K(π, 1) for Artin Groups of Finite Type

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K$(\pi, 1)$'S FOR ARTIN GROUPS OF FINITE TYPE

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1. Introduction.

This paper is a continuation of a programme to construct new K$(\pi, 1)$'s for Artin groups of finite type which began in [4] with Artin groups on 2 and 3 generators and was extended to braid groups in [3]. These K$(\pi, 1)$'s differ from those in [6] in that their universal covers are simplicial complexes.

In [4] a complex is constructed whose top-dimensional cells correspond to minimal factorizations of a Coxeter element as a product of reflections in a finite Coxeter group. Asphericity is established in low dimensions using a metric of non-positive curvature. Since the non-positive curvature condition is difficult to check in higher dimensions a combinatorial approach is used in [3] in the case of the braid groups.

It is clear from [3] that the techniques used can be applied to any finite Coxeter group W. When W is equipped with the partial order given by reflection length and γ is a Coxeter element in W, the construction of the K$(\pi, 1)$'s is exactly analogous provided that the interval $[I, \gamma]$ forms a lattice. In dimension 3, see [4], establishing this condition amounts to observing that two planes through the origin meet in a unique line. In the braid group case, see [3], where the reflections are transpositions and the Coxeter element is an n-cycle this lattice property is established by identifying $[I, \gamma]$ with the lattice of noncrossing partitions of $\{1, 2, \ldots, n\}$.

In this paper, we consider the Artin groups of type C_n and D_n. Thus, for each finite reflection group W of type C_n or D_n, partially ordered by reflection length, we identify a lattice inside W and use it to construct a finite aspherical complex $K(W)$. In the C_n case this lattice coincides with the lattice of noncrossing partitions of $\{1, 2, \ldots, n, -1, \ldots, -n\}$ studied in [8]. The final ingredient is to prove that $\pi_1(K(W))$ is isomorphic to $A(W)$, the associated finite type Artin group. As in [4] and [3] this involves a lengthy check that the obvious maps between the two presentations are well-defined.

David Bessis has independently obtained similar results which can be seen at [1]. His approach exploits in a clever way the extra structure given by viewing these groups as complex reflection groups. In addition, he has verified that in the exceptional cases that the interval $[I, \gamma]$ forms a lattice and that the corresponding poset groups are isomorphic to the respective Artin groups of finite type. Combined with the results of our section 5 below this provides the new $\mathrm{K}(\pi, 1)$'s in these cases and we thank him for drawing our attention to this fact.

In section 2 we collect some general facts about the reflection length function on finite reflection groups and the induced partial order. In section 3 we study the cube group C_n and its index two subgroup D_n. In section 4 we identify the subposets of interest in C_n and D_n and show that they are lattices. In section 5 we define the poset group $\Gamma(W, \alpha)$ associated to the interval $[I, \alpha]$ for $\alpha \in W$. In the case where $[I, \alpha]$ is a lattice we construct the complexes $K(W, \alpha)$ and show that they are $\mathrm{K}(\pi, 1)$'s. Section 6 shows that the groups $\Gamma(C_n, \gamma_C)$ and $\Gamma(D_n, \gamma_D)$ are indeed the Artin groups of the appropriate type when γ_C and γ_D are the respective Coxeter elements.

2. A partial order on finite reflection groups.

Let W be a finite reflection group with reflection set $\mathcal{R}$ and identity element I. We let $d : W \times W \to \mathbf{Z}$ be the distance function in the Cayley graph of W with generating set $\mathcal{R}$ and define the *reflection length function* $l : W \to \mathbf{Z}$ by $l(w) = d(I, w)$. So $l(w)$ is the length of the shortest product of reflections yielding the element w. It follows from the triangle inequality for d that $l(w) \leq l(u) + l(u^{-1}w)$ for any $u, w \in W$.

Definition 2.1. *We introduce the relation $\leq$ on W by declaring*

$$u \leq w \qquad \Leftrightarrow \qquad l(w) = l(u) + l(u^{-1}w).$$

Thus $u \leq w$ if and only if there is a geodesic in the Cayley graph from I to w which passes through u. Alternatively, equality occurs if and only if there is a shortest factorisation of u as a product of reflections which is a prefix of a shortest factorisation of w. It is readily shown that $\leq$ is reflexive, antisymmetric and transitive so that $(W, \leq)$ becomes a partially ordered set.

Since $(u^{-1}w)^{-1}w = w^{-1}uw$ is conjugate to u it follows that $u^{-1}w \leq w$ whenever $u \leq w$. Furthermore, whenever $\alpha \leq \beta \leq \gamma$ we have

$$l(\gamma) = l(\alpha) + (l(\alpha^{-1}\beta) + l(\beta^{-1}\gamma)),$$

so that $\alpha^{-1}\beta \le \alpha^{-1}\gamma$.

We recall some general facts about orthogonal transformations from [5]. If $A \in O(n)$, we associate to A two subspaces of $\mathbf{R}^n$, namely

$$M(A) = \mathrm{im}(A - I) \quad \text{and} \quad F(A) = \ker(A - I).$$

We recall that $M(A)^{\perp} = F(A)$. We use the notation $|V|$ for $\dim(V)$ when V is a subspace of $\mathbf{R}^n$. It is shown in [5] that

$$|M(AC)| \le |M(A)| + |M(C)|$$

We define a partial order on $O(n)$ by

$$A \le_o B \quad \Leftrightarrow \quad |M(B)| = |M(A)| + |M(A^{-1}B)|$$

and we note that $A \le_o B$ if and only if $M(B) = M(A) \oplus M(A^{-1}B)$. In particular $A \le_o B$ implies that $M(A) \subseteq M(B)$ or equivalently $F(B) \subseteq F(A)$. The main result we will use from [5] is that for each $A \in O(n)$ and each subspace V of $M(A)$ there exists a unique $B \in O(n)$ with $B \le_o A$ and $M(B) = V$.

Our finite reflection group W is a subgroup of $O(n)$, so the results of [5] can be applied to the elements of W. We begin with a geometric interpretation of the length function l on W.

Proposition 2.2. *$l(\alpha) = |M(\alpha)| = n - |F(\alpha)|$, for $\alpha \in W$.*

Proof. First note that the proposition holds when $\alpha = I$ so we will assume $\alpha \ne I$ and let $k = |M(\alpha)| > 0$.

To establish the inequality $l(\alpha) \le k$ we show that α can be expressed as a product of k reflections. We will use induction on k noting that the case $k = 1$ is immediate. Consider the subspace $F(\alpha) \ne \mathbb{R}^n$. Recall from part (d) of Theorem 1.12 of [7] that the subgroup W' of W of elements which fix $F(\alpha)$ pointwise is generated by those reflections R in W satisfying $F(\alpha) \subset F(R)$. Since $\alpha \ne I$ there exists at least one such reflection R. Since $M(A) = F(A)^{\perp}$ we have $M(R) \subset M(\alpha)$. The unique orthogonal transformation induced on $M(R)$ by α must be R by Corollary 3 of [5]. Hence $R \le_o \alpha$ and

$$|M(R\alpha)| = |M(\alpha)| - |M(R)| = k - 1.$$

By induction $R\alpha$ can be expressed as a product of $k - 1$ reflections and hence there is an expression $\alpha = R_1 \dots R_k$ for α as a product of k reflections. We note that by construction each of these reflections R_i satisfies $M(R_i) \subset M(\alpha)$.

To establish the other inequality suppose $\alpha = S_1 S_2 \dots S_m$ is an expression for α as a product of m reflections realizing $l(\alpha) = m$. Repeated use of the identity $|M(AC)| \leq |M(A)| + |M(C)|$ gives

$$k = |M(\alpha)| \leq |M(S_1)| + \cdots + |M(S_m)| = m = l(\alpha). \qquad \text{q.e.d.}$$

In particular the partial order $\leq$ on W is a restriction of the partial order $\leq_o$ on $O(n)$ and we will drop the subscript from $\leq_o$ from now on. The following lemma is immediate.

Lemma 2.3. *Let W be a finite Coxeter group with reflection set $\mathcal{R}$ and let W_1 be a subgroup generated by a subset $\mathcal{R}_1$ of $\mathcal{R}$. Then the length function for W_1 is equal to the restriction to W_1 of the length function for W.*

Definition 2.4. *For each $\delta \in W$ we define the reflection set of δ, S_δ, by $S_\delta = \{R \in \mathcal{R} \mid r \leq \delta\}$.*

Repeated application of $A \leq B \Rightarrow |M(B)| = |M(A)| + |M(A^{-1}B)|$ gives $M(\delta) = \mathrm{Span}\{M(R) \mid R \leq \delta\}$ so that S_δ determines $M(\delta)$. However, in the case where $\delta \leq \gamma$, δ itself is determined by γ and S_δ since δ is the unique orthogonal transformation induced on $M(\delta)$ by γ. The following results are consequences of this fact.

Lemma 2.5. *If $\alpha, \beta \leq \gamma$ in W and $S_\alpha \subseteq S_\beta$ then $\alpha \leq \beta$.*

Proof. $M(\alpha) \subset M(\beta) \subset M(\gamma)$ and by uniqueness the transformation induced on $M(\alpha)$ by β is the same as the transformation induced by γ, namely α. q.e.d.

Lemma 2.6. *Suppose $\alpha, \beta \leq \gamma$ in W. If there is an element $\delta \in W$ with $\delta \leq \gamma$ and $S_\delta = S_\alpha \cap S_\beta$ then δ is the greatest lower bound of α and β in W, that is, if $\tau \in W$ satisfies $\tau \leq \alpha, \beta$ then $\tau \leq \delta$.*

3. The Cube groups C_n and D_n.

For general facts about the groups C_n and D_n see [2] or [7]. Let $I = [-1, 1]$ and let C_n denote the group of isometries of the cube I^n in $\mathbb{R}^n$. That is

$$C_n = \{\alpha \in O(n) : \alpha(I^n) = I^n\}$$

Let $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$ and let $x_1, \ldots, x_n$ denote the corresponding coordinates. The set $\mathcal{R}_c$ of all reflections in C_n consists of the following n^2 elements. For each $i = 1, \ldots, n$, reflection in the hyperplane $x_i = 0$ is denoted $[i]$ and also by $[-i]$. For each $i \neq j$, reflection in the hyperplane $x_i = x_j$ is denoted by any one of the four expressions $(\!(i, j)\!)$, $(\!(j, i)\!)$, $(\!(-i, -j)\!)$ and $(\!(-j, -i)\!)$, while reflection in the plane $x_i = -x_j$ is denoted by any one of the four expressions $(\!(i, -j)\!)$, $(\!(-i, j)\!)$, $(\!(j, -i)\!)$, and $(\!(-j, i)\!)$. The set of these $n(n-1)$ reflections, in hyperplanes of the form $x_i = \pm x_j$, is denoted $\mathcal{R}_d$ and the subgroup they generate, D_n, is well known to be an index two subgroup of C_n. The group C_n acts on the set $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ in the obvious manner and this action satisfies $\alpha \cdot (-e_i) = -(\alpha \cdot e_i)$ for each i and each $\alpha \in C_n$. Thus we obtain an injective homomorphism p from C_n into the group Σ_{2n} of permutations of the set $\{1, 2, \dots, n, -1, -2, \dots, -n\}$. Note that for each i, $p([i])$ is a transposition in Σ_{2n}, while each element of $\mathcal{R}_d$ is mapped to a product of two disjoint transpositions. Thus $p(D_n)$ is contained in the subgroup of even permutations.

