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A Linearised Singularly Perturbed Convection-Diffusion Problem with an Interior Layer

Eugene O'Riordan
Dublin City University, eugene.oriordan@dcu.ie

Jason Quinn
Dublin Institute of Technology, jason.quinn@dit.ie

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E. O'Riordan[1]

School of Mathematical Sciences, Dublin City University, Ireland.

J. Quinn

School of Mathematical Sciences, Dublin Institute of Technology, Ireland.

Abstract

A linear time dependent singularly perturbed convection-diffusion problem is examined. The convective coefficient contains an interior layer (with a hyperbolic tangent profile), which in turn induces an interior layer in the solution. A numerical method consisting of a monotone finite difference operator and a piecewise-uniform Shishkin mesh is constructed and analysed. Neglecting logarithmic factors, first order parameter uniform convergence is established.

Keywords: singularly perturbed, interior layer, convection diffusion, parabolic

1. Introduction

To construct layer adapted meshes (such as the piecewise-uniform Shishkin mesh [3]) for a class of singularly perturbed problems, whose solutions contain boundary layers, it is necessary to identify both the location and the width of any boundary layers present in the solution. In addition to boundary layers, interior layers can also appear in the solutions of singularly perturbed problems. In the context of time dependent problems, an additional issue with interior layers is that the location of the layer can move with time. Here we focus on parabolic problems with interior layers, whose location is approximately known at all time.

Consider singularly perturbed parabolic problems of convection-diffusion type, which take the form: Find u such that

$$-\varepsilon u_{xx} + a u_x + bu + cu_t = f, \quad (x,t) \in (0,1) \times (0,T], \quad b \geq 0, c > 0; \tag{1a}$$

$$0 < \varepsilon \ll 1, \qquad u(0,t), u(1,t), u(x,0) \quad \text{specified.} \tag{1b}$$

Email addresses: `eugene.oriordan@dcu.ie` (E. O'Riordan), `jason.quinn@dit.ie` (J. Quinn)

[1]Corresponding author

In [1, 11], interior layers appeared in the solution of (1), in the special case where the convective coefficient $a(x)$ was assumed to be time independent, discontinuous across a curve $\Gamma_1 := \{(d(t), t) | t \in [0, T], 0 < d(t) < 1\}$ and to have the particular sign pattern $a(x) > 0, x < d(t); a(x) < 0, x > d(t)$. In [11], by mapping this curve Γ_1 to the vertical line $x = d(0)$, a piecewise-uniform Shishkin mesh [3] was constructed to align the fine mesh with this curve. This mesh enabled a parameter-uniform numerical method [3] for problem (1) to be constructed. In [4], interior layers appeared in the solution of (1), in the case where the initial condition $u(x, 0)$, contained it's own interior layer. In the case of [4], the convective coefficient $a(t)$ was assumed to be space independent, smooth and of one sign. The reduced initial condition (set $\varepsilon = 0$) was discontinuous at some point $x = d$ and this discontinuity was transported along the characteristic curve $\Gamma_2 := \{(d(t), t) | t \in [0, T], d'(t) = a(t), d(0) = d.\}$, associated with the reduced hyperbolic problem $av_x + bv + cv_t = f$. Again, a parameter-uniform numerical method (akin to the method analysed in [1]) was shown [4] to be (essentially) first order uniformly convergent. In the current paper, an interior layer appears in the solution of (1) due to the fact that the convective coefficient $a_\varepsilon(x, t)$ is assumed to be smooth, but to contain a layer and to smoothly change from positive to negative values within the domain. In the limiting case of $\varepsilon = 0$, the convective coefficient of the reduced differential equation will be discontinuous. This problem may be viewed as a time dependent version of the ordinary differential equation examined in [9].

Under certain conditions [12] the solution of the quasilinear problem

$$-\varepsilon y_{xx} + yy_x + by + y_t = 0, \quad x \in (0, 1), t > 0, \quad b \geq 0; \tag{2a}$$

$$y(0, t) > 0, \ y(1, t) < 0, \quad y(x, 0) \text{ specified}; \tag{2b}$$

will exhibit an interior layer [5] centered along some curve $\Gamma^* := \{(q(t), t), t > 0\}$, which has a hyperbolic tangent profile. In the case of the corresponding Cauchy problem posed on the unbounded domain $(x, t) \in (-\infty, \infty) \times (0, \infty)$ with a smooth initial condition $y(x, 0) = g(x), x \in (-\infty, \infty)$, there will be an initial phase before the interior layer is fully formed [6]. After this initial phase, the solution always exhibits a sharp interior layer and the location of the center of this layer will vary with time. Our interest is in studying numerical methods that will track the solution, after the formative phase has elapsed. Hence, we wish to consider the behavior of the solution of the boundary/initial value problem (2), when the initial condition already contains an interior layer.

The location of this curve Γ^* (across which the reduced solution is discontinuous) can be estimated using asymptotic expansions [8, 12]. To the left of Γ^*, the solution can be viewed as being the sum of two components v_L, w_L, where the regular component v_L is composed of an asymptotic expansion of the form $v_L = v_0^L + \varepsilon v_1^L + \varepsilon^2 v_2^L + ..$; and v_0^L satisfies the reduced nonlinear first order differential equation (set $\varepsilon = 0$) and $v_L(0, t) = y(0, t), v_L(x, 0) = y(x, 0)$. The regular components v_L, v_R are constructed so that v_L (v_R) satisfies the quasilinear differential equation when $x < d(t)$ ($x > d(t)$) and their partial derivatives (up to a certain order) are bounded independently of ε. However, in general,

$v_L(d(t),t) \neq v_R(d(t),t)$. To the left of Γ^*, the decomposition is also designed so that the singular component w_L satisfies bounds [12] of the following form

$$\Big|\frac{\partial^{i+j} w_L(x,t)}{\partial x^i \partial t^j}\Big| \leq C\varepsilon^{-i} e^{-\theta(d(t)-x)/\varepsilon}, \quad \theta > 0; \quad x < d(t), t > 0.$$

In this paper, we formulate a linearized version of the above quasilinear problem (2). The definition of the linearized problem is motivated by the above decomposition of the solution into regular and singular components.

In §2 we state the continuous problem (3) examined in this paper and impose constraints (4) on the convective coefficient a that mimic that character of the continuous solution itself. These assumptions on a confine the location of the interior layer to an $O(\varepsilon)$ neighbourhood of its initial location, The continuous solution is decomposed into the sum of a discontinuous regular component and a discontinuous interior layer component. Pointwise bounds on both components and on their derivatives are established. In §3, based on the bounds established on the layer component, a piecewise-uniform Shishkin mesh is constructed. In §4, the numerical approximations, generated from using a simple finite difference operator on this layer-adapted mesh, are shown to converge ε–uniformly. Some numerical results are presented and discussed in the final section.

Notation: Throughout this paper C denotes a generic constant which is independent of ε and all mesh parameters. Also $\|\cdot\|$ denotes the pointwise maximum norm, which will be subscripted when the norm is restricted to a subdomain.

2. Continuous problem

Consider the following singularly perturbed linear parabolic problem posed on the domain $\Omega := (0,1) \times (0,T]$

$$\begin{gathered} L_\varepsilon u := (-\varepsilon u_{xx} + (a_\varepsilon + \varepsilon g)u_x + bu + cu_t)(x,t) = f(x,t),\ (x,t) \in \Omega, \\ |g(x,t)| \leq C_1,\ b(x,t) \geqslant \beta \geqslant 0, \quad c(x,t) \geqslant \gamma > 0, \quad (x,t) \in \overline{\Omega}; \end{gathered} \tag{3a}$$

subject to the following boundary and initial conditions

$$u(0,t) = \phi_L(t),\ u(1,t) = \phi_R(t),\ 0 < t \leqslant T; \tag{3b}$$

$$u(x,0) = \phi(x) + C_2 a_\varepsilon(x,0),\ 0 \leqslant x \leqslant 1. \tag{3c}$$

If $C_2 = 0$ then the initial condition is independent of ε; and, on the other hand, if $\phi(x) \equiv 0, C_2 = 1$ then the initial condition can contain an interior layer, which can have the same layer character in space as the solution $u(x,t)$.

Motivated by the properties of the solution to the quasilinear problem (2), we consider problems where $a_\varepsilon \in \mathcal{C}^{4+\gamma}(\overline{\Omega})^2$ and

$$|a_\varepsilon|_k \le C\varepsilon^{-k}, \quad |a_\varepsilon|_{k+\gamma} \le C\varepsilon^{-(k+\gamma)}, \quad \text{for all} \quad k \leqslant 4; \tag{4a}$$

$$\begin{aligned}(d(t)-x)a_\varepsilon(x,t) > 0,\ x \neq d(t), \quad a_\varepsilon(d(t),t) = 0; \quad t > 0; \quad \text{where}\\ 0 < d(t) < 1;\ |d'(t)| \le C_3\varepsilon,\ |d(t)|_{i+\gamma} \le C,\ i = 2,3; \quad t \ge 0;\end{aligned} \tag{4b}$$

$$\begin{aligned}&|a_\varepsilon(x,t)| \geqslant |\alpha_\varepsilon(x,t)|,\ t \ge 0; \quad \text{where}\\ &\alpha_\varepsilon(x,t) := \left\{\begin{array}{ll} \theta(1-e^{-\frac{r}{\varepsilon}(d(t)-x)}), & x \leqslant d(t) \\ -\theta(1-e^{-\frac{r}{\varepsilon}(x-d(t))}), & x > d(t) \end{array}\right., \qquad r \geqslant 2\theta > 0.\end{aligned} \tag{4c}$$

Note that the convective coefficient a_ε depends on the singular perturbation parameter and is both space and time dependent; but the time variation possible is limited by the constraint $|d'(t)| \le C_3\varepsilon$. We introduce the limiting discontinuous convective coefficient, defined for any $t \ge 0$ by

$$a_0(x,t) := \lim_{\varepsilon\to 0} a_\varepsilon(x,t), x \neq d(t); \qquad |a_0(d(t)^\pm,t)| := \lim_{x\to d(t)^\pm} |a_0(x,t)| \ge \theta.$$

The condition $r \geqslant 2\theta > 0$ on the parameters r, θ, ensures that the convective coefficient approaches the value of zero rapidly (relative to the magnitude of the jump 2θ in the reduced coefficient $a_0(x,t)$) from either side of the curve

$$\Gamma := \{(d(t),t) | 0 \le t \le T\}.$$

In relation to the quasilinear problem (2) the convective coefficient a_ε takes the place of y and a_0 may be viewed as the reduced solution of the first order nonlinear hyperbolic problem (with appropriate boundary/initial conditions) either side of Γ^*. Hence, in the case of the linear problem (3), we make the following

