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Weyl-Titchmarsh Type Formula for Periodic Schrodinger Operator with Wigner-von Neumann Potential

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ABSTRACT. Schrödinger operator on the half-line with periodic background potential perturbed by a certain potential of Wigner-von Neumann type is considered. The asymptotics of generalized eigenvectors for $\lambda \in \mathbb{C}_+$ and on the absolutely continuous spectrum is established. Weyl-Titchmarsh type formula for this operator is proven.

1. Introduction

Consider the one dimensional Schrödinger operator with the real potential which can be represented as a sum of three terms: a certain periodic function, Wigner-von Neumann potential and a certain absolutely integrable function. More precisely, let q be a real periodic function with period a such that $q \in L_1(0; a)$ and let $q_1 \in L_1(\mathbb{R}_+)$. Then the Schrödinger operator $\mathcal{L}_\alpha$ is defined by the differential expression

$$(1) \qquad \mathcal{L}_\alpha := -\frac{d^2}{dx^2} + q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x),$$

on the set of functions satisfying the boundary condition

$$(2) \qquad \psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0,$$

where $c, \omega, \delta, \in \mathbb{R}$, $\alpha \in [0, \pi)$, $\gamma \in \left(\frac{1}{2}; 1\right]$. As it was shown by the first author and Naboko in [16], the absolutely continuous spectrum of this

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operator has multiplicity one and coincides as a set with the spectrum
of the corresponding periodic operator on $\mathbb{R}$,

$$(3) \qquad\qquad \mathcal{L}_{per} = -\frac{d^2}{dx^2} + q(x).$$

Note that the spectrum of $\mathcal{L}_{per}$ has multiplicity two. Let $\psi_+(x, \lambda)$ and
$\psi_-(x, \lambda)$ be Bloch solutions for $\mathcal{L}_{per}$ and $\varphi_\alpha(x, \lambda)$ be the solution of
Cauchy problem

$$-\varphi_\alpha''(x, \lambda) + \left(q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x) \right) \varphi_\alpha(x, \lambda) = \lambda\varphi_\alpha(x, \lambda),$$
$$\varphi_\alpha(0, \lambda) = \sin\alpha,$$
$$\varphi_\alpha'(0, \lambda) = \cos\alpha.$$

The main result of the present paper is the following theorem that
relates the spectral density ρ_α' of the operator $\mathcal{L}_\alpha$ and the asymptotics
of the solution φ_α. We call it the Weyl-Titchmarsh formula.

Theorem 1. *Let $\frac{2a\omega}{\pi} \notin \mathbb{Z}$ and $q_1 \in L_1(\mathbb{R}_+)$, then for almost all $\lambda \in \sigma(\mathcal{L}_{per})$ there exists $A_\alpha(\lambda)$ such that*

$$(4) \qquad \varphi_\alpha(x, \lambda) = A_\alpha(\lambda)\psi_-(x, \lambda) + \overline{A_\alpha(\lambda)}\psi_+(x, \lambda) + o(1) \ as \ x \to \infty$$

and

$$\rho_\alpha'(\lambda) = \frac{1}{2\pi|W(\psi_+(\lambda), \psi_-(\lambda))| \, |A_\alpha(\lambda)|^2}.$$

Weyl-Titchmarsh formulas form an efficient tool to study the be-
havior of the spectral density. The absolutely continuous spectrum
of the operator $\mathcal{L}_\alpha$ contains infinitely many critical (resonance) points
(see (5)) where the type of the asymptotics of generalized eigenvectors
changes and is not given by a linear conbination of ψ_+ and ψ_- (as in
(4)). Precisely at these points the embedded eigenvalues of $\mathcal{L}_\alpha$ may
occur. In the generic case no eigenvalue occurs, but it is natural to
suspect that the spectral density of $\mathcal{L}_\alpha$ vanishes at these points.

Vanishing of the spectral density divides the absolutely continuous
spectrum into independent parts and has a clear physical meaning.
This phenomenon is called pseudogap. In the forthcoming paper we
intend to study zeros of the spectral density in more detail.

The study of Schrödinger operators with Wigner-von Neumann po-
tentials began from the classical paper [19] where it was observed for
the first time that the potential $\dfrac{c\sin(2\omega x + \delta)}{x + 1}$ may produce an eigen-
value inside the absolutely continuous spectrum. Later on such opera-
tors attracted attention of many authors [1],[17],[18],[6],[2],[3],[4],[14],[15],[12],[13].
The phenomenon of this nature, an embedded eigenvalue ("bound state

in the continuum"), was even observed experimentally in semiconductor geterostructures [7].

Weyl-Titchmarsh formula for the spectral density in the case of zero periodic background potential follows directly from the results of [17]. This formula was proved once again in [2] where the method of Harris-Lutz transformations [11] was used. In the present paper we also use a modification of this method. We would like to mention that another one approach was suggested in [5], but again in the case of zero periodic background potential.