For each cycle $c = (i_1, \dots, i_r)$ in Σ_{2n}, we define the cycle $\bar{c}$ by

$$\bar{c} = (-i_1, \dots, -i_r)$$

Note that $\bar{c} = z_0 c z_0$ where $z_0 = (1, -1)(2, -2) \dots (n, -n)$ has order two. Note also that $z_0 = p(\zeta_0)$ where $\zeta_0 = [1][2] \cdots [n]$ is the nontrivial element in the centre of C_n.

Proposition 3.1. *The image $p(C_n)$ is the centraliser $Z(z_0)$ of z_0 in Σ_{2n}. It consists of all products of disjoint cycles of the form*

$$c_1\bar{c}_1 \dots c_k\bar{c}_k\gamma_1 \dots \gamma_r \quad \textit{where} \quad \gamma_j = \bar{\gamma}_j \quad \forall\ j = 1, \dots, r. \tag{1}$$

The image $p(D_n)$ consists of all elements of the form (1) with r even.

Proof. Since z_0 has order 2 and $z_0 c_1 c_2 \dots c_k z_0 = \bar{c}_1\bar{c}_2 \dots \bar{c}_k$ for any product of cycles in Σ_{2n}, it follows that the centraliser $Z(z_0)$ consists of those products of disjoint cycles $c_1 c_2 \dots c_k$ for which

$$c_1 c_2 \dots c_k = \bar{c}_1 \bar{c}_2 \dots \bar{c}_k$$

By uniqueness (up to reordering) of cycle decomposition in Σ_{2n}, for each i either $c_i = \bar{c}_j$ for some $j \neq i$ or else $c_i = \bar{c}_i$. It follows that the centraliser of z_0 is precisely the set of elements in Σ_{2n} of the form (1).
For each $\alpha \in C_n$, the identity $\zeta_0 \alpha \zeta_0 = \alpha$ implies that $p(\alpha)$ lies in the centraliser of z_0. Thus $p(C_n) \subset Z(z_0)$. In the reverse direction, if $c = (i_1, \dots, i_k)$ is disjoint from $\bar{c}$, one may readily verify that

$$c\bar{c} = p\left(\langle\!\langle i_1, i_2\rangle\!\rangle\langle\!\langle i_2, i_3\rangle\!\rangle \dots \langle\!\langle i_{q-1}, i_q\rangle\!\rangle\right) \tag{2}$$

Likewise, if $c = \bar{c}$ then c must be the form $c = (i_1, \dots, i_k, -i_1, \dots, -i_k)$ for some $-n \leq i_1, i_2, \dots, i_k \leq n$ and one may verify that

$$c = (i_1, -i_1)(i_1, i_2)(-i_1, -i_2) \dots (i_{k-1}, i_k)(-i_{k-1}, -i_k) \tag{3}$$

$$= p\left([i_1]\langle\!\langle i_1, i_2\rangle\!\rangle \dots \langle\!\langle i_{k-1}, i_k\rangle\!\rangle\right) \tag{4}$$

It follows that any element of the form (1) lies in $p(C_n)$ and hence $p(C_n) = Z(z_0)$.
Let $\alpha \in D_n$ and write $p(\alpha) = c_1\bar{c}_1 \cdots c_k\bar{c}_k\gamma_1 \cdots \gamma_r$. Since $p(\alpha)$ and each $c_i\bar{c}_i$ is an even permutation while each γ_j is an odd permutation, r must

be even. To show that every element of the form (1) with r even is in $p(D_n)$, we need only note the following facts.

- If the cycle c is disjoint from $\bar{c}$ then equation (2) implies that $c\bar{c} \in p(D_n)$.
- If $i \neq j$ then $[i][j] = (\!(i,j)\!)(\!(i,-j)\!)$ and hence is an element of $p(D_n)$. It now follows from equation (3) that if $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ are disjoint cycles then $c_1c_2 \in p(D_n)$. q.e.d.

Notation. From now on we will identify C_n and D_n with their respective images in Σ_{2n}. If a cycle $c = (i_1, \dots, i_k)$ is disjoint from $\bar{c}$ then we write

$$(\!(i_1, \dots, i_k)\!) = c\bar{c} = (i_1, \dots, i_k)(-i_1, \dots, -i_k)$$

and we call $c\bar{c}$ a *paired cycle*. If $k = 1$ then $c = (i_1)$ and the paired cycle $c\bar{c} = (\!(i_1)\!)$ fixes the vector e_{i_1}. If $c = \bar{c} = (i_1, \dots, i_r, -i_1, \dots, -i_r)$ then we say that c is a balanced cycle and we write

$$c = [i_1, \dots, i_k].$$

This notation is consistent with that introduced earlier for the elements of the generating set $\mathcal{R}_c$. With these conventions, proposition 3.1 states that each element of C_n may be written as a product of disjoint paired cycles and balanced cycles. If $\alpha \in C_n$ fixes the standard basis vector e_i then we will assume that the paired cycle $(\!(i)\!)$ appears in the corresponding expression (1) for α.

Denote the length function for C_n with respect to the generating set $\mathcal{R}_c$ by l. Lemma 2.3 allows us to use the same symbol l for the length function of D_n with respect to the set $\mathcal{R}_d$. The length function for Σ_{2n} with respect to the set T of all transpositions is denoted by L.

Lemma 3.2. *The fixed space $F((\!(i_1, \dots, i_k)\!))$ has dimension $n - k + 1$ and is given by*

$$\{x \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k}\}$$

where x_i means $-x_{|i|}$ for $i < 0$. The fixed space $F([i_1, \dots, i_k])$ has dimension $n - k$ and is given by

$$\{x \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}$$

Proof. By inspection. q.e.d.

Lemma 3.3. *The l-length of a paired cycle $c\bar{c} = (\!(i_1, \dots, i_k)\!)$ is $k - 1$. Moreover, no minimal length factorisation of $c\bar{c}$ as a product of elements of $\mathcal{R}_c$ contains a generator of the form $[i]$.*

Proof. The fixed space $F(c\bar{c})$ has dimension $n-k+1$ by lemma 3.2 and thus $l(c\bar{c}) = n-(n-k+1) = k-1$.
If a minimal l-length factorisation of $c\bar{c}$ contained a term of the form $[i]$, we would obtain a factorisation of $c\bar{c}$ as a product of fewer than $2(k-2)+1 = 2k-3$ transpositions. As $L(c\bar{c}) = 2k-2$ this is impossible. q.e.d.

Lemma 3.4. *The l-length of $\gamma = [j_1, \dots, j_r]$ as a product of elements of $\mathcal{R}_c$ is r. Moreover any minimal length factorisation of γ as a product of elements of $\mathcal{R}_c$ contains exactly one generator of the form $[i]$.*

Proof. As the fixed space $F(\gamma)$ is $(n-r)$-dimensional by lemma 3.2, we find $l(\gamma) = n-(n-r) = r$.
As $L(\gamma) = 2r-1$, any factorisation of γ as a product of r elements of $\mathcal{R}_c$ can contain at most one generator of the form $[i]$. If such a factorisation contained no element of this form, we would have an expression for γ as a product of an even number of transpositions. But this contradicts the fact that the $2r$-cycle γ has odd parity in Σ_{2n}. q.e.d.

Proposition 3.5. *If $\alpha = c_1\bar{c}_1 \dots c_a\bar{c}_a\gamma_1 \dots \gamma_b \in C_n$ is a product of disjoint cycles then*

$$l(\alpha) = \sum_{i=1}^{a} l(c_i\bar{c}_i) + \sum_{j=1}^{b} l(\gamma_j)$$

Proof. By choosing a new basis from $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ if necessary, we may assume that $c_i = (j_{i-1}+1, j_{i-1}+2, \dots, j_i)$ and $\gamma_i = [k_{i-1}+1, k_{i-1}+2, \dots, k_i]$ where $1 = j_0 < j_1 < \cdots < j_a < j_a + 1 = k_0 < k_1 < \cdots < k_b = n$. Then $c_i\bar{c}_i$ (resp. γ_j) maps $U_i = \mathrm{span}(e_{j_{i-1}+1}, e_{j_{i-1}+2}, \dots, e_{j_i})$ (resp. $V_i = \mathrm{span}(e_{k_{i-1}+1}, e_{k_{i-1}+2}, \dots, e_{k_i})$) to itself and leaves all the other U's and V's pointwise fixed. As $c_i\bar{c}_i$ (resp. γ_j) fixes a 1 (resp. 0) dimensional subspace of U_i (resp. V_j), we see that α fixes an a-dimensional subspace of $\mathbb{R}^n$. Therefore $l(\alpha) = n-a$. Since $\sum(1 + l(c_i\bar{c}_i)) + \sum l(\gamma_j) = n$ by lemmas 3.3 and 3.4, the result follows. q.e.d.

Consider now the effect of multiplying $\alpha \in C_n$ on the right by a reflection $R = (\!(i, j)\!)$ or $R = [i]$. It is clear that only those cycles which contain an integer of R will be affected. The following example lists the possibilities and the corresponding changes in lengths.

Example 3.6. *The following four identities can be verified directly.*

$$\begin{aligned}
[i_1, i_2, \dots, i_k][i_k] &= (\!(i_1, i_2, \dots, i_k)\!) \\
[i_1, i_2, \dots, i_k](\!(i_j, i_k)\!) &= [i_1, \dots, i_j](\!(i_{j+1}, i_{j+2}, \dots, i_k)\!) \\
(\!(i_1, i_2, \dots, i_k)\!)(\!(i_j, i_k)\!) &= (\!(i_1, \dots, i_j)\!)(\!(i_{j+1}, i_{j+2}, \dots, i_k)\!) \\
[i_1, \dots, i_j][i_{j+1}, \dots, i_k](\!(-i_j, i_k)\!) &= (\!(i_1, i_2, \dots, i_k)\!)
\end{aligned}$$

Since each reflection has order 2*, the following identities are immediate.*

$$\begin{aligned}
[i_1, i_2, \dots, i_k] &= (\!(i_1, i_2, \dots, i_k)\!)[i_k] \\
[i_1, i_2, \dots, i_k] &= [i_1, \dots, i_j](\!(i_{j+1}, i_{j+2}, \dots, i_k)\!)(\!(i_j, i_k)\!) \\
(\!(i_1, i_2, \dots, i_k)\!) &= (\!(i_1, \dots, i_j)\!)(\!(i_{j+1}, i_{j+2}, \dots, i_k)\!)(\!(i_j, i_k)\!) \\
[i_1, \dots, i_j][i_{j+1}, \dots, i_k] &= (\!(i_1, i_2, \dots, i_k)\!)(\!(-i_j, i_k)\!)
\end{aligned}$$

By proposition 3.5, we see that

$$\begin{aligned}
l([i_1, i_2, \dots, i_n]) &= l((\!(i_1, i_2, \dots, i_n)\!)) + 1 \\
l([i_1, i_2, \dots, i_n]) &= l([i_1, \dots, i_j](\!(i_{j+1}, i_{j+2}, \dots, i_n)\!)) + 1 \\
l((\!(i_1, i_2, \dots, i_n)\!)) &= l((\!(i_1, \dots, i_j)\!)(\!(i_{j+1}, i_{j+2}, \dots, i_n)\!)) + 1 \\
l([i_1, \dots, i_j][i_{j+1}, \dots, i_k]) &= l((\!(i_1, i_2, \dots, i_k)\!)) + 1
\end{aligned}$$

Definition 3.7. *Let $\sigma = c_1c_2\cdots c_k$ and $\tau = d_1d_2\cdots d_l$ be two products of disjoint cycles in Σ_{2n}. We say that σ is contained in τ (and write $\sigma \subset \tau$) if for each i we can find j such that the set of integers in the cycle c_i is a subset of the set of integers in the cycle d_j. This notion restricts to give a notion of containment for elements of C_n.*
A reflection $(\!(i, j)\!)$ is s-contained in $\alpha = c_1\bar{c}_1 \dots c_a\bar{c}_a\gamma_1 \dots \gamma_b \in C_n$ (and we write $(\!(i, j)\!) \sqsubset \alpha$) if i is contained in γ_k and j is contained in γ_l for some $k \neq l$.