---

[2]The space $C^{0+\gamma}(D)$ is the set of all functions that are Hölder continuous of degree γ with respect to the metric $\|\cdot\|_p$, where for all $\mathbf{u} = (u_1,u_2),\ \mathbf{v} = (v_1,v_2) \in \mathbb{R}^2$

$$\|\mathbf{u}-\mathbf{v}\|_p^2 := (u_1-v_1)^2 + |u_2-v_2|.$$

For f to be in $C^{0+\gamma}(D)$ then $f \in C^0(D)$ and the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma,D} := \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in\mathbf{D}} \frac{|f(\mathbf{u})-f(\mathbf{v})|}{\|\mathbf{u}-\mathbf{v}\|_p^\gamma}.$$

The space $C^{n+\gamma}(D)$ is the set of all functions, whose derivatives of order n are Hölder continuous of degree $\gamma > 0$ in the domain D. That is,

$$C^{n+\gamma}(D) := \{z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in C^\gamma(D),\ 0 \le i+2j \le n\}.$$

Also $\|\cdot\|_{n+\gamma}$ and $\lceil\cdot\rceil_{n+\gamma}$ are the associated Hölder norms and semi-norms defined by

$$\|v\|_{n+\gamma} := \sum_{0\le k\le n} |v|_k + \lceil v \rceil_{n+\gamma},\ |v|_k := \sum_{k=i+2j} \Big\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \Big\|,\ \lceil v \rceil_{n+\gamma} := \sum_{i+2j=n} \Big\lceil \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \Big\rceil_{0+\gamma}.$$

additional assumption on the limiting nature of the convective coefficient. For all $i+2j \leqslant 4$, we assume that

$$\left|\frac{\partial^{i+j}}{\partial x^i \partial t^j}(a_\varepsilon - a_0)(x,t)\right| \leqslant C\varepsilon^{-i}(1+\varepsilon^{1-j})e^{-\frac{\theta}{2\varepsilon}|x-d(t)|};\ t \geq 0; \tag{4d}$$

which ensures that $a_\varepsilon \to a_0$ at all points outside of an $O(\varepsilon \ln(1/\varepsilon))$ neighbourhood of the curve Γ. The problem data for problem (3) is assumed to be sufficiently smooth and sufficiently compatible so that $u \in \mathcal{C}^{4+\gamma}(\overline{\Omega})$ and the analysis presented below is valid. In the case where $C_2 = 1$, given that $u \in \mathcal{C}^{4+\gamma}(\overline{\Omega})$ and (4a), in order that problem (3) is indeed a linearized version of the quasilinear problem (2) (in the case where the interior layer has fully formed), it is natural then to make the following further assumption on the initial condition

$$\left|\Big(-\varepsilon\frac{\partial^2}{\partial x^2} + a_\varepsilon(x,0)\frac{\partial}{\partial x}\Big)^j a_\varepsilon(x,0)\right| \leqslant C(1+\varepsilon^{1-j});\ j=0,1. \tag{4e}$$

The differential operator associated with the linear problem (3) satisfies a maximum principle. From this, we deduce bounds on the solution of problem (3),(4).

Lemma 1. *For the solution u of (3),(4) we have the following bounds*

$$\begin{aligned}
\|u\| &\leqslant \frac{\|f\|}{\theta}\Big(1+\frac{T}{\gamma}(\varepsilon\|g\|+\|c\|\|d'\|)\Big) + \max_{\overline{\Omega}\setminus\Omega}|u(x,t)|,\\
\left\|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right\| &\leqslant C\varepsilon^{-(i+j)}, \quad 0 \leq i+2j \leq 4.
\end{aligned}$$

Proof. Note, from (4) and $r \geq 2\theta > 0$, we can easily show that

$$-\varepsilon\frac{\partial \alpha_\varepsilon}{\partial x}(x,t) + \alpha_\varepsilon^2(x,t) \geqslant \theta^2, \quad \forall (x,t) \in \bar{\Omega}. \tag{5}$$

Define the barrier function

$$B_1(x,t) := \frac{M_1}{\theta^2}\left(\int_{d(t)}^x \alpha_\varepsilon(s,t)\ ds + \theta\right) + \frac{\varepsilon M_2}{\gamma}t + \max_{\overline{\Omega}\setminus\Omega}|u(x,t)|.$$

Using (3), (4) and (5) we see that

$$\begin{aligned}
L_\varepsilon(B_1 \pm u) &\geqslant \frac{M_1}{\theta^2}(\alpha_\varepsilon^2 - \varepsilon\frac{\partial \alpha_\varepsilon}{\partial x}) + \varepsilon g\frac{\partial B_1}{\partial x} + c\frac{\partial B_1}{\partial t} - \|f\|\\
&\geqslant M_1 + \varepsilon M_2 - \varepsilon M_1\frac{C_1}{\theta} - \varepsilon\frac{M_1\|c\|C_3}{\theta} - \|f\| \ \geqslant 0.
\end{aligned}$$

This is used to establish the bound on $\|u\|$. To obtain bounds on the derivatives of the solution, we introduce the stretched variables $\zeta := (x-d(0))/\varepsilon, \eta := t/\varepsilon$. Note that if $\tilde{a}_\varepsilon(\zeta,\eta) := a_\varepsilon(x,t)$ then for $n = i+2j$, we have that

$$\|\tilde{a}_\varepsilon\|_{n+\gamma} = \sum_{k=0}^n \varepsilon^k |a_\varepsilon|_k + \varepsilon^{n+\gamma}\lceil a_\varepsilon\rceil_{n+\gamma}.$$

Using this relationship, the bounds (4a) and the *a priori* bounds [7, pg. 320, Theorem 5.2], one deduces the bounds $\|\tilde{u}\|_n \le \|\tilde{u}\|_{n+\gamma} \le C$. From these bounds, one deduces that

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\| \le C\varepsilon^{-(i+j)},\ 0 \le i + 2j \le 4.$$

□

Remark 1. *To avoid having a bound that depends exponentially on T, we choose not to use the standard change of variables $u(x,t) =: v(x,t)e^{\theta t}$. Note that if we assume the strict lower bound $b(x,t) \ge \beta > 0$, then we easily establish that*

$$\|u\| \leqslant \frac{\|f\|}{\beta} + \max_{\overline{\Omega}\setminus\Omega} |u(x,t)|.$$

We next decompose the solution into the sum of a discontinuous regular component v_ε and a discontinuous singular component w_ε. Define the differential operator (which is obviously related to L_ε)

$$\mathcal{L}_\varepsilon u := -\varepsilon u_{xx} + (a_0(x,t) + \varepsilon g(x,t))u_x + b(x,t)u + c(x,t)u_t, \quad x \ne d(t),\ t > 0.$$

Observe that the convective coefficient $(a_0+\varepsilon g)$ is discontinuous across the curve Γ, with $(a_0 + \varepsilon g) > 0,\ x < d(t)$ and $(a_0 + \varepsilon g) < 0,\ x > d(t)$.

Lemma 2. *For sufficiently small ε, there exists functions $r^\pm(t)$ such that the solutions $v^\pm$ of the problems*

$$\begin{gathered}\mathcal{L}_\varepsilon v^- = f(x,t), \quad (x,t) \in \Omega^- := (0, d(t)) \times (0, T],\\ v^-(x,0) = \phi(x) + C_2 a_0(x,0),\ 0 \le x \le d(t),\\ v^-(0,t) = \phi_L(t),\ v^-(d(t),t) = r^-(t),\ 0 < t \le T,\end{gathered}$$

$$\begin{gathered}\mathcal{L}_\varepsilon v^+ = f(x,t), \quad (x,t) \in \Omega^+ := (d(t), 1) \times (0, T],\\ v^+(x,0) = \phi(x) + C_2 a_0(x,0),\ d(t) \leqslant x \leqslant 1,\\ v^+(1,t) = \phi_R(t),\ v^+(d(t),t) = r^+(t),\ 0 < t \le T,\end{gathered}$$

are, respectively, in $\mathcal{C}^{4+\gamma}(\overline{\Omega}^\pm)$ and satisfy the bounds

$$\left\| \frac{\partial^{i+j} v^\pm}{\partial x^i \partial t^j} \right\|_{\overline{\Omega}^\pm} \le C(1 + \varepsilon^{2-(i+j)}), \quad 0 \le i + 2j \le 4.$$

Proof. Consider the extended rectangular domain

$$\Omega^{-,*} := \{(x,t) \in (0, d^*) \times (0,1) | d^* > d(t),\ \forall t > 0\},$$

and $a_0^*, b^*, c^*, g^*, f^*$ are smooth extensions of a_0, b, c, g, f to this extended domain. The first order reduced operator $\mathcal{L}_0^*$ is defined by

$$\mathcal{L}_0^* z := a_0^* z_x + b^* z + c^* z_t.$$

The left regular component $v^{-,*} := v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^*$ is composed of the reduced solution v_0^* and the higher order terms v_1^*, v_2^*, where

$$\mathcal{L}_0^* v_0^* = f^*, \quad (x,t) \in (0, d^*] \times (0, T],$$
$$v_0^*(x,0) = \phi^*(x) + C_2 a_0^*(x,0), \quad x \in [0, d^*]; \quad v_0^*(0,t) = \phi_L(t), \quad t \in [0,T];$$

$$(a_0^* + \varepsilon g^*)(v_1^*)_x + b^* v_1^* + c^*(v_1^*)_t = (v_0^*)_{xx} - g^*(v_0^*)_x, \quad (x,t) \in (0, d^*] \times (0, T],$$
$$v_1^*(x,0) = 0, \quad x \in [0, d^*]; \quad v_1^*(0,t) = 0, \quad t \in [0,T];$$

$$\mathcal{L}_\varepsilon^* v_2^* = (v_1^*)_{xx}, \quad (x,t) \in \Omega^{-,*}, \qquad v_2^* = 0, \text{ on } \overline{\Omega}^{-,*} \setminus \Omega^{-,*}.$$

The bounds on the derivatives of $v^{-,*}$ (and hence v^-) are then easily deduced. An analogous argument is applied over the domain Ω^+ to establish the bounds on the right regular component v^+. □

We now define the interior layer components $w^\pm \in \mathcal{C}^{4+\gamma}(\overline{\Omega}^\pm)$ as

$$w^\pm(x,t) := u(x,t) - v^\pm(x,t), \quad (x,t) \in \overline{\Omega}^\pm,$$

which satisfy the problems

$$L_\varepsilon w^\pm(x,t) = (a_0(x,t) - a_\varepsilon(x,t)) v_x^\pm(x,t), \quad (x,t) \in \Omega^\pm, \tag{6a}$$

$$w^-(0,t) = 0, \quad w^-(d(t),t) = (u - v^-)(d(t),t), \ t > 0, \tag{6b}$$
$$w^-(x,0) = C_2(a_\varepsilon(x,0) - a_0(x,0)), \quad 0 < x < d(t),$$
$$w^+(1,t) = 0, \quad w^+(d(t),t) = (u - v^+)(d(t),t), \ t > 0, \tag{6c}$$
$$w^+(x,0) = C_2(a_\varepsilon(x,0) - a_0(x,0)), \quad d(t) < x < 1.$$

In the next lemma, we show that the partial derivatives of the interior layer components depend inversely on powers of ε within the layer, but are small external to an $O(\varepsilon \ln(1/\varepsilon))$ neighbourhood of Γ.