2. Preliminaries

The spectrum of $\mathcal{L}_{per}$ consists of infinitely many intervals [9, Theorem 2.3.1]

$$\sigma(\mathcal{L}_{per}) := \bigcup_{n=0}^{\infty}([\lambda_{2n}; \mu_{2n}] \cup [\mu_{2n+1}; \lambda_{2n+1}]),$$

where

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_2 \le \lambda_4 < ...,$$

where λ_j and μ_j are the eigenvalues of the Schrödinger differential equation on the interval $[0, a]$ with periodic and antiperiodic boundary conditions. Spectral properties of $\mathcal{L}_{per}$ are related to the entire function $D(\lambda)$ (discriminant) and the function $k(\lambda)$ (quasi-momentum)

$$k(\lambda) := -i \ln\left(\frac{\mathrm{tr}D(\lambda) + \sqrt{\mathrm{tr}^2 D(\lambda) - 4}}{2}\right).$$

We can choose the branch of $k(\lambda)$ so that (this follows from the properties of $D(\lambda)$, see [9, Theorem 2.3.1])

$$k(\lambda_0) = 0, k(\mu_0) = k(\mu_1) = \pi, k(\lambda_1) = k(\lambda_2) = 2\pi, ...,$$
$$k(\lambda) \in \mathbb{R}, \text{ if } \lambda \in \sigma(\mathcal{L}_{per}),$$
$$k(\lambda) \in \mathbb{C}_+, \text{ if } \lambda \in \mathbb{C}_+.$$

The eigenfunction equation for $\mathcal{L}_{per}$,

$$-\psi''(x) + q(x)\psi(x) = \lambda\psi(x),$$

has two solutions (Bloch solutions) $\psi_+(x, \lambda)$ and $\psi_-(x, \lambda)$ satisfying quasiperiodic conditions:

$$\psi_+(x + a, \lambda) \equiv e^{ik(\lambda)}\psi_+(x, \lambda),$$
$$\psi_-(x + a, \lambda) \equiv e^{-ik(\lambda)}\psi_-(x, \lambda).$$

They are determined uniquely up to multiplication by coefficients depending on λ. It is possible to choose these coefficients so that Bloch solutions have the following properties:

(1) $\psi_+(x, \lambda), \psi_-(x, \lambda)$ for every $x \geq 0$ and their Wronskian $W(\psi_+(\lambda), \psi_-(\lambda))$ are analytic functions of λ in $\mathbb{C}_+$ and continuous up to $\sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, n \geq 0\}$.

(2) For $\lambda \in \sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, n \geq 0\}$,
$$\psi_+(x, \lambda) \equiv \overline{\psi_-(x, \lambda)}.$$

(3) The Wronskian does not have zeros and for $\lambda \in \sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, n \geq 0\}$
$$W(\psi_+(\lambda), \psi_-(\lambda)) \in i\mathbb{R}_+.$$

Bloch solutions can also be written in the form
$$\psi_+(x, \lambda) = e^{ik(\lambda)\frac{x}{a}}p_+(x, \lambda),$$
$$\psi_-(x, \lambda) = e^{-ik(\lambda)\frac{x}{a}}p_-(x, \lambda),$$

where the functions $p_+(x, \lambda)$ and $p_-(x, \lambda)$ have period a in the variable x and the same properties as $\psi_+(x, \lambda)$ and $\psi_-(x, \lambda)$ with respect to the variable λ.

As we mentioned earlier, the operator $\mathcal{L}_\alpha$ was studied in [16], where the asymptotics of the generalized eigenvectors was obtained. The authors showed that in every zone of $\sigma(\mathcal{L}_{per})$ ($[\lambda_n; \mu_n]$ if n is even and $[\mu_n; \lambda_n]$ if n is odd) there exist two critical points λ_n^+ and λ_n^- determined by the equalities

(5)
$$k(\lambda_n^+) = \pi\left(n + 1 - \left\{\frac{a\omega}{\pi}\right\}\right),$$
$$k(\lambda_n^-) = \pi\left(n + \left\{\frac{a\omega}{\pi}\right\}\right).$$

They do not coincide with each other and with the ends of zones, if

(6)
$$\frac{2a\omega}{\pi} \notin \mathbb{Z}.$$

3. Reduction of the spectral equation to the discrete linear system of Levinson form

In this section we transform the eigenufnction equation for $\mathcal{L}_\alpha$ to a linear 2×2 system with the coefficient matrix being a sum of the diagonal and summable matrices.

