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Derivatives Pricing with Accelerated Trinomial Trees

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Abstract

Accelerated Trinomial Trees (ATTs) are a derivatives pricing lattice method that circumvent the restrictive time step condition inherent in standard trinomial trees and explicit finite difference methods (FDMs) in which the time step must scale with the square of the spatial step. ATTs consist of L uniform supersteps each of which contains an inner lattice/trinomial tree with N non-uniform subtime steps. Similarly to implicit FDMs, the size of the superstep in ATTs, a function of N, are constrained primarily by accuracy demands. ATTs can price options up to N times faster than standard trinomial trees (explicit FDMs).

ATTs can be interpreted as using risk neutral extended probabilities; extended in the sense that values can lie outside the range $[0, 1]$ on the substep scale but aggregate to probabilities within the range $[0, 1]$ on the superstep scale. Hence it is only strictly at the end of each superstep that a practically meaningful solution may be extracted from the tree. We demonstrate that ATTs with L supersteps are more efficient than competing implicit methods which use L time steps in pricing Black-Scholes American put options and 2-dimensional American basket options. Crucially this performance is achieved using an algorithm that requires only a modest modification of a standard trinomial tree. This is in contrast to implicit FDMs which may be relatively complex in their implementation.

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1 Introduction

Trinomial trees are equivalent to explicit finite difference methods (FDMs) if spatial boundary conditions are applied and the full lattice is populated. Trinomial trees and explicit FDMs suffer from a well-known stability constraint, known as the Courant-Friedrichs-Lewy (CFL) condition, in which the time step must scale with the square of the spatial step. This constraint motivates many users to switch to implicit FDMs which are unconditionally stable but are more complex to implement, especially in multifactor derivative pricing models. The main contribution of this work is to present accelerated trinomial trees (ATTs) which circumvent the CFL condition. We demonstrate that ATTs are as efficient, and sometimes more efficient, in terms of their computation time and accuracy relative to competing numerical methods for the test cases pricing American put options under the Black-Scholes model and two-dimensional American basket options.

Heston and Zhou (2000) and Rubinstein (2000) regarded binomial trees as a special case of trinomial trees with the middle probability set to zero. This paper builds on this observation by considering the equivalence between a one-step binomial tree and a multi-step trinomial tree. We solve for a set of non-uniform time steps that enable us to replicate the stencil of one-step binomial tree with a multi-step trinomial tree with non-uniform time steps. The one-step binomial tree will have a larger spatial step, $2N\Delta x$ as opposed to Δx (where Δx is the spatial step size and N is the number of time steps in the trinomial tree), hence the CFL condition will be less restrictive. However, to obtain option prices on the finer trinomial tree, the solution is damped for the non-uniform time steps in the embedded trinomial tree from their optimal values thereby converting the one-step binomial tree with spatial step $2N\Delta x$ into a multinomial tree which retains the original spatial step Δx. Damping causes most of the probability in the multi-step trinomial (one-step multinomial) tree to lie in the outer nodes with some small probability distribution on the inner nodes. These multi-step trinomial trees (or one-step multinomial trees) are convolved into a set of L supersteps to create ATTs.

ATTs thus consist of L uniform supersteps each of which contains a trinomial tree with N non-uniform subtime steps. The L uniform supersteps in ATTs are analogous to the standard time steps used in implicit FDMs as the time step can be chosen independently of the spatial step. The computations required at each of the N substeps in the inner trinomial tree are somewhat analogous to the calculations required at each time step of an implicit FDM to numerically solve a system of equations directly or indirectly. ATTs can price options up to N times faster than standard trinomial trees with the same spatial step. We derive the conditions required for ATTs to produce convergent option prices. We also demonstrate with numerical experiments that ATTs price options more accurately and are often faster than benchmark implicit FDMs when they are compared using the same number of (super) time steps L.

When the underlying asset has no risk-neutral drift then ATTs are equivalent to an accelerated FDM known as the Super-Time-Stepping (STS) scheme, see Alexiades *et al* (1996); O'Sullivan and O'Sullivan (2011, 2013), that applies to PDEs with symmetric positive definite spatial operators. However ATTs are easily generalised to the pricing of derivatives on assets with small non-zero risk neutral drift (weakly non-symmetric spatial operators). Furthermore, ATTs have a number of advantages over the formal STS scheme: ATTs provide a pedagogical explanation of the STS scheme providing insight into how the scheme circumvents the CFL condition; ATTs inherit the probabilistic interpretation of the standard trinomial tree (albeit with modifications with the introduction of extended probabilities, whose values may lie outside the interval $[0, 1]$ in the subtime steps of the trinomial tree; and ATTs obviate the need for boundary conditions thereby greatly simplifying the analysis (however the whole lattice can be populated if appropriate boundary conditions are used).

The financial economic interpretation of a trinomial tree considers an option price at a given time level on the tree as the discounted risk neutral expectation of the price one step into the future. This interpretation can be applied to ATTs leading to the concept of risk neutral extended probabilities at the substep scale, however the risk neutral probabilities can be interpreted as standard probabilities on the larger superstep scale. Therefore, it is only strictly at the end of each superstep that a solution may be extracted from the tree.

The outline of our article is as follows: in the next section we briefly review previous lattice methods; in section 3 we review the constraint on the time step known as the Courant-Friedrichs-Lewy condition in standard trinomial trees; section 4 introduces ATTs as an approach to overcome the restrictive time step constraint in standard trinomial trees; section 5 discusses how ATTs can be used in derivatives pricing; section 6 discusses some of the technical issues that arise with ATTs; section 7 presents numerical experiments to illustrate the performance of ATTs relative to standard trinomial trees (explicit FDMs) and implicit FDMs; section 8 presents conclusions.

2 Review

Lattice methods were first developed in Parkinson (1977) and Cox *et al* (1979) and the application of FDMs to option pricing was first proposed in Brennan and Schwartz (1977) and Brennan and Schwartz (1978). The equivalence of explicit FDMs to trinomial trees and implicit FDMs to multinomial trees where the asset price can jump to any node at the next time level was outlined in Brennan and Schwartz (1978) and Geske and Shastri (1985). Explicit FDMs were used in Hull and White (1990) to express the option price as a discounted risk neutral expectation of its price one step ahead using only three nodes in the expectation thereby emphasising the equivalence between trinomial trees and explicit FDMs. These papers pointed out that non-negative risk neutral probabilities were required to produce convergent numerical methods. This results in a minimum time step that must be used in the trinomial tree to ensure non-negative probabilities. The minimum time step condition for non-negative probabilities is equivalent to the CFL condition that ensures stability in explicit FDMs. The negative probability constraint was also discussed in Boyle (1988) and Kamrad and Ritchken (1991) in their respective parameterizations of a trinomial tree which were derived by means of moment matching. An additional parameter was introduced in both parameterizations to increase the size of the spatial step in proportion to the square root of the time step thus ensuring the transition probabilities remained in the interval $[0, 1]$.

Increasing the efficiency of lattice and tree methods, often in the context of American option pricing, has long been a topic of research. Techniques considered to increase efficiency have included: Richardson extrapolation Breen (1991); modification of the trinomial tree parameters to enhance accuracy and convergence properties, Tian (1993); adding fine high resolution lattice sections to the trinomial tree in regions of critical importance, for example, regions of the tree close to a barrier for barrier options, Figlewski and Gao (1999); simplifying and extending this approach to multi-dimensions, Dai *et al* (2013); pruning the trees so that lower resolution lattice sections are used for the low probability wings in the tree, Baule and Wilkens (2004); expanding the multinomial tree proportional to the square root of time to avoid the unnecessary computations at the low probability wings of the tree, Curran (2001). However none of these methods consider breaking the non-negative probability constraint.

Negative coefficients which may be interpreted as negative probabilities for mathematical convenience, arise in solving two-factor option pricing PDEs in the work of Zvan *et al* (2003). These authors demonstrated that FDMs remain stable and consistent in the presence of negative discretization coefficients. It is conjectured in Haug (2007) that negative probabilities in binomial and trinomial trees may provide additional flexibility to these lattice models. This paper introduces ATTs which result in the use of extended probabilities in the context of a one-factor option pricing PDE.

As previously mentioned when trinomial trees are symmetric then ATTs are equivalent to the STS finite difference scheme. The STS scheme has been successfully applied to the pricing of European and American put options in the one-factor Black-Scholes model in O'Sullivan and O'Sullivan (2011) and in the two-factor Heston model in O'Sullivan and O'Sullivan (2013). While the behaviour of the STS scheme was originally formally established for symmetric operators, see Alexiades *et al* (1996), the implementation of a novel splitting approach to treat non-symmetric operators is described by O'Sullivan and O'Sullivan (2011) for the one-factor Black-Scholes model.

This paper extends this work by introducing a trinomial tree interpretation of the STS scheme and applying the scheme to asymmetric operators, without the use of operator splitting methods.

3 Standard Trinomial Trees

There are a number of different ways to parameterize a trinomial tree by moment matching, eg, Boyle (1988), Kamrad and Ritchken (1991) and Tian (1993). However, we follow the approach taken in Brennan and Schwartz (1978), Geske and Shastri (1985) and Hull and White (1990), by parameterizing the trinomial tree via a discretization of the option price PDE. Parameterization by moment matching and PDE discretization may be extended to more general option pricing models, for example, the stochastic volatility option pricing PDE given in Heston (1993). We note that, in the case of the Heston model, only the latter approach will yield a recombining two-dimensional trinomial tree. As a consequence, discretization of the PDE can sometimes be the least costly of the two approaches in computational terms for multifactor problems.