Lemma 3. *The solutions $w^\pm$ of the problems specified in (6), (4) satisfy the following pointwise bounds*

$$\left| \frac{\partial^{i+j} w^\pm(x,t)}{\partial x^i \partial t^j} \right|_{\overline{\Omega}^\pm} \le C(1 + \varepsilon^{-j}) \varepsilon^{-i} e^{-\frac{\theta}{2\varepsilon}|d(t)-x|}, \quad 0 \le i + 2j \le 4.$$

Proof. We outline below how to establish the bounds in the region Ω^-. The bounds in the region Ω^+ are established in an analogous fashion. We first define the transformation $Y(x,t) = (y,t)$ by $d(t)y = d(0)x$ and

$$Y : \Omega^- \to G^- := (0, d(0)) \times (0, 1]; \quad \bar{w}(y,t) := w^-(x,t);$$

so that the transformed domain is rectangular. The function $\bar{w}$ satisfies the differential equation

$$
\begin{aligned}
\bar{L}_\varepsilon \bar{w} &= (\bar{a}_0(y,t) - \bar{a}_\varepsilon(y,t))(\frac{d(t)}{d(0)})\bar{v}_y^-(y,t), \quad (y,t) \in G^- \\
\text{where} \quad \bar{L}_\varepsilon \bar{w} &:= -\varepsilon \bar{w}_{yy} + A_\varepsilon(y,t)\bar{w}_y + \frac{d^2(t)}{d^2(0)}\big(\bar{b}\bar{w} + \bar{c}\bar{w}_t\big) \\
\text{and} \quad A_\varepsilon(y,t) &:= \frac{d(t)}{d(0)}\Big(\bar{a}_\varepsilon + \varepsilon\bar{g} - \bar{c}d'(t)\frac{y}{d(0)}\Big) = \bar{a}_\varepsilon + O(\varepsilon).
\end{aligned}
$$

In the above, we have used the following

$$
d(0)\bar{u}_y = d(t)u_x, \quad u_t = \bar{u}_t - y\frac{d'(t)}{d(t)}\bar{u}_y \quad \text{and} \quad |d(t) - d(0)| \leq C_2 t\varepsilon.
$$

The singular component $\bar{w}^-(y,t)$ can be further decomposed as follows

$$
\bar{w}^-(y,t) = (g_R(t) - g_R(0))\Psi(y,t) + \bar{w}^-(y,0) + \varepsilon R(y,t),
$$

where $g_R(t) := (u - v^-)(d(t),t)$, and for each value of t, the unit boundary layer function Ψ satisfies

$$
-\varepsilon\Psi_{yy} + A_\varepsilon(y,t)\Psi_y = 0, \quad \Psi(0,t) = 0, \quad \Psi(d(0),t) = 1.
$$

Note that

$$
\Psi(y,t) = \frac{B(y,t)}{B(d(0),t)}, \quad \text{where} \quad B(y,t) := \varepsilon^{-1}\int_{p=0}^{y} e^{-\int_{s=p}^{d(0)} \frac{A_\varepsilon(s,t)}{\varepsilon} ds}\, dp.
$$

Using the strict inequality $((1-\theta)z)^m e^{-z} \leqslant m! e^{-\eta z}$, $0 < \eta < 1$, $z \geqslant 0$ and the lower bound (4c), we have that

$$
\begin{aligned}
Ce^{-\frac{\|a_\varepsilon\|}{\varepsilon}(d(0)-p)} &\leqslant e^{-\frac{1}{\varepsilon}\int_{s=p}^{d(0)} A_\varepsilon(s,t)\ ds} \leqslant Ce^{-\frac{\theta}{\varepsilon}(d(0)-p)} \\
B(y,t) &\leq Ce^{-\theta(d(0)-y)/\varepsilon}(1 - e^{-\theta y/\varepsilon}); \quad B(d(0),t) \geq C > 0; \\
\Big|\frac{\partial B}{\partial t}(y,t)\Big| &\leq \varepsilon^{-1}\Big|\int_{p=0}^{y}(\int_{s=p}^{d(0)} \frac{|A_t|}{\varepsilon} ds)e^{-\int_{s=p}^{d(0)} \frac{A_\varepsilon(s,t)}{\varepsilon} ds}\ dp\Big| \\
&\leq C\varepsilon^{-1}\Big|\int_{p=0}^{y} \frac{d(0)-p}{\varepsilon} e^{-\frac{\theta(d(0)-p)}{\varepsilon}} dp\Big| \leq Ce^{-\eta\theta(d(0)-y)/\varepsilon}.
\end{aligned}
$$

Using these bounds, one can deduce the following bounds

$$
\Big|\frac{\partial^m \Psi(y,t)}{\partial t^m}\Big| \leq C(1 + \varepsilon^{1-m})e^{-\theta(d(0)-y)/2\varepsilon}, \quad 0 \leq m \leq 2; \qquad (y,t) \in \bar{G}.
$$

For the remainder term, $R(y,t) = 0, (y,t) \in \partial G$ and for all $(y,t) \in G$

$$
\begin{aligned}
\varepsilon\bar{L}_\varepsilon R = &- \frac{d^2(t)}{d^2(0)}\big(g_R(t)\bar{b}(y,t) + g_R'(t)\bar{c}(y,t)\big)\Psi(y,t) - \bar{L}_\varepsilon(\bar{w}^-(y,0)) \\
&- \frac{d^2(t)}{d^2(0)} g_R(t)\bar{c}(y,t)\Psi_t(y,t) + (\bar{a}_0 - \bar{a}_\varepsilon)(y,t)\frac{d(t)}{d(0)}\bar{v}_y^-(y,t).
\end{aligned}
$$

Hence, since $|g_R'(t)| \leq C\varepsilon^{-1}$ by Lemma 1, it follows that

$$|\varepsilon \bar{L}_\varepsilon R(y,t)| \leq C\varepsilon^{-1} e^{-\theta(d(0)-y)/2\varepsilon}, \qquad (y,t) \in G.$$

Consider the following barrier function

$$B_2(y,t) := e^{-\frac{1}{2\varepsilon}\int_y^{d(0)} \bar{\alpha}_\varepsilon(s,t)\ ds}, \quad y \leq d(0),\ t \geq 0$$

which satisfies

$$\begin{aligned}
\frac{\partial B_2(y,t)}{\partial t} &= \frac{B_2(y,t)}{2\varepsilon} \int_{d(0)}^{y} \frac{\partial \bar{\alpha}_\varepsilon(s,t)}{\partial t}\ ds \\
&= \frac{-\theta d'(t)}{2d(t)\varepsilon} B_2(y,t) \int_y^{d(0)} \frac{rd(t)(d(0)-s)}{\varepsilon d(0)} e^{-\frac{rd(t)(d(0)-s)}{\varepsilon d(0)}}\ ds \\
\left|\frac{\partial B_2(y,t)}{\partial t}\right| &\leq CB_2(y,t).
\end{aligned}$$

Then, for ε sufficiently small, and since $r \geq \theta > 0$ we have that

$$\begin{aligned}
\bar{L}_\varepsilon B_2 &\geq -\varepsilon \frac{\partial^2 B_2(y,t)}{\partial y^2} + A_\varepsilon(y,t)\frac{\partial B_2(y,t)}{\partial y} + \frac{d^2(t)}{d^2(0)}\bar{c}\frac{\partial B_2(y,t)}{\partial t} \\
&\geq \tfrac{1}{2\varepsilon}(-\varepsilon \frac{\partial \bar{\alpha}_\varepsilon}{\partial y} + \bar{\alpha}_\varepsilon A_\varepsilon - \tfrac{1}{2}\bar{\alpha}_\varepsilon^2 - C\varepsilon)B_2 \geqslant \tfrac{\theta^2}{8\varepsilon} B_2.
\end{aligned}$$

Note also that, for $y \leq d(0),\ t > 0$,

$$e^{-\frac{\theta}{2\varepsilon}(d(0)-y)} \leq B_2(y,t) \leq Ce^{-\frac{\theta}{2\varepsilon}(d(0)-y)}.$$

From this and a maximum principle, we deduce that

$$|\varepsilon R(y,t)|_{\bar{G}} \leq Ce^{-\frac{\theta}{2\varepsilon}(d(0)-y)}.$$

Using the stretched variables $(d(0)-y)/\varepsilon, t/\varepsilon$ and the localized bounds on the derivatives [7, pg. 352, (10.5)] one can deduce the bounds

$$\varepsilon\left|\frac{\partial^{i+j} R(y,t)}{\partial y^i \partial t^j}\right| \leq C\varepsilon^{-(i+j)} e^{-\theta(d(0)-y)/2\varepsilon}, \quad 0 \leq i+2j \leq 4.$$

Hence,

$$\left|\frac{\partial^{i+j} w^-(x,t)}{\partial x^i \partial t^j}\right| \leq C(1+\varepsilon^{-j})\varepsilon^{-i} e^{-\frac{\theta}{2\varepsilon}(d(t)-x)}, \quad 0 \leq i+2j \leq 4.$$

□

In the next result, we sharpen the bounds on the time derivatives of the solution.