Consider the eigenfunction equation for $\mathcal{L}_\alpha$:

(7)
$$-\psi''(x) + \left(q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x)\right)\psi(x) = \lambda\psi(x).$$

For every
$$\lambda \in \mathbb{C}_+ \cup (\sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, n \geq 0\})$$

let us make the following substitution

$$\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} = \begin{pmatrix} \psi_-(x,\lambda) & \psi_+(x,\lambda) \\ \psi'_-(x,\lambda) & \psi'_+(x,\lambda) \end{pmatrix} u(x),$$

(8)
$$\text{or}$$

$$u(x) := \frac{1}{W(\psi_+(\lambda),\psi_-(\lambda))} \begin{pmatrix} \psi'_+(x,\lambda) & -\psi_+(x,\lambda) \\ -\psi'_-(x,\lambda) & \psi_-(x,\lambda) \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}.$$

Writing (7) as

$$\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x) - \lambda & 0 \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}$$

and substituting (8) into it, we get:
(9)
$$u'(x) = \frac{\frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x)}{W(\psi_+(\lambda),\psi_-(\lambda))} \begin{pmatrix} -\psi_+(x,\lambda)\psi_-(x,\lambda) & -\psi_+^2(x,\lambda) \\ \psi_-^2(x,\lambda) & \psi_+(x,\lambda)\psi_-(x,\lambda) \end{pmatrix} u(x).$$

Let us introduce another one vector valued function v

$$v(x) := \begin{pmatrix} e^{-ik(\lambda)\frac{x}{a}} & 0 \\ 0 & e^{ik(\lambda)\frac{x}{a}} \end{pmatrix} u(x),$$

(10)
$$\text{or}$$

$$u(x) = \begin{pmatrix} e^{ik(\lambda)\frac{x}{a}} & 0 \\ 0 & e^{-ik(\lambda)\frac{x}{a}} \end{pmatrix} v(x)$$

and the matrix
(11)
$$R^{(1)}(x,\lambda) := \frac{q_1(x)}{W(\psi_+(\lambda),\psi_-(\lambda))} \begin{pmatrix} -p_+(x,\lambda)p_-(x,\lambda) & -p_+^2(x,\lambda) \\ p_-^2(x,\lambda) & p_+(x,\lambda)p_-(x,\lambda) \end{pmatrix}.$$

Then the system (9) is equivalent to

$$(12) \quad v'(x) = \left[\begin{pmatrix} -\frac{ik(\lambda)}{a} & 0 \\ 0 & \frac{ik(\lambda)}{a} \end{pmatrix} + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+(\lambda),\psi_-(\lambda))} \right.$$
$$\times \left. \begin{pmatrix} -p_+(x,\lambda)p_-(x,\lambda) & -p_+^2(x,\lambda) \\ p_-^2(x,\lambda) & p_+(x,\lambda)p_-(x,\lambda) \end{pmatrix} + R^{(1)}(x,\lambda) \right] v(x).$$

Let us search for a differentiable matrix-valued function $Q(x)$ such that $Q(x), Q'(x) = O\left(\frac{1}{x^\gamma}\right)$ as $x \to \infty$ and such that the substitution

$$(13) \qquad\qquad v(x) = e^{Q(x)}\widetilde{v}(x)$$

leads to a system for the vector valued function $\widetilde{v}$ of the form

$$\widetilde{v}'(x) = \left[\begin{pmatrix} -\frac{ik}{a} & 0 \\ 0 & \frac{ik}{a} \end{pmatrix} + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+, \psi_-)} \right.$$

$$\left. \times \begin{pmatrix} -p_+(x)p_-(x) & 0 \\ 0 & p_+(x)p_-(x) \end{pmatrix} + R^{(2)}(x) \right] \widetilde{v}(x),$$

where the remainder $R^{(2)}(x)$ also belongs to $L_1(0, \infty)$. Using that

$$e^{\pm Q(x)} = I \pm Q(x) + O\left(\frac{1}{x^{2\gamma}}\right),$$
$$\left(e^{\pm Q(x)}\right)' = \pm Q'(x) + O\left(\frac{1}{x^{2\gamma}}\right)$$

as $x \to \infty$ we obtain

$$(14) \quad \widetilde{v}'(x) = \left[\begin{pmatrix} -\frac{ik}{a} & 0 \\ 0 & \frac{ik}{a} \end{pmatrix} + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+, \psi_-)} \right.$$

$$\times \begin{pmatrix} -p_+(x)p_-(x) & -p_+^2(x) \\ p_-^2(x) & p_+(x)p_-(x) \end{pmatrix} - Q'(x)$$

$$\left. - \left[Q(x), \begin{pmatrix} -\frac{ik}{a} & 0 \\ 0 & \frac{ik}{a} \end{pmatrix} \right] + R^{(1)}(x) + O\left(\frac{1}{x^{2\gamma}}\right) \right] \widetilde{v}(x),$$

where

$$\left[Q(x), \begin{pmatrix} -\frac{ik}{a} & 0 \\ 0 & \frac{ik}{a} \end{pmatrix} \right]$$

