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Inverse Scattering Solutions for Side-Band Signals

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Abstract — **When a signal is recorded that has been physically generated by some scattering process (the interaction of electromagnetic waves with an inhomogeneous dielectric, for example), the 'standard model' for the signal (i.e. information content convolved with a characteristic Impulse Response Function) is usually based on a single scattering approximation. An additive noise term is introduced into the model to take into account a range of non-deterministic factors including multiple scattering that, along with electronic noise and other background noise sources, is assumed to be relatively weak. Thus, the standard model is based on a 'weak field condition' and the inverse scattering problem is often reduced to the deconvolution of a signal in the presence of additive noise. Attempts at solving the exact inverse scattering problem which take into account multiple scattering effects often prove to be intractable, particularly with regard to the goal of implementing algorithms that are computationally stable and/or compatible with standard signal analysis methods and Digital Signal Processing 'toolkits'. This paper provides an approach to solving the multiple scattering problem for narrow side-band systems (typically, electromagnetic signal processing systems) that is compounded in the introduction of a single extra term to the standard model.**

Keywords — **Inverse Scattering, Multiple Scattering, Side-band Systems, Signal Reconstruction, Synthetic Aperture Radar**

I INTRODUCTION

We consider a one-dimensional electromagnetic wave equation of the form (the inhomogeneous Helmholtz equation)

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) u(x,k) = -k^2 \gamma(x) u(x,k) \quad (1)$$

where u is the electric wavefield, k is the wavenumber and $\gamma = \epsilon_r - 1$, $\epsilon_r > 1$ being the relative permittivity. The function ϵ_r is taken to be real whereas the wavefield u is taken to be a complex function (for generality) with variations as a function of x and k in both amplitude and phase. On the basis of this model, the forward scattering problem is defined as: Given γ obtain a solution for u. The inverse scattering problem is: Given u derive a solution for γ, e.g. [1], [2] and [3].

Let $u(x,k)$, $x \in (-\infty, \infty)$ be given by the sum of an incident wavefield $u_i(x,k)$ and a scattered wavefield $u_s(x,k)$, u_i being given by a solution of

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) u_i(x,k) = 0. \quad (2)$$

Thus, with $u = u_i + u_s$,

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) u_s(x,k) = -k^2 \gamma(x)[u_i(x,k)+u_s(x,k)]$$

where it is assumed that γ is of compact support, i.e. $\gamma(x) \exists \forall x \in [-X, X]$. The conventional approach to solving the inverse problem is as follows: (i) derive a solution for the scattered field u_s; (ii) use this solution to obtain a solution for γ. The solution for u_s is [4], [5]

$$u_s(x,k) = k^2 g(\mid x \mid, k) \otimes \gamma(x)[u_i(x,k) + u_s(x,k)]$$

where $\otimes$ denotes the convolution integral and g is the Green's function given by

$$g(\mid x \mid, k) = \frac{i}{2k} \exp(ik \mid x \mid).$$

This equation requires an iterative procedure to be adopted in order determine u_s. However, under the weak field condition when $\|u_s\| << \|u_i\|$ we consider the solution

$$u_s(x,k) = k^2 g(|\,x\,|,k) \otimes \gamma(x) u_i(x,k).$$

This is known as the Born approximation and has an asymptotic solution (for $x \to \infty$ and with $u_i(x,k) = \exp(-ikx)$) given by [6]

$$u_s(x,k) = \frac{ik}{2} \exp(ikx) \int\limits_{-\infty}^{\infty} \gamma(x) \exp(-2ikx) dx.$$

Thus, in the far field, the relationship between the scattering function and the scattered field is characterised by the Fourier transform, a relationship that is specific to the weak field condition. The scattering amplitude spectrum $A_s(k)$ (the 'reflection coefficient') can be defined as (with $k \to k/2$)

$$A_s(k) = \frac{ik}{4} \widetilde{\gamma}(k)$$

where

$$\widetilde{\gamma}(k) = \int\limits_{-\infty}^{\infty} \gamma(x) \exp(-ikx) dx$$

and

$$\gamma(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \widetilde{\gamma}(k) \exp(ikx) dk$$