Theorem 1. *The solution of problem* (3), (4) *satisfies the bounds*

$$\left|\frac{\partial^{i+j} u}{\partial x^i \partial t^j}(x,t)\right| \leq C\varepsilon^{-i}(1+\varepsilon^{1-j}) e^{-\frac{\theta}{2\varepsilon}|d(t)-x|}, \quad 0 \leq i+2j \leq 4.$$

Proof. From the boundary/initial conditions, the assumptions (4a),(4e) and the fact that $u \in \mathcal{C}^{4+\gamma}(\overline{\Omega})$, we can deduce that

$$\Big\|\frac{\partial^m u}{\partial t^m}\Big\|_{\overline{\Omega}\setminus\Omega} \le C(1+\varepsilon^{1-m}), \quad m=1,2.$$

Let $p := u_t, q := u_{tt}$. By differentiating with respect to time, both sides of the differential equation (3a), we have that

$$\begin{aligned} L_\varepsilon p + c_t p &= f_t - b_t u - (a+\varepsilon g)_t u_x; \\ L_\varepsilon q + (c_t + c_{tt})q &= f_{tt} - (b_t u)_t - (a+\varepsilon g)_{tt} u_x - 2(a+\varepsilon g)_t u_{xt} - b_t u_t. \end{aligned}$$

Hence we have that

$$|(L_\varepsilon + c_t)p(x,t)| \leqslant C(1+\varepsilon^{-1}e^{-\frac{\theta}{2\varepsilon}|d(0)-x|}).$$

Let $\beta_1 < (b+c_t)(x,t), (x,t) \in \Omega$, $p = \hat{p}e^{-\beta_1 t}$ and then

$$-\varepsilon\hat{p}_{xx} + a_\varepsilon\hat{p}_x + (b+c_t-\beta_1)\hat{p} + c\hat{p}_t = e^{\beta_1 t}(f_t - b_t u - (a+\varepsilon g)_t u_x), \ (x,t) \in \Omega.$$

Use the barrier function

$$Ct + Ce^{-\frac{1}{2\varepsilon}\int_x^{d(0)} \alpha_\varepsilon(s,0)\ ds} + \max_{\overline{\Omega}\setminus\Omega} |u_t(x,t)|$$

to deduce the bound $\|u_t\| \le C$. In addition, as in the proof of Lemma 1, we use stretched variables to establish that

$$\Big\|\frac{\partial^{i+j}}{\partial x^i \partial t^j}\Big(\frac{\partial u}{\partial t}\Big)\Big\| \le C\varepsilon^{-(i+j)}, \quad 0 \le i+2j \le 2.$$

An analogous argument is used to establish the bound $|u_{tt}(d(t),t)| \le C\varepsilon^{-1}$. Consider the function $\bar{w}^-(y,t) - \bar{w}^-(y,0)$ for which we have the following bounds

$$\begin{aligned} \Big\|\frac{\partial^m}{\partial t^m}(\bar{w}^-(y,t) - \bar{w}^-(y,0))\Big\|_{\bar{\Omega}^-\setminus\Omega^-} &\le C(1+\varepsilon^{1-m}), \quad m=1,2 \\ \Big|\bar{L}_\varepsilon \frac{\partial^m}{\partial t^m}(\bar{w}^-(y,t) - \bar{w}^-(y,0))\Big|_{\Omega^-} &\le C\varepsilon^{-m}e^{-\frac{\theta}{2\varepsilon}|d(0)-y|}, \quad m=1,2 \end{aligned}$$

Repeat the earlier argument (used to bound u_t and its derivatives) to obtain the bounds

$$\Big|\frac{\partial^{i+j} w^-}{\partial x^i \partial t^j}(x,t)\Big| \le C\varepsilon^{-i}(1+\varepsilon^{1-j})e^{-\frac{\theta}{2\varepsilon}|d(t)-x|}, \quad 0 \le i+2j \le 4.$$

□

Remark 2. *Comparing the bounds in Theorem 1 with the bounds in (4d), we see that the solution u and the convective coefficient $a_\varepsilon + \varepsilon g$ satisfy the same bounds. However, although $a_\varepsilon(d(t),t) = 0$, in general $u(d(t),t) \neq 0$.*

Remark 3. *The explicitly defined function $|\alpha_\varepsilon|$ acts as a pointwise lower bound for the convective coefficient $|a_\varepsilon|$. All the bounds on the solution and its components established in this section can also be derived for any other function α^*_ε (with $|a_\varepsilon| \geq |\alpha^*_\varepsilon|$) that has the following properties:*

$$\alpha^*_\varepsilon \in C^{2+0}(\Omega); \quad \alpha^*_\varepsilon(d(t),t) = 0; \quad (d(t)-x)\alpha^*_\varepsilon(x,t) > 0,\ x \neq d(t);$$
$$\left\| \frac{\partial^j \alpha^*_\varepsilon}{\partial t^j} \right\| \leq C(1+\varepsilon^{1-j}),\ x \neq d(t); \quad j = 1,2;$$

and there exists some $\theta > 0$ such that

$$\|\alpha^*_\varepsilon\| \leq \theta;\ -\varepsilon \frac{\partial \alpha^*_\varepsilon}{\partial x} + 0.5\alpha^*_\varepsilon \alpha^*_\varepsilon \geq \theta^2;\ \Big| \int_x^{d(t)} (\theta - |\alpha^*_\varepsilon(s,t)|)\ ds \Big| \leq C\varepsilon.$$

3. Discrete problem

Given the bounds in Lemma 3 on the layer component, it is natural to refine the mesh in the vicinity of the curve $(d(t),t)$. We examine such a mesh below. Moreover, in the case of problem (3), the trajectory where $a_\varepsilon(x,t) = 0$ is explicitly known, but due to the presence of the function g the point where the convective coefficient changes sign can only be estimated. This is also the case for the quasilinear problem (2), where the location of the inflection point will be, at best, approximated with an asymptotic expansion. With this in mind, we consider the effect of centering the mesh along some vertical line $x = d^*$, located near the curve $(d(t),t)$.

The discrete problem is: Find a mesh function U such that:

$$L^{N,M}_\varepsilon U(x_i,t_j) = f(x_i,t_j), \quad (x_i,t_j) \in \Omega^{N,M}_\varepsilon, \tag{7a}$$
$$U(0,t_j) = u(0,t_j), \quad U(1,t_j) = u(1,t_j), \quad U(x_i,0) = u(x_i,0), \tag{7b}$$
$$L^{N,M}_\varepsilon := -\varepsilon\delta^2_x + (a_\varepsilon + \varepsilon g)D_x + bI + cD_t \tag{7c}$$
$$\delta^2_x Z(x_i,t_j) := \frac{D^+_x Z(x_i,t_j) - D^-_x Z(x_i,t_j)}{(h_{i+1}+h_i)/2}, \quad h_i := x_i - x_{i-1} \tag{7d}$$
$$(AD_x Z)(x_i,t_j) := \tfrac{1}{2}\left((A+|A|)D^-_x + (A-|A|)D^+_x\right) Z(x_i,t_j), \tag{7e}$$

where D^+_x and D^-_x are the standard forward and backward finite difference operators in space, respectively. The fine mesh will be centered at some point d^* (independent of time). We define the piecewise-uniform Shishkin mesh $\Omega^{N,M}_\varepsilon$

as follows

$$|d^* - d(t)| \leqslant C\varepsilon, \quad \forall t \in [0,T], \tag{8a}$$

$$\sigma_1 := \min\{\frac{d^*}{2}, \tfrac{2\varepsilon}{\theta}\ln N\}, \ \sigma_2 := \min\{\frac{1-d^*}{2}, \tfrac{2\varepsilon}{\theta}\ln N\}, \tag{8b}$$

$$H_0 := \tfrac{4}{N}(d^* - \sigma_1), \ h := \tfrac{2}{N}(\sigma_1+\sigma_2), \ H_1 := \tfrac{4}{N}(1-d^*-\sigma_2), \ k = \tfrac{T}{M}, \tag{8c}$$

$$\overline{\Omega}_\varepsilon^{N,M} := \left\{ (x_i,t_j) \left| \begin{array}{cc} x_i = H_0 i, & 0 \leqslant i \leqslant \frac{N}{4}, \\ x_i = x_{\frac{N}{4}} + h(i - \frac{N}{4}), & \frac{N}{4} < i \leqslant \frac{3N}{4}, \\ x_i = x_{\frac{3N}{4}} + H_1(i - \frac{3N}{4}), & \frac{3N}{4} < i \leqslant N, \\ t_j = jk, & 0 \leqslant j \leqslant M, \end{array} \right. \right\}, \tag{8d}$$

$$\Omega_\varepsilon^{N,M} := \overline{\Omega}_\varepsilon^{N,M} \cap \Omega_\varepsilon; \ M = CN; \tag{8e}$$

where the parameter θ in (8b) appears in (4). Note that if $d(t) = d(0), \forall t$ then we can select $d^* = d(0)$. If $d'(t) \neq 0$, then the mesh is not always centered at the point $d(0)$, but we can again choose to set $d^* = d(0)$. However, we will choose d^* such that there exists some $t^* \in [0,T]$ so that $d(t^*) = d^*$. We identify the nearest mesh point to the left of the fixed point $x = d^*$ as x_Q and for each time level t_j, we identify the nearest mesh point to the left of $x = d(t_j)$ as x_{Q_j}. That is,

$$x_Q := \max_i \{x_i | \ x_i \leqslant d^*\} \quad \text{and} \quad x_{Q_j} := \max_i \{x_i | \ x_i \leqslant d(t_j)\}. \tag{9}$$

The finite difference operator (7) is the standard upwind operator and hence it satisfies a discrete comparison principle, which ensures existence of the discrete solution. In the next Lemma, we establish a discrete stability result by using the time dependent barrier function

$$\frac{\|f\|}{\gamma} t_j + \max_{\overline{\Omega}_\varepsilon^{N,M} \setminus \Omega_\varepsilon^{N,M}} |u(x,t)|.$$

Lemma 4. *The solution U of the discrete problem* (7),(8) *satisfies*

$$\|U\|_{\overline{\Omega}_\varepsilon^{N,M}} \leqslant CT$$

4. Error analysis

The discrete solution can be decomposed into the sum $U = V^\pm + W^\pm$, where the discrete regular components satisfy the problems

$$\mathcal{L}_\varepsilon^{N,M} V^- = f, \quad (x_i,t_j) \in \Omega_\varepsilon^{N,M}; \ x_i < x_Q,$$
$$V^-(0,t_j) = u(0,t_j), \quad V^-(x_Q,t_j) = v^-(x_Q,t_j), \quad V^-(x_i,0) = v^-(x_i,0);$$

$$\mathcal{L}_\varepsilon^{N,M} V^+ = f, \quad (x_i,t_j) \in \Omega_\varepsilon^{N,M}; \ x_i > x_Q,$$
$$V^+(x_Q,t_j) = v^+(x_Q,t_j), \quad V^+(1,t_j) = u(1,t_j), \quad V^+(x_i,0) = v^+(x_i,0);$$

where

$$\mathcal{L}_\varepsilon^{N,M} Z(x_i,t_j) := \left(-\varepsilon\delta_x^2 + (a_0+\varepsilon g)D_x^* + b + cD_t^-\right) Z(x_i,t_j)$$

and

$$D_x^* Z(x_i,t_j) := \begin{cases} D_x^- Z(x_i,t_j), & \text{if} \quad (a_\varepsilon+\varepsilon g)(x_i,t_j) \geq 0, \\ D_x^+ Z(x_i,t_j),, & \text{if} \quad (a_\varepsilon+\varepsilon g)(x_i,t_j) < 0 \end{cases}$$

The finite difference operator D_x^* may not correspond to upwinding only within the fine mesh region. Hence, $\mathcal{L}_\varepsilon^{N,M}$ retains the property of discrete stability (as it is an M-matrix for N sufficiently large).