We proceed by examining ATTs in the context of the Black-Scholes model. The results presented in this paper may be applied to more general diffusion driven processes with time dependent parameters by considering the universal lattice procedure of Chen and Yang (1999). We consider an option price $u(t, S)$ at time t maturing at a later time T written on an underlying stock S which follows geometric Brownian motion (GBM). The Black-Scholes partial differential equation governing the option price under the usual assumptions of perfect markets and no arbitrage opportunities is then given by:

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - ru = 0 \tag{1}$$

where r is the risk-free rate, σ is the instantaneous volatility of the stock price and the option payoff is given by $u(T, S) = h(S)$ for some payoff function $h(\cdot)$. Using Itô's lemma to express the PDE in terms of the log stock price, $x = \ln S$ yields:

$$\frac{\partial u}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0 \tag{2}$$

with $u(T, x) = h(e^x)$. To solve this PDE numerically a grid for t and x is set up with the grid in the time axis denoted by $\{t = t_0, t_1, \ldots, t_L = T\}$ where Δt is a uniform time interval. The grid for the $x-$axis is denoted by $\{x_0, x_0 \pm \Delta x, \ldots, x_0 \pm L\Delta x\}$ where Δx is the uniform spatial step and x_0 is the initial log stock price. The option price $u(t_i, x_j)$ at time t_i and the log stock price x_j is denoted by $u_{i,j}$ for ease of notation. Spatial derivative terms are approximated via second order accurate finite differences. The temporal derivative is approximated with a one-sided first order accurate finite difference. Substituting the finite difference approximations into the PDE in Equation 2 and evaluating the discount term implicitly at time level t_i yields the well-known result:

$$u_{i,j} = \left(\frac{1}{1+r\Delta t}\right)\{q_u u_{i+1,j+1} + q_m u_{i+1,j} + q_d u_{i+1,j-1}\} \tag{3}$$

where

$$q_u = w_u \Delta t, \; q_m = 1 - \frac{\sigma^2}{\Delta x^2}\Delta t, \text{ and } q_d = w_d \Delta t \tag{4}$$

and

$$w_{u,d} = \frac{1}{2}\frac{\sigma^2}{\Delta x^2} \pm \frac{r - \frac{1}{2}\sigma^2}{2\Delta x}.$$

Hence we can express the option price as a discounted risk neutral expectation of its price one step ahead where the q's, under certain constraints to be discussed next, can be interpreted as risk neutral probabilities since $q_u + q_m + q_d = 1$. We note here that in later numerical experiments, the discrete discount factor $1/(1 + r\Delta t)$ in Equation 3 is replaced with the continuously compounded discount factor: $e^{-r\Delta t}$. In order to obtain a convergent trinomial tree, the risk neutral probabilities must remain in the interval $[0, 1]$. This is true when

$$|r - \frac{1}{2}\sigma^2| \leq \frac{\sigma^2}{\Delta x} \tag{5}$$

and the time step satisfies

$$\Delta t \leq \Delta t_{\text{crit}} = \frac{\Delta x^2}{\sigma^2} \tag{6}$$

A modification to the geometry of the branches in the trinomial tree is carried out by Hull and White (1990) to ensure the first condition is satisfied in a trinomial tree approximating a diffusion process for the interest rate r. A sufficiently small time step ensures the second condition is satisfied, see Brennan and Schwartz (1978), Geske and Shastri (1985) and Hull and White (1990), among others.

[Figure 1 about here.]

Under the Black-Scholes model where a stock follows GBM and the interest rate is constant the first condition is more likely to be satisfied. Hence in this article we assume the first condition is met. The remaining condition given by Equation 6 is equivalent to a non-negativity requirement on the middle branch probability. For a fine resolution trinomial tree with a small spatial step, $\Delta x << 1$, this condition may be highly restrictive. As a direct consequence, more complex implicit FDMs are frequently chosen by practitioners over explicit FDMs and trinomial trees, especially in the context of multifactor models. In the next section we introduce a methodology for relaxing the CFL time step constraint whilst retaining the original spatial step.

4 Accelerated Trinomial Trees

In this section we apply ATTs to the Black-Scholes PDE. Consider rolling together N substeps of different sizes, $\Delta t_1, \ldots, \Delta t_N$, such that the branch probabilities at the end of the combined time step $\Delta\tau = \sum_{k=1}^{N} \Delta t_k$ fall in the range $[0, 1]$ but the transition probabilities at each substep are free to fall outside this range. A superstep may be interpreted as an N-order multinomial tree over a time step $\Delta\tau \equiv \sum_{k=1}^{N} \Delta t_k$. We choose the superstep length so that the sum of the supersteps is equal to the maturity of the option, i.e. $L\Delta\tau = T - t$ where L is the number of supersteps.

To aid exposition we consider two substeps. The generalisation to N substeps is direct. For the $N = 2$ case the option price can be written as a discounted risk neutral expectation of its price two steps ahead by applying Equation 3 twice using

$$u_{i,j} = e^{-r\Delta\tau} \sum_{n=-2}^{2} q_n^{(2)} u_{i+2,j+n} \tag{7}$$

where $\Delta\tau = \sum_{k=1}^{2} \Delta t_k$, $q_n^{(2)}$ is the risk neutral probability of moving n nodes from the original node (i, j) to node $(i + 2, j + n)$ for $n = -2, \ldots, 2$ and the superscript (2) signifies that this is a probability evaluated over two substeps and should not be confused with a power index. Henceforth we will refer to these probabilities as SPs (superstep-probabilities). Figure 2 contains an example of an ATT with $N = 2$ substeps and $L = 2$ supersteps.

In order to gain a computational advantage from this procedure, the superstep must be greater than twice the critical explicit time step Δt_{crit}. We want to maximise the size of the superstep subject to the constraint that the SPs are greater than or equal to zero and less than or equal to one. This can be written as a constrained optimization problem:

$$\max_{\Delta t_1, \Delta t_2} \Delta\tau \text{ subject to } 0 \leq q_n^{(2)} \leq 1 \text{ for } n = -2, \ldots, +2. \tag{8}$$

However, we do not require the extended transition probabilities given by Equation 4 inside the superstep, denoted by $q_{u,k}$, $q_{m,k}$ and $q_{d,k}$, lie in the interval $[0, 1]$.

It is instructive to consider the symmetric trinomial tree with $q_{u,k} = q_{d,k} \equiv q_k = w\Delta t_k$ and

$q_{m,k} = 1 - 2w\Delta t_k$ where $w = \frac{1}{2}\frac{\sigma^2}{\Delta x^2}$. In the Black-Scholes model this corresponds to the case where the continuously compounded risk neutral return of the stock is zero with $r - \frac{1}{2}\sigma^2 = 0$.

[Figure 2 about here.]

4.1 Symmetric Trinomial Tree with $N = 2$ Substeps

The solution to the optimization problem in Equation 8 for $N = 2$ is achieved by replicating a one-step binomial tree stencil with a two-step trinomial tree (or equivalently a pentanomial tree) stencil. The symmetric trinomial tree over two substeps will replicate a symmetric binomial tree if and only if the outer node SPs are equal to $\frac{1}{2}$ and the inner node SPs are zero i.e. the solution to the optimization problem in Equation 8 is given by

$$
\begin{aligned}
q_2^{(2)} &= q_{-2}^{(2)} = \frac{1}{2} \\
q_1^{(2)} &= q_0^{(2)} = q_{-1}^{(2)} = 0
\end{aligned}
$$

We shall proceed by seeking the conditions for the set of substep extended probabilities, $q_{u,k}, q_{m,k}$ and $q_{d,k}$, and hence the size of the substeps, Δt_k, to satisfy these relations, for $k = 1, 2$.

Proposition 1 *To replicate a one-step binomial stencil with a two-step trinomial stencil the non-uniform substeps of the trinomial tree must be chosen as follows:*

$$\Delta t_k = \Delta t_{\text{crit}} \left[1 - \zeta_k\right]^{-1}$$

where ζ_k, for $k = 1, 2$, are the roots of the Chebyshev polynomials (of the first kind) of degree 2:

$$\zeta_{1,2} = \pm\frac{1}{\sqrt{2}}$$

We introduce time dependent parameters ζ_k, for $k = 1, 2$, and relate them to the extended probabilities as follows:

$$
\begin{aligned}
q_{u,k} = q_{d,k} = q_k &= \frac{1}{2}\left(\frac{1}{1-\zeta_k}\right) \\
q_{m,k} = 1 - \left(\frac{1}{1-\zeta_k}\right) &= \frac{-\zeta_k}{1-\zeta_k}
\end{aligned}
$$

where $-\infty \le \zeta_k < 1$. Clearly for $0 < \zeta_k < 1$ acceleration is achieved with the consequence that values for $q_{m,k}$ that lie outside the interval $[0, 1]$. Hence, we may define a set of transformed extended probabilities via

$$\tilde{q}_k = q_k\left(1-\zeta_k\right) = \frac{1}{2}$$
$$\tilde{q}_{m,k} = q_{m,k}\left(1-\zeta_k\right) = -\zeta_k$$

with associated transformed SPs

$$\tilde{q}_j^{(2)} = q_j^{(2)} \prod_{k=1}^{2} (1-\zeta_k) \text{ for } j = -2, \ldots, +2$$

Maximum acceleration is achieved when the inner node SPs (and hence the inner node transformed SPs) are equal to zero. The required result immediately follows with

$$\Delta t_1 \approx 3.414 \Delta t_{\text{crit}} \quad \Delta t_2 \approx 0.586 \Delta t_{\text{crit}}$$

using the fact that $\Delta t_{\text{crit}} = \frac{1}{2w}$.

4.1.1 Discussion of Proposition 1

The first substep Δt_1 is superstable in the sense that $\Delta t_1 > \Delta t_{\text{crit}}$ with associated extended probabilities having values $q_{u,1} = q_{d,1} \approx 1.707$ and $q_{m,1} \approx -2.414$, see Khrennikov (1995) and Burgin (2009) for more on extended probabilities. The second substep Δt_2 is substable in the sense that $\Delta t_2 < \Delta t_{\text{crit}}$ with extended probabilities having values $q_{u,2} = q_{d,2} \approx 0.293$ and $q_{m,2} \approx 0.414$.

If the two substeps in the two-step trinomial tree are chosen according to proposition 1 then the superstep size is given by

$$\Delta\tau = \sum_{k=1}^{2} \Delta t_k = 4\Delta t_{\text{crit}}$$

Noting that 2 time steps in a standard trinomial tree, each of length Δt_{crit}, cover time $2\Delta t_{\text{crit}}$ we see that executing a superstep consisting of 2 substeps covers a time interval 2 times longer. Thus the accelerated trinomial tree is 2 times faster than the standard trinomial tree but at the same computational cost.