The purpose of this paper is to explore a solution for γ and thereby u_s when $\|u_s\| \sim \|u_i\|$ that is based on the result, from equation (1):

$$\gamma(x) = -\frac{u^*}{|\,u(x,k)\,|^2} \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) u_s(x,k)$$

II   Inverse Solutions

Given equation (1), we can write

$$\gamma(x) = -\frac{1}{k^2 u(x,k)} \left(k^2 + \frac{\partial^2}{\partial x^2}\right) u(x,k)$$

which, noting that $\gamma = \epsilon_r - 1$ can be written in the form

$$\epsilon_r(x) = -\frac{u^*(x,k)}{k^2 \,|\,u(x,k)\,|^2} \frac{\partial^2}{\partial x^2} u(x,k)$$

and since $u = u_i + u_s$ we can write

$$\gamma(x) = -\frac{u_i^*(x,k) + u_s^*(x,k)}{k^2 \,|\,u_i(x,k) + u_s(x,k)\,|^2}$$

$$\times \left(k^2 + \frac{\partial^2}{\partial x^2}\right) u_s(x,k).$$

Note that when $\|u_s\| << \|u_i\|$,

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) u_s(x,k) = -k^2 \gamma(x) u_i(x,k),$$

the equivalent expression for $\gamma(x)$ being

$$\gamma(x) = -\frac{u_i^*(x,k)}{k^2 \,|\,u_i(x,k)\,|^2} \left(k^2 + \frac{\partial^2}{\partial x^2}\right) u_s(x,k)$$

$$= -\frac{u_i^*(x,k)}{k^2} \left(k^2 + \frac{\partial^2}{\partial x^2}\right) u_s(x,k)$$

given that $u_i(x,k) = \exp(\pm ikx)$ is a solution of equation (2). In comparison with the solution under the Born approximation, this solution for γ includes $|\,u(x,k)\,|^{-2}$ which describes the (inverse square) amplitude envelop of the sum of the incident and scattered fields, i.e. with $u(x,k) = A_u(x,k) \exp[i\theta_u(x,k)]$, it is clear that $|\,u(x,k)\,|^2 = [A_u(x,k)]^2$.

a)   Solution for Side-band Signals

If u is taken to be a narrow side-band signal that is dominated by a high frequency and carrier wave characterised by k_0, which, in turn, is determined by a narrow side-band incident wavefield u_i, then we can consider the condition where $|\,u(x,k_0)\,|^2 \sim 1 \forall x$ (in general, a constant). This condition is equivalent to considering the case where u is a phase only field $u(x,k_0) = \exp[i\theta_u(x,k_0)]$ and is imposed in respect of the fact that, for narrow side-band signals, the contribution of $|\,u\,|^{-2}$ to the reconstruction of γ from u is not significant when compared to $(u_i^* + u_s^*)(k_0^2 + \partial_x^2)u_s$. However, strictly speaking, the condition being imposed implies that, with $u_i = \exp(\pm ik_0 x)$,

$$u_i^* u_s + (u_i^* u_s)^* + |\,u_s\,|^2 = 0. \tag{3}$$

The principal difference between the reconstruction of γ using the Born approximation and the inverse solution considered here is compounded in the addition of a single term, i.e. for a weak field where $\|u_s\| << \|u_i\|$

$$\gamma(x) = -u_i^*(x,k_0) \left(1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial x^2}\right) u_s(x,k_0) \tag{4}$$

and for a strong field where $\|u_s\| \sim \|u_i\|$

$$\gamma(x) = -u_i^*(x,k_0) \left(1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial x^2}\right) u_s(x,k_0)$$

$$-u_s^*(x,k_0) \left(1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial x^2}\right) u_s(x,k_0). \tag{5}$$