Lemma 3.8. *Let $\alpha \in C_n$ and $R \in \mathcal{R}_c$. Then $R \leq \alpha$ if and only if $R \subset \alpha$ or $R \sqsubset \alpha$.*

Proof. By proposition 3.5 and the calculations in example 3.6 we see that $l(\alpha R) < l(\alpha)$ if and only if $R \subset \alpha$ or $R \sqsubset \alpha$. Since $R \leq \alpha$ if and only if $l(\alpha R) < l(\alpha)$, the lemma follows. q.e.d.

4. The lattice property

In this section we show that the interval $[1, \gamma]$ in $(W \leq)$ is a lattice for $W = C_n, D_n$ and γ a Coxeter element in W. Since all Coxeter elements in W are conjugate we can choose our favourite one in each case.

Definition 4.1. *We choose the Coxeter elements γ_C in C_n and γ_D in D_n given by $\gamma_C = [1, 2, \dots, n]$ and $\gamma_D = [1][2, 3, \dots, n]$.*

Proposition 4.2. *Write the Coxeter element $\gamma_C \in C_n$ (resp. $\gamma_D \in D_n$) as $\gamma_C = R_1 R_2 \dots R_n$ (resp. $\gamma_D = R_1 R_2 \dots R_n$) for reflections $R_1, \dots, R_n$ in $\mathcal{R}_c$ (resp. $\mathcal{R}_d$) and let b_i denote the number of balanced cycles in $R_1 R_2 \cdots R_i$. Then there exists i_0 such that $b_i = 0$ for $i < i_0$ and $b_i = 1$ (resp. $b_i = 2$) for $i \geq i_0$. In the D_n case, if $b_i = 2$ then one of the balanced cycles in $R_1 \cdots R_i$ must be* $[1]$.

Proof. By example 3.6, if the multiplication of $\alpha \in C_n$ by $R \in \mathcal{R}_c$ increases the number of balanced cycles then $l(\alpha R) = l(\alpha) + 1$ and αR contains either 1 or 2 balanced cycles more than α. Conversely, if multiplication of α by R decreases either the number of balanced cycles or the size of a balanced cycle, then $l(\alpha R) = l(\alpha) - 1$. Since $l(R_1 \cdots R_i) + 1 = l(R_1 \cdots R_{i+1})$ it follows that $b_{i+1} - b_i \in \{0, 1, 2\}$. As γ_C consists of a single balanced cycle, the claim for C_n is immediate. For γ_D, none of the R_i can be of the form $[j]$ and hence $b_{i+1} - b_i$ cannot be 1. As the passage from $R_1 \cdots R_i$ to $R_1 \cdots R_{i+1}$ cannot decrease the size of any balanced cycle and as γ_D contains the balanced cycle $[1]$, this cycle must be present in $R_1 \cdots R_i$ for each $i \geq i_0$. q.e.d.

Corollary 4.3. *If $\alpha \leq \gamma_C$ in C_n then α has at most one balanced cycle. If $\beta \leq \gamma_D$ in D_n then β has either no balanced cycles or two balanced cycles. In the latter case, one of these balanced cycles is* $[1]$.

4.1. **The C_n lattice.** Set $\gamma = \gamma_C = [1, 2, \dots, n]$.

Definition 4.4. *The action of γ defines a cyclic order on the set $A = \{1, \dots, n, -1, \dots, -n\}$ in which the successor of i is $\gamma(i)$ (thus 1 is the successor of $-n$). An ordered set of elements $i_1, i_2, \dots, i_s$ in A is oriented consistently (with the cyclic order on A) if there exist integers $0 < r_2 < \dots < r_s \leq 2n - 1$ such that $i_j = \gamma^{r_j}(i_1)$ for $j = 2, \dots, s$. A cycle $(\!(i_1, \dots, i_s)\!)$ or $[i_1, \dots, i_s]$ is oriented consistently if the ordered set $i_1, \dots, i_s, -i_1, \dots, -i_s$ in A is oriented consistently.*

Definition 4.5. *Two disjoint reflections $R_1 = (\!(i, j)\!)$ and $R_2 = (\!(k, l)\!)$ (resp. $R_2 = [k]$) are said to cross if one of the following four ordered sets is oriented consistently in A: i, k, j, l or $i, -k, j, -l$ or k, i, l, j or $k, -i, l, -j$ (resp. $i, k, j, -k$ or $i, -k, j, k$ or $k, i, -k, j$ or $-k, i, k, j$). Two disjoint cycles ζ_1 and ζ_2 in C_n are said to cross if there exist crossing reflections R_1 and R_2 which are contained in ζ_1 and ζ_2 respectively. An element $\sigma \in C_n$ is called* crossing *if some pair of disjoint cycles of σ cross. Otherwise σ is* non-crossing.

Proposition 4.6. *If $\sigma \in C_n$ satisfies $\sigma \leq \gamma$ then the cycles of σ are oriented consistently and are noncrossing.*

Proof. We will proceed by induction on $n - l(\sigma)$. If $l(\sigma) = n$ then $\sigma = \gamma$ and the two conditions of the conclusion are satisfied.
We assume therefore that the proposition is true for $\tau \in C_n$ with $n - l(\tau) = 0, 1, \ldots, k-1$ and that $\sigma \leq \gamma$ satisfies $l(\sigma) = n - k$. By definition there is an expression for γ as a product of n reflections $\gamma = R_1 R_2 \ldots R_{n-k} R R_{n-k+2} \ldots R_n$ with $\sigma = R_1 R_2 \ldots R_{n-k}$. We define $\tau = \sigma R$ so that $l(\tau) = l(\sigma) + 1$ and $\tau \leq \gamma$. By induction, the cycles of τ are noncrossing and oriented consistently with γ.
We know that R is either of the form $(\!(i, j)\!)$ or $[i]$ and that $R \leq \tau \leq \gamma$. Lemma 3.8 thus implies that R is contained in some paired cycle or some balanced cycle of τ. The effect of multiplying this cycle by R is thus described by one of the first three equations in Example 3.6. Since the cycles of τ are noncrossing and oriented consistently with γ, we see that the same is true for σ. q.e.d.

Proposition 4.7. *Let $\sigma \in C_n$. If the cycles of σ are oriented consistently and are noncrossing then $\sigma \leq \gamma$.*

Proof. Assume that $\sigma \in C_n$ satisfies the two hypotheses of the proposition. Write $\sigma = c_1 \bar{c}_1 \ldots c_a \bar{c}_a \gamma_1 \ldots \gamma_b$ and set $t(\sigma) = a + b$. We proceed by induction on $t(\sigma)$. If $t(\sigma) = 1$ then either σ consists of a single balanced cycle or a single paired cycle. In the former case, consistent orientation implies that $\sigma \leq \gamma$. In the latter case, consistent orientation implies that $\sigma = (\!(i, i+1, \ldots, n, -1, \ldots, -i+1)\!)$ for some i. As $l(\sigma) = n - 1$ and $\sigma[i-1] = \gamma$, we see that $\sigma \leq \gamma$.

Assume now that $t(\sigma) \geq 2$ and that the proposition is true for each element $\theta \in C_n$ with $t(\theta) < t(\sigma)$. If σ contains a balanced cycle, the non-crossing hypothesis implies that there can be only one which we denote $\tau = [i_1, \ldots, i_r]$. Otherwise let $\tau = (\!(i_1, \ldots, i_r)\!)$ be some paired cycle of σ. As $\sigma \neq \tau$, there exists an i_k whose successor does not lie in $\{\pm i_1, \ldots, \pm i_r\}$. By choosing one of the other $2r - 1$ cycle expressions for τ if necessary, we may assume that the successor j_1 of i_r does not lie in $\{\pm i_1, \ldots, \pm i_r\}$. Let $\rho = (\!(j_1, \ldots, j_s)\!)$ be the paired cycle of σ which contains j_1 and let $R = (\!(i_r, j_s)\!)$. Then $\sigma = \tau \rho \sigma_1 \ldots \sigma_k$ for some disjoint paired cycles $\sigma_1, \ldots, \sigma_k$ (some $k \geq 0$) and

$$\sigma R = \begin{cases} [i_1, \ldots, i_r, j_1, \ldots, j_s] \sigma_1 \ldots \sigma_k & \text{or} \\ (\!(i_1, \ldots, i_r, j_1, \ldots, j_s)\!) \sigma_1 \ldots \sigma_k. & \end{cases}$$

Note that $t(\sigma R) = t(\sigma) - 1$. As the cycles τ and ρ do not cross and each is oriented consistently, our choice of j_1 ensures that the ordered set $i_1, \ldots, i_r, j_1, \ldots, j_s, -i_1, \ldots, -i_r, -j_1, \ldots, -j_s$ is also oriented consistently.

Assume now that one of the cycles σ_e crosses the cycle $\tau\rho R$ of σR. Then there exist crossing reflections R_1 and R_2 contained in $\tau\rho R$ and σ_e respectively. Since σ_e is paired, R_2 is necessarily paired; $R_2 = ((c, d))$ say. Since σ is non-crossing, R_1 cannot be contained in τ or in ρ. There are three cases to consider

(1) $R_1 = ((i_a, j_b))$ for some $1 \leq a \leq r$ and $1 \leq b \leq s$.
(2) $R_1 = ((j_b, -j_b))$ for some $1 \leq b \leq s$ (τ is necesarily balanced).
(3) $R_1 = ((i_a, -j_b))$ for some $1 \leq b \leq s$ (τ is necesarily balanced).