However at maximal acceleration, since the ATT replicates a single step binomial stencil, the spatial grid of the ATT becomes odd-even decoupled. To recouple the spatial grids we damp the acceleration with the addition of a small positive parameter ν by setting

$$\zeta_{1,2} = -\nu \mp (-1+\nu)\frac{1}{\sqrt{2}}$$

with associated time steps given by

$$\Delta t_{1,2} = \Delta t_{\text{crit}} \left[1 + \nu \pm (-1 + \nu) \frac{1}{\sqrt{2}} \right]^{-1}$$

Acceleration is then controlled by adjusting ν. In the limit $\nu \to 0$ the size of the superstep $\Delta\tau \to 4\Delta t_{\text{crit}}$.

4.2 Symmetric Accelerated Trinomial Tree with N Substeps

In this section we generalise to N substeps in each superstep. We roll together N substeps, $\Delta t_1, \ldots, \Delta t_N$, such that the branch probabilities at the end of the combined time step $\Delta\tau = \sum_{k=1}^{N} \Delta t_k \in [0, 1]$ but the extended probabilities at each substep are not necessarily $\in [0, 1]$. In this N-level multinomial tree the option price is given by

$$u_{i,j} = e^{-\Delta\tau} \sum_{n=-N}^{N} q_n^{(N)} u_{i+N,j+n}$$

where $q_n^{(N)}$ is the risk neutral probability of moving n nodes from the starting node (i, j) to the final node $(i+N, j+n)$ for $n = -N, -N+1, \ldots, +N$ over the N substeps $\Delta t_1, \ldots, \Delta t_N$. Applying the same approach as before we maximise the superstep by setting the SPs of reaching the upper and lower nodes equal to one-half and the SPs of reaching the inner nodes equal to zero.

Proposition 2 *To replicate a one-step binomial stencil with an N-step trinomial stencil the non-uniform substeps of the trinomial tree must be chosen as follows:*

$$\Delta t_k = \Delta t_{\text{crit}} \left[1 - \zeta_k \right]^{-1}$$

where ζ_k are the roots of the Chebyshev polynomials (of the first kind) of degree N:

$$\zeta_k = \cos\left(\frac{2k-1}{N} \frac{\pi}{2} \right)$$

for $k = 1, \ldots, N$[1].

The objective is to prove that if the risk neutral extended probabilities are chosen as follows:

$$q_{u,k} = q_{d,k} = w\Delta t_k, \;\; q_{m,k} = 1 - 2w\Delta t_k$$

[1]The ordering of the substeps is not unique.

where $\Delta t_k = \Delta t_{\text{crit}} \left[1 - \zeta_k\right]^{-1}$ for $k = 1, \ldots, N$ then the SPs will satisfy the following:

$$q_{\pm N}^{(N)} = \frac{1}{2}, \text{ and } q_j^{(N)} = 0$$

for j $= -N + 1, \ldots, N - 1$.

We prove this result by induction. When $N = 1$ the degenerate solution is $\zeta_k = 0$ (where 0 is the root of the Chebyshev polynomial of degree 1) hence $\Delta t_k = \Delta t_{\text{crit}}$ and the trinomial tree becomes a binomial tree. For $N = 2$ the proof of the result is given in proposition 1. Now assume that proposition 2 holds for all $n = 1, 2, \ldots, N - 1$. Hence the SPs and transformed SPs satisfy the following equations:

$$\begin{aligned} q_{\pm n}^{(n)} &= \frac{1}{2}, \; q_j^{(n)} = 0 \\ \Rightarrow \tilde{q}_{\pm n}^{(n)} &= \frac{1}{2^n}, \; \tilde{q}_j^{(n)} = 0 \end{aligned} \tag{9}$$

where $\tilde{q}_j^{(n)} = q_j^{(n)} \prod_{k=1}^{n} (1 - \zeta_k)$ for $j = -n + 1, \ldots, n - 1$ and $n = 1, \ldots, N - 1$. We also assume $q_j^{(n)} = \tilde{q}_j^{(n)} = 0$ when the node j is outside the range of the trinomial tree at a particular time level t_n i.e. when $j > n$ or $j < -n$ for $n = 1, \ldots, N - 1$.

The following recursive relation exists between the transformed SPs in an n level substep tree and the transformed SPs in an $n - 1$ level substep tree:

$$\tilde{q}_j^{(n)} = \frac{1}{2} \left(\tilde{q}_{j-1}^{(n-1)} + \tilde{q}_{j+1}^{(n-1)} \right) - \frac{1}{4} \tilde{q}_j^{(n-2)} \tag{10}$$

for $j = -n, \ldots, n$ and $n = 1, \ldots, N - 1$. If the N-order transformed SPs are constructed using the time steps in proposition 2 they will also satisfy the recursive relation in Equation 10 hence the N-order transformed SPs will satisfy Equation 9 for $n = N$. The resulting SPs replicate a binomial tree since:

$$\begin{aligned} q_j^{(N)} &= \tilde{q}_j^{(N)} \prod_{k=1}^{N} \frac{1}{1 - \zeta_k} = \tilde{q}_j^{(N)} 2^{N-1} \\ &= \frac{1}{2^N} 2^{N-1} = \frac{1}{2} \text{ for } j = \pm N \\ &= 0 \text{ for } j = -N + 1, \ldots, N - 1 \end{aligned}$$

using the fact that $\prod_{k=1}^{N} \frac{1}{1-\zeta_k} = 2^{N-1}$ when ζ_k are chosen to be the roots of the Chebyshev polynomials (of the first kind) of degree N. Solving for the substeps yields the required result.

4.2.1 Discussion of Proposition 2

To recouple the grid damping is introduced via a small positive parameter ν and the subtime steps are given by:

$$\Delta t_k = \Delta t_{\text{crit}} \left[1 + \nu + (-1+\nu)\cos\left(\frac{2k-1}{N}\frac{\pi}{2} \right) \right]^{-1} \tag{11}$$

for $k = 1, \ldots, N$. This is exactly the same prescription given in Alexiades *et al* (1996) for the substeps in the STS finite difference scheme. It can be shown that the superstep is equal to

$$\Delta\tau = \sum_{k=1}^{N} \Delta t_k = \Delta t_{\text{crit}} \frac{N}{2\sqrt{\nu}} \left(\frac{(1+\sqrt{\nu})^{2N} - (1-\sqrt{\nu})^{2N}}{(1+\sqrt{\nu})^{2N} + (1-\sqrt{\nu})^{2N}} \right)$$

which yields

$$\Delta\tau \to N^2 \Delta t_{\text{crit}} \text{ as } \nu \to 0$$

Noting that N time steps in a standard trinomial tree, each of length Δt_{crit}, cover a time $N\Delta t_{\text{crit}}$ we see that executing a superstep consisting of N substeps covers a time interval up to N times longer.

4.3 Asymmetric Accelerated Trinomial Trees

In this section we consider the asymmetric accelerated trinomial trees given by Equations 3 and 4 when $r \neq \frac{1}{2}\sigma^2$. The up and down risk neutral probabilities are no longer equal but the middle probability is the same as before. All three probabilities are given by:

$$\begin{aligned} q_{u,k} &= w_u \Delta t_k = w\,(1+\alpha)\,\Delta t_k \\ q_{d,k} &= w_d \Delta t_k = w\,(1-\alpha)\,\Delta t_k \\ q_{m,k} &= 1 - 2w\Delta t_k, \end{aligned}$$

for $k = 1, \ldots, N$ where

$$w = \frac{1}{2}\frac{\sigma^2}{\Delta x^2}, \text{ and } \alpha = \frac{r - \frac{1}{2}\sigma^2}{\sigma^2}\Delta x \tag{12}$$

In the Black-Scholes model this corresponds to the case where the continuously compounded risk neutral return of the stock is not zero i.e. $r - \frac{1}{2}\sigma^2 \neq 0$.

We apply the same procedure as before by rolling together N substeps, $\Delta t_1, \ldots, \Delta t_N$, such that the branch probabilities at the end of the combined time step $\Delta\tau = \sum_{k=1}^{N} \Delta t_k \in [0, 1]$ but the extended probabilities at each substep are not necessarily $\in [0, 1]$.

Proposition 3 *To replicate a one-step asymmetric binomial stencil with an N-step asymmetric trinomial stencil the non-uniform subtime steps of the trinomial tree must be chosen as follows:*

$$\Delta t_k = \Delta t_{\text{crit}} \left[1 - \gamma\zeta_k\right]^{-1}$$

where $\gamma = \sqrt{1 - \alpha^2}$, $\alpha = \frac{r - 1/2\sigma^2}{\sigma^2}\Delta x$ is a parameter that measures the extent of asymmetry in the trinomial tree, and ζ_k are the roots of the Chebyshev polynomials (of the first kind) of degree N:

$$\zeta_k = \cos\left(\frac{2k-1}{N}\frac{\pi}{2}\right)$$

for $k = 1, \ldots, N$.

This is proved by induction using a recursive relation, analogous to that in proposition 2, that takes into account the asymmetry in the trinomial tree.

4.3.1 Discussion of Proposition 3

In the asymmetric tree the superstep probabilities at full acceleration are given by

$$q_{\pm N}^{(N)} = \frac{(1 \pm \alpha)^N}{(1+\alpha)^N + (1-\alpha)^N} \text{ and } q_j^{(N)} = 0 \text{ for } j = -N+1, \ldots, N-1$$

As in the symmetric case, damping is introduced to recouple the spatial grid resulting in the following expression for substep size:

$$\Delta t_k = \Delta t_{\text{crit}} \left[1 + \nu + (-1+\nu)\,\gamma\cos\left(\frac{2k-1}{N}\frac{\pi}{2}\right)\right]^{-1} \tag{13}$$

for $k = 1, \ldots, N$.