is the commutator of the two matrices. Our aim is to cancel the anti-diagonal entries of

$$\begin{pmatrix} -p_+(x)p_-(x) & -p_+^2(x) \\ p_-^2(x) & p_+(x)p_-(x) \end{pmatrix}$$

in (14) by properly choosing Q. To this end Q has to satisfy the following equation:

$$(15)$$
$$Q'(x) + \left[Q(x), \begin{pmatrix} -\frac{ik}{a} & 0 \\ 0 & \frac{ik}{a} \end{pmatrix} \right] = \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+, \psi_-)} \begin{pmatrix} 0 & -p_+^2(x) \\ p_-^2(x) & 0 \end{pmatrix}.$$

The latter is equivalent (after multiplication by $\begin{pmatrix} e^{-ik\frac{x}{a}} & 0 \\ 0 & e^{ik\frac{x}{a}} \end{pmatrix}$ from the right and by its inverse from the left) to

$$(16) \quad \left(\begin{pmatrix} e^{ik\frac{x}{a}} & 0 \\ 0 & e^{-ik\frac{x}{a}} \end{pmatrix} Q(x) \begin{pmatrix} e^{-ik\frac{x}{a}} & 0 \\ 0 & e^{ik\frac{x}{a}} \end{pmatrix} \right)'$$

$$= \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+, \psi_-)} \begin{pmatrix} 0 & -p_+^2(x)e^{2ik\frac{x}{a}} \\ p_-^2(x)e^{-2ik\frac{x}{a}} & 0 \end{pmatrix}.$$

For every

$$\mu \in \sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}$$

and for the values of λ from some neighbourhood of the point μ (which we will specify later) let us take the following solution of (16):

$$\left(\begin{pmatrix} e^{ik\frac{x}{a}} & 0 \\ 0 & e^{-ik\frac{x}{a}} \end{pmatrix} Q(x, \lambda, \mu) \begin{pmatrix} e^{-ik\frac{x}{a}} & 0 \\ 0 & e^{ik\frac{x}{a}} \end{pmatrix}\right) = \frac{c}{W(\psi_+(\lambda), \psi_-(\lambda))}$$

$$\times \begin{pmatrix} 0 & \int\limits_x^\infty \frac{\sin(2\omega t+\delta)p_+^2(t,\lambda)e^{2ik(\lambda)\frac{t}{a}}dt}{(t+1)^\gamma} \\ \int\limits_0^x \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)e^{-2ik(\lambda)\frac{t}{a}}dt}{(t+1)^\gamma} - \int\limits_0^\infty \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)e^{-2ik(\mu)\frac{t}{a}}dt}{(t+1)^\gamma} & 0 \end{pmatrix}$$

(this is our choice of constants of integration that depend on μ). This leads to

$$(17) \quad Q(x, \lambda, \mu) := \frac{c}{W(\psi_+(\lambda), \psi_-(\lambda))}$$

$$\times \begin{pmatrix} 0 & e^{-2ik(\lambda)\frac{x}{a}} \int\limits_x^\infty \frac{\sin(2\omega t+\delta)p_+^2(t,\lambda)e^{2ik(\lambda)\frac{t}{a}}dt}{(t+1)^\gamma} \\ e^{2ik(\lambda)\frac{x}{a}} \left(\int\limits_0^x \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)e^{-2ik(\lambda)\frac{t}{a}}dt}{(t+1)^\gamma} - \int\limits_0^\infty \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)e^{-2ik(\mu)\frac{t}{a}}dt}{(t+1)^\gamma}\right) & 0 \end{pmatrix}.$$

In particular, for $\lambda = \mu$,

$$(18) \quad Q(x, \mu, \mu) = \frac{c}{W(\psi_+(\mu), \psi_-(\mu))}$$

$$\times \begin{pmatrix} 0 & e^{-2ik(\mu)\frac{x}{a}} \int\limits_x^\infty \frac{\sin(2\omega t+\delta)\psi_+^2(t,\mu)dt}{(t+1)^\gamma} \\ -e^{2ik(\mu)\frac{x}{a}} \int\limits_x^\infty \frac{\sin(2\omega t+\delta)\psi_-^2(t,\mu)dt}{(t+1)^\gamma} & 0 \end{pmatrix}.$$

Formula (18) does not make sense if $\mu \in \mathbb{C}_+$ due to the divergence of the integral in the lower entry. But it has analytic continuation in its second argument from the point μ.