By a suitable choice of the representative $R = ((c, d)) = ((d, c)) = ((-c, -d)) = ((-d, -c))$, the first case splits into two essential subcases: *(a)* the ordered set i_a, c, j_b, d is oriented consistently and *(b)* the ordered set c, i_a, d, j_b is oriented consistently. We know that c is not in $\{\pm i_1, \ldots, \pm i_r, \pm j_1, \ldots, \pm j_s\}$. In particular $c \neq i_r,\ j_1$. In case $(1a)$, if c precedes i_r, then $S = ((i_1, i_r))$ is contained in τ and crosses R_2, contradicting the fact that σ is non-crossing. Likewise, if c follows i_r then c follows j_1 and $S = ((j_1, j_b))$ is contained in ρ and crosses R_2, again contradicting the fact that σ is non-crossing. Thus case $(1a)$ is impossible. A similar argument shows that case $(1b)$ is also impossible.
As in case 1, case 2 splits into two subcases: *(a)* the ordered set $j_b, c, -j_b, d$ is oriented consistently and *(b)* the ordered set $c, j_b, d, -j_b$ is oriented consistently. In case $(2a)$, if c precedes $-i_r$ then the ordered set $i_r, j_b, c, -i_r, d$ is oriented consistently and hence $((c, d))$ crosses $[-i_r] \subset \tau$. But this contradicts the fact that σ is non-crossing. If c follows $-i_r$, then c necessarily succeeds $-j_1$ and we find that the ordered set $-j_1, c, -j_b, d$ is consistently oriented. Thus $((c, d))$ crosses $((-j_1, -j_b)) \subset \rho$, again contradicting the fact that σ is non-crossing. Thus case $(2a)$ is impossible. A similar argument shows that case $(2b)$ is also impossible.
Finally, case 3 also splits into two subcases: *(a)* the ordered set $i_a, c, -j_b, d$ is oriented consistently and *(b)* the ordered set $c, i_a, d, -j_b$ is oriented consistently. We show that $(3b)$ is impossible (the proof that case $(3a)$ is impossible is similar). We are given that the ordered set $c, i_a, d, -j_b$ is oriented consistently. If d precedes $-i_a$ then $((c, d))$ crosses $[i_a]$ in σ, a contradiction. Therefore d follows $-i_a$. If d now precedes $-i_r$, then the ordered set $c, -i_a, d, -i_r$ is oriented consistently. Hence $((-i_a, -i_r))$ crosses $((c, d))$ in σ, a contradiction. Therefore d follows $-i_r$ and hence $-j_1$. But now $((-j_1, -j_b))$ crosses $((c, d))$ in σ, a contradiction. Thus case $(3b)$ is impossible.
We conclude that the cycles $\tau\rho R$ and σ_e do not cross. Since no two distinct elements of $\sigma_1, \ldots, \sigma_k$ cross (because σ is assumed non-crossing), it follows that σR is non-crossing. As $t(\sigma R) = t(\sigma) - 1$ and the cycles

of σR are oriented consistently, it follows by induction that $\sigma R \leq \gamma$. Thus there exist reflections $R_1, \ldots, R_k$ with $k = n - l(\sigma R)$ and

$$\sigma R R_1 \ldots R_k = \gamma \tag{5}$$

As $l(\sigma R) = l(\sigma) + 1$ by lemmas 3.3 and 3.4 and proposition 3.5, we see that $k + 1 = n - l(\sigma)$. Hence equation (5) also implies that $\sigma \leq \gamma$. q.e.d.

Lemma 4.8. *If $\sigma \leq \gamma$ and $\tau \leq \gamma$ then $\sigma \leq \tau$ if and only if $\sigma \subset \tau$.*

Proof. Follows from Lemma 2.5 and lemma 3.8. q.e.d.

Combining the previous three results yields the following Theorem.

Theorem 4.9. *Let NCP denote Reiner's non-crossing partition lattice for the C_n group from* [8]*. The mapping*

$$: \{\alpha \in C_n : \alpha \leq \gamma\} \longrightarrow NCP$$

which takes α to the noncrossing partition defined by its cycle structure is a bijective poset map. In particular, $\{\alpha \in C_n : \alpha \leq \gamma\}$ is a lattice.

4.2. **The D_n lattice.** Set $\gamma = \gamma_D = [1][2, 3, \ldots, n]$ and suppose $\alpha \leq \gamma$. Recall from Corollary 4.3 that for such an α either $[1][k] \leq \alpha$ for some $k \in \{2, 3, \ldots, n\}$ or l and $-l$ are in different α orbits for all $l \in \{1, 2, \ldots, n\}$. In the former case we will call α *balanced* and in the latter case we will call α *paired*.

We note that lattices are associated to the groups C_n and D_n in [8]. We have shown the Reiner C_n lattices are isomorphic to ours. However the Reiner D_n lattices are not the same as the ones we consider. In particular, the Reiner D_n lattices are subposets of the Reiner C_n lattices.

To show that the interval $[I, \gamma]$ in D_n is a lattice we will compute $\alpha \wedge \beta$ for $\alpha, \beta \leq \gamma$. Since the poset is finite the existence of least upper bounds follows. We will consider different cases depending on the types of α and β. In all cases we will construct a candidate σ for $\alpha \wedge \beta$ and show that $\sigma \in D_n$, $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$. Since the reverse inclusion is immediate it follows from Lemma 2.6 that $\sigma = \alpha \wedge \beta$.

Note 4.10. *In this section we will frequently pass between the posets determined by C_n, D_n and several other finite reflection subgroups of C_n. As the partial order on each of these groups is the restriction of the partial order on $O(n)$, we can use the same symbol $\leq$ to denote the partial order in each case. The reflection subgroup in question should be clear from the context.*

Suppose first that both α and β are balanced. Since $D_n \subset C_n$ and C_{n-1} can be identified with the subgroup of C_n which fixes 1, each balanced element of D_n can be used to define a balanced element of C_{n-1}, that is, an element containing a balanced cycle. Thus we define the balanced C_{n-1} elements α' and β' by

$$\alpha = [1]\alpha' \quad \text{and} \quad \beta = [1]\beta'$$

and the C_{n-1} element $\sigma' = \alpha' \wedge \beta'$, where the meet is taken in C_{n-1}. Now σ' may or may not be balanced. If σ' is balanced define the C_n element σ by $\sigma = [1]\sigma'$. If σ' is not balanced set $\sigma = \sigma'$.

Proposition 4.11. *If α and β are balanced and σ is defined as above then $\sigma \in D_n$, $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. We show that $\sigma \in D_n$ and $\sigma \leq \alpha$. The proof that $\sigma \leq \beta$ is completely analogous. First consider the case where σ' is balanced. Thus $[k] \leq \sigma' \leq \alpha'$ in C_{n-1} for some k satisfying $2 \leq k \leq n$. So we can find reflections $R_1, \ldots, R_s$ in C_{n-1} with

$$\alpha' = R_1 R_2 \ldots R_s, \qquad \sigma' = R_1 R_2 \ldots R_t, \qquad R_1 = [k],$$

where $l(\alpha') = s \geq t = l(\sigma')$. Since $\alpha' \in C_{n-1}$, Lemma 3.4 gives $R_2, \ldots, R_s$ all of the form $(\!(i,j)\!)$ or $(\!(i,-j)\!)$ for $2 \leq i < j \leq n$. In particular, these reflections lie in D_n. Now α is of length $s+1$ in C_n and

$$\begin{aligned} \alpha &= [1] R_1 R_2 \ldots R_t R_{t+1} \ldots R_s \\ &= [1][k] R_2 \ldots R_t R_{t+1} \ldots R_s \\ &= (\!(1,k)\!)(\!(1,-k)\!) R_2 \ldots R_t R_{t+1} \ldots R_s. \end{aligned}$$

This last expression only uses D_n reflections so that

$$\sigma = (\!(1,k)\!)(\!(1,-k)\!) R_2 \ldots R_t \leq \alpha \quad \text{in } D_n.$$

Next we consider the case where σ' is paired. Here $\sigma' \leq \alpha'$ and α' is balanced so we can find reflections $R_1, \ldots, R_s$ in C_{n-1} with

$$\alpha' = R_1 R_2 \ldots R_s, \qquad \sigma' = R_1 R_2 \ldots R_t,$$

where $l(\alpha') = s > t = l(\sigma')$ and exactly one of $R_{t+1}, \ldots, R_s$ is of form $[k]$. Since $R[k] = [k]([k]R[k])$, we can assume $R_{t+1} = [k]$. Note also that $R_1, \ldots, R_t$ are each of the form $(\!(i,j)\!)$ or $(\!(i,-j)\!)$ for $2 \leq i < j \leq n$ and hence commute with $[1]$ in C_n. Thus we can write the following identities in C_n.

$$\begin{aligned} \alpha &= [1] R_1 R_2 \ldots R_t [k] R_{t+2} \ldots R_s \\ &= R_1 \ldots R_t [1][k] R_{t+2} \ldots R_s \\ &= R_1 \ldots R_t (\!(1,k)\!)(\!(1,-k)\!) R_{t+2} \ldots R_s. \end{aligned}$$

This last expression only uses D_n reflections so that $\sigma \leq \alpha$ in D_n.

Finally we show that $S_\alpha \cap S_\beta \subset S_\sigma$. First suppose σ' is balanced and $R \in S_\alpha \cap S_\beta$. Thus R is a reflection satisfying $R \leq \alpha, \beta$. If R is of the form $(\!(1,k)\!)$, then $[1][k] \leq \alpha, \beta$ since k must belong to a balanced cycle of both α and β. Thus $[k] \leq \alpha', \beta'$ so that $[k] \leq \sigma'$ and $[1][k] \leq \sigma$, which gives $(\!(1,k)\!) \leq \sigma$ as required. If R is not of form $(\!(1,k)\!)$ then $R \leq \alpha, \beta$ implies $R \leq \alpha', \beta'$ so that $R \leq \sigma'$ and $R \leq \sigma$.

In the case where σ' is paired, $R \leq \alpha, \beta$ implies R must be of form $(\!(i,j)\!)$ or $(\!(i,-j)\!)$ for $2 \leq i < j \leq n$ so that $R \leq \alpha', \beta'$ giving $R \leq \sigma' = \sigma$. q.e.d.

Since we have completed the case where both α and β are balanced we will assume from now on that α is paired. We note some consequences of this fact which will apply in the remaining cases. The fact that α is paired means that $\alpha \leq (\!(1,k)\!)\gamma$ or $\alpha \leq (\!(1,-k)\!)\gamma$ for some $k \in \{2,3,\ldots,n\}$. Since conjugation by the C_{n-1} element $[2,\ldots,n]$ is a poset isomorphism of the interval $[I,\gamma]$ in D_n, we may assume for convenience of notation that $k=-2$ so that

$$\alpha \leq (\!(1,-2)\!)[1][2,\ldots,n] = (\!(1,2,\ldots,n)\!).$$

If we let $\delta = (\!(1,2,\ldots,n)\!)$ then a reflection R in D_n satisfies $R \leq \delta$ if and only if $R \subset \delta$. Thus we can identify the interval $[I,\delta]$ in D_n with the set of non-crossing partitions of $\{1,2,\ldots,n\}$. Recall that a non-crossing partition of the ordered set $\{a_1,a_2,\ldots,a_n\}$ is a partition with the property that whenever

$$1 \leq i < j < k < l \leq n$$

with a_i, a_k belonging to the same block B_1 and a_j, a_l belonging to the same block B_2 we have $B_1 = B_2$. If $\alpha \wedge \beta$ exists, it will satisfy

$$\alpha \wedge \beta \leq \alpha \leq (\!(1,2,\ldots,n)\!)$$

and so will correspond to a noncrossing partition of $\{1,2,\ldots,n\}$. Accordingly, we define a reflexive, symmetric relation on $\{1,2,\ldots,n\}$ by

$$i \sim j \quad \Leftrightarrow \quad i = j \quad \text{or} \quad (\!(i,j)\!) \leq \alpha, \beta.$$

We need to show that $\sim$ is transitive and hence is an equivalence relation. We then show that the resulting partition of $\{1,2,\ldots,n\}$ is non-crossing and determines an element σ of D_n which satisfies $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.