Given the constraint in Equation 5 the asymmetric term $\gamma = \sqrt{1-\alpha^2}$ lies in the interval $0 \leq \gamma \leq 1$. When there is no asymmetry in the trinomial tree $\alpha = 0 \Rightarrow \gamma = 1$ thus the solution is identical to the symmetric solution with the approach achieving full acceleration when no damping is applied. The maximum possible level of asymmetry is achieved when $|\alpha| = 1 \Rightarrow \gamma = 0$ which implies no acceleration is achieved relative to the standard trinomial tree. The maximal asymmetric case results in the standard trinomial tree having all of its probability concentrated in the upper or lower node branches and zero probability in the other two branches when $\Delta\tau = \Delta\tau_{\text{crit}}$.

In numerical experiments carried out by O'Sullivan and O'Sullivan (2011) it was found that in weakly asymmetric cases (i.e. cases where the asymmetry term $0 < |\alpha| << 1$) the use of the standard symmetric prescription to construct the substeps, with the addition of some slight damping, works perfectly well for accelerating asymmetric trees. That is you can construct accelerated trees with the following equations for the extended probabilities:

$$q_{u,k} = w(1+\alpha)\Delta t_k, \quad q_{d,k} = w(1-\alpha)\Delta t_k, \quad q_{m,k} = 1 - 2w\Delta t_k$$

where the substeps, Δt_k for $k = 1, \ldots, N$, are chosen according to Equation 11 rather than Equation 13. The cost of using the symmetric substeps in an asymmetric ATT is that the SPs will be negative if used at, or very close to, full acceleration hence the numerical solution will no longer be guaranteed to be stable for every level of damping.

4.3.2 Moment Matching

In this subsection we demonstrate that ATTs used at full acceleration match the first two moments of the log stock price. The log stock price follows an arithmetic Brownian motion with $\mathrm{d}x = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}z$ where $\mathrm{d}z$ is a standard Wiener process. Denote the trinomial tree approximating distribution as δx. The first two moments of the approximating distribution δx in a standard trinomial tree (STT) match the moments of the Brownian motion

$$E_{\mathrm{STT}}[\delta x] = (q_u - q_d)\,\Delta x = 2w\alpha\Delta t\Delta x = (r - \frac{1}{2}\sigma^2)\Delta t$$
$$V_{\mathrm{STT}}[\delta x] = (q_u + q_d)\,\Delta x^2 = 2w\Delta t\Delta x^2 = \sigma^2\Delta t$$

When the STT is at full acceleration then the mean and variance are given by

$$E_{\mathrm{STT}}[\delta x] = (r - \frac{1}{2}\sigma^2)\Delta t_{\mathrm{crit}}$$
$$V_{\mathrm{STT}}[\delta x] = \sigma^2\Delta t_{\mathrm{crit}} = \sigma^2\frac{\Delta x^2}{\sigma^2} = \Delta x^2$$

Hence the mean and variance after N steps of a fully accelerated STT are given by

$$E^N_{\mathrm{STT}}[\delta x] = N(r - \frac{1}{2}\sigma^2)\Delta t_{\mathrm{crit}}$$
$$V^N_{\mathrm{STT}}[\delta x] = N\sigma^2\Delta t_{\mathrm{crit}} = N\Delta x^2$$

Similarly the first two moments of the approximating distribution δx in the ATT match the moments of the underlying Brownian motion to $\mathcal{O}\left(\Delta x^2\right)$ but over the larger superstep. To demonstrate this we consider the mean of the fully accelerated ATT over one superstep:

$$
\begin{aligned}
E_{\mathrm{ATT}}[\delta x] &= N\Delta x \left(q_N^{(N)} - q_{-N}^{(N)}\right) \\
&= N\Delta x w^N \left(\frac{1}{2w}\right)^N \prod_{k=1}^{N} \frac{1}{1-\gamma\zeta_k} \left[(1+\alpha)^N - (1-\alpha)^N\right] \\
&= N\Delta x \frac{(1+\alpha)^N - (1-\alpha)^N}{(1+\alpha)^N + (1-\alpha)^N} \\
&= N^2\alpha\Delta x + \mathcal{O}\left(\Delta x^2\right) \\
&\approx N^2 \left(r - \frac{1}{2}\sigma^2\right) \frac{\Delta x^2}{\sigma^2} \\
&= N^2 \left(r - \frac{1}{2}\sigma^2\right) \Delta t_{\mathrm{crit}}.
\end{aligned}
$$

The above reasoning uses the fact that $\prod_{k=1}^{N} \frac{1}{1-\gamma\zeta_k} = \frac{2^N}{(1+\alpha)^N + (1-\alpha)^N}$ for $0 \le \alpha \le 1$ when ζ_k are the roots of the Chebyshev polynomials of degree N.

The variance of the fully accelerated ATT over one superstep

$$
V_{\mathrm{ATT}}[\delta x] = \left(q_N^{(N)} + q_{-N}^{(N)}\right) N^2\Delta x^2 = N^2\Delta x^2 = \sigma^2\Delta\tau
$$

where $\Delta\tau = N^2\Delta t_{\mathrm{crit}}$. Both the mean and the variance of the N-order ATT after one superstep are N times larger than the mean and variance of an N-step STT in the limit as $\Delta x \to 0$ (and hence $\alpha \to 0$) thereby confirming the acceleration achieved by the ATT. For a non-zero value of Δx the acceleration achieved is given by:

$$
\frac{E_{\mathrm{ATT}}[\delta x]}{E_{\mathrm{STT}}^{N}[\delta x]} = \frac{1}{\alpha} \frac{(1+\alpha)^N - (1-\alpha)^N}{(1+\alpha)^N + (1-\alpha)^N} \to N \text{ as } \alpha \to 0 \tag{14}
$$

It can also be shown that the moments of a damped ATT match the moments of the log stock price but over a slightly reduced superstep however a formal proof is not provided in this article.

4.3.3 Example

Consider an example with $\Delta x = 0.01$, $\sigma = 0.2$, $N = 5, r = 0.05$ hence $\alpha = 0.00375$ and $\Delta t_{\mathrm{crit}} = \frac{\Delta x^2}{\sigma^2} = 0.0025$. Table I depicts the extended probability of reaching a particular node conditional on starting from the root node in the inner substep trinomial tree with damping applied. After 5 substeps the inner trinomial tree extended probabilities regain non-negative values although the outer node probabilities are no longer equal reflecting the positive drift in the stock price. The time covered is given by $\Delta\tau = 0.0151$ which is 24.2063 times the explicit time step of 0.0025. Accounting for the five steps taken this results in an acceleration of a factor of 4.8413 relative to a standard trinomial tree and agrees with the theoretical prediction given in Equation 14. The inclusion of damping results in small but positive probabilities being associated with the inner nine nodes which recouples the spatial grids.

[Table 1 about here.]

5 Option Pricing with ATTs

In ATTs the number of supersteps L can be chosen independently of the number of spatial intervals M. The number of substeps N in each superstep must satisfy $LN^2 > L_{\text{crit}}$ where L_{crit} is the minimum number of time steps needed in an explicit FDM, or equivalently, a standard trinomial tree. The choice of L and N should be made to ensure that L is sufficiently large to recouple all grid points while N is sufficiently large to generate substantive acceleration. Once N and L are chosen by the user the size of the superstep can be solved to fit the required maturity of the option by optimising over the damping parameter ν to ensure that the sum of the substeps in Equation 13 is equal to the superstep length specified by the user.

ATT option prices are computed using backward induction. There are two approached to carrying this procedure out. The first is equivalent to a multinomial tree approach where the substeps are skipped and the option price is calculated according to Equation 7 over $2N+1$-nomial tree steps. Assuming constant coefficients in the PDE this means the SPs are calculated once at the beginning of the numerical scheme and the method will take a total of L supersteps to compute the option price. The approach can also accommodate PDE's coefficients that are piecewise constant however the SPs will need to be re-calculated as the PDE coefficients change. We will refer to this approach as the accelerated multinomial tree (AMT) method. AMTs can be faster than standard trinomial trees by a factor greater than N, the number of substeps used.

The second, and preferred, method in this work, is to calculate the option prices at every time step in the tree, including the levels inside the supersteps, by the recursive application of Equation 3. As previously discussed, for this ATT approach, it is only at the end of each superstep that a meaningful solution may be extracted from the tree to perform calculations such as early exercise decisions or other required adjustments such as barrier or dividend computations. While not as efficient as AMTs in general, the ATT approach is simpler to implement since a narrow local computational stencil is used. As a consequence, extension to multifactor lattices is straightforward thus providing a competitive simple alternative to multifactor implicit PDE solvers.

In practical applications of ATTs, substeps may be chosen according to

$$\Delta t_k = \Delta t_{\text{ref}} \left[1 + \nu + (-1 + \nu)\, \gamma \cos\left(\frac{2k-1}{N} \frac{\pi}{2} \right) \right]^{-1} \tag{15}$$

for $k = 1, \ldots, N$, with a reference time step $\Delta t_{\text{ref}} \lesssim \Delta t_{\text{crit}}$. This may occur when fitting an options time-to-maturity to an integer number of supersteps L with some finite damping parameter value $\nu > 0$, the approach used in this work, or when a non-uniform grid spacing is employed. As a result, analogously to numerical oscillations in FDMs, transient negative SPs may arise which are rapidly damped.

When pricing American options the maximum operator is applied only at the superstep level and since this operator is applied explicitly no projection of the early exercise condition is needed.

To achieve second order temporal accuracy, we use Richardson extrapolation (RE), Richardson (1910). ATTs are run on a high resolution grid with L supersteps and N substeps and on a low resolution grid with $\frac{L}{2}$ supersteps and N substeps and the RE price is set equal to twice the fine grid price minus the coarse grid price. The work load associated with ATT-REs is higher by a factor of 1.5 relative to ATTs that do not use RE. The acceleration of ATT-REs is limited by $\frac{1}{2}LN^2 > L_{\text{crit}}$.

6 Numerical Experiments

In this section the performance of ATTs are examined in terms of timings and accuracies in pricing European and American put options under the one-factor Black-Scholes model. ATTs are tested against explicit FDMs to determine the empirical acceleration achieved. We also compare ATTs to a number of well-known implicit FDMs where the time step is not constrained by the spatial step. In all tests we use ATTs as accelerated lattices with the inclusion of boundary conditions at x_{max} and x_{min}. Boundary conditions are applied in all schemes 5 standard deviations. We have carried out convergence tests to ensure that any error introduced by the boundary conditions did not impact the reported results.