Let us denote

$$\varepsilon(\mu) := \frac{1}{2} \min_{n\in\mathbb{Z}} \left\{ \left|\frac{2k(\mu)}{a} + 2\omega + \frac{2\pi n}{a}\right|, \left|\frac{2k(\mu)}{a} - 2\omega + \frac{2\pi n}{a}\right| \right\}.$$

Consider some $\beta > 0$ and the set

$$U(\beta, \mu) := \{\lambda \in \overline{\mathbb{C}_+} : 2\varepsilon(\lambda) \geq \varepsilon(\mu), 0 \leq \text{Im } 2k(\lambda)/a \leq 1,$$
$$|\text{Re } k(\lambda) - k(\mu)| \leq \beta\text{Im } k(\lambda)\}.$$

The set $U(\beta, \mu)$ is compact and contains the point μ. For every $\beta_1 < \beta$ it contains some neighbourhood of the vertex of the sector

$$|\text{Re } \lambda - \mu| \leq \frac{\beta_1}{k'(\mu)} \text{Im } \lambda.$$

Note that $k'(\mu)$ is positive for $\mu \in \sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, n \geq 0\}$.

Theorem 2. *Let $\beta > 0$ and*

$$\mu \in \sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}.$$

Then there exists $c_1(\beta, \mu, \gamma)$ such that for every $x \geq 0$ and $\lambda \in U(\beta, \mu)$ holds:

$$\|Q(x, \lambda, \mu)\|, \|Q'(x, \lambda, \mu)\| < \frac{c_1(\beta, \mu, \gamma)}{(x+1)^\gamma}.$$

PROOF. Note first that

$$k(\lambda) \in \mathbb{R} \text{ for } \lambda \in \sigma(\mathcal{L}_{per})$$

and

$$k(\lambda) \in \mathbb{C}_+ \text{ for } \lambda \in \mathbb{C}_+.$$

Let us denote the entries of $Q(x, \lambda, \mu)$ as follows

$$Q(x, \lambda, \mu) = \begin{pmatrix} 0 & Q_{12}(x, \lambda) \\ Q_{21}(x, \lambda, \mu) & 0 \end{pmatrix}.$$

Let us estimate first the entry Q_{12}. Let f be a periodic function with period a such that $f \in L_1(0; a)$, its Fourier coefficients will be denoted by f_n

$$f_n := \frac{1}{a} \int_0^a f(x) e^{-2\pi i n \frac{x}{a}} dx.$$

Lemma 1. *If*

$$\{f_n\}_{n=-\infty}^{+\infty} \in l^1(\mathbb{Z})$$

and $\xi \in \overline{\mathbb{C}_+}$ is such that

$$\frac{a\xi}{2\pi} \notin \mathbb{Z},$$

then

$$(19) \quad \left| e^{-i\xi x} \int_x^\infty \frac{e^{i\xi t} f(t) dt}{(t+1)^\gamma} \right| \leq 2 \left(\sum_{n=-\infty}^{+\infty} \frac{|f_n|}{\left| \xi - \frac{2\pi n}{a} \right|} \right) \frac{1}{(x+1)^\gamma}$$

(i.e. the expression on the left-hand side exists and is estimated by the right-hand side).

PROOF. Consider $x_1 > x$. Since the Fourier series converges absolutely, we have

(20)
$$e^{-i\xi x} \int_x^{x_1} \frac{e^{i\xi t}}{(t+1)^\gamma} \left(\sum_{n=-\infty}^{+\infty} f_n e^{2\pi n \frac{t}{a}} \right) dt = \sum_{n=-\infty}^{+\infty} f_n e^{-i\xi x} \int_x^{x_1} \frac{e^{i\left(\xi + \frac{2\pi n}{a}\right)t}}{(t+1)^\gamma} dt.$$

Integrating by parts and estimating the absolute value we get

$$\left| e^{-i\xi x} \int_x^{x_1} \frac{e^{i\left(\xi + \frac{2\pi n}{a}\right)t}}{(t+1)^\gamma} dt \right| \leq \frac{1}{\left| \xi + \frac{2\pi n}{a} \right|}$$
$$\times \left(\frac{1}{(x+1)^\gamma} + \frac{1}{(x_1+1)^\gamma} + \gamma \int_x^{x_1} \frac{dt}{(t+1)^{\gamma+1}} \right) = \frac{2}{\left| \xi + \frac{2\pi n}{a} \right| (x+1)^\gamma}.$$

Substituting into (20) yields:

$$\left| e^{-i\xi x} \int_x^{x_1} \frac{e^{i\xi t} f(t) dt}{(t+1)^\gamma} \right| \leq 2 \left(\sum_{n=-\infty}^{+\infty} \frac{|f_n|}{\left| \xi - \frac{2\pi n}{a} \right|} \right) \frac{1}{(x+1)^\gamma}.$$