The error analysis argument concentrates on dealing with the case where the mesh is piecewise uniform and

$$\sigma_1 = \sigma_2 = \frac{2\varepsilon}{\theta}\ln N. \tag{10}$$

Using a classical truncation error bound separately either side of x_Q, we derive the following bound on the error in the regular component

$$\left|V^\pm - v^\pm\right| \leq CN^{-1}, \quad (x_i,t_j) \in \Omega_\varepsilon^{N,M}. \tag{11}$$

Note, we assume that the domains $\Omega^\pm$ are sufficiently extended in Lemma 2 so that $v^\pm(x_Q,t)$ are well defined. The discrete interior layer functions are the solutions of the problems

$$L_\varepsilon^{N,M} W^- = (a_0 - a_\varepsilon)(x_i,t_j) D_x^* V^-, \ (x_i,t_j) \in \Omega_\varepsilon^{N,M}, \ x_i < x_Q, \tag{12a}$$
$$W^-(0,t_j) = 0, \quad W^-(x_i,0) = w^-(x_i,0), \tag{12b}$$
$$W^-(x_Q,t_j) = (U - V^-)(x_Q,t_j); \tag{12c}$$

$$L_\varepsilon^{N,M} W^+ = (a_0 - a_\varepsilon)(x_i,t_j) D_x^* V^+, \ (x_i,t_j) \in \Omega_\varepsilon^{N,M}, \ x_i > x_Q, \tag{12d}$$
$$W^+(1,t_j) = 0, \quad W^+(x_i,0) = w^+(x_i,0), \tag{12e}$$
$$W^+(x_Q,t_j) = (U - V^+)(x_Q,t_j). \tag{12f}$$

We proceed to bound the discrete interior layer components outside the fine mesh. The discrete barrier function $\hat{Z}$ defined and analysed in the Appendix is the key component in the proof.

Lemma 5. *Assume (10). If $W^\pm$ are the solutions of* (12) *then for all $t_j \geq 0$ we have*

$$|W^\pm(x_i,t_j)| \leqslant CN^{-1}, \quad \text{if } x_i \in [0, d^* - \sigma_1] \cup [d^* + \sigma_2, 1].$$

Proof. It suffices to confine the discussion to the mesh points $x_i \in [0, x_Q]$ as the argument for $x_i > x_Q$ is analogous. First note that

$$\begin{aligned} |D_x^* V^-(x_i,t_j)| &\leqslant |D_x^* v^-(x_i,t_j)| + |D_x^*(V^- - v^-)(x_i,t_j)| \\ &\leqslant \begin{cases} C, & x_i \leq x_{\frac{N}{4}}, \\ C(1+(\varepsilon\ln N)^{-1}), & x_{\frac{N}{4}} < x_i \leq x_Q. \end{cases} \end{aligned}$$

and using (4d) and (8), for all $t_j \geqslant 0$ we have

$$|(a_\varepsilon - a_0)(x_i,t_j)| \leqslant \begin{cases} CN^{-1}, & x_i \leq x_{\frac{N}{4}}, \\ Ce^{-\frac{\theta}{2\varepsilon}(d^*-x_i)}, & x_{\frac{N}{4}} < x_i < x_Q. \end{cases}$$

Consider the following barrier function to complete the proof

$$B_4(x_i,t_j) := C\hat{Z}(x_i,t_j) + CN^{-1}x_i$$

where $\hat{Z}$ is defined in Lemma 9 in the appendix. Note first that

$$\hat{Z}(x_Q,t_j) \geq e^{-\frac{\theta}{2\varepsilon}|x_{Q_j}-x_Q|} \geq C,\ \hat{Z}(0,t_j) \geq 0,\ \hat{Z}(x_i,0) \geq e^{-\frac{\theta}{2\varepsilon}(x_Q-x_i)}.$$

For all $0 < x_i < x_Q$,

$$\begin{aligned} L_\varepsilon^{N,M}(B_4 \pm W^-)(x_i,t_j) &\geq CL_\varepsilon^{N,M}\hat{Z} + CN^{-1}\alpha_\varepsilon(x_i,t_j) - \|(a_\varepsilon - a_0)D_x^* V^-\| \\ &\geq \begin{cases} \frac{C}{\varepsilon}e^{-\frac{\theta}{2\varepsilon}(d^*-x_i)} - \|(a_\varepsilon - a_0)D_x^* V^-\|, & i > \frac{N}{4}, \\ CN^{-1} - \|(a_\varepsilon - a_0)D_x^- V^-\|, & i \leq \frac{N}{4}, \end{cases} \\ &\geq 0. \end{aligned}$$

Use (b) from Lemma 9 in the appendix to finish. □

The error analysis is completed in the final theorem.

Theorem 2. *Assume $M = CN$. The solutions u and U of the problems (3),(4) and* (7),(8)*, respectively, satisfy the bound*

$$\left\|\bar{U} - u\right\|_{\bar{\Omega}} \leqslant CN^{-1}(\ln N)^2,$$

where $\bar{U}$ is the bilinear interpolant of U over the Shishkin mesh $\overline{\Omega}_\varepsilon^{N,M}$.

Proof. Given the error bound (11), it remains to bound the error in approximating the layer components. We first establish the error bound at the mesh points and also consider the case where (10) applies. From the previous lemma and the pointwise bound on w^-, we have that

$$|(W^- - w^-)(x_i,t_j)| \leq CN^{-1},\ x_i \leq x_{\frac{N}{4}};\ |(W^+ - w^+)(x_i,t_j)| \leq CN^{-1},\ x_i \geq x_{\frac{3N}{4}}.$$

Combine this with the bound (11) to establish the nodal error bound outside the fine mesh. We next bound the nodal error within the fine mesh region.

The truncation error within the interior layer region is bounded as follows

$$\begin{aligned} |L^{N,M}(U-u)(x_i,t_j)| &\leqslant Ch(1 + \tfrac{1}{\varepsilon^2}e^{-\frac{\theta}{2\varepsilon}|d^*-x_i|}) + Ck(1 + \tfrac{1}{\varepsilon}e^{-\frac{\theta}{2\varepsilon}|d^*-x_i|}), \\ &\leqslant CN^{-1} + C\frac{N^{-1}\ln N}{\varepsilon}e^{-\frac{\theta}{2\varepsilon}|d^*-x_i|}, \quad x_{N/4} < x_i < x_{3N/4}; \\ |(U-u)(x_{\frac{N}{4}},t_j)| &\leqslant CN^{-1}, \quad |(U-u)(x_{\frac{3N}{4}},t_j)| \leqslant CN^{-1}. \end{aligned}$$

Complete the proof of the nodal bound in the case of (10), using Lemma 9 from the appendix with the barrier function

$$B_5(x_i,t_j) = C(N^{-1}\ln N)\hat{Z}(x_i,t_j) + CN^{-1}t_j.$$

The proof of the nodal error bound in the case where (10) does not apply, is completed using the above truncation error/barrier function argument across the entire domain, while also noting that $\varepsilon^{-1} \leqslant C\ln N$ in this case. Follow the arguments in [3, §3.5] applied separately over Ω^- and Ω^+ to extend this nodal error bound to the global error bound. □

5. Numerical results

Example 1: Consider the following particular sample problem over the region $[0,1]\times[0,1]$

$$-\varepsilon u_{xx} + (1+t^2)\tanh\left(\tfrac{2}{\varepsilon}(\tfrac{1}{3}-x)\right)u_x + x(1-x)u + u_t = (1+t)\cos(\pi x), \quad (13a)$$

$$u(0,t) = 1+\tanh(t), u(1,t) = 1-\tanh(t), u(x,0) = (x(1-x))^2+1. \quad (13b)$$

Note, we choose $\theta = 1$ and $d^* = 1/3$ in (8) for this example. We estimate the order of convergence using the double mesh principle [3]. The linear interpolants of the numerical solutions on the coarse and fine mesh will be denoted by $\bar{U}^{N,M}$ and $\bar{U}^{2N,2M}$ respectively. We compute the maximum global two-mesh differences $d_\varepsilon^{N,M}$ and the uniform global differences $d^{N,M}$ from

$$d_\varepsilon^{N,M} := \max_{\Omega^{N,M}\cup\Omega^{2N,2M}} \left|(\bar{U}^{N,M} - \bar{U}^{2N,2M})(x_i,t_j)\right|, \quad d^{N,M} := \max_{S_\varepsilon} d_\varepsilon^{N,M},$$

where $S_\varepsilon = \{2^0, 2^{-1}, \dots, 2^{-20}\}$. From these values we calculate the corresponding computed orders of global convergence $q_\varepsilon^{N,M}$ and the computed orders of uniform global convergence $q^{N,M}$ using

$$q_\varepsilon^{N,M} := \log_2\left(d_\varepsilon^{N,M}/d_\varepsilon^{2N,2M}\right), \quad q^{N,M} := \log_2\left(d^{N,M}/d^{2N,2M}\right). \quad (14)$$

The computed orders of uniform convergence for test problem (13) for sample values of N and ε are given in Table 1.

Example 2: In Example 1 the continuous convection coefficient rapidly changes sign within the domain (as $a_\varepsilon(x,t) = (1+t^2)\tanh\left(\frac{2}{\varepsilon}(\frac{1}{3}-x)\right)$). We compare the solution of (13) to the solution of a problem with the same boundary and initial conditions, but with the discontinuous function $a_0(x,t)$ as the convection coefficient. That is, consider the problem

$$\begin{aligned} -\varepsilon z_{xx} + (1+t^2)z_x + x(1-x)z + z_t &= (1+t)\cos(\pi x), \quad x < \tfrac{1}{3}, && (15a)\\ [z_x](\tfrac{1}{3},t) &= 0, && (15b)\\ -\varepsilon z_{xx} - (1+t^2)z_x + x(1-x)z + z_t &= (1+t)\cos(\pi x), \quad x > \tfrac{1}{3}. && (15c) \end{aligned}$$

$q_\varepsilon^{N,M}$						
ε	**N=32**	**N=64**	**N=128**	**N=256**	**N=512**	**N=1024**
$\mathbf{2^{-0}}$	0.90	0.95	0.94	0.86	0.84	0.89
$\mathbf{2^{-2}}$	0.91	0.95	0.97	0.99	0.99	1.00
$\mathbf{2^{-4}}$	0.82	0.86	0.92	0.95	0.98	0.99
$\mathbf{2^{-6}}$	0.58	0.55	0.61	0.71	0.74	0.78
$\mathbf{2^{-8}}$	0.61	0.55	0.63	0.75	0.78	0.82
$\mathbf{2^{-10}}$	0.62	0.55	0.63	0.75	0.78	0.82
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$\mathbf{2^{-20}}$	0.62	0.56	0.63	0.75	0.78	0.82
$\mathbf{q^{N,M}}$	**0.72**	**0.56**	**0.44**	**0.92**	**0.81**	**0.82**

Table 1: Computed rates of convergence, (14), generated from applying the numerical method (7,8) to test problem (13) for sample values of (N,ε).