The numerical schemes reported in tests include the explicit FDM (EXP), the accelerated trinomial tree (ATT), and two implicit FDMs. The first implicit method is the indirect (projected) successive over relaxation method for European (American) options, denoted by IMP-(P)SOR for the fully implicit scheme and CN-(P)SOR for the Crank-Nicolson scheme. The successive over relaxation parameter is fixed to 1.9 and the tolerance level is set to 1×10^{-5}, see for example Cryer (1971) and Crank (1984). The second implicit method uses a direct LU decomposition method, see Ikonen and Toivanen (2007a) which is itself a variant of the algorithm introduced by Brennan and Schwartz (1977). This scheme also projects the early exercise condition onto the solution for

American options and is denoted as IMP-(P)LU for the fully implicit scheme and CN-(P)LU for the Crank-Nicolson scheme. The LU method requires that the spatial operator of the discretized PDE is an $\mathbf{M}-$matrix (a diagonally dominant matrix with positive diagonal elements and non-positive off-diagonal elements). Crucially, the projection employed by the Brennan-Schwartz algorithm for American style payoff functions is only effective when the location of the optimal exercise boundary is single valued. However, as noted in Ikonen and Toivanen (2007a), it may be possible to relax this restriction at the expense of efficiency. The latter condition is not satisfied for the two-factor American basket option problems, as considered in this work.

We use both fully implicit FDMs and Crank-Nicolson FDMs as benchmarks as fully implicit FDMs have first order temporal error and second order spatial error so are directly comparable to ATTs. Whereas Crank-Nicolson FDMs, with Rannacher time stepping[2], have second order temporal and spatial errors and so are directly comparable to ATTs using RE extrapolation.

The spatial grid used in the numerical option prices in all cases is given by

$$x = \left\{x_0 - K\sigma\sqrt{T}, x_0 - K\sigma\sqrt{T} + \Delta x, \ldots, x_0 + K\sigma\sqrt{T}\right\}$$

where $K = 5$ and $x_0 = \ln S_0$. The grid spacing Δx is given by

$$\Delta x = \frac{2K\sigma\sqrt{T}}{M}$$

where M is the number of steps in the spatial grid. This results in a minimum number of time steps L_{crit} in the standard trinomial tree (explicit FDM):

$$L_{\text{crit}} = \frac{T}{\Delta t_{\text{crit}}} = \frac{\sigma^2 T}{\Delta x^2} = \left(\frac{M}{2K}\right)^2.$$

Hence the number of substeps used in ATTs must satisfy the stability condition

$$N \geq \frac{1}{\sqrt{L}}\frac{M}{2K}.$$

Assuming the user chooses $L = \mathcal{O}(M)$, the number of substeps should follow

$$N \geq \mathcal{O}\left(\frac{\sqrt{M}}{2K}\right).$$

[2]The first four backward induction time steps use the fully implicit scheme with a time step of $\frac{\Delta t}{2}$ and the remaining $L-4$ time steps use the Crank Nicolson scheme with a time step of Δt.

6.1 Convergence

The first set of numerical experiments test the convergence of ATTs and ATT-REs. These tests use a Black-Scholes model with exercise price $E = 100$, time-to-maturity $T = 1$, interest rate $r = 5\%$ and volatility $\sigma = 20\%$. A total of M option prices $\hat{u}_j$ are output by each numerical method corresponding to log stock prices x_j for $j = 1, \ldots, M$. The maximum absolute error (MAE) and the root mean square error (RMSE) are used as error measures with

$$MAE = \max |\hat{u}_j - u_j|$$

$$RMSE = \sqrt{\frac{1}{M}\sum_{i=j}^{M} (\hat{u}_j - u_j)^2}$$

where u_j is the benchmark option price and $\hat{u}_j$ is the numerical option price for $j = 1, \ldots, M$.

Benchmark European option prices are computed using the Black-Scholes formula. Benchmark American option prices are calculated using a high resolution explicit FDM with $M = 12,800$ and $L = 1,843,200$. A high resolution CN-PLU scheme was also used for benchmark American put prices however the nature of the results did not change significantly and are not presented. The benchmark American option prices are interpolated using a cubic spline interpolation. The numerical schemes are run at increasingly finer resolutions for the (super-) time steps L and the number of spatial steps is given by $M = 4L$. The number of substeps in the ATTs is fixed at $N = 20$ (except when $M = 6,400$ where $N = 30$) and the damping parameter is set to $\nu = 0.0005$, unless otherwise stated.

The results of the convergence tests are depicted in tables II and III. These tables illustrate the considerable speed-up achieved by ATTs relative to standard trinomial trees (EXP) (at the cost of increased error). Column 5 of table II (table III) depict the empirical acceleration achieved by ATTs relative to the EXP method in pricing European (American) options. When pricing European options the empirical acceleration achieved is very similar to the ratio of the total number of time steps taken in each scheme given by $L_{\text{EXP}}/(LN)$. For American options the empirical acceleration achieved by ATTs is typically more than twice the ratio of the time steps. The application of the early exercise constraint at each time step in the EXP method slows the method down considerably relative to the other schemes which require a lower number of time steps for stability.

Comparing the error measures in table II confirms that ATTs are first order schemes whilst ATT-REs are second order schemes. Despite also being first order in time, the error in the explicit scheme (EXP) is close to those of the second order integrations. This is due to the substantially larger number of time steps required to maintain a stable solution resulting in a dominant second order spatial error.

Table III demonstrates that the convergence for American options is less than first (second) order for all first (second) order schemes due to the presence of the early exercise boundary. However the convergence rates of ATTs and ATT-REs are in line with the benchmark implicit methods.

When comparing the computation time from implicit FDMs to ATTs, we are comparing overhead in numerically solving a system of equations (indirectly or directly) at each time step in an implicit method to the time taken to compute the full set of N inner substep values in an ATT superstep. Accuracy is of course an equally significant component of the comparison. The second order schemes are the most efficient hence we focus on these schemes in the following analysis.

ATT-REs are slightly less efficient than CN-LU methods in pricing European options as evidenced by columns 4 and 6 in table II which depict the timings and RMSE errors for all schemes. However columns 4 and 6 in table III highlights that ATT-REs are more efficient at attaining an acquired accuracy level in less time than CN-PLU methods when pricing American put options. Thus the lower computation times of CN-PLU becomes less of an advantage when the early exercise condition is taken into account.

Figures 3(a) and 3(b) plot the RMSE errors versus the number of time steps L and the computation time for European options from table II. Figure 3(a) illustrates clearly that ATTs are first order schemes and ATT-REs are second order schemes. Figure 3(b) demonstrates that, as mentioned above, ATT-REs are comparable to the CN-LU scheme in terms of error versus computation time however the CN-LU scheme is the most efficient scheme for pricing European options.

Figures 3(c) and 3(d) plot the RMSE errors versus the number of time steps L and the computation time for American options from table III. We note that the early exercise boundary reduces the convergence of all second order schemes as is clearly evident from figure 3(c). ATT-REs do however have very similar convergence rates to the other second order schemes tested. These plots emphasise that ATT-REs are the most efficient schemes considered in terms of accuracy versus computation time for pricing American options when the grid is sufficiently dense (on the coarse grid the EXP method is the most efficient).

[Table 2 about here.]

[Table 3 about here.]

[Figure 3 about here.]

6.2 Robustness

In this section the number of supersteps L and substeps N are varied to determine their effect on the performance of ATTs/ATT-REs. Table IV depicts a number of different acceleration settings on a high resolution lattice to demonstrate the flexibility and robustness of ATTs and ATT-REs. In this table the number of spatial steps is fixed at $M = 3,200$ and the damping parameter is fixed at $\nu = 0.0003$. The number of substeps N and the number of supersteps L are varied to produce different acceleration factors (and different error values). As before, benchmark American put option prices are calculated using a high resolution EXP scheme with $M = 6,400$ and $L = 1,843,200$ with resulting prices interpolated using a cubic spline. The IMP-PLU and CN-PLU schemes are used as the benchmark first and second order schemes. The table demonstrates that if RMSE is used as the error measure, ATT-REs are more efficient than the CN-PLU scheme. The table also demonstrates that different acceleration levels can be used so that ATTs/ATT-REs can be readily adjusted.

[Table 4 about here.]

To test the robustness of ATTs and ATT-REs with respect to different American put option parameter values we follow the approach of Broadie and Detemple (1996), Leisen (1998), Baule and Wilkens (2004) and Chan *et al* (2009) for testing numerical methods for option pricing. These performance tests are appropriate for evaluating tree/lattice methods or other numerical methods where only one option price returned by the method is tested against a benchmark price. The test involves generating 200 random American put option parameter sets at which to evaluate the numerical option price and comparing the numerical price obtained to a benchmark price. The random sample of parameter values are chosen as follows:

- the exercise price is fixed at 100;
- the initial stock price is uniformly distributed between 70 and 130;
- time to maturity is uniformly distributed between 0.1 and 1 year with probability 0.75 and uniformly distributed between 1 and 5 years with probability 0.25;
- volatility is uniformly distributed between 10% and 60%;
- interest rate is uniformly distributed between 0% and 10%.

Since we are dealing with only one option from each scheme as opposed to a vector of options the maximum relative error (MRE) and the root mean square relative error (RMSRE) are used as error measures with

$$MRE = \max\left(\frac{|\hat{u}_i - u_i|}{u_i}\right)$$

$$RMSRE = \sqrt{\frac{1}{P}\sum_{i=1}^{P}\left(\frac{\hat{u}_i - u_i}{u_i}\right)^2}$$

where u_i is the benchmark price from the i^{th} parameter setting, $\hat{u}_i$ is the numerical option price from the i^{th} parameter setting and P is the number of different parameter sets used in the error measures. Only benchmark prices greater than or equal to 0.5 are used in the error measures to make relative error values more meaningful. Hence P reduces from 200 to 190 in the tests on American put options. We measure computation time as the number of seconds a numerical scheme takes to complete. The tests were conducted for European and American option prices at three different grid resolutions, however, only results for American put prices are reported in table V for compactness. The results are similar to those obtained in table III with ATT-RE being the most efficient numerical method when accuracy and computation times are taken into account. This illustrates that the performance of ATTs and ATT-REs are not sensitive to choice of American put option parameter values considered.