Suppose that α is paired and β is balanced. Recall that β has two balanced cycles, one of which is $[1]$. For convenience of terminology we will call the other balanced cycle the second balanced cycle of β. As above we will have occasion to use the balanced element $\beta' \leq [2, \ldots, n]$ in C_{n-1} defined by $\beta = [1]\beta'$.

Proposition 4.12. *If α is paired and β is balanced then the relation $\sim$ above determines an element σ of D_n satisfying $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. First we establish the transitivity of the $\sim$ relation. Suppose i, j, k are distinct elements of $\{1, 2, \ldots, n\}$ with $i \sim j$ and $j \sim k$. Since $(\!(i, j)\!), (\!(j, k)\!) \leq \alpha$ we get $(\!(i, k)\!) \leq \alpha$ since α corresponds to a partition of $\{1, 2, \ldots, n\}$. If $1 \notin \{i, j, k\}$ then $(\!(i, j)\!), (\!(j, k)\!) \subset \beta$ (s-containment cannot arise) and it follows that $(\!(i, k)\!) \leq \beta$. If $i = 1$, then $(\!(i, j)\!) \leq \beta$ means that $(\!(i, j)\!) \sqsubset \beta$ so that j belongs to the second balanced cycle of β. Since $j \sim k \neq 1$, k also belongs to this second balanced cycle and $(\!(i, k)\!) \leq [1][j, k] \leq \beta$. If $j = 1$, then both i and k belong to the second balanced cycle of β. Hence $(\!(i, k)\!) \leq \beta$. The case $k = 1$ is analogous to the case $i = 1$.

To show that the partition of $\{1, \ldots, n\}$ defined by $\sim$ is non-crossing suppose $1 \leq i < j < k < l \leq n$ with

$$(\!(i, k)\!), (\!(j, l)\!) \leq \alpha, \beta.$$

Since α corresponds to a noncrossing partition we have $(\!(i, j, k, l)\!) \leq \alpha$. If $i = 1$, then k belongs to the second balanced cycle and $[k] \leq \beta'$ in C_{n-1}. Since $1 < j < k < l$, $(\!(j, l)\!) \leq \beta'$ and $\beta' \leq [2, \ldots, n]$ in C_{n-1}, the crossing pair consisting of (j, l) and $(k, -k)$ must lie in the same β' cycle. Thus $[j, k, l] \leq \beta'$ and $(\!(1, j, k, l)\!) \leq [1][j, k, l] \leq \beta$. If $i \neq 1$, then $(\!(i, k)\!), (\!(j, l)\!) \leq \beta'$ and since $\beta' \leq [2, \ldots, n]$ in C_{n-1}, $(\!(i, j, k, l)\!) \leq \beta'$ by proposition 4.6, giving $(\!(i, j, k, l)\!) \leq \beta$.

Thus the relation $\sim$ defines a noncrossing partition of $\{1, 2, \ldots, n\}$ and hence determines an element σ of D_n. By the definition of $\sim$ the element σ satisfies $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$. q.e.d.

Finally we consider the case where both α and β are paired.

Proposition 4.13. *If α and β are paired then the relation $\sim$ above determines an element σ of D_n satisfying $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. To establish the transitivity of $\sim$ in this case let i, j, k be distinct elements of $\{1, 2, \ldots, n\}$ with $i \sim j$ and $j \sim k$. As in the previous proposition, $(\!(i, k)\!) \leq \alpha$ follows immediately. Since β is paired, $i \sim j$ and $j \sim k$ mean that i, j, k belong to the same cycle of β so that $(\!(i, k)\!) \leq \beta$ also.

To show that the partition of $\{1,\ldots,n\}$ defined by $\sim$ is noncrossing suppose $1 \le i < j < k < l \le n$ with

$$((i,k)), ((j,l)) \le \alpha, \beta.$$

Since α corresponds to a noncrossing partition we have $((i,j,k,l)) \le \alpha$. The element β is paired so we can assume $\beta \le \tau = ((1,m))\gamma$ or $\beta \le \tau = ((1,-m))\gamma$, for some $m \in \{2,3,\ldots,n\}$. Looking at the case $\tau = ((1,m))\gamma$ first we get

$$\tau = ((1,-m,-m-1,\ldots,-n,2,3,\ldots,m-1)).$$

Since $\beta \le \tau$ the element β corresponds to a noncrossing partition of the ordered set $\{1,-m,-m-1,\ldots,-n,2,3,\ldots,m-1\}$. Since $1 \le i < j < k < l \le n$, we deduce that either

$$1 \le i < j < k < l \le m-1 \quad \text{or} \quad m \le i < j < k < l \le n.$$

Since β corresponds to a noncrossing partition of the ordered set

$$\{1,-m,-m-1,\ldots,-n,2,3,\ldots,m-1\}$$

and $((i,k)), ((j,l)) \le \beta$ it follows in either case that $((i,j,k,l)) \le \beta$. The case $\tau = ((1,-m))\gamma$ is similar. Here

$$\tau = ((1,m,m+1,\ldots,n,-2,-3,\ldots,-m+1)),$$

and again we can deduce $((i,j,k,l)) \le \beta$.

Thus $\sim$ defines a noncrossing partition of $\{1,2,\ldots,n\}$ and hence an element σ in D_n satisfying $\sigma \le \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$ as in previous proposition. q.e.d.

Combining the results of this subsection we obtain the following theorem.

Theorem 4.14. *The interval $[I,\gamma]$ in D_n is a lattice.*

5. Poset groups and K$(\pi,1)$'s.

Definition 5.1. *If W is a finite Coxeter group and $\gamma \in W$ we define the* poset group $\Gamma = \Gamma(W,\gamma)$ *to be the group with the following presentation. The generating set for Γ consists of a copy of the set of non-identity elements in $[I,\gamma]$. We will denote by $\{w\}$ the generator of Γ corresponding the element $w \in (I,\gamma]$. The relations in Γ are all identities of the form $\{w_1\}\{w_2\} = \{w_3\}$, where w_1, w_2 and w_3 lie in $(I,\gamma]$ with $w_1 \le w_3$ and $w_2 = w_1^{-1}w_3$.*

Since none of the relations involve inverses of the generators, there is a semigroup, which we will denote by $\Gamma_+ = \Gamma_+(W,\gamma)$, with the same presentation. As in section 5 of [3], we define a *positive* word in Γ to be a word in the generators that does not involve the inverses of the generators. We say two positive words A and B are *positively equal*, and we write $A \doteq B$, if A can be transformed to B through a sequence of positive words, where each word in the sequence is obtained from the previous one by replacing one side of a defining relator by the other side. Since the interval $(I, \gamma]$ inherits the reflection length from W we use this to associate a *length* to each generator of $\Gamma(W, \gamma)$ and hence a length $l(A)$ to each positive word A. It is immediate that positively equal words have the same length.

From now on we only consider those pairs (W, γ) with the property that *the interval* $[I, \gamma]$ *in* W *forms a lattice*. It is clear that the results stated for the braid group in sections 5 and 6 of [3] apply to poset groups under this extra assumption. We will review them briefly below.

In [3] it is shown that this lattice condition is satisfied when W is a Coxeter group of type A_n and γ is a Coxeter element. In section 4 above we have shown that the lattice condition is satisfied when W is a Coxeter group of type C_n or D_n and γ is a Coxeter element. When the Coxeter group is generated by two reflections the lattice condition is automatic for any γ. When the Coxeter group is generated by three reflections the lattice condition reduces to checking the only case where

$$\alpha \wedge \beta \notin \{\alpha, \beta, \gamma\}.$$

This occurs when α and β are distinct reflections and have at least one common upper bound of length 2. Any such length 2 element δ must have $F(\delta)$ coinciding with the unique line of intersection of the two reflection planes. Hence δ is unique. This is precisely the ingredient which makes the metric constructed in [4] have non-positive curvature. The following result is taken from [3]. Its proof is the same.

Lemma 5.2. *Assume that the interval* $[I, \gamma]$ *forms a lattice and suppose* $a, b, c \leq \gamma$. *We define nine elements* d, e, f, g, h, k, l, m *and* n *in* $[I, \gamma]$ *by the equations*

$$a \vee b = ad = be, \quad b \vee c = bf = cg, \quad c \vee a = ch = ak$$

and

$$a \vee b \vee c = (a \vee b)l = (b \vee c)m = (c \vee a)n.$$

Then we can deduce

$$e \vee f = el = fm, \quad d \vee k = dl = kn, \quad h \vee g = hn = gm.$$

The statements and proofs of the results of section 5 and section 6 of [3] generalize in a straightforward manner to the current setting. In particular, we have the following definitions and results.

Lemma 5.3. *The semigroup associated to* Γ *has right and left cancellation properties.*

Lemma 5.4. *Suppose* $a_1, a_2, \ldots, a_k \leq \gamma$ *in* W*,* P *is positive and*

$$P \doteq X_1\{a_1\} \doteq \ldots \doteq X_k\{a_k\}$$

with X_i *all positive. Then there is a positive word* Z *satisfying*

$$P \doteq Z\{a_1 \vee \cdots \vee a_k\}.$$

Theorem 5.5. *In* Γ*, if two positive words are equal they are positively equal. In other words, the semigroup* Γ_+ *embeds in* Γ*.*

As in [3] we define an abstract simplicial complex $X(W, \gamma)$ for each $\Gamma(W, \gamma)$.

Definition 5.6. *We let* $X = X(W, \gamma)$ *be the abstract simplicial complex with vertex set* Γ*, which has a* k*-simplex on the subset* $\{g_0, g_1, \ldots, g_k\}$ *if and only if* $g_i = g_0\{w_i\}$ *for* $i = 1, 2, \ldots, k$ *where*

$$I < w_1 < \cdots < w_k \leq \gamma \quad \text{in } W.$$

There is an obvious simplicial action of Γ on X given by

$$g \cdot \{g_0, g_1, \ldots, g_k\} = \{gg_0, gg_1, \ldots, gg_k\}.$$

The main result of section 6 of [3] also holds for these poset groups.

Theorem 5.7. $X(W, \gamma)$ *is contractible.*

If we define $K = K(W, \gamma)$ to be the quotient space $K = \Gamma\backslash X$, then K is a $K(\Gamma, 1)$.

We finish this section with an example of a poset group $\Gamma(W, \gamma)$, with $[I, \gamma]$ a lattice but γ not a Coxeter element in W.