Table 2 displays the quantities:

$$E_\varepsilon^N := \max_{i,j} |(U-Z)(x_i,t_j)| \qquad \text{and} \qquad E^N := \max_\varepsilon E_\varepsilon^N,$$

where U and Z are the numerical approximations to the solution of (13) and (15) respectively. We observe that as $\varepsilon \to 0$, the solutions to the two problems remain distinct.

E_ε^N							
$\varepsilon\backslash$**N**	**32**	**64**	**128**	**256**	**512**	**1024**	**2048**
$\mathbf{2^{-0}}$	0.075	0.089	0.091	0.095	0.095	0.096	0.096
$\mathbf{2^{-2}}$	0.082	0.061	0.066	0.082	0.083	0.087	0.088
$\mathbf{2^{-4}}$	0.155	0.068	0.085	0.078	0.075	0.080	0.079
$\mathbf{2^{-6}}$	0.012	0.031	0.050	0.079	0.093	0.079	0.077
$\mathbf{2^{-8}}$	0.009	0.029	0.048	0.061	0.069	0.074	0.077
$\mathbf{2^{-10}}$	0.008	0.028	0.048	0.061	0.069	0.074	0.077
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$\mathbf{2^{-40}}$	0.008	0.028	0.048	0.061	0.068	0.074	0.077
$\mathbf{E^N}$	**0.008**	**0.028**	**0.048**	**0.061**	**0.068**	**0.074**	**0.077**

Table 2: Computed differences between the numerical solutions of (13) and (15) for some sample values of (N,ε).

Example 3: Consider the quasilinear problem

$$-\varepsilon u_{xx} + 2uu_x + u_t = 0, \;\; (x,t) \in (0,1) \times (0,t], \tag{16a}$$

$$u(0,t) = u(0,0), u(1,t) = u(1,1), \quad u(x,0) = \tanh\left(\tfrac{0.5-x}{\varepsilon}\right). \tag{16b}$$

Note that this problem has the exact solution $u(x,t) = \tanh\left(\frac{0.5-x}{\varepsilon}\right)$ and, hence, the location of the interior layer does not vary with time.

We examine the numerical performance of the linearized numerical method

$$(-\varepsilon\delta_x^2 + 2Y(x_i,t_{j-1})D_x + D_t^-)Y(x_i,t_j) = 0, \quad (x_i,t_j) \in \Omega_\varepsilon^{N,M}; \tag{17a}$$

$$\begin{aligned} Y(x_i,0) &= \tanh\left(\tfrac{0.5-x_i}{\varepsilon}\right),\ x_i \in [0,1] \\ Y(0,t_j) = Y(0,0), &\quad Y(1,t_j) = Y(1,0),\ t_j > 0; \end{aligned} \tag{17b}$$

where $\Omega_\varepsilon^{N,M}$ is the grid (8b) centered at $d^* = 0.5$ with $\theta = 1$, $T = 1$ and $M = N$. The exact rates of convergence (14) are displayed in Table 3, which indicate parameter-uniform convergence.

			$p_\varepsilon^{N,M}$			
$\varepsilon\backslash$**N**	**32**	**64**	**128**	**256**	**512**	**1024**
$\mathbf{2^{-0}}$	1.11	1.06	1.03	1.02	1.01	1.00
$\mathbf{2^{-1}}$	1.28	1.16	1.09	1.05	1.02	1.01
$\mathbf{2^{-2}}$	1.58	1.20	1.12	1.06	1.03	1.02
$\mathbf{2^{-3}}$	1.30	1.28	1.21	1.13	1.08	1.04
$\mathbf{2^{-4}}$	1.00	1.19	1.30	1.31	1.23	1.14
$\mathbf{2^{-5}}$	0.67	0.76	0.85	1.06	1.32	1.33
$\mathbf{2^{-6}}$	0.67	0.75	0.80	0.86	0.98	1.12
$\mathbf{2^{-7}}$	0.67	0.75	0.80	0.83	0.87	0.98
$\mathbf{2^{-8}}$	0.67	0.75	0.80	0.83	0.85	0.88
$\mathbf{2^{-9}}$	0.67	0.75	0.80	0.83	0.85	0.86
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$\mathbf{2^{-19}}$	0.67	0.75	0.80	0.83	0.85	0.86
$\mathbf{p^{N,M}}$	**0.67**	**0.75**	**0.80**	**0.83**	**0.85**	**0.86**

Table 3: Exact rates of convergence $p_\varepsilon^{N,M}$ computed from the known solution of (16) for sample values of ε and N generated from the numerical solutions of (17).

A computed sample solution of the numerical method (17) is displayed in Figure 1, where the interior layer is evidently fixed in time. Note that the particular linearisation used in (17) is important. Alternative linearisations affect the accuracy of the scheme, which can be seen in Figure 2. Observe how the linearisations indicated by A and B on Figure 17 generate approximate solutions with shocks occuring outside the computational layer region. The linearisation indicated by C, motivated by the finite difference scheme described by Osher in [2], appears to also produce accurate approximations.

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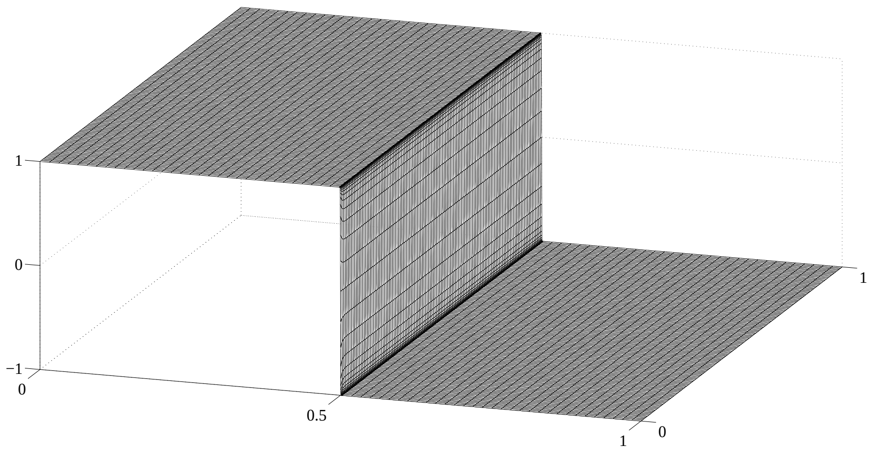

Figure 1: Numerical solution of (17) for $\varepsilon = 2^{-12}$ and $N = 128$.

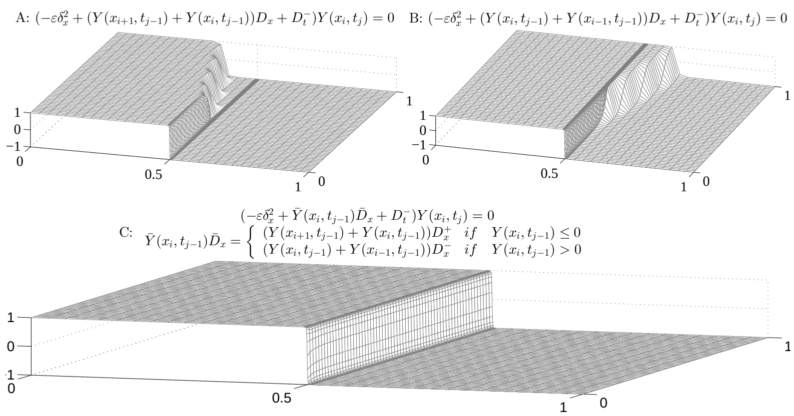

Figure 2: Numerical solutions for $\varepsilon = 2^{-12}$ and $N = 128$ of schemes related to (17) but with alternative linearisations.

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Appendix A. Technical results

We establish a discrete analogue of the inequality (5) and other properties of the convective coefficient function α_ε in the following lemma.

Lemma 6. *For sufficiently large N and for each time level $t_j \geq 0$, the function $\alpha_\varepsilon(x,t)$ defined in (4c) satisfies:*

$$(a)\ |\alpha_\varepsilon(x_i,t_j) - \alpha_\varepsilon(x_{i-1},t_j)| \leq CN^{-1}\ln N, \quad 0 < x_i \leqslant 1,$$

$$(b)\ \varepsilon|D_x^-\alpha_\varepsilon(x_i,t_j) - \frac{\partial}{\partial x}\alpha_\varepsilon(x_i,t_j)| \leq CN^{-1}\ln N, \quad 0 < x_i \leqslant 1$$

$$(c)\ \alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i-1},t_j) - \varepsilon D_x^-\alpha_\varepsilon(x_i,t_j) \geqslant \tfrac{\theta^2}{2}, \quad x_i \in (0,1)$$

$$(d)\ \min\{\alpha_\varepsilon(x_{\frac{N}{4}},t_j), |\alpha_\varepsilon(x_{\frac{3N}{4}},t_j)|\} \geqslant \tfrac{\theta}{\sqrt{2}}, \quad \textit{when } h \neq H_0, H_1.$$

Proof. (a) The argument is split into three subcases: (i) $x_i \leq d(t_j)$, (ii) $x_{i-1} \geq d(t_j)$ and (iii) $d(t_j) \in (x_{i-1}, x_i)$.