[Table 5 about here.]

6.3 Basket Options

ATTs are used to price a basket option to emphasize the generality of the method in this section. We consider a two-dimensional (2d) American put option with a payoff based on the minimum of two assets given by $\max(E - \min(S_1, S_2), 0)$ where the stocks follow a bivariate lognormal process with volatilities σ_1, σ_2 and correlation ρ. Symmetric ATTs/ATT-REs are used to price this basket option and damping is applied to ensure the method remains stable.

The benchmark method used is a componentwise splitting projected LU decomposition scheme as presented in Ikonen and Toivanen (2007b). We note that this scheme is found to be the most efficient, in terms of computation time and accuracy, amongst a number of FD schemes evaluated when used to price options under the two-factor Heston model. The 2d Black-Scholes PDE is discretized using a Crank Nicolson scheme with a 7-point computational stencil. The spatial operator of the 2d discretized PDE is decomposed into an x-component (log stock 1), a y-component (log stock 2) and an xy-component (correlation term). Each of these directional components are then solved incrementally using a series of 1d LU decomposition schemes. Furthermore, Strang symmetrization is used to retain a second order scheme after splitting. This involves taking a half time step in the

x-direction, a half time step in the y-direction, a full time step in the xy-direction, a half time step in the y-direction and a half time step in the x-direction. Hence 5 1d directional sweeps are carried out where each sweep invokes a number of 1d LU solvers. Dirichlet boundary conditions are used at the nearfield boundaries $x_{\min}$ and $y_{\min}$ and Von Nuemann boundary conditions are used at the farfield boundaries $x_{\max}$ and $y_{\max}$ $\left(\partial^2 u/\partial^2 z = 0 \text{ for } z \in (x, y, xy)\right)$.

American basket option prices are calculated at 25 reference stock prices $(S_1, S_2) \in (30, 35, \ldots, 50)$ using the following schemes: an explicit scheme (EXP); a first order ATT; a second order ATT (ATT-RE) and a componentwise splitting projected LU decomposition scheme (CS-PLU). Table VI illustrates the basket option prices for each scheme for two different correlation values $\rho = 0.5$ and $\rho = 0.95$ where all schemes were run with a high resolution grid with the number of spatial steps in the x and y-axes given by $M_1 = M_2 = 1028$. It is clear that the prices from all schemes are in good agreement with each other when $\rho = 0.5$ with the exception that the CS-PLU scheme breaks down for $\rho = 0.95$. We note that the LU scheme should strictly only be applied to problems where the early exercise region and the continuation region are separated by a single valued early exercise boundary, however, the early exercise boundary of this basket option is bifurcated.

Table VII depicts the computation times and RMSE errors associated with each scheme for the above parameter set with $\rho = 0.50$. The benchmark scheme is the explicit scheme with $M_1 = M_2 = 1,028$ and $L = 23,250$. Reference American basket option prices from EXP are calculated at the 25 reference stock prices using a 2d spline interpolation. The $RMSE$ errors for each scheme are calculated using these 25 reference prices. ATTs are faster than ATT-REs and CS-PLU at all grid resolutions but have a larger RMSE and a lower order of convergence. ATT-REs are faster than CS-PLU on all but the final grid resolution. The error measures from the ATT-RE and CS-PLU schemes are very similar. However, CS-PLU fails to converge under certain parameter configurations, including for $\rho > 0.6$ with the above parameters. We conclude that ATT-REs are competitive PDE solvers in two-factor option pricing problems and are more robust to parameter variation than alternative methods.

[Table 6 about here.]

[Table 7 about here.]

[Table 8 about here.]

7 Conclusion

This paper introduces accelerated trinomial trees, a novel efficient lattice method for the numerical pricing of derivative securities. The time step can be chosen independently of the spatial step in ATTs for appropriate choices of N and ν. ATTs are shown to be more efficient than selected state-of-the-art implicit methods when pricing one-factor American options. In pricing two-factor American options ATTs show comparable efficiency but are more robust and have a substantially lower complexity in implementation. ATTs inherit the simplicity of trinomial trees whilst achieving high accuracy at low computational cost. It is convenient that the ATT approach inherits the probabilistic interpretation of trinomial trees with the modification of allowing risk neutral extended probabilities. We conclude that when faced with complex numerical pricing problems, ATTs offer a compelling alternative to conventional implicit techniques. Models involving multiple factors, non-uniform meshes, moving boundaries, or meshes which are distributed in parallel over several processors will be particularly amenable to ATTs.

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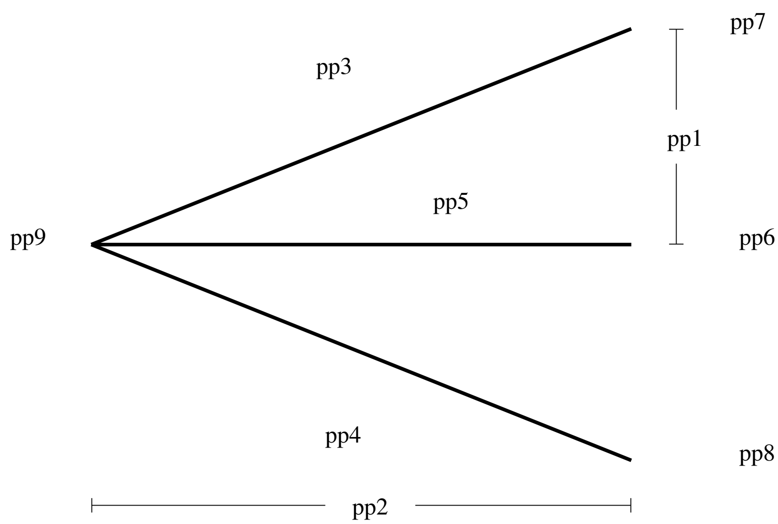


Figure 1: Standard trinomial tree

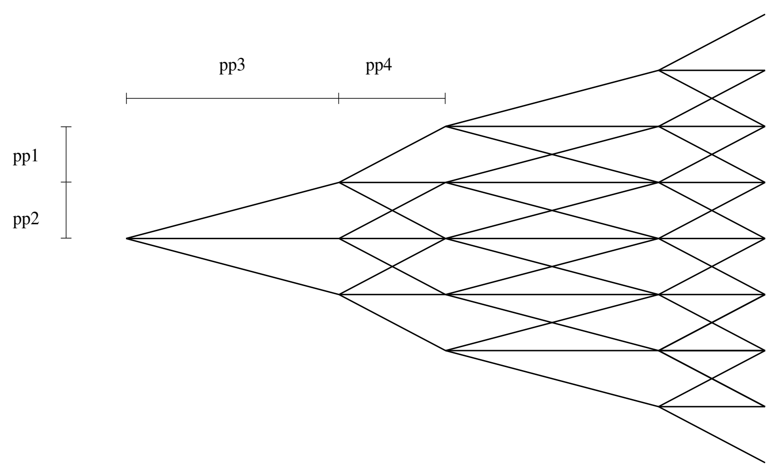
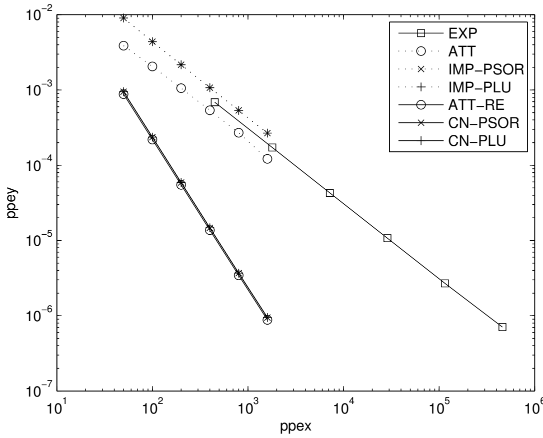
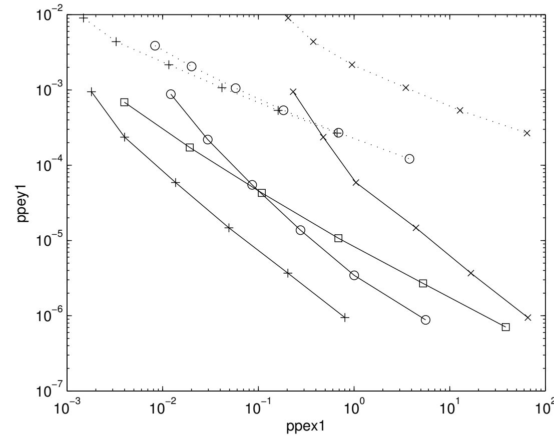


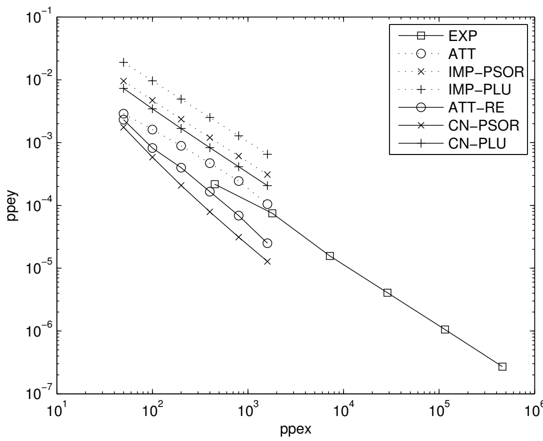
Figure 2: Illustration of the accelerated trinomial tree where the number of non-uniform substeps is $N = 2$ and the number of supersteps is $L = 2$.