Note that under the strict definition of a phase only field which leads to equation (3), for $\|u_s\| \sim \|u_i\|$

$$\gamma(x) = u_i(x,k_0) u_s^*(x,k_0)$$

$$-\frac{[u_i(x,k_0) + u_s(x,k_0)]^*}{k_0^2} \frac{\partial^2}{\partial x^2} u_s(x,k_0)$$

$$= -u_i(x,k_0) u_s^*(x,k_0), \quad k_0 \to \infty$$

In this paper, we relax this condition and utilze equations (4) and (5) given that $|\,u_i + u_s\,|^2 \sim 1$. This is the 'key' to the results that follow.

b) Numerical Simulation

We consider a numerical simulation based on computing the scattered field using a forward differencing approach to equation (1). For a fixed wavenumber k_0 (which defines a continuous wave mode), the vector u_n computed over a uniformly sampled discrete array composed of N samples, each of length Δ, is given by

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta^2} + k_0^2 u_n = -k_0^2 \gamma_n u_n, \quad n \in [1, N]$$

which has the solution

$$u_n = \frac{u_{n+1} + u_{n-1}}{2 - \Delta^2 k_0^2 (1 + \gamma_n)} = \frac{u_{n+1} + u_{n-1}}{2 - \Delta^2 k_0^2 \epsilon_{r,n}}. \quad (6)$$

A solution to equation (6) can be obtained that is based on the following iteration:

$$u_n^{m+1} = \frac{u_{n+1}^m + u_{n-1}^m}{2 - \Delta^2 k_0^2 \epsilon_{r,n}} \quad (7)$$

where u_n^1 is taken to be a discrete representation of the (unit amplitude) incident field which we define with a fixed number of periods p over $n \in [1, N]$, i.e.

$$u_n^1 = \exp\left(\frac{2\pi i p (n-1)}{(N-1)}\right).$$

A necessery condition that must be applied to equation (7) is that

$$\Delta k_0 < \sqrt{\frac{2}{\|\epsilon_{r,n}\|_\infty}}$$

where $\| \bullet \|_\infty$ defines the 'uniform norm' and

$$k_0 = \frac{2\pi p}{N}.$$

Since a solution to the equation (1) is not conditional on the amplitude of the wavefield (i.e. the incident field can be $A\exp(\pm ik_0 x)$ where A is an arbitrary value), the amplitude of the array u_n^m, after M iterations is normalised on output, i.e. $u_n^M \rightarrow u_n^M / \|u_n^M\|_\infty$.

For the discrete case, equations (4) and (5) transform to (for M iterations used to compute the scattered field)

$$\epsilon_{r,n} = 1 - u_{i,n}^*[u_{s,n} + (\Delta k_0)^{-2} u_{s,n} \otimes (1, -2, 1)]$$

$$= 1 - (u_n^1)^*[(u_n^M - u_n^1) + (\Delta k_0)^{-2}(u_n^M - u_n^1) \otimes (1, -2, 1)] \quad (8)$$

and

$$\epsilon_{r,n} = 1 - u_n^*[u_{s,n} + (\Delta k_0)^{-2} u_{s,n} \otimes (1, -2, 1)]$$

$$= 1 - (u_n^M)^*[(u_n^M - u_n^1) + (\Delta k_0)^{-2}(u_n^M - u_n^1) \otimes (1, -2, 1)] \quad (9)$$