(i) Assume $x_i \le d(t_j)$, then $x_i \le d^* + \sigma_2$ and

$$\begin{aligned} |\alpha_\varepsilon(x_i,t_j) - \alpha_\varepsilon(x_{i-1},t_j)| &\le \theta e^{-r(d(t_j)-x_i)/\varepsilon}(1-e^{-rh_i/\varepsilon}) \\ &\le \theta e^{-r(d(t_j)-x_i)/\varepsilon} \min\{1, \frac{rh_i}{\varepsilon}\} \ \le CN^{-1}\ln N, \end{aligned}$$

where we have used the inequalities

$$\begin{aligned} \frac{rh_i}{\varepsilon} &\le CN^{-1}\ln N, \quad \forall h_i \quad \text{if} \quad \varepsilon \ln N \ge C, \\ \frac{rh_i}{\varepsilon} &\le CN^{-1}\ln N, \quad \text{if} \quad x_i \in (d^*-\sigma_1, d^*+\sigma_2,] \\ e^{-r(d(t_j)-x_i)/\varepsilon} &\le e^{-r(d(t_j)-d^*)/\varepsilon} e^{-r(d^*-x_i)/\varepsilon} \le Ce^{-r\sigma_1/\varepsilon}, \quad x_i \in (0, d^*-\sigma_1]. \end{aligned}$$

(ii) Assume $x_{i-1} \ge d(t_j)$, then $x_{i-1} \ge d^* - \sigma_1$ and as in (i),

$$|\alpha_\varepsilon(x_i,t_j) - \alpha_\varepsilon(x_{i-1},t_j)| \le \theta e^{-r(x_{i-1}-d(t_j))/\varepsilon}(1-e^{-rh_i/\varepsilon}) \ \le CN^{-1}\ln N.$$

(iii) Assume $d(t_j) \in (x_{i-1}, x_i)$, then $x_i \in (d^*-\sigma_1, d^*+\sigma_2)$ and

$$|\alpha_\varepsilon(x_i,t_j) - \alpha_\varepsilon(x_{i-1},t_j)| \le 2\theta(1-e^{-rh_i/\varepsilon}) \le CN^{-1}\ln N.$$

(b) The argument is again split into three subcases.
(i) Assume $x_i \le d(t_j)$, then $x_i \le d^* + \sigma_2$ and let $\rho_i := \frac{rh_i}{\varepsilon}$

$$\begin{aligned} \varepsilon|D_x^-\alpha_\varepsilon(x_i,t_j) - \frac{\partial}{\partial x}\alpha_\varepsilon(x_i,t_j)| &= \frac{\theta r}{h_i} e^{-rd(t_j)/\varepsilon} \int_{s=x_{i-1}}^{x_i} e^{rx_i/\varepsilon} - e^{rs/\varepsilon}\ ds \\ &= \theta r e^{-r(d(t_j)-x_i)/\varepsilon}(1 - \frac{1-e^{-\rho_i}}{\rho_i}), \\ &\le C\theta r e^{-r(d^*-x_i)/\varepsilon}\min\{1, \frac{rh_i}{\varepsilon}\} \\ &\le CN^{-1}\ln N. \end{aligned}$$

(ii) Assume $x_{i-1} \ge d(t_j)$, then $x_{i-1} \ge d^* - \sigma_1$ and as in (i),

$$\begin{aligned} \varepsilon|D_x^-\alpha_\varepsilon(x_i,t_j) - \frac{\partial}{\partial x}\alpha_\varepsilon(x_i,t_j)| &= \theta r e^{-r(x_{i-1}-d(t_j))/\varepsilon}(e^{-\rho_i} - \frac{1-e^{-\rho_i}}{\rho_i}) \\ &\le CN^{-1}\ln N. \end{aligned}$$

(iii) Assume $d(t_j) \in (x_{i-1}, x_i)$, then $x_i \in (d^*-\sigma_1, d^*+\sigma_2)$ and

$$\begin{aligned} &\varepsilon|D_x^-\alpha_\varepsilon(x_i,t_j) - \frac{\partial}{\partial x}\alpha_\varepsilon(x_i,t_j)| \\ = \ &\theta r\Big|e^{-\frac{\rho_i(x_i-d(t_j))}{h_i}} - \frac{1-e^{-\frac{\rho_i(x_i-d(t_j))}{h_i}}}{\rho_i} - \frac{1-e^{-\rho_i(d(t_j)-x_{i-1})/h_i}}{\rho_i})\Big| \\ \le \ &\frac{rh_i}{\varepsilon} \le CN^{-1}\ln N. \end{aligned}$$

Use (a),(b) and inequality (5) to establish (c). The final inequality (d) is easily checked by a simple evaluation. □

In the next lemma, we establish properties of the main barrier function used in the error analysis in §4. Recall the definitions

$$x_Q := \max_i \{x_i|\ x_i \leqslant d^*\} \quad \text{and} \quad x_{Q_j} := \max_i \{x_i|\ x_i \leqslant d(t_j)\}.$$

In the proof of the various inequalities established in the next lemma, we identify the following special case

$$i \in [\tfrac{N}{4}, \tfrac{3N}{4}],\ \text{if (10) holds;} \quad \text{or} \quad 0 \leqslant i \leqslant N,\ \text{otherwise.} \tag{A.1}$$

Lemma 7. *Assume $M = CN$. For sufficiently large N, the mesh function $\hat{Z}$ defined below*

$$\hat{Z}(x_i, t_j) := \begin{cases} \prod_{k=i+1}^{Q_j} \left(1 + \frac{\alpha_\varepsilon(x_k,t_j)}{2\varepsilon} h_k\right)^{-1}, & 0 \le i < Q_j, \\ 1, & i = Q_j, \\ \prod_{k=Q_j+1}^{i} \left(1 + \frac{|\alpha_\varepsilon(x_k,t_j)|}{2\varepsilon} h_k\right)^{-1}, & Q_j < i \le N, \end{cases};$$

satisfies the following bounds

(a) $\hat{Z}(x_i, t_j) \ge e^{-\frac{\theta}{2\varepsilon}|x_{Q_j} - x_i|}$;

(b) $\hat{Z}(x_i, t_j) \leqslant Ce^{-\frac{\theta}{2\varepsilon}|x_{Q_j} - x_i|}$, *if (A.1) holds*;

(c) $|D_t^- \hat{Z}(x_i, t_j)| \leqslant CN^{-1}(\ln N)^2 e^{-\frac{\theta}{2\varepsilon}|x_{Q_j} - x_i|}$, *if (A.1) holds*;

(d) $|D_t^- \hat{Z}(x_i, t_j)| \leqslant Ce^{-\frac{\theta}{2\varepsilon}|x_{Q_j} - x_i|}$, $0 \leqslant i \leqslant N$;

(e) $L_\varepsilon^{N,M} \hat{Z}(x_i, t_j) \geqslant \frac{C}{\varepsilon} e^{-\frac{\theta}{2\varepsilon}|x_{Q_j} - x_i|}$ *if (A.1) holds*;

(f) $|L_\varepsilon^{N,M} \hat{Z}(x_i, t_j)| \le CN^{-1}$; *if* (10) *holds and* $\{i \le N/4$ *or* $i \ge 3N/4\}$.

Proof. (a) Using the bounds $(1+s)^{-1} \geqslant e^{-s}$, $s \geqslant 0$ and $|\alpha_\varepsilon| \leqslant \theta$, we can establish the lower bound on $\hat{Z}(x_i, t_j)$.

(b) For $\frac{N}{4} \leqslant i < Q_j$ (when (10) holds) or for all $0 \leqslant i < Q_j$ (otherwise), we have $h_{i+1}/\varepsilon \leqslant CN^{-1} \ln N$. Then

$$\begin{aligned} \int_{x_i}^{x_{Q_j}} \alpha_\varepsilon(s, t_j)\ ds &= \theta(x_{Q_j} - x_i) - \theta\tfrac{\varepsilon}{r}[e^{-\frac{r}{\varepsilon}(d(t_j) - x_{Q_j})} - e^{-\frac{r}{\varepsilon}(d(t_j) - x_i)}] \\ &\leqslant \theta(x_{Q_j} - x_i) + C\varepsilon, \quad \text{and} \end{aligned}$$

$$\begin{aligned} |\sum_{k=i+1}^{Q_j} \alpha_\varepsilon(x_k, t_j) h_k - \int_{x_i}^{x_{Q_j}} \alpha_\varepsilon(s, t_j)\ ds| &= |\sum_{k=i}^{Q_j - 1} \int_{x_k}^{x_{k+1}} \alpha_\varepsilon(x_k, t_j) - \alpha_\varepsilon(s, t_j)\ ds| \\ &= \sum_{k=i}^{Q_j-1} \int_{x_k}^{x_{k+1}} e^{-\frac{r}{\varepsilon}(d(t_j) - x_k)}(e^{\frac{r}{\varepsilon}(s - x_k)} - 1)\ ds \le C\varepsilon N^{-1}(\ln N); \end{aligned}$$

where we note that, within the uniform mesh spacing,

$$\sum_{k=i+1}^{Q_j} e^{-\frac{r}{\varepsilon}(d(t_j)-x_k)} = e^{-\frac{r}{\varepsilon}(d(t_j)-x_{Q_j})} \frac{1-e^{-\frac{r}{\varepsilon}x_i}}{1-e^{-\frac{r}{\varepsilon}h}} \leq C. \tag{A.2}$$

Using the bound $(1+s)^{-1} \leqslant e^{s^2/2}e^{-s}, s > 0$, we have

$$\hat{Z}(x_i,t_j) \leqslant Ce^{-\frac{1}{2\varepsilon}\sum_{k=i+1}^{Q_j}\alpha_\varepsilon(x_k,t_j)h_k} \leqslant Ce^{-\frac{\theta}{2\varepsilon}(x_{Q_j}-x_i)}.$$

The upper bounds on $\hat{Z}(x_i,t_j)$ for $x_i > x_{Q_j}$ are established in an analogous manner.