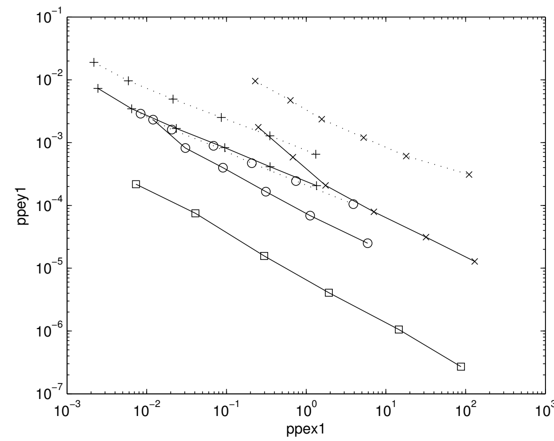
(a) Error versus time steps: European options



(b) Error versus computation time: European options



(c) Error versus time steps: American options



(d) Error versus computation time: American options

Figure 3: Error versus refinement (number of time steps) and computation time for European and American options.

Table I: Inner substep trinomial tree extended probabilities of reaching a particular node conditional on starting from the root node for $N = 5$ and $\nu = 0.001$.

Substep k		1	2	3	4	5
						0.4874
					1.8949	0.0049
				5.9964	-3.5721	0.0048
			11.9599	-18.3317	4.9075	0.0048
		9.8611	-36.5869	31.0632	-5.7360	0.0048
	1	-18.6485	50.1587	-36.3632	6.0182	0.0048
		9.7874	-36.3135	30.8311	-5.6932	0.0048
			11.7818	-18.0588	4.8345	0.0048
				5.8630	-3.4926	0.0047
					1.8389	0.0047
						0.4695
$\sum_1^k \Delta t$	0	0.0123	0.0138	0.0144	0.0148	0.0151

Table II: Performance of ATTs/ATT-REs versus benchmark numerical methods in pricing European put options. M denotes the number of spatial steps and L denotes the number of (super) time steps. In ATTs/ATT-REs the number of substeps is fixed at $N = 20$ (except when $M = 6,400$ where $N = 30$) and the damping parameter is set to $\nu = 0.0005$. **Time** is the computation time (in seconds) it takes to complete the scheme in MATLAB and return a vector of option prices: one option price for each log stock price in the lattice. **Accel** is the acceleration of a scheme given by the ratio of the computation time for the explicit scheme relative to the computation time for the alternative numerical scheme where both schemes use the same number of spatial grid points. **RMSE** is the root mean square error, **RMSE Ratio** is the ratio of the RMSEs, **MAE** is the maximum absolute error and **Ratio MAE** is the ratio of the MAEs. Black-Scholes prices are used as the benchmark European put option prices.

Numerical Method	M	L	**Time**	**Accel**	**RMSE** $\times 10^{-4}$	**Ratio RMSE**	**MAE** $\times 10^{-4}$	**Ratio MAE**
EXP	200	450	0.00	–	6.87	–	17.34	–
	400	1,800	0.02	–	1.72	4.00	4.33	4.00
	800	7,200	0.11	–	0.43	4.00	1.08	4.00
	1,600	28,800	0.69	–	0.11	4.00	0.27	4.00
	3,200	115,200	5.28	–	0.03	3.99	0.07	4.00
	6,400	460,800	38.59	–	0.01	3.81	0.02	2.90
ATT	200	50	0.01	0.48	38.80	–	90.61	–
	400	100	0.02	0.96	20.55	1.89	46.14	1.96
	800	200	0.06	1.88	10.58	1.94	24.49	1.88
	1,600	400	0.18	3.75	5.37	1.97	12.60	1.94
	3,200	800	0.69	7.65	2.71	1.99	6.39	1.97
	6,400	1,600	3.81	10.14	1.21	2.23	2.88	2.22
IMP-SOR	200	50	0.20	0.02	90.52	–	213.42	–
	400	100	0.38	0.05	43.94	2.06	101.00	2.11
	800	200	0.95	0.11	21.65	2.03	49.08	2.06
	1,600	400	3.48	0.20	10.75	2.01	24.18	2.03
	3,200	800	12.84	0.41	5.36	2.01	12.00	2.01
	6,400	1,600	64.42	0.60	2.68	2.00	5.99	2.01
IMP-LU	200	50	0.00	2.68	90.52	–	213.42	–
	400	100	0.00	5.89	43.94	2.06	101.00	2.11
	800	200	0.01	9.39	21.65	2.03	49.08	2.06
	1,600	400	0.04	16.49	10.75	2.01	24.18	2.03
	3,200	800	0.16	32.81	5.36	2.01	12.00	2.01
	6,400	1,600	0.66	58.09	2.67	2.00	5.98	2.01
ATT-RE	200	50	0.01	0.33	8.79	–	24.02	–
	400	100	0.03	0.65	2.20	4.00	5.99	4.01
	800	200	0.09	1.26	0.55	4.00	1.50	4.00
	1,600	400	0.28	2.50	0.14	4.00	0.37	4.00
	3,200	800	1.01	5.25	0.03	3.99	0.09	4.00
	6,400	1,600	5.61	6.87	0.01	3.90	0.02	4.01
CN-SOR	200	50	0.23	0.02	9.50	–	26.20	–
	400	100	0.48	0.04	2.36	4.02	6.52	4.02
	800	200	1.05	0.10	0.59	4.01	1.63	4.01
	1,600	400	4.44	0.15	0.15	4.00	0.41	4.00
	3,200	800	16.65	0.32	0.04	4.00	0.10	4.00
	6,400	1,600	65.72	0.59	0.01	3.89	0.03	4.00
CN-LU	200	50	0.00	2.21	9.50	–	26.20	–
	400	100	0.00	4.81	2.36	4.02	6.52	4.02
	800	200	0.01	7.98	0.59	4.01	1.63	4.01
	1,600	400	0.05	13.96	0.15	4.00	0.41	4.00
	3,200	800	0.20	25.88	0.04	4.00	0.10	4.00
	6,400	1600	0.80	48.05	0.01	3.89	0.03	4.00

Table III: Performance of ATTs/ATT-REs versus benchmark numerical methods in pricing American put options. Benchmark American put option prices are calculated using a high resolution EXP scheme with $M = 12,800$ and $L = 1,843,200$ with resulting prices interpolated using a cubic spline to obtain prices at the same set of log stock prices as the lower resolution numerical schemes.

Numerical Method	M	L	Time	Accel	RMSE $\times 10^{-4}$	Ratio RMSE	MAE $\times 10^{-4}$	Ratio MAE
EXP	200	450	0.01	–	2.18	–	6.42	–
	400	1,800	0.04	–	0.75	2.91	6.42	1.00
	800	7,200	0.30	–	0.16	4.78	0.60	10.73
	1,600	28,800	1.93	–	0.04	3.86	0.19	3.23
	3,200	115,200	14.52	–	0.01	3.86	0.07	2.57
	6,400	460,800	87.54	1.00	0.00	3.90	0.01	6.30
ATT	200	50	0.01	0.87	28.86	–	188.98	–
	400	100	0.02	1.98	16.17	1.79	81.52	2.32
	800	200	0.07	4.31	8.88	1.82	54.69	1.49
	1,600	400	0.21	9.27	4.70	1.89	28.34	1.93
	3,200	800	0.74	19.50	2.45	1.92	13.67	2.07
	6,400	1,600	3.90	22.46	1.05	2.33	7.13	1.92
IMP-PSOR	200	50	0.23	0.03	95.70	–	289.98	–
	400	100	0.63	0.06	47.12	2.03	142.75	2.03
	800	200	1.57	0.19	23.68	1.99	71.70	1.99
	1,600	400	5.25	0.37	11.99	1.97	36.27	1.98
	3,200	800	17.94	0.81	6.09	1.97	18.41	1.97
	6,400	1,600	110.35	0.79	3.10	1.96	9.36	1.97
IMP-PLU	200	50	0.00	3.37	189.88	–	520.22	–
	400	100	0.01	6.93	96.55	1.97	262.95	1.98
	800	200	0.02	13.92	49.39	1.95	133.54	1.97
	1,600	400	0.09	22.33	25.22	1.96	67.83	1.97
	3,200	800	0.35	41.37	12.84	1.96	34.42	1.97
	6,400	1,600	1.32	66.35	6.52	1.97	17.44	1.97
ATT-RE	200	50	0.01	0.61	23.22	–	191.45	–
	400	100	0.03	1.34	8.19	2.83	89.75	2.13
	800	200	0.09	3.29	3.99	2.05	48.94	1.83
	1,600	400	0.31	6.18	1.65	2.41	28.25	1.73
	3,200	800	1.12	12.92	0.69	2.40	11.87	2.38
	6,400	1,600	5.90	14.83	0.25	2.76	6.42	1.85
CN-PSOR	200	50	0.25	0.03	17.55	–	48.94	–
	400	100	0.68	0.06	5.87	2.99	16.15	3.03
	800	200	1.75	0.17	2.08	2.82	5.69	2.84
	1,600	400	7.07	0.27	0.79	2.63	2.15	2.65
	3,200	800	31.85	0.46	0.31	2.53	0.85	2.54
	6,400	1,600	129.76	0.67	0.13	2.45	0.35	2.45
CN-PLU	200	50	0.00	3.00	73.14	–	246.59	–
	400	100	0.01	6.29	34.55	2.12	122.75	2.01
	800	200	0.02	12.63	16.83	2.05	63.45	1.93
	1,600	400	0.10	20.24	8.31	2.03	30.95	2.05
	3,200	800	0.35	41.29	4.14	2.01	15.46	2.00
	6,400	1,600	1.35	64.93	2.07	2.00	7.80	1.98

Table IV: Performance of ATTs/ATT-REs at a number of different acceleration settings versus benchmark numerical methods in pricing American put options. In ATTs/ATT-REs the damping parameter is set to $\nu = 0.0003$ but the number of substeps N and the number of supersteps L are varied to achieve different acceleration values. The number of spatial steps is fixed at $M = 3,200$. Benchmark American put option prices are calculated using a high resolution EXP scheme with $M = 12,800$ and $L = 1,843,200$.