By Cauchy's criterion the integral

$$\int_x^\infty \frac{e^{i\xi t} f(t) dt}{(t+1)^\gamma}$$

exists and the desired estimate (19) follows. $\square$

Formula (17) implies

(21) $\quad Q_{12}(x, \lambda)$
$$= \frac{c e^{2i\omega x + i\delta}}{2iW(\psi_+(\lambda), \psi_-(\lambda))} e^{-i\left(\frac{2k(\lambda)}{a} + 2\omega \right)x} \int_x^\infty \frac{p_+^2(t, \lambda) e^{i\left(\frac{2k(\lambda)}{a} + 2\omega \right)t} dt}{(t+1)^\gamma}$$
$$- \frac{c e^{-2i\omega x - i\delta}}{2iW(\psi_+(\lambda), \psi_-(\lambda))} e^{-i\left(\frac{2k(\lambda)}{a} - 2\omega \right)x} \int_x^\infty \frac{p_+^2(t, \lambda) e^{i\left(\frac{2k(\lambda)}{a} - 2\omega \right)t} dt}{(t+1)^\gamma}.$$

Denote the Fourier coefficients of the function $p_+^2(\cdot, \lambda)$ by $b_n(\lambda)$,

$$b_n(\lambda) := \frac{1}{a} \int_0^a p_+^2(x, \lambda) e^{-2\pi i n \frac{x}{a}} dx.$$

Then Lemma 1 applied to to (21) gives

$$(22) \quad |Q_{12}(x,\lambda)| \leq \frac{|c|}{|W(\psi_+(\lambda),\psi_-(\lambda))|} \frac{1}{(x+1)^\gamma}$$

$$\times \sum_{n=-\infty}^{+\infty} |b_n(\lambda)| \left(\frac{1}{\left| \frac{2k(\lambda)}{a} + 2\omega + \frac{2\pi n}{a} \right|} + \frac{1}{\left| \frac{2k(\lambda)}{a} - 2\omega + \frac{2\pi n}{a} \right|} \right)$$

$$\leq \frac{2|c|}{\varepsilon(\mu)|W(\psi_+(\lambda),\psi_-(\lambda))|} \left(\sum_{n=-\infty}^{+\infty} |b_n(\lambda)| \right) \frac{1}{(x+1)^\gamma}.$$

Let us estimate now the entry Q_{21}. Formula (17) implies

$$Q_{21}(x,\lambda,\mu) = \frac{ce^{2ik(\lambda)\frac{x}{a}}}{W(\psi_+(\lambda),\psi_-(\lambda))} \left(\int_0^x \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)e^{-2ik(\lambda)\frac{t}{a}}dt}{(t+1)^\gamma} \right.$$

$$\left. - \int_0^\infty \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)e^{-2ik(\mu)\frac{t}{a}}dt}{(t+1)^\gamma} \right)$$

$$= \frac{ce^{2ik(\lambda)\frac{x}{a}}}{W(\psi_+(\lambda),\psi_-(\lambda))} \int_0^x \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)\left(e^{-2ik(\lambda)\frac{t}{a}} - e^{-2ik(\mu)\frac{t}{a}}\right)dt}{(t+1)^\gamma}$$

$$- \frac{ce^{2ik(\lambda)\frac{x}{a}}}{W(\psi_+(\lambda),\psi_-(\lambda))} \int_x^\infty \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)e^{-2ik(\mu)\frac{t}{a}}dt}{(t+1)^\gamma}.$$

Denote

$$Q_{21}^I(x,\lambda,\mu) := \frac{ce^{2ik(\lambda)\frac{x}{a}}}{W(\psi_+(\lambda),\psi_-(\lambda))}$$

$$\times \int_0^x \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)\left(e^{-2ik(\lambda)\frac{t}{a}} - e^{-2ik(\mu)\frac{t}{a}}\right)dt}{(t+1)^\gamma}$$

and

$$Q_{21}^{II}(x,\lambda,\mu) := -\frac{ce^{2ik(\lambda)\frac{x}{a}}}{W(\psi_+(\lambda),\psi_-(\lambda))} \int_x^\infty \frac{\sin(2\omega t + \delta)p_-^2(t,\lambda)e^{-2ik(\mu)\frac{t}{a}}dt}{(t+1)^\gamma},$$

so that

$$Q_{21}(x,\lambda,\mu) = Q_{21}^I(x,\lambda,\mu) + Q_{21}^{II}(x,\lambda,\mu).$$

The second term can be estimated in the same manner as $Q_{12}(x, \lambda)$ using Lemma 1. Denote by

$$\hat{b}_n(\lambda) := \frac{1}{a} \int_0^a p_-^2(x, \lambda) e^{-2\pi i n \frac{x}{a}} dx$$

the Fourier coefficients of $p_-^2(\cdot, \lambda)$. Then

$$(23) \quad |Q_{21}^{II}(x, \lambda, \mu)| \leq \frac{|c|}{|W(\psi_+(\lambda), \psi_-(\lambda))|} \frac{1}{(x+1)^\gamma}$$

$$\times \sum_{n=-\infty}^{+\infty} |\hat{b}_n(\lambda)| \left(\frac{1}{\left| \frac{2k(\lambda)}{a} - 2\omega - \frac{2\pi n}{a} \right|} + \frac{1}{\left| \frac{2k(\lambda)}{a} + 2\omega - \frac{2\pi n}{a} \right|} \right)$$

$$\leq \frac{2|c|}{\varepsilon(\mu)|W(\psi_+(\lambda), \psi_-(\lambda))|} \left(\sum_{n=-\infty}^{+\infty} |\hat{b}_n(\lambda)| \right) \frac{1}{(x+1)^\gamma}$$