respectively where $\otimes$ now denotes the discrete convolution sum. Figures 1-2 show comparisons between the numerical results given by equations (8) and (9), illustrating the superiority of the inverse scattering solution considered over that given by the weak field solution for the case when $k_0\Delta = 0.01$. These results are based on applying the initial solution $u_n^1 = \sin[2\pi i p (n-1)/(N-1)]$ and Hilbert transforming the output arrays before computation of equations (8) and (9) through use of the MATLAB function *hilbert*. The profile of the reconstruction for $\epsilon_{r,n}$ based on equation (9) is preserved inclusive of characteristic 'ringing' due to the discontinuities associated with the model for $\epsilon_{r,n}$. In comparison, the weak field solution given by equation (8) has a relatively narrow dynamic range and provides a poor reconstruction particularly with regard to resolving the discontinuities associated with $\epsilon_{r,n}$.
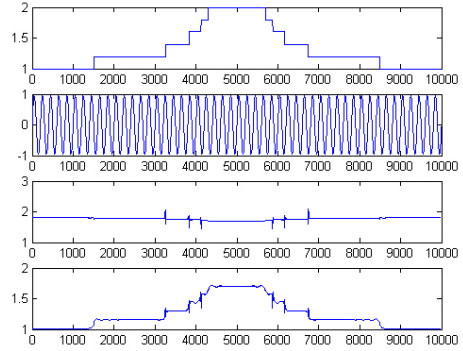

Fig. 1: Comparison between the weak and strong field inverse scattering solutions for the case when $N = 10000$, $M = 100$, $\Delta k_0 = 0.01$ with $p = 50$. From top to bottom: Relative permittivity function model $\epsilon_{r,n}$; real part of wavefield computed via equation (7); inverse solution (real part) computed using equation (8); inverse scattering solution (real part) computed using equation (9).
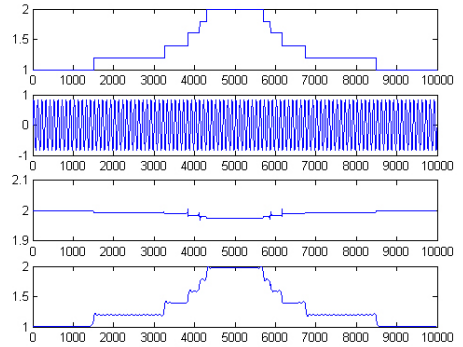

Fig. 2: Comparison between the weak and strong field solutions for the case when $N = 10000$, $M = 100$, $\Delta k_0 = 0.01$ with $p = 100$. The descriptions of each plot follow those as given in Figure 1.

III Solution for Narrow Side-Band Pulse-Echo Systems

The inverse solutions derived and demonstrated numerically in the previous section have little practical value with regard to engineering a system designed to recover $\epsilon_r(x)$ from information on the scattered field measured 'outside' the scatterer, i.e. when $\mid x \mid > \mid X \mid$. This is because the solutions considered require that the scattered field is known $\forall x$ including $\forall x \in [-X, X]$. Practically applicable systems typically measure the scattered field at a fixed point x in the far field (i.e. as $x \to \infty$) by recording the spectrum of u_s over a range of values of k. In practice, this requires the application of pulse-echo mode methods.

Pulse-echo methods involve the emission of a pulse and a recording of the back-scattered wavefield (echo). This approach is consistent with the physical nature a system if: (i) the scattering function is a one-dimensional function; (ii) the incident wavefield is a 'pencil-line beam'. However, since all physical systems are intrinsically three-dimensional, this model is idealised. Nevertheless, a variety of electromagnetic information and imaging systems can be 'cast' in terms of problems involving layered materials (e.g. the response of light, radio and microwaves to layered dielectric materials including the propagation of electromagnetic waves along transmission lines such as an optical fiber).

For a narrow side-band system with carrier frequency $k_0 \gg \mid k \mid$, the inverse scattering solution is given by (for $\mid A_u \mid \sim 1$)

$$\gamma(x) = -\frac{u^*(x,k)}{k_0^2}\left(k_0^2 + \frac{\partial^2}{\partial x^2}\right)u_s(x,k)$$

which, for a weak field where $\|u_s\| \ll \|u_i\|$ reduces to

$$\gamma(x) = -\frac{u_i^*(x,k)}{k_0^2}\left(k_0^2 + \frac{\partial^2}{\partial x^2}\right)u_s(x,k).$$