Example 5.8. *Let* $W = C_2$ *and* $\gamma = [1][2]$*. The group* $\Gamma(C_2, \gamma)$ *has presentation*

$$\langle a, b, c, d, x \mid x = ab = ba = cd = dc \rangle$$

where $a = \{[1]\}$*,* $b = \{[2]\}$*,* $c = \{(\!(1, 2)\!)\}$*,* $d = \{(\!(1, -2)\!)\}$ *and* $x = \{[1][2]\}$*. From the presentation we see that* Γ *is an amalgamated free product of a copy* $\mathbb{Z} \times \mathbb{Z}$ *generated by* a *and* b *with a copy* $\mathbb{Z} \times \mathbb{Z}$ *generated by* c *and* d *over the infinite cyclic subgroup generated by* x*. The above construction gives a two-dimensional contractible universal cover for the presentation 2-complex which can be shown to be simplicially isomorphic to* $X(C_2, [1, 2])$*.*

6. Group Presentations.

In this section we prove that the poset groups $\Gamma(W, \gamma)$ of section 5 are isomorphic to the Artin groups $A(W)$ for W of type C_n or D_n and γ the appropriate Coxeter element. The proof is based on the following surprising property that these Artin groups share with the braid group. If $X = x_1 x_2 \ldots x_n$ is the product of the standard Artin generators then there is a finite set of elements in $A(W)$ which is invariant under conjugation by X. Moreover under the canonical surjection from $A(W)$ to W this set is taken bijectively to the set of reflections in W. The following lemma is a straightforward generalisation of Lemma 4.5 of [3].

Lemma 6.1. *The poset group $\Gamma(W, \gamma)$ is isomorphic to the abstract group generated by the set of all $\{R\}$, for R a reflection in $[I, \gamma]$, subject to the relations*

$$\{R_1\}\{R_2\} \ldots \{R_n\} = \{S_1\}\{S_2\} \ldots \{S_n\},$$

for R_i, S_j reflections satisfying

$$\gamma = R_1 R_2 \ldots R_n \quad \text{and} \quad \gamma = S_1 S_2 \ldots S_n,$$

where $n = l(\gamma)$.

We will refer to $\{w\} \in \Gamma(W, \gamma)$ as the lift of $w \in W$ whenever $w \leq \gamma$. In particular, we will refer to $\{w\}$ as a reflection lift whenever w is a reflection.

Since the Artin groups of type C_n and D_n both contain copies of the n-strand braid group B_n we collect here some facts about the braid group which will be useful. We recall that B_n is the group with generating set $x_2, x_3, \ldots x_n$ and defining relations

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad \text{for} \quad 2 \leq i \leq n-1,$$

$$x_i x_j = x_j x_i \quad \text{for} \quad |j - i| \geq 2.$$

We define $x_{i,j}$ and $Y_{i,j}$, for $1 \leq i < j \leq n$ by

$$Y_{i,j} = x_{i+1} \ldots x_j, \quad \text{and} \quad Y_{i,j} = Y_{i+1,j} x_{i,j}.$$

Then Lemma 4.2 of [3] gives, for $1 \leq i < j < k \leq n$,

$$x_{i,j} x_{j,k} = x_{j,k} x_{i,k} = x_{i,k} x_{i,j}.$$

Since $x_k = x_{k-1,k}$ it follows that $x_{i,j} Y_{i,j-1} = Y_{i,j}$ and that

$$x_k Y_{i,j} = Y_{i,j} x_{k-1} \quad \text{for} \quad i + 2 \leq k \leq j.$$

When $k = i + 1$ we have $x_{i+1} Y_{i,j} = x_{i+1} Y_{i+1,j} x_{i,j} = Y_{i,j} x_{i,j}$.

6.1. **The C_n case.** The Artin group $A(C_n)$ has a presentation with generating set $x_1, x_2, \dots x_n$, subject to the relations

$$x_1x_2x_1x_2 = x_2x_1x_2x_1$$

$$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$$

whenever $1 < i < n$ and

$$x_ix_j = x_jx_i$$

whenever $|j-i| \geq 2$.

Definition 6.2. *We define a function ϕ from the generators of $A(C_n)$ to $\Gamma(C_n, \gamma)$ by*

$$x_1 \mapsto \{[1]\}, x_2 \mapsto \{(\!(1,2)\!)\}, x_3 \mapsto \{(\!(2,3)\!)\}, \dots, x_n \mapsto \{(\!(n-1,n)\!)\}$$

Lemma 6.3. *The function ϕ determines a well-defined and surjective homomorphism.*

Proof: The relations involving $\phi(x_1)$ hold in $\Gamma(C_n, \gamma)$ by virtue of the following identities in $\Gamma(C_n, \gamma)$.

$$\begin{aligned} \{[1]\}\{(\!(1,2)\!)\}\{[1]\}\{(\!(1,2)\!)\} &= \{[1,2]\}\{[1,2]\} \\ &= \{(\!(1,2)\!)\}\{[2]\}\{(\!(1,-2)\!)\}\{[1]\} \\ &= \{(\!(1,2)\!)\}\{[1,2]\}\{[1]\} \\ &= \{(\!(1,2)\!)\}\{[1]\}\{(\!(1,2)\!)\}\{[1]\} \end{aligned}$$

$$\{[1]\}\{(\!(i,i+1)\!)\} = \{(\!(i,i+1)\!)\}\{[1]\}, \quad \text{for} \quad i \geq 2.$$

The image of the subgroup generated by $\{x_2, \dots, x_n\}$ lies in the copy of the braid group corresponding to $\Sigma_n < C_n$ so that the relations not involving $\phi(x_1)$ hold by Lemma 4.2 and Lemma 4.4 of [3]. Thus ϕ is well-defined.

To establish surjectivity, first note that

$$\{(\!(i, i+1, \dots, j)\!)\} = \phi(Y_{i,j}) \quad \text{and} \quad \{(\!(i,j)\!)\} = \phi(x_{i,j})$$

for $1 \leq i < j \leq n$ all lie in $\mathrm{im}(\phi)$. Next $\{[j]\} \in \mathrm{im}(\phi)$ since

$$\phi(x_1x_{1,j}) = \{[1]\}\{(\!(1,j)\!)\} = \{[1,j]\} = \{(\!(1,j)\!)\}\{[j]\}.$$

Finally, $\{(\!(i,-j)\!)\} \in \mathrm{im}(\phi)$ for $1 \leq i < j \leq n$ since

$$\{(\!(i,j)\!)\}\{[j]\} = \{[i,j]\} = \{[j]\}\{(\!(i,-j)\!)\}.$$

q.e.d.

To construct an inverse to ϕ we will use the presentation for $\Gamma(C_n, \gamma)$ given by lemma 6.1.

Definition 6.4. *We define a function θ from the generators of $\Gamma(C_n,\gamma)$ to $A(C_n)$ by*

$$\begin{array}{rcl} \{[1]\} & \mapsto & x_1\,,\ \{\!(\!(i,j)\!)\!\} \mapsto x_{i,j} \quad \text{for} \quad 1 \le i < j \le n, \\ \{[j]\} & \mapsto & y_j \ \text{ for } \ 2 \le j \le n, \quad \{\!(\!(i,-j)\!)\!\} \mapsto z_{i,j} \ \text{ for } \ 1 \le i < j \le n, \end{array}$$

where y_j is the unique element of $A(C_n)$ satisfying

$$x_1x_2\dots x_j = x_2\dots x_jy_j$$

and $z_{i,j}$ is the unique element of $A(C_n)$ satisfying

$$z_{i,j}y_i = y_ix_{i,j}.$$

The homomorphism determined by θ will be surjective since each x_i is the image of some reflection lift. We note that $Y_{i,j}y_j = y_iY_{i,j}$ for $1 \le i < j \le n$ if we define $y_1 = x_1$. To show that θ determines a well-defined homomorphism we first define the special element $X = x_1x_2\dots x_n$ in $A(C_n)$ and establish the following result.

Proposition 6.5. *For any reflection R in C_n,*

$$X\theta(\{R\})X^{-1} = \theta(\{\gamma R\gamma^{-1}\}).$$

Proof. Since $X = x_1Y_{1,n}$ and x_1 commutes with $x_3,\dots,x_n$, it follows that $Xx_i = x_{i+1}X$ for $2 \le i < n$ and $Xx_{i,j} = x_{i+1,j+1}X$ for $1 \le i < j < n$. This establishes the proposition for R of the form $(\!(i,j)\!)$ for $1 \le i < j < n$.

The identity $Xy_j = y_{j+1}X$ for $1 \le j < n$ is a consequence of the following calculation.

$$\begin{array}{rcl} Y_{2,j+1}Xy_j & = & x_2Y_{3,j+1}Xy_j = x_2XY_{2,j}y_j = x_2Xx_1Y_{2,j} \\ & = & x_2x_1x_2Y_{3,n}x_1Y_{2,j} = x_2x_1x_2x_1Y_{3,n}Y_{2,j} \\ & = & x_1x_2x_1x_2Y_{3,n}Y_{2,j} = x_1x_2XY_{2,j} = x_1x_2Y_{3,j+1}X \\ & = & x_1Y_{2,j+1}X = Y_{2,j+1}y_{j+1}X \end{array}$$

This establishes the proposition for R of the form $[j]$ for $1 \le i < n$.

Conjugating y_n by X gives x_1, since

$$Xy_n = (x_1x_2\dots x_n)y_n = x_1(x_2\dots x_ny_n) = x_1(x_1\dots x_n).$$

This establishes the proposition for the reflection $[n]$.

Next we show $Xx_{i,n} = z_{1,i+1}X$.

$$\begin{array}{rcl} z_{1,i+1}X & = & z_{1,i+1}x_1Y_{1,n} = x_1x_{1,i+1}Y_{1,n} = x_1x_{1,i+1}Y_{1,i}Y_{i,n} \\ & = & x_1Y_{1,i}x_{i+1}Y_{i,n} + x_1Y_{1,i}x_{i+1}Y_{i+1,n}x_{i,n} = x_1Y_{1,n}x_{i,n} = Xx_{i,n} \end{array}$$

This establishes the proposition for R of the form $(\!(i,n)\!)$ for $1 \le i < n$.

The identity $Xz_{i,j} = z_{i+1,j+1}X$ for $1 \leq i < j < n$ follows from the definition of $z_{i,j}$ and the corresponding identities for $x_{i,j}$ and y_i, which establishes the proposition for R of the form $⟪i, -j⟫$ for $1 \leq i < j < n$.

Next we observe that, for $3 \leq j \leq n$, $y_j z_{1,j} = x_{1,j} y_j$ because

$$\begin{aligned} Y_{1,j} y_j z_{1,j} x_1 &= x_1 Y_{1,j} z_{1,j} x_1 = x_1 Y_{1,j} x_1 x_{1,j} = x_1 x_2 Y_{2,j} x_1 x_{1,j} \\ &= x_1 x_2 x_1 Y_{2,j} x_{1,j} = x_1 x_2 x_1 Y_{1,j} = x_1 x_2 x_1 x_2 Y_{2,j} \\ &= x_2 x_1 x_2 x_1 Y_{2,j} = x_2 x_1 x_2 Y_{2,j} x_1 = x_2 x_1 Y_{1,j} x_1 \\ &= x_2 Y_{1,j} y_j x_1 = Y_{1,j} x_{1,j} y_j x_1. \end{aligned}$$

Since $Xz_{i,n} y_i = X y_i x_{i,n} = y_{i+1} z_{1,i+1} X = x_{1,i+1} y_{i+1} X = x_{1,i+1} X y_i$, it follows that $Xz_{i,n} = x_{1,i+1}X$ and hence the proposition is established for the final case, R of the form $⟪i, -n⟫$ for $1 \leq i < n$. q.e.d.