(using that $k(\mu) \in \mathbb{R}$ and $k(\lambda) \in \overline{\mathbb{C}_+}$).

To estimate Q_{21}^{I} we shall need the following lemma:

Lemma 2. *Let $\varepsilon, \beta > 0$, then there exists $c_2(\varepsilon, \beta, \gamma)$ such that for every ξ_1 and ξ_2 such that*

$$0 \leq Im \ \xi_1 \leq 1, \ |\xi_1| \geq \varepsilon,$$
$$\xi_2 \in \mathbb{R}, \ |\xi_2| \geq \varepsilon,$$
$$|Re \ \xi_1 - \xi_2| \leq \beta Im \ \xi_1$$

and for every $x \geq 0$ holds:

$$\left| e^{i\xi_1 x} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^\gamma} \right| < \frac{c_2(\varepsilon, \beta, \gamma)}{(x+1)^\gamma}.$$

PROOF. Integrating by parts we get

$$e^{i\xi_1 x} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^\gamma} = \frac{i e^{i\xi_1 x}(\xi_1 - \xi_2)}{i\xi_1\xi_2} + \frac{e^{i(\xi_1 - \xi_2)x}}{i\xi_2(x+1)^\gamma}$$

$$+ \frac{i}{\xi_1(x+1)^\gamma} + \frac{\gamma e^{i\xi_1 x}(\xi_1 - \xi_2)}{i\xi_1\xi_2} \int_0^x \frac{e^{-i\xi_2 t} dt}{(t+1)^{\gamma+1}}$$

$$- \frac{\gamma e^{i\xi_1 x}}{i\xi_1} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^{\gamma+1}}.$$

Consider the new constant

$$c_3(\gamma) := \max_{x \geq 0} x^\gamma e^{-x}.$$

For every $x \geq 0$ and ξ_1 considered,

$$(\text{Im } \xi_1)^\gamma e^{-\text{Im } \xi_1 x} \leq \frac{c_3(\gamma) e^{\text{Im } \xi_1}}{(x+1)^\gamma} \leq \frac{e c_3(\gamma)}{(x+1)^\gamma}.$$

Using that

$$|\text{Re } \xi_1 - \xi_2| \leq \beta \text{Im } \xi_1,$$

which is equivalent to

$$|\xi_1 - \xi_2| \leq \sqrt{\beta^2 + 1} \text{Im } \xi_1,$$

and the integral can be estimated as

$$\left| e^{i\xi_1 x} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^\gamma} \right| \leq \frac{2 e c_3(\gamma) \sqrt{\beta^2 + 1}}{\varepsilon^2 (x+1)^\gamma} + \frac{2}{\varepsilon (x+1)^\gamma}$$

$$+ \frac{\gamma}{\varepsilon} \left| e^{i\xi_1 x} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^{\gamma+1}} \right|.$$

The last term can be split into three parts as follows:

$$\left| e^{i\xi_1 x} \int_0^x \frac{\left(e^{-i\xi_1 t} - e^{-i\xi_2 t} \right) dt}{(t+1)^{\gamma+1}} \right|$$

$$\leq e^{-\text{Im } \xi_1 x} \left[\int_0^{\frac{1}{\text{Im } \xi_1}} + \int_{\frac{1}{\text{Im } \xi_1}}^{\frac{x}{2}} + \int_{\frac{x}{2}}^x \right] \frac{\left| e^{-i\xi_1 t} - e^{-i\xi_2 t} \right| dt}{(t+1)^{\gamma+1}}.$$

Let us estimate these three integrals separately.