These results apply for all values of $x \in (-\infty, \infty)$. As discussed in Section I, under the Born approximation, for $x \to \infty$, a mapping is obtained between the scattered field u_s and γ that is based on the Fourier transform $\widetilde{\gamma}(k)$ of $\gamma(x)$, i.e.

$$u_s(x,k) = \frac{ik_0}{2}\exp(ikx)\widetilde{\gamma}(k), \quad \mid k \mid \ll k_0$$

which is a solution of

$$\gamma(x) = -\frac{u_i^*(x,k)}{k_0^2}\left(k_0^2 + \frac{\partial^2}{\partial x^2}\right)u_s(x,k).$$

This result suggests taking the Fourier transform of the equation

$$\gamma(x) =$$

$$-[u_i^*(x,k_0) + u_s^*(x,k_0)]\left(1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial x^2}\right)u_s(x,k_0)$$

giving

$$\widetilde{\gamma}(k) =$$

$$-[\widetilde{u}_i^*(k,k_0) + \widetilde{u}_s^*(k,k_0)]\odot\left(1 - \frac{\mid k \mid^2}{k_0^2}\right)\widetilde{u}_s(k,k_0)$$

$$= -[\widetilde{u}_i^*(k,k_0) + \widetilde{u}_s^*(k,k_0)]\odot\widetilde{u}_s(k), \quad \mid k \mid \ll k_0$$

where $\widetilde{\gamma}$, $\widetilde{u}_i^*$, $\widetilde{u}_s^*$ and $\widetilde{u}_s$ are the Fourier transforms of γ, u_i^*, u_s^* and u_s respectively and $\odot$ denotes the correlation integral. Noting that, for $u_i(x,k_0) = \exp(-ik_0 x)$, $\widetilde{u}_i^*(k,k_0) = 2\pi\delta(k_0 - k)$ we have

$$\widetilde{\gamma}(k) = 2\pi\widetilde{u}_s(k - k_0) - \widetilde{u}_s^*(k)\odot\widetilde{u}_s(k), \quad \mid k \mid \ll k_0$$

or (ignoring scaing by 2π)

$$\widetilde{u}_s(k) =$$

$$\widetilde{p}(k)\widetilde{\gamma}(k + k_0) + \widetilde{p}(k)[\widetilde{u}_s^*(k + k_0)\odot\widetilde{u}_s(k + k_0)], \quad \forall k$$

where $\widetilde{p}(k)$ is some lowpass filter with a bandwidth significantly less than k_0. Thus, upon takling the inverse Fourier transform, we obtain an expression for the (demodulated) scattered field u_s given by

$$u_s(x,k_0) = p(x)\otimes[\gamma(x)\exp(-ik_0 x)]$$

$$+p(x)\otimes[\mid u_s(x,k_0)\mid^2\exp(-ik_0 x)]$$

where $p(x)$ is the inverse Fourier transform of $\widetilde{p}(k)$. Here, $p(x)$ described the bandlimited pulse that is incident on a layered dielectric described by $\gamma(x)$. The first term is a description for the weak scattered field and the second term describes the effects of multiple scattering. In terms of the 'standard model' for a stationary signal $s(t)$ as a function of time t where

$$s(t) = p(t)\otimes f(t) + n(t) \tag{10}$$

it is now clear that the Impulse Response Function (IRF) f is given by

$$f(t) = \gamma(t)\exp(-i\omega_0 t)$$

and the noise $n(t)$ is given by

$$n(t) = p(t)\otimes[\mid s(t)\mid^2\exp(-i\omega_0 t)]$$

where ω_0 is the angular (carrier) frequency. Note that in practice, $n(t)$ will include additional background noise and in this sense, we have extract a component of the noise term that is attributed to multiple scattering effects, albeit under the conditions that: (i) $\mid u_i + u_s\mid^{-2}\sim 1$; (ii) $\mid k \mid \ll k_0$.