Definition 6.6. *We define a lift of γ to $A(C_n)$ to be an element of the form*

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

where the R_i are reflections in C_n satisfying $R_1 R_2 \dots R_n = [1, 2, 3, \dots, n]$.

We note that one lift of γ to $A(C_n)$ is

$$X = x_1 x_2 \dots x_n = \theta(\{[1]\})\theta(\{⟪1, 2⟫\}) \dots \theta(\{⟪n-1, n⟫\}).$$

To show that θ is well-defined it suffices, by Lemma 6.1, to prove the following.

Proposition 6.7. *For any lift E of γ to $A(C_n)$ we have $E = X$.*

Proof. Given a lift $E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\})$ of γ to $A(C_n)$, we know that $R_1 R_2 \dots R_n = [1, 2, \dots, n]$ and by Lemma 3.4 exactly one of the R_k is of the form $[j]$. Since $E = X$ if and only if $X^l E X^{-l} = X$ for any integer l, we may assume by the previous proposition that $R_k = [1]$. We will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\{[1]\})\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R'.

To simplify notation we set $R_{k-1} = T$ so that $R_{k-1} R_k = T[1]$. Since $T[1] \leq \gamma$ we know that $T \leq \gamma[1]$ or

$$T \leq ⟪1, -2, -3, \dots, -n⟫$$

so that T has the form $⟪1, -p⟫$ for $2 \leq p \leq n$ or T has the form $⟪i, j⟫$ with $2 \leq i < j \leq n$. In the latter case $\theta(\{T\})$ lies in the subgroup of $A(C_n)$ generated by $\{x_3, x_4, \dots, x_n\}$ and so commutes with $\theta(\{[1]\}) = x_1$.

Thus we can use $R' = T$. In the former case, $\theta(\{T\}) = z_{1,p}$ and E' can be constructed using

$$\theta(\{T\})\theta(\{[1]\}) = z_{1,p}x_1 = x_1x_{1,p} = \theta(\{[1]\})\theta(\{(\!(1,p)\!)\}).$$

After $k-1$ such steps we get $E = x_1\theta(\{S_2\})\ldots\theta(\{S_n\})$, where the product on the right is a lift of γ to $A(C_n)$. However, this means $S_2S_3\ldots S_n = (\!(1,2,\ldots,n)\!)$ in C_n so that $S_i \in \Sigma_n < C_n$ and

$$\theta(\{S_2\})\ldots\theta(\{S_n\}) = x_2x_3\ldots x_n,$$

by Lemma 4.6 of [3]. q.e.d.

Combining the results in this subsection we get the following theorem.

Theorem 6.8. *The poset group $\Gamma(C_n,\gamma)$ is isomorphic to the Artin group $A(C_n)$ for γ a Coxeter element in C_n.*

6.2. **The D_n case.** In this case our approach will be exactly as in the C_n case. However, the computations are more numerous and more complicated. The Artin group $A(D_n)$ has a presentation with generating set $x_1, x_2, \ldots x_n$, subject to the relations

$$\begin{array}{rcll} x_1x_2 &=& x_2x_1, & \\ x_1x_3x_1 &=& x_3x_1x_3, & \\ x_1x_i &=& x_ix_1, & \text{for} \quad i \geq 4 \\ x_ix_{i+1}x_i &=& x_{i+1}x_ix_{i+1}, & \text{for} \quad 1 < i < n \quad \text{and} \\ x_ix_j &=& x_jx_i, & \text{for} \quad |j-i| \geq 2 \quad \text{and} \quad i,j \neq 1. \end{array}$$

Definition 6.9. *We define a function ϕ from the generators of $A(D_n)$ to $\Gamma(D_n,\gamma)$ by*

$$x_1 \mapsto \{(\!(1,-2)\!)\}, x_2 \mapsto \{(\!(1,2)\!)\}, x_3 \mapsto \{(\!(2,3)\!)\}, \ldots, x_n \mapsto \{(\!(n-1,n)\!)\}$$

Lemma 6.10. *The function ϕ determines a well-defined surjective homomorphism.*

Proof: The relations involving $\phi(x_1)$ hold in $\Gamma(D_n,\gamma)$ by virtue of the following identities in $\Gamma(D_n,\gamma)$.

$$\{(\!(1,-2)\!)\}\{(\!(1,2)\!)\} = \{[1][2]\} = \{(\!(1,2)\!)\}\{(\!(1,-2)\!)\}$$

$$\begin{array}{rcl} \{(\!(1,-2)\!)\}\{(\!(2,3)\!)\}\{(\!(1,-2)\!)\} &=& \{(\!(1,-2,-3)\!)\}\{(\!(1,-2)\!)\} \\ &=& \{(\!(2,3)\!)\}\{(\!(1,-3)\!)\}\{(\!(1,-2)\!)\} \\ &=& \{(\!(2,3)\!)\}\{(\!(1,-2,-3)\!)\} \\ &=& \{(\!(2,3)\!)\}\{(\!(1,-2)\!)\}\{(\!(2,3)\!)\} \end{array}$$

$$\{(\!(1,-2)\!)\}\{(\!(i,i+1)\!)\} = \{(\!(i,i+1)\!)\}\{(\!(1,-2)\!)\}, \quad \text{for} \quad i \geq 3.$$

The image of the subgroup generated by $\{x_2, \dots, x_n\}$ again lies in the copy of the braid group corresponding to $\Sigma_n < D_n$ so that the relations not involving $\phi(x_1)$ hold by Lemma 4.2 and Lemma 4.4 of [3]. Thus ϕ is well-defined.
To establish surjectivity, note that both $\{((i, j))\}$ and $\{((i, i+1, \dots, j))\}$ lie in $\mathrm{im}(\phi)$, for $1 \le i < j \le n$ as in the C_n case. To find the other reflection lifts in $\mathrm{im}(\phi)$ first note that

$$\phi(x_1 x_2 \dots x_j) = \{[1][2, 3 \dots, j]\} = \{((1, -2))\}\{((1, 2, \dots, j))\} \in \mathrm{im}(\phi),$$

and $\{((1, -j))\} \in \mathrm{im}(\phi)$ for $j \ge 3$ since

$$\{((1, 2, \dots, j))\}\{((1, -j))\} = \{[1][2, \dots, j]\}.$$

Reflection lifts of the form $\{((2, -j))\}$ for $j \ge 3$ lie in $\mathrm{im}(\phi)$ since

$$\{((1, -2))\}\{((1, j))\} = \{((1, j, -2))\} = \{((2, -j))\}\{((1, -2))\}$$

and reflection lifts of the form $\{((i, -j))\}$ for $3 \le i < j \le n$ lie in $\mathrm{im}(\phi)$ since

$$\{((i, -j))\}\{((1, i))\}\{((1, -i))\} = \{[1][i, j]\} = \{((1, i))\}\{((1, -i))\}\{((i, j))\}.$$

q.e.d.

To construct an inverse to ϕ we will use the presentation for $\Gamma(D_n, \gamma)$ given by Lemma 6.1.

Definition 6.11. *We define a function θ from the generators of $\Gamma(D_n, \gamma)$ to $A(D_n)$ by*

$$\{((1, -2))\} \mapsto x_1, \quad \{((i, j))\} \mapsto x_{i,j} \quad \text{and} \quad \{((i, -j))\} \mapsto z_{i,j},$$

for $1 \le i < j \le n$, where $z_{i,j}$ is the unique element of $A(D_n)$ satisfying

$$\begin{aligned} z_{1,j} x_1 &= x_1 x_{2,j} \quad \text{when} \quad j \ge 3 \\ z_{2,j} x_1 &= x_1 x_{1,j} \quad \text{when} \quad j \ge 3 \\ z_{i,j} x_{1,i} z_{1,i} &= x_{1,i} z_{1,i} x_{i,j} \quad \text{when} \quad 3 \le i < j \le n \end{aligned}$$

We note that $z_{1,2} = x_1$. Since each $x_{i,j}$ lies in the copy of B_n generated by $\{x_2, \dots x_n\}$ the elements $x_{i,j}$ satisfy the same identities as in the C_n case. The homomorphism determined by θ will be surjective since each x_i is the image of some reflection lift. To show that θ determines a well-defined homomorphism we define the special element $X = x_1 x_2 \dots x_n$ in $A(D_n)$ and establish the D_n analogue of Proposition 6.5 .

Proposition 6.12. *For any reflection R in D_n,*

$$X\theta(\{R\})X^{-1} = \theta(\{\gamma R \gamma^{-1}\}).$$

Proof. Since $X = x_1Y_{1,n}$ and x_1 commutes with $x_4, \dots, x_n$ it follows that $Xx_i = x_{i+1}X$ for $3 \le i < n$ and $Xx_{i,j} = x_{i+1,j+1}X$ for $3 \le i < j < n$. This establishes the proposition in the case $R = (\!(i, j)\!)$ for $3 \le i < j < n$.

For some of the later cases we will require the identities $x_{2,j}z_{1,j} = x_1x_{2,j}$ and $x_{1,j}z_{2,j} = x_1x_{1,j}$ for $3 \le j \le n$. The first follows from

$$\begin{aligned} Y_{3,j}x_{2,j}z_{1,j}x_1 &= x_3Y_{3,j}z_{1,j}x_1 = x_3Y_{3,j}x_1x_{2,j} = x_3x_1Y_{3,j}x_{2,j} \\ &= x_3x_1x_3Y_{3,j} = x_1x_3x_1Y_{3,j} = x_1x_3Y_{3,j}x_1 = x_1Y_{3,j}x_{2,j}x_1 \\ &= Y_{3,j}x_1x_{2,j}x_1, \end{aligned}$$

while the second follows from

$$\begin{aligned} x_1Y_{2,j}x_{1,j}z_{2,j}x_1 &= x_1x_2Y_{2,j}z_{2,j}x_1 = x_1x_2Y_{2,j}x_1x_{1,j} = x_1x_2x_3Y_{3,j}x_1x_{1,j} \\ &= x_1x_2x_3x_1Y_{3,j}x_{1,j} = x_2x_1x_3x_1Y_{3,j}x_{1,j} = x_2x_3x_1x_3Y_{3,j}x_{1,j} \\ &= x_2x_3x_1Y_{2,j}x_{1,j} = x_2x_3x_1x_2Y_{2,j} = x_2x_3x_1x_2Y_{2,j} \\ &= x_2x_3x_2x_1Y_{2,j} = x_3x_2x_3x_1Y_{2,j} = x_3x_2x_3x_1x_3Y_{3,j} \\ &= x_3x_2x_1x_3x_1Y_{3,j} = x_3x_2x_1x_3Y_{3,j}x_1 = x_3x_2x_1Y_{2,j}x_1 \\ &= x_3x_1x_2Y_{2,j}x_1 = x_3x_1Y_{2,j}x_{1,j}x_1 = x_3x_1x_3Y_{3,j}x_{1,j}x_1 \\ &= x_1x_3x_1Y_{3,j}x_{1,j}x_1 = x_1x_3Y_{3,j}x_1x_{1,j}x_1. \end{aligned}$$

The conjugation action of X on x_1 is given by $Xx_1 = x_{1,3}X$ since

$$\begin{aligned} x_3Xx_1 &= x_3x_1x_2x_3Y_{3,n}x_1 = x_3x_1x_2x_3x_1Y_{3,n} = x_3x_2x_1x_3x_1Y_{3,n} \\ &= x_3x_2x_3x_1x_3Y_{3,n} = x_2x_3x_2x_1x_3Y_{3,n} = x_2x_3x_1x_2x_3Y_{3,n} \\ &= Y_{1,3}X = x_3x_{1,3}X. \end{aligned}$$

A similar calculation gives $x_3Xx_2 = x_1x_3X$. Since

$$x_1x_3X = x_1x_{2,3}X = x_{2,3}z_{1,3}X$$

we get $Xx_2 = z_{1,3}X$. This establishes the proposition in the cases $R = (\!(1, -2)\!)$ and $R = (\!(1, 2)\!)$.