(c) By design, $d(t)$ is within the fine mesh for all t. Since $|d'(t)| \leq C\varepsilon$ we have that

$$|d(t_k) - d(t_{k-1})| \leq C\varepsilon M^{-1}.$$

Hence $Q_{j-1} \in \{Q_j, Q_j-1, Q_j+1\}$

(i) Let us first consider the case when $Q_{j-1} = Q_j$. Then, for all i, by using $1-e^{-s} \leq s, s \geq 0$, we have the bound

$$|\alpha_\varepsilon(x_i,t_j) - \alpha_\varepsilon(x_i,t_{j-1})| \leq CM^{-1}e^{-\frac{r}{\varepsilon}|d(t_j)-x_i|}.$$

Note the following identity for all $a_i, b_i \neq 0$

$$\prod_{i=1}^{N} a_i - \prod_{i=1}^{N} b_i = \sum_{p=1}^{N}(\prod_{i=1}^{p} a_i(\frac{1}{b_p}-\frac{1}{a_p})\prod_{i=p}^{N} b_i).$$

For all $i \leq Q_j$, we have that

$$|D_t^- \hat{Z}(x_i,t_j)| \quad \leq \quad C \sum_{p=i+1}^{Q_j} \frac{\theta h_p}{2\varepsilon} e^{-\frac{r}{\varepsilon}(d(t_j)-x_p)} (\prod_{k=i+1}^{p} a_k \prod_{k=p}^{Q_j} b_k),$$

where

$$a_k := \left(1+\tfrac{\alpha_\varepsilon(x_k,t_j)}{2\varepsilon}h_k\right)^{-1}, \quad b_k := \left(1+\tfrac{\alpha_\varepsilon(x_k,t_{j-1})}{2\varepsilon}h_k\right)^{-1}.$$

As in (b), we can show that for all i

$$\prod_{k=i+1}^{p-1} a_k \prod_{k=p}^{Q_j} b_k \leq Ce^{-\frac{\theta}{2\varepsilon}(x_{Q_j}-x_i)}.$$

Hence, we have that

$$|D_t^- \hat{Z}(x_i,t_j)| \leq C \sum_{p=i+1}^{Q_j} \frac{\theta h_p}{2\varepsilon} e^{-\frac{r}{\varepsilon}(d(t_j)-x_p)} e^{-\frac{\theta}{2\varepsilon}(x_{Q_j}-x_i)}. \tag{A.3}$$

In the fine mesh region, when $N/4 \leq i \leq Q_j$ if (10) holds (or $1 \leq i \leq Q_j$ otherwise) we use (A.2) to deduce the desired bound on $|D_t^- \hat{Z}(x_i,t_j)|$.

(ii) We now consider the case where $Q_{j-1} = Q_j - 1$, then from above we have that, for $i \le Q_j$,

$$|\hat{Z}(x_i,t_j) - \Big(1 + \tfrac{\alpha_\varepsilon(x_{Q_j},t_{j-1})}{2\varepsilon} h_{Q_j}\Big)^{-1} \hat{Z}(x_i,t_{j-1})| \le CM^{-1}(N^{-1}\ln N)e^{-\frac{\theta}{2\varepsilon}|x_{Q_j}-x_i|}.$$

Noting that $|\alpha_\varepsilon(x_{Q_j},t_{j-1})| \le CN^{-1}\ln N$, we can deduce that in this case for all i, we have that

$$|D_t^- \hat{Z}(x_i,t_j)| \le C(1 + M(N^{-1}\ln N))(N^{-1}\ln N)e^{-\frac{\theta}{2\varepsilon}|x_{Q_j}-x_i|}.$$

The case where $Q_{j-1} = Q_j + 1$ is managed in an analogous fashion.

(d) Using (A.3) in the coarse mesh region, where $i \le N/4$, we note that

$$\frac{\theta h_i}{2\varepsilon} e^{-rh_i/\varepsilon} \le C, \quad e^{-\frac{r}{2\varepsilon}(d(t_j)-x_{i+1})} \le CN^{-1}.$$

Hence, for all i, we have that $|D_t^- \hat{Z}(x_i,t_j)| \le Ce^{-\frac{\theta}{2\varepsilon}|x_{Q_j}-x_i|}$.

(e) Observe that

$$\begin{aligned}
\hat{Z}(x_i,t_j) &= \Big(1 + \tfrac{\alpha_\varepsilon(x_i,t_j)}{2\varepsilon} h_i\Big) \hat{Z}(x_{i-1},t_j), \quad i \le Q_j \\
\hat{Z}(x_i,t_j) &= \Big(1 + \tfrac{|\alpha_\varepsilon(x_i,t_j)|}{2\varepsilon} h_i\Big)^{-1} \hat{Z}(x_{i-1},t_j), \quad i > Q_j.
\end{aligned}$$

Note the following differences:

$$\begin{aligned}
D_x^- \hat{Z}(x_i,t_j) &= \left\{ \begin{array}{ll} \frac{\alpha_\varepsilon(x_i,t_j)}{2\varepsilon} \hat{Z}(x_{i-1},t_j), & i \leqslant Q_j, \\ \frac{-|\alpha_\varepsilon(x_i,t_j)|}{2\varepsilon} \hat{Z}(x_i,t_j), & i > Q_j, \end{array} \right. ; \\
D_x^+ \hat{Z}(x_i,t_j) &= \left\{ \begin{array}{ll} \frac{\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} \hat{Z}(x_i,t_j), & i < Q_j, \\ \frac{-|\alpha_\varepsilon(x_{i+1},t_j)|}{2\varepsilon} \hat{Z}(x_{i+1},t_j), & i \geqslant Q_j, \end{array} \right. ;
\end{aligned}$$

which are used to establish that

$$\varepsilon\delta_x^2 \hat{Z}(x_i,t_j) = \left\{ \begin{array}{ll} \frac{h_i}{\Sigma_h}[\frac{\alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} + \frac{h_{i+1}}{h_i} D_x^+ \alpha_\varepsilon(x_i,t_j)]\hat{Z}(x_{i-1},t_j), & i < Q_j, \\ \frac{1}{\Sigma_h}[\alpha_\varepsilon(x_{i+1},t_j)\hat{Z}(x_{i+1},t_j) - \alpha_\varepsilon(x_i,t_j)\hat{Z}(x_{i-1},t_j)], & i = Q_j, \\ \frac{h_{i+1}}{\Sigma_h}[\frac{\alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} + D_x^+ \alpha_\varepsilon(x_i,t_j)]\hat{Z}(x_{i+1},t_j), & i > Q_j. \end{array} \right.$$

with $\Sigma_h := h_i + h_{i+1}$. Consider the mesh points where $i < Q_j$, then

$$\begin{aligned}
(a_\varepsilon + \varepsilon g) D_x^+ \hat{Z}(x_i,t_j) &\ge \tfrac{\alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} \hat{Z}(x_i,t_j) - C\hat{Z}(x_i,t_j), \\
(a_\varepsilon + \varepsilon g) D_x^- \hat{Z}(x_i,t_j) &\ge \tfrac{\alpha_\varepsilon^2(x_i,t_j)}{2\varepsilon} \hat{Z}(x_{i-1},t_j) - C\hat{Z}(x_{i-1},t_j);
\end{aligned}$$

and hence, for $i < Q_j$,

$$(a_\varepsilon + \varepsilon g) D_x \hat{Z}(x_i,t_j) \ge \tfrac{\alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} \hat{Z}(x_{i-1},t_j) - C\hat{Z}(x_i,t_j)$$

Likewise, we have that for $i > Q_j$,

$$(a_\varepsilon + \varepsilon g) D_x \hat{Z}(x_i, t_j) \geq \tfrac{\alpha_\varepsilon(x_i,t_j)\alpha_\varepsilon(x_{i+1},t_j)}{2\varepsilon} \hat{Z}(x_{i+1}, t_j) - C\hat{Z}(x_i, t_j)$$

where we note that if $(a_\varepsilon + \varepsilon g)(x_i, t_j) > 0$ for $i > Q_j$ then $|(a_\varepsilon + \varepsilon g)| \leq C\varepsilon$. For $i \leqslant Q_j$, we have $h_i \geqslant h_{i+1}$ and for $i \geqslant Q_j$, we have $h_i \leqslant h_{i+1}$. Using the previous lemma, we have for $i < Q_j$,

$$\begin{aligned} L_\varepsilon^{N,M} \hat{Z}(x_i, t_j) &\geq \tfrac{h_{i+1}}{2\varepsilon \Sigma_h} [\alpha_\varepsilon(x_i, t_j)\alpha_\varepsilon(x_{i+1}, t_j) - 2\varepsilon D_x^+ \alpha_\varepsilon(x_i, t_j)] \hat{Z}(x_{i-1}, t_j) \\ &\qquad -C|D_t^- \hat{Z}(x_i, t_j)| - C\hat{Z}(x_i, t_j) \\ &\geq \tfrac{h_{i+1}}{\Sigma_h} \tfrac{\theta^2}{4\varepsilon} \hat{Z}(x_{i-1}, t_j) - C|D_t^- \hat{Z}(x_i, t_j)| - C\hat{Z}(x_i, t_j) \end{aligned}$$

and for $i > Q_j$,

$$\begin{aligned} L_\varepsilon^{N,M} \hat{Z}(x_i, t_j) &\geq \tfrac{h_i}{2\varepsilon \Sigma_h} [\alpha_\varepsilon(x_i, t_j)\alpha_\varepsilon(x_{i+1}, t_j) - 2\varepsilon \tfrac{h_{i+1}}{h_i} D_x^+ \alpha_\varepsilon(x_i, t_j)] \hat{Z}(x_{i+1}, t_j) \\ &\qquad -C|D_t^- \hat{Z}(x_i, t_j)| - C\hat{Z}(x_i, t_j) \\ &\geq \tfrac{h_i}{\Sigma_h} \tfrac{\theta^2}{4\varepsilon} \hat{Z}(x_{i+1}, t_j) - C|D_t^- \hat{Z}(x_i, t_j)| - C\hat{Z}(x_i, t_j) \end{aligned}$$

If $i \leqslant Q_j$ then for $i > \frac{N}{4}$ when (10) holds (or for $i > 1$ otherwise) and for sufficiently large N we have

$$\hat{Z}(x_{i-1}, t_j) \geq e^{-\frac{\theta}{2\varepsilon}(x_{Q_j} - x_{i-1})} = e^{-\frac{\theta}{2\varepsilon}(x_{Q_j} - x_i)} e^{-\frac{\theta}{2\varepsilon} h_i} \geqslant \tfrac{1}{2} e^{-\frac{\theta}{2\varepsilon}(x_{Q_j} - x_i)}$$

Similar bounds can be established for $\hat{Z}(x_{i+1}, t_j)$ when $i \geqslant Q_j + 1$ and the lower bound for $L_\varepsilon^{N,M} \hat{Z}(x_i, t_j)$, $i \neq Q_j$ follows.

For sufficiently large N, we have $(1 \pm \frac{\alpha_\varepsilon(x_{i\mp 1})}{2\varepsilon} h)^{-1} \geqslant 1 - CN^{-1} \ln N \geqslant \frac{1}{2}$, and so for $i = Q_j$ we have

$$L_\varepsilon^{N,M} \hat{Z}(x_i, t_j) \geq \tfrac{\theta^2}{16\varepsilon} \hat{Z}(x_i, t_j) - C|D_t^- \hat{Z}(x_i, t_j)| - C\hat{Z}(x_i, t_j).$$

Collecting all these lower bounds on $L_\varepsilon^{N,M} \hat{Z}(x_i, t_j)$ completes the argument in the case of (e).

(f) Use the earlier bounds on $\hat{Z}, D_t^- \hat{Z}$ and bound the expression $L_\varepsilon^{N,M} \hat{Z}(x_i, t_j)$ as above in (e). □