IV Linear FM Signal Processing

A linear FM 'chirp' signal, which is taken to be of compact support $t \in [-T/2, T/2]$, is given by (in complex form and for a unit amplitude)

$$p(t) = \exp(i\alpha t^2), \quad \mid t \mid \le \frac{T}{2}$$

where α is a constant which defines the 'chirp rate' and T is the length of the 'pulse'. The phase of this signal is given by αt^2 (i.e. it has a quadratic phase factor) and its instantaneous frequency is therefore given by $2\alpha t$ which varies linearly with time t. The frequency modulations are linear, the signal therefore being referred to as a 'linear' FM pulse.

a) Inversion under the Weak Field Condition

Consider a signal generated by the reflection of a linear FM pulse. Application of the weak field condition yields the following single scattering model:

$$s(t) = \exp(i\alpha t^2) \otimes f(t), \quad \mid t \mid \le \frac{T}{2}.$$

Using a pulse of this type provides a simple but effective way of recovering the IRF by correlating the signal with the complex conjugate of the pulse to give the reconstruction

$$\hat{f}(t) = \exp(-i\alpha t^2) \odot \exp(i\alpha t^2) \otimes f(t), \quad \mid t \mid \le \frac{T}{2}$$

$$= T \exp(-i\alpha t^2)\text{sinc}(\alpha Tt) \otimes f(t)$$

$$\simeq T\text{sinc}(\alpha Tt) \otimes f(t), \quad T >> 1.$$

This simplification, under the condition that $T >> 1$, is not only practically applicable, but systems desirable, especially with regard to electromagnetic pulse generation[1] and allows the result for $\hat{f}$ to be easily analysed in Fourier space. Using the convolution theorem we can write (ignoring scaling by π/α)

$$\hat{F}(\omega) = \begin{cases} F(\omega), & \mid \omega \mid \le \alpha T; \\ 0, & \mid \omega \mid > \alpha T. \end{cases}$$

which describes $\hat{f}$ as being a band-limited version of f (given that f is not a band-limited function) where the bandwidth is determined by αT.

b) Inversion Under the Strong Field Condition

For a side-band FM system, the results discussed above imply that, under the single scattering approximation

$$s(t) = p(t) \otimes f(t)$$

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[1]A 'long pulse' provides greater signal energy and thus, the recorded data has a higher Signal-to-Noise Ratio.

with inverse scattering estimate

$$\hat{f}(t) = p^*(t) \odot s(t)$$

where, for a side-band system $p(t) = \exp(i\alpha t^2)$. Here, $s(t)$ is taken to be the (complex) analytic signal obtained by taking the Hilbert transform of the real signal $\text{Re}[s(t)]$. For side-band systems this is typically undertaken by quadrature detection during demodulation from side-band to base-band [5].

The strong field inverse scattering solution for the signal, given by equation (10), implies that for the same (side-band) system

$$\hat{f}(t) = T\text{sinc}(\alpha Tt) \otimes f(t)$$

$$+T\text{sinc}(\alpha Tt) \otimes [\mid s(t) \mid^2 \exp(-i\omega_0 t)]. \qquad (11)$$

V Applications to Synthetic Aperture Radar

Synthetic Aperture Radar (SAR) is a side-band pulse-echo system which utilizes the response of a scatterer as it passes through the radar beam to synthesize the lateral (azimuth) resolution [8], [9]. This allows relatively high resolution images to be obtained at a long range. The antenna emits a pulse of microwave radiation toward the ground and the return signal is recorded at fixed time intervals along the flight path.

