truth is a more consistent approach exploiting both argumentation semantics and multi-valued logics.

Regarding other works investigating gradualism in argumentation, we mention Pollock's work on degrees of justification [3]. Pollock considers the strengths of arguments as cardinal quantities that can be subtracted. The accrual of arguments is denied and it is the argument with the maximum strength that defines the attack. It is interesting to notice how Pollock's computation is not grounded in any logic systems, but his attack function behaves like our framework using Łukasiewicz's logic, while his accrual behaves like Zadeh's and Godel's logics. The *vs-defence* model, by Cayrol [4], is an extension of abstract argumentation where attacks have a strength associated with them. Argument admissibility status is the result of the comparisons of attack strengths, as we mentioned in example 4 in section 5. However, there is no description of the nature and the computation of such strengths. We also mention [18] that first extended Dung's framework introducing different levels of strength for the attacks. [5] proposed weighted argument systems, where attacks have weights, and such weights might have different interpretations: an agent-based voting, or a measure of how many premises of the attacked argument are compromised.

7. CONCLUSIONS

In this paper we explored how Dung's abstract argumentation framework can be extended to handle arguments built with gradual evidence. We studied some basic properties and provided examples using Godel's, Łukasiewicz's and Zadeh's multi-valued logics. The findings are a contribution to the field of approximate reasoning and they also represent a well-grounded proposal of an argumentation system able to handle gradualism. We believe we have provided a novel synthesis between argumentation semantics and multi-valued logic, providing the theoretical foundation of a framework for reasoning under uncertainty that has both the soundness of argumentation semantics (w.r.t. conflict resolution) and the ability to handle gradual properties proper of multi-valued logics. Regarding the practical applications of our framework, we mention several applications where a decision has to be made using gradual and conflicting information, such as a trust-based decision in a MAS environment ([19] [20]), medical diagnosis ([21] [22] [23] [24]) and cognitive science ([25] [26]).

Several future directions are open. First, the extension of our frameworks to other semantics. Second, further studies have to be carried out in investigating the various multi-value logics proposed here; in particular, the meaning of the degrees of truth computed by each logic and what kind of vagueness each logic system is more suitable to model. Finally, work has to be done in investigating how to handle situations in which probabilistic uncertainty and vagueness coexist.

8. REFERENCES

- [1] P.M. Dung, "On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games", *Artif. Intell.*, vol. 77, no. 2, pp. 321–357, 1995.
- [2] M.W. Caminada and D.M. Gabbay, "A logical account of formal argumentation", *Stud. Logica*, vol. 93, no. 2–3, pp. 109–145, 2009.
- [3] J.L. Pollock, "Defeasible reasoning with variable degrees of justification", *Artif. Intell.*, vol. 133, no. 1, pp. 233–282, 2001.
- [4] C. Cayrol, C. Devred, and M.-C. Lagasque-Schiex, "Acceptability semantics accounting for strength of attacks in argumentation", in *ECAI*, 2010, pp. 995–996.
- [5] P.E. Dunne, A. Hunter, P. McBurney, S. Parsons and M. Wooldridge, "Inconsistency tolerance in weighted argument systems", in *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems*, vol. 2, 2009, pp. 851–858.
- [6] H. Li, N. Oren, and T.J. Norman, "Probabilistic argumentation frameworks", in *International Workshop on Theorie and Applications of Formal Argumentation*, 2011.
- [7] A. Hunter, "A probabilistic approach to modelling uncertain logical arguments", *Int. J. Approx. Reason.*, vol. 54, no. 1, pp. 47–81, 2013.
- [8] T. Lukasiewicz and U. Straccia, "Managing uncertainty and vagueness in description logics for the semantic web", *Web Semant. Sci. Serv. Agents World Wide Web*, vol. 6, no. 4, pp. 291–308, 2008.
- [9] D. Dubois and H. Prade, "What are fuzzy rules and how to use them", *Fuzzy Sets Syst.*, vol. 84, no. 2, pp. 169–185, 1996.
- [10] P. M. Dung, R.A. Kowalski and F. Toni, "Assumption-based argumentation", in *Argumentation in Artificial Intelligence*, Springer, 2009, pp. 199–218.
- [11] P. Dondio, "Toward a computational analysis of probabilistic argumentation frameworks", *Cybern. Syst.*, vol. 45, no. 3, pp. 254–278, 2014.
- [12] P.M. Dung and P.M. Thang, "Towards (Probabilistic) Argumentation for Jury-based Dispute Resolution", *COMMA*, vol. 216, pp. 171–182, 2010.
- [13] M. Thimm, "A Probabilistic Semantics for abstract Argumentation", in *ECAI*, 2012, pp. 750–755.
- [14] P. Dondio, "Multi-Valued and Probabilistic Argumentation Frameworks", in *COMMA*, 2014, pp. 253–260.
- [15] P. Dondio, "Multi-valued Argumentation Frameworks", in *International Workshop on Rules and Rule Markup Languages for the Semantic Web*, pp. 142–156, 2014.
- [16] J. Janssen, M. De Cock, and D. Vermeir, "Fuzzy argumentation frameworks", in *Information Processing and Management of Uncertainty in Knowledge-based Systems*, 2008, pp. 513–520.
- [17] D.M. Gabbay, "Equational approach to argumentation networks", *Argum. Comput.*, vol. 3, no. 2–3, pp. 87–142, 2012.
- [18] D.C. Martinez, A.J. Garcia and G.R. Simari, "An abstract argumentation framework with varied-strength attacks", in *Proceedings of the Eleventh International Conference on Principles of Knowledge Representation and Reasoning (KR'08)*, pp. 135–144, 2008.

- [19] P. Dondio and L. Longo, "Computing Trust as a form of Presumptive Reasoning", in *Proceedings of the 2014 IEEE/WIC/ACM International Joint Conferences on Web Intelligence (WI) and Intelligent Agent Technologies (IAT)-Volume 02*, pp. 274–281, 2014.
- [20] P. Dondio and S. Barrett, "Presumptive selection of trust evidence", in *Proceedings of the 6th international joint conference on Autonomous agents and multiagent systems*, p. 166, 2007.
- [21] P. Dondio, *Trust as a form of Defeasible Reasoning*. Trinity College Dublin, 2008.
- [22] L. Longo and P. Dondio, "Defeasible reasoning and argument-based systems in medical fields: An informal overview," in *2014 IEEE 27th International Symposium on Computer-Based Medical Systems*, pp. 376–38, 2014.
- [23] L. Rizzo, P. Dondio, S.J. Delany, and L. Longo, "Modeling Mental Workload Via Rule-Based Expert System: A Comparison with NASA-TLX and Workload Profile", in *IFIP International Conference on Artificial Intelligence Applications and Innovations*, pp. 215–229, 2016.
- [24] P. Dondio, S. Barrett, and S. Weber, "Calculating the trustworthiness of a Wikipedia article using dante methodology", in *IADIS eSociety conference*, Dublin, Ireland, 2006.
- [25] L. Longo, "A defeasible reasoning framework for human mental workload representation and assessment", *Behav. Inf. Technol.*, vol. 34, no. 8, pp. 758–786, 2015.
- [26] P. Dondio and L. Longo, "Trust-based techniques for collective intelligence in social search systems", in *Next Generation Data Technologies for Collective Computational Intelligence*, Springer, pp. 113–135, 2011.
- [27] Dubois, Prade, Smets, *Gradual properties vs. uncertainty: Fuzzy logic vs. possibilistic logic*, Technical report, from http://iridia.ulb.ac.be/~psmets/Gradual_vs_Uncert.pdf (2000).