Next we establish $Xx_n = z_{2,n}X$.

$$Xx_n = x_1Y_{1,n}x_n = x_1x_{1,n}Y_{1,n-1}x_n = z_{2,n}x_1Y_{1,n} = z_{2,n}X$$

which takes care of the case $R = (\!(n-1, n)\!)$. To obtain the identity $Xx_{1,j} = z_{1,j+1}X$ we note that

$$Y_{1,n}x_{1,j}Y_{1,j-1} = Y_{1,n}Y_{1,j} = Y_{2,j+1}Y_{1,n} = x_{2,j+1}Y_{2,j}Y_{1,n} = x_{2,j+1}Y_{1,n}Y_{1,j-1}$$

giving $Y_{1,n}x_{1,j} = x_{2,j+1}Y_{1,n}$ so that

$$Xx_{1,j} = x_1Y_{1,n}x_{1,j} = x_1x_{2,j+1}Y_{1,n} = z_{1,j+1}x_1Y_{1,n} = z_{1,j+1}X.$$

This completes the case $R = (\!(1, j)\!)$ for $2 \le j < n$.

For the identity $Xx_{1,n} = x_2X$ we compute

$$Xx_{1,n} = x_1x_2(x_3 \dots x_n)x_{1,n} = x_1x_2(x_2x_3 \dots x_n) = x_2X,$$

which establishes the case $R = (\!(1, n)\!)$.

For $2 \le i < n$ we have

$$\begin{aligned} Xx_{i,n} &= x_1Y_{1,i+1}Y_{i+1,n}x_{i,n} = x_1Y_{1,i+1}x_{i+1}Y_{i+1,n} \\ &= x_1x_{1,i+1}Y_{1,i}x_{i+1}Y_{i+1,n} = z_{2,i+1}x_1Y_{1,n} = z_{2,i+1}X \end{aligned}$$

and hence the proposition is true for $R = (\!(i, n)\!)$ with $2 \le i < n$.

The identity $Xz_{1,j} = x_{1,j+1}X$ for $3 \le j < n$ follows from

$$Xz_{1,j}x_1 = Xx_1x_{2,j} = x_{1,3}x_{3,j+1}X = x_{1,j+1}x_{1,3}X = x_{1,j+1}Xx_1,$$

while the identity $Xz_{1,n} = z_{1,2}X = x_1X$ follows from

$$Xz_{1,n}x_1 = Xx_1x_{2,n} = x_{1,3}z_{2,3}X = x_1x_{1,3}X = x_1Xx_1.$$

This establishes the proposition for $R = (\!(1, -j)\!)$ with $2 \le j \le n$.

The identity $Xz_{i,n} = x_{2,i+1}X$ for $2 \le i < n$ follows from

$$\begin{aligned} Xz_{i,n}x_{1,i}z_{1,i} &= Xx_{1,i}z_{1,i}x_{i,n} = z_{1,i+1}x_{1,i+1}z_{2,i+1}X \\ &= z_{1,i+1}x_1x_{1,i+1}X = x_1x_{2,i+1}x_{1,i+1}X \\ &= x_{2,i+1}z_{1,i+1}x_{1,i+1}X = x_{2,i+1}Xx_{1,i}z_{1,i}. \end{aligned}$$

This establishes the proposition for $R = (\!(i, -n)\!)$ with $2 \le i < n$.

Finally we note that $x_{1,i}z_{1,i} = z_{1,i}x_{1,i}$ since

$$\begin{aligned} x_{2,i}x_{1,i}z_{1,i} &= x_2x_{2,i}z_{1,i} = x_2x_1x_{2,i} = x_1x_2x_{2,i} \\ &= x_1x_{2,i}x_{1,i} = x_{2,i}z_{1,i}x_{1,i}. \end{aligned}$$

From this we deduce that $Xz_{i,j} = z_{i+1,j+1}X$ for $2 \le i < j < n$ since

$$\begin{aligned} Xz_{i,j}x_{1,i}z_{1,i} &= Xx_{1,i}z_{1,i}x_{i,j} = z_{1,i+1}x_{1,i+1}x_{i+1,j+1}X \\ &= z_{i+1,j+1}z_{1,i+1}x_{1,i+1}X = z_{i+1,j+1}Xx_{1,i}z_{1,i}. \end{aligned}$$

This establishes the proposition for the remaining cases $R = (\!(i, -j)\!)$ with $2 \le i < j < n$. q.e.d.

Definition 6.13. *We define a lift of γ to $A(D_n)$ to be an element of the form*

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

where the R_i are reflections in D_n satisfying $R_1R_2 \dots R_n = [1][2, 3, \dots, n]$.

We note that one lift of γ to $A(D_n)$ is

$$X = x_1 x_2 \dots x_n = \theta(\{⟬1,-2⟭\})\theta(\{⟬1,2⟭\}) \dots \theta(\{⟬n-1,n⟭\}).$$

To show that θ determines a well-defined homomorphism it suffices, by Lemma 6.1, to prove the following.

Proposition 6.14. *For any lift E of γ to $A(D_n)$ we have $E = X$.*

Proof: Given a lift E of γ to $A(D_n)$, where

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

we know that $R_1 R_2 \dots R_n = [1][2, \dots, n]$. It follows for the proof of proposition 4.2 that one of the R_k is of the form $⟬1, \pm j⟭$. Since $E = X$ if and only if $X^l E X^{-l} = X$ for any integer l, we may assume $R_k = ⟬1, \pm 2⟭$. We treat these two cases separately.

Suppose that $R_k = ⟬1, -2⟭$. We will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\{⟬1,-2⟭\})\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R'.

To simplify notation we set $R_{k-1} = T$ so that $R_{k-1}R_k = T⟬1,-2⟭$. Since $T⟬1,-2⟭ \le [1][2, \dots, n]$ we know that

$$T \le ⟬1, -3, -4, \dots, -n, 2⟭$$

so that T has one of the forms

(1) $⟬1, 2⟭$,
(2) $⟬i, j⟭$ for $3 \le i < j \le n$,
(3) $⟬1, -p⟭$ for $3 \le p \le n$ or
(4) $⟬2, -p⟭$ for $3 \le p \le n$.

In the first case $\theta(\{T\}) = x_2$, which commutes with $\theta(\{⟬1,-2⟭\}) = x_1$. In the second case, $\theta(\{T\}) = x_{i,j}$ lies in the subgroup generated by $\{x_4, \dots x_n\}$ and hence also commutes with x_1. In the third case E' can be constructed using

$$\theta(\{T\})\theta(\{⟬1,-2⟭\}) = z_{1,p}x_1 = x_1 x_{2,p} = \theta(\{⟬1,-2⟭\})\theta(\{⟬2,p⟭\})$$

and in the fourth case using

$$\theta(\{T\})\theta(\{⟬1,-2⟭\}) = z_{2,p}x_1 = x_1 x_{1,p} = \theta(\{⟬1,-2⟭\})\theta(\{⟬1,p⟭\}).$$

After $k-1$ such steps we get $E = x_1\theta(\{S_2\}) \dots \theta(\{S_n\})$, where the product on the right is a lift of γ to $A(D_n)$. However, this means $S_2 S_3 \dots S_n = ⟬1, 2, \dots, n⟭$ in C_n so that $S_i \in \Sigma_n < C_n$ and

$$\theta(\{S_2\}) \dots \theta(\{S_n\}) = x_2 x_3 \dots x_n,$$

by Lemma 4.6 of [3].

Next suppose $R_k = (\!(1,2)\!)$. As in the previous case, we will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\{(\!(1,2)\!)\})\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R'. To simplify notation we again set $R_{k-1} = T$ so that $R_{k-1}R_k = T(\!(1,2)\!)$. Since $T(\!(1,2)\!) \leq [1][2,\dots,n]$ we know that

$$T \leq (\!(1,3,4,\dots,n,-2)\!)$$

so that T has one of the forms

(1) $(\!(1,-2)\!)$,
(2) $(\!(i,j)\!)$ for $3 \leq i < j \leq n$,
(3) $(\!(1,p)\!)$ for $3 \leq p \leq n$ or
(4) $(\!(2,-p)\!)$ for $3 \leq p \leq n$.

In the first case $\theta(\{T\}) = x_1$, which commutes with $\theta(\{(\!(1,2)\!)\}) = x_2$. In the second case, $\theta(\{T\}) = x_{i,j}$ lies in the subgroup generated by $\{x_4, \dots x_n\}$ and hence also commutes with x_2. In the third case E' can be constructed using

$$\theta(\{T\})\theta(\{(\!(1,2)\!)\}) = x_{1,p}x_{1,2} = x_{1,2}x_{2,p} = \theta(\{(\!(1,2)\!)\})\theta(\{(\!(2,p)\!)\}).$$

In the fourth case E' is constructed using

$$\theta(\{T\})\theta(\{(\!(1,2)\!)\}) = z_{2,p}x_2 = x_2 z_{1,p} = \theta(\{(\!(1,2)\!)\})\theta(\{(\!(1,-p)\!)\}).$$

The middle equality holds since

$$z_{2,p}x_2x_1 = z_{2,p}x_1x_2 = x_1x_{1,p}x_2 = x_1x_2x_{2,p} = x_2x_1x_{2,p} = x_2z_{1,p}x_1.$$

After $k-1$ such steps we get $E = x_2\theta(\{S_2\}) \dots \theta(\{S_n\})$, where the product on the right is a lift of γ to $A(D_n)$. However, this means $S_2S_3 \dots S_n = (\!(1,-2,\dots,-n)\!)$ in C_n so that S_i lie in the copy of Σ_n generated $\{(\!(1,-2)\!), (\!(2,3)\!), \dots, (\!(n-1,n)\!)\}$ and

$$\theta(\{S_2\}) \dots \theta(\{S_n\}) = x_1x_3 \dots x_n,$$

by Lemma 4.6 of [3]. Finally

$$E = x_2x_1x_3 \dots x_n = x_1x_2x_3 \dots x_n.$$

q.e.d.

Combining the results in this subsection we get the following theorem.

Theorem 6.15. *The poset group $\Gamma(D_n, \gamma)$ is isomorphic to the Artin group $A(D_n)$ for γ a Coxeter element in D_n.*

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