By studying the response of a SAR to a point scatterer in range x, and then in azimuth y, the Point Spread Function of the system is established which is given by [7]

$$p(x,y) = \text{sinc}(\alpha Lx)\text{sinc}(\beta k_0 y)$$

where L is the pulse length and β is the angle of divergence of the radar beam. Thus, the (post-processed) SAR image data $\hat{f}(x,y)$ is given by the convolution of the object function $f(x,y)$ (e.g. the 'ground truth') with the appropriate point spread function, i.e.

$$\hat{f}(x,y) = p(x,y) \otimes_2 f(x,y)$$

where $\otimes_2$ denotes the two-dimensional convolution integral and $\hat{f}$ is taken to be complex data generated by f. A SAR image is usually generated by displaying the amplitude modulations of the data, i.e.

$$I_{SAR}(x,y) = \mid \hat{f}(x,y) \mid.$$

The object function describes the imaged properties of the ground or sea surface. A conventional model for this function is the point scattering model that is typically used in SAR simulations [10]. This is where the object function is taken to be a distribution of point scatterers each of which reflects a replica of the emitted pulse and responds identically in azimuth. Here, nothing is said about the true physical nature of the

object function in terms of material (dielectric) properties. Moreover, this standard convolution model for the data is based on application weak field approximation where multiple scattering effects (which generate 'speckle') are taken to be part of an additive noise function [7]. SAR images are coherent images (i.e. based on complex data containing magnitude and phase information) and consequently, contain noise that is characteristic of speckle patterns (coherent noise).

Based on equation (11), we consider a model where

$$\hat{f}(x,y) = p(x,y) \otimes_2 [f(x,y) + | s(x,y) |^2 \exp(-ik_0 x)]$$
$$(12)$$

where the second term (on the right hand side) is taken to be the multiple scattering term[2]. Figure 3 shows the effect of applying a Gaussian lowpass filter to the complex data $\hat{f}(x,y)$ before generation of the image $| \hat{f}(x,y) |$ using data obtained from the Sandia National Laboratories SAR database [11]. Filtering the complex data is undertaken in order to attempt to suppress the term $p(x,y) \otimes_2 [| s(x,y) |^2 \exp(-ik_0 x)]$. The more usual practice is to filter the amplitude image $| \hat{f} |$. However, within the context of equation (12), applying a lowpass filter prior to computing an amplitude image of the complex data reduces the influence of the cross terms inherent in $| \hat{f}(x,y) |$ and it is this effect that reduces the speckle as illustrated in Figure 3.
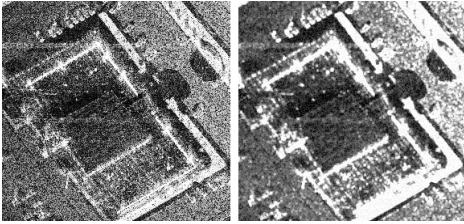

Fig. 3: Example of an original SAR image (left) and the same image after lowpass filtering the complex data (right). In both cases, the images have been histogram equalized utilizing the MATLAB function *histeq*.

VI   CONCLUSION

The extension of the standard (stationary) model $s(t) = p(t) \otimes f(t)$ to include multiple scattering effects that are compounded in a single additional term provides a practical method of processing signals that is compatible with a standard signal processing 'toolkit'. The inverse scattering problem is usually formulated by first solving the forward scattering problem. Once the relationship between the scattered field and the scattering function has been established by solving the Helmholtz equation, an inverse solution is then attempted. The approach taken in this paper has

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[2]After application of conventional SAR signal processing.

been to work directly with the Helmholtz equation to produce an expression for the scattering function. The example numerical simulations given in Section II(b) provide evidence of the expected superiority of this solution over the weak scattering solution. However, it should be noted that this result is based on the application of the narrow side-band condition which is required in order to set $| u_i(x,k) + u_s(x,k) |^2 \sim 1$. In this sense, the solution developed is not based on an entirely 'exact' inverse scattering solution, but rather a modified version of a solution tailored for applicability to narrow side-band systems. Such systems are consistent with the applications of electromagnetic signal processing.

A further extension of the inverse scattering approach considered here that is independent of the condition $| u_i(x,k) + u_s(x,k) |^{-2} \sim 1$ can be developed using the expansion

$$| u_i(x,k) + u_s(x,k) |^{-2} = 1 - u_i u_s^* - u_i^* u_s - | u_s |^2 + ...$$

details of which lie beyond the scope of this paper.

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