Numerical Method	$\boldsymbol{L}$	$\boldsymbol{N}$	**Time**	**Accel**	**RMSE** $\times\mathbf{10^{-4}}$	**MAE** $\times\mathbf{10^{-4}}$
EXP	115,200	–	8.98	–	0.01	0.07
ATT	600	20	0.38	23.36	3.48	17.43
	700	19	0.43	20.86	3.02	15.46
	800	18	0.47	19.21	2.68	13.51
	900	17	0.50	17.93	2.41	10.92
	1,000	16	0.53	16.96	2.20	10.45
IMP-PLU	600	–	0.18	49.39	16.97	45.56
	700	–	0.21	42.07	14.62	39.21
	800	–	0.24	37.37	12.84	34.42
	900	–	0.27	32.89	11.46	30.69
	1,000	–	0.31	29.24	10.34	27.69
ATT-RE	600	20	0.61	14.71	1.10	16.96
	700	19	0.68	13.20	0.92	15.46
	800	18	0.74	12.10	0.80	13.42
	900	17	0.79	11.32	0.70	11.68
	1,000	16	0.84	10.71	0.64	10.73
CN-PLU	600	–	0.19	46.08	5.50	20.81
	700	–	0.22	40.18	4.72	17.62
	800	–	0.26	34.68	4.14	15.46
	900	–	0.29	31.05	3.68	13.73
	1,000	–	0.32	27.86	3.32	12.41

Table V: Performance of ATTs/ATT-REs versus benchmark numerical methods in pricing American put options at 200 different American put option parameter settings. In ATTs/ATT-REs the number of substeps is fixed at $N = 20$ and the damping parameter is set to $\nu = 0.0005$. **RMSRE** is the root mean square relative error and **MRE** is the maximum relative error. Benchmark American put option prices are calculated using a high resolution EXP scheme with $M = 6,400$ and $L = 460,800$.

Numerical Method	$\boldsymbol{M}$	$\boldsymbol{L}$	**Time**	**Accel**	**RMSRE** $\boldsymbol{\times 10^{-4}}$	**MRE** $\boldsymbol{\times 10^{-4}}$
EXP	800	7,200	0.26	-	0.19	0.60
	1,600	28,800	1.83	-	0.04	0.10
	3,200	115,200	13.05	-	0.01	0.03
ATT	800	200	0.07	3.83	4.68	9.20
	1,600	400	0.19	9.62	2.46	4.73
	3,200	800	0.74	17.60	1.27	2.32
IMP-PSOR	800	200	0.50	0.52	10.76	24.69
	1,600	400	1.49	1.23	5.53	13.10
	3,200	800	8.29	1.57	3.56	9.48
IMP-PLU	800	200	0.02	12.50	20.24	55.07
	1,600	400	0.08	23.75	10.39	28.87
	3,200	800	0.31	42.26	5.30	14.88
ATT-RE	800	200	0.10	2.70	1.20	6.99
	1,600	400	0.29	6.54	0.21	0.82
	3,200	800	1.13	11.62	0.07	0.29
CN-PSOR	800	200	0.65	0.40	0.87	2.49
	1,600	400	2.72	0.69	0.37	1.31
	3,200	800	12.90	1.02	0.15	0.54
CN-PLU	800	200	0.02	11.36	6.74	25.75
	1,600	400	0.09	21.50	3.39	13.14
	3,200	800	0.34	38.28	1.69	6.56

Table VI: American basket put option prices calculated at 25 reference stock price pairs using 2-dimensional EXP, ATT, ATT-RE and CS-PLU schemes on a high resolution grid where the number of spatial steps in both stock price axes are $M_1 = M_2 = 1,028$. The number of time steps in the EXP method is $23,250$ while the other schemes use $L = 160$ time steps. In ATTs/ATT-REs the number of substeps is $N = 20$ and the damping parameter is $\nu = 0.001$. The volatilities of the bivariate lognormal process are: $\sigma_1 = 0.20$ and $\sigma_2 = 0.30$. Correlation $\rho = 0.5$ in the upper panel and $\rho = 0.95$ in the lower panel. The initial stock prices and strike price are given by $S_1 = S_2 = E = 40$, the interest rate is $r = 0.0488$ and the time-to-maturity is $T = 7/12$.

	$\rho = 0.50$	Stock Price S_1				
	Stock Price S_2	30	35	40	45	50
EXP	30	11.5873	10.2877	10.0454	10.0221	10.0200
	35	10.1835	7.2747	6.2080	5.9635	5.9255
	40	10.0000	5.8288	3.8964	3.3091	3.1895
	45	10.0000	5.3120	2.7417	1.8228	1.5988
	50	10.0000	5.1764	2.2544	1.1004	0.7827
ATT	30	11.5880	10.2879	10.0450	10.0213	10.0192
	35	10.1835	7.2763	6.2094	5.9645	5.9264
	40	10.0000	5.8300	3.8988	3.3111	3.1913
	45	10.0000	5.3122	2.7436	1.8245	1.6001
	50	10.0000	5.1762	2.2557	1.1013	0.7831
ATT-RE	30	11.5874	10.2878	10.0460	10.0213	10.0192
	35	10.1836	7.2749	6.2081	5.9636	5.9256
	40	10.0000	5.8290	3.8966	3.3092	3.1896
	45	10.0000	5.3121	2.7418	1.8229	1.5988
	50	10.0000	5.1765	2.2546	1.1005	0.7828
CS-PLU	30	11.5871	10.2874	10.0450	10.0218	10.0196
	35	10.1834	7.2745	6.2076	5.9632	5.9251
	40	10.0000	5.8286	3.8961	3.3088	3.1892
	45	10.0000	5.3117	2.7414	1.8226	1.5986
	50	10.0000	5.1761	2.2542	1.1003	0.7826

	$\rho = 0.95$	Stock Price S_1				
	Stock Price S_2	30	35	40	45	50
EXP	30	10.4877	10.0207	10.0198	10.0198	10.0198
	35	10.0000	6.2094	5.9214	5.9207	5.9207
	40	10.0000	5.1502	3.2466	3.1699	3.1698
	45	10.0000	5.1371	2.1434	1.5641	1.5535
	50	10.0000	5.1371	1.9941	0.8191	0.7088
ATT	30	10.4871	10.0199	10.0190	10.0190	10.0190
	35	10.0000	6.2100	5.9223	5.9216	5.9216
	40	10.0000	5.1499	3.2484	3.1717	3.1716
	45	10.0000	5.1369	2.1445	1.5654	1.5548
	50	10.0000	5.1369	1.9951	0.8196	0.7091
ATT-RE	30	10.4878	10.0199	10.0190	10.0190	10.0190
	35	10.0000	6.2094	5.9214	5.9207	5.9207
	40	10.0000	5.1503	3.2467	3.1700	3.1699
	45	10.0000	5.1372	2.1435	1.5641	1.5536
	50	10.0000	5.1372	1.9942	0.8192	0.7088
CS-PLU	30	10.0078	10.0000	10.0000	10.0000	10.0000
	35	10.0000	5.0008	5.0000	5.0000	5.0000
	40	10.0000	5.0000	0.2053	-0.0000	0.0000
	45	10.0000	5.0000	0.0000	-2.4201	-2.4204
	50	10.0000	5.0000	0.0000	-2.8858	5.2194

Table VII: Performance of ATTs/ATT-REs versus a benchmark numerical method, CS-PLU, in pricing a 2d American basket put option. The number of spatial steps in the x and y-axes are denoted by M_1 and M_2. The number of substeps and supersteps used in the ATT/ATT-RE schemes are denoted by N and L respectively. The damping parameter used in the ATT/ATT-RE schemes is $\nu = 0.001$. Correlation $\rho = 0.5$ with other parameters the same as in table VI. Benchmark American basket put option prices are calculated using a high resolution EXP scheme with $M_1 = M_2 = 1,028$ and $L = 23,250$.

Numerical Method	$M1 = M2$	L	N	**Time**	**RMSE**
EXP	128	361	–	0.14	0.0014
ATT	128	20	15	0.11	0.0083
ATT-RE	128	20	15	0.17	0.0029
CS-PLU	128	20	–	0.76	0.0039
EXP	256	1,442	–	2.56	3.6905×10^{-4}
ATT	256	40	15	1.05	0.0046
ATT-RE	256	40	15	1.57	0.0018
CS-PLU	256	40	–	4.36	0.0014
EXP	512	5,768	–	80.85	9.6819×10^{-5}
ATT	512	80	15	16.87	0.0025
ATT-RE	512	80	15	25.07	6.3804×10^{-4}
CS-PLU	512	80	–	28.47	5.8479×10^{-4}
EXP	1,028	23,250	–	1,370.90	-
ATT	1,028	160	20	185.42	0.0012
ATT-RE	1,028	160	20	280.99	2.8268×10^{-4}
CS-PLU	1,028	160	–	203.81	2.6303×10^{-4}

Table VIII: Performance comparison of C implementations of ATT-RE and CN-OSLU (the operator splitting method considers the early exercise constraint of the LCP through a separate fractional timestep) applied to a 2d American basket put option pricing problem. The number of spatial steps in the x and y-axes are denoted by M_1 and M_2. The number of (super)steps used is denoted by L. The damping parameter used in the ATT-RE scheme is $\nu = 0.001$ with $N = 15$ in all cases. Correlation is set to $\rho = 0.5$ with other parameters the same as in table VI. Benchmark American basket put option prices are calculated using the resepective schemes at high resolutions. The walltime, RMSE, and MAE errors are presented with the errors calculated over a box of size E and 17×11 points centred on the strike price E.

Numerical Method	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$	$\boldsymbol{L}$	**Time**	**RMSE**	**MAE**
ATT-RE	4096	2728	64	300.29	2.75×10^{-4}	1.86×10^{-3}
CS-OSLU				286.69	6.96×10^{-4}	2.39×10^{-3}
	3072	2048	32	84.98	7.35×10^{-4}	4.35×10^{-3}
				95.67	1.50×10^{-3}	3.70×10^{-3}
	2048	1364	16	18.91	1.38×10^{-3}	6.18×10^{-3}
				20.71	4.50×10^{-3}	1.09×10^{-2}
	1536	1024	8	5.98	4.11×10^{-3}	1.27×10^{-2}
				6.65	1.08×10^{-2}	3.08×10^{-2}
	1024	682	4	1.49	1.44×10^{-2}	5.13×10^{-2}
				1.90	2.21×10^{-2}	6.61×10^{-2}
	768	512	2	0.55	4.77×10^{-2}	1.50×10^{-1}
				0.75	3.58×10^{-2}	9.68×10^{-2}