(1)

$$e^{-\text{Im } \xi_1 x} \int_0^{\frac{1}{\text{Im } \xi_1}} \frac{\left| e^{-i\xi_1 t} - e^{-i\xi_2 t} \right| dt}{(t+1)^{\gamma+1}} = e^{-\text{Im } \xi_1 x} \int_0^{\frac{1}{\text{Im } \xi_1}} \frac{\left| e^{-i(\xi_1 - \xi_2) t} - 1 \right| dt}{(t+1)^{\gamma+1}}.$$

Introduce the constant

$$c_4(\beta) := \max_{|x| \leq \sqrt{\beta^2 + 1}} \frac{|e^x - 1|}{|x|}.$$

Since for the first interval

$$| - i(\xi_1 - \xi_2) t| \leq \frac{|\xi_1 - \xi_2|}{\text{Im } \xi_1} \leq \sqrt{\beta^2 + 1},$$

we have:

$$e^{-\mathrm{Im}\ \xi_1 x}\int_0^{\frac{1}{\mathrm{Im}\ \xi_1}}\frac{\left|e^{-i\xi_1 t}-e^{-i\xi_2 t}\right|dt}{(t+1)^{\gamma+1}}\leq e^{-\mathrm{Im}\ \xi_1 x}\int_0^{\frac{1}{\mathrm{Im}\ \xi_1}}\frac{c_4(\beta)\sqrt{\beta^2+1}\mathrm{Im}\ \xi_1 t dt}{(t+1)^{\gamma+1}}$$

$$\leq\frac{ec_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}(\mathrm{Im}\ \xi_1)^{1-\gamma}}{(x+1)^\gamma}\int_0^{\frac{1}{\mathrm{Im}\ \xi_1}}\frac{dt}{(t+1)^\gamma}$$

$$=\frac{ec_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}((1+\mathrm{Im}\ \xi_1)^{1-\gamma}-(\mathrm{Im}\ \xi_1)^{1-\gamma})}{(x+1)^\gamma(1-\gamma)}$$

$$\leq\frac{2^{1-\gamma}ec_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}}{(1-\gamma)(x+1)^\gamma}.$$

(2) For $x\geq\frac{2}{\mathrm{Im}\ \xi_1}$ we have

$$e^{-\mathrm{Im}\ \xi_1 x}\int_{\frac{1}{\mathrm{Im}\ \xi_1}}^{\frac{x}{2}}\frac{\left|e^{-i\xi_1 t}-e^{-i\xi_2 t}\right|dt}{(t+1)^{\gamma+1}}$$

$$\leq e^{-\mathrm{Im}\ \xi_1\frac{x}{2}}\int_{\frac{1}{\mathrm{Im}\ \xi_1}}^{\frac{x}{2}}\frac{\left(e^{\mathrm{Im}\ \xi_1\left(t-\frac{x}{2}\right)}+e^{-\mathrm{Im}\ \xi_1\frac{x}{2}}\right)dt}{(t+1)^{\gamma+1}}$$

$$\leq 2e^{-\mathrm{Im}\ \xi_1\frac{x}{2}}\int_{\frac{1}{\mathrm{Im}\ \xi_1}}^\infty\frac{dt}{t^{\gamma+1}}\leq\frac{2^{\gamma+1}ec_3(\gamma)}{\gamma(x+2)^\gamma}.$$

For $x<\frac{2}{\mathrm{Im}\ \xi_1}$ the integral is negative.

(3)

$$e^{-\mathrm{Im}\ \xi_1 x}\int_{\frac{x}{2}}^x\frac{\left|e^{-i\xi_1 t}-e^{-i\xi_2 t}\right|dt}{(t+1)^{\gamma+1}}\leq\int_{\frac{x}{2}}^x\frac{2dt}{(t+1)^{\gamma+1}}<\frac{2^{\gamma+1}}{\gamma(x+2)^\gamma}.$$

Combining these estimates we get

$$\left|e^{i\xi_1 x}\int_0^x\frac{\left(e^{-i\xi_1 t}-e^{-i\xi_2 t}\right)dt}{(t+1)^{\gamma+1}}\right|<\frac{c_2(\varepsilon,\beta,\gamma)}{(x+1)^\gamma}$$

with

$$c_2(\varepsilon,\beta,\gamma):=\frac{2ec_3(\gamma)\sqrt{\beta^2+1}}{\varepsilon^2}+\frac{2}{\varepsilon}$$

$$+\frac{\gamma}{\varepsilon}\left(\frac{2^{1-\gamma}ec_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}}{1-\gamma}+\frac{2^{\gamma+1}ec_3(\gamma)}{\gamma}+\frac{2^{\gamma+1}}{\gamma}\right).$$

This completes the proof of the lemma. $\square$

Let us continue to estimate Q_{21}^I

$$Q_{21}^I(x, \lambda, \mu) = \sum_{n=-\infty}^{+\infty} \frac{c\hat{b}_n(\lambda)e^{i\delta+2i\omega+2\pi in\frac{x}{a}}}{2iW(\psi_+(\lambda), \psi_-(\lambda))}e^{i\left(\frac{2k(\lambda)}{a}-2\omega-\frac{2\pi n}{a}\right)x}$$

$$\times \int_0^x \frac{\left(e^{-i\left(\frac{2k(\lambda)}{a}-2\omega-\frac{2\pi n}{a}\right)t} - e^{-i\left(\frac{2k(\mu)}{a}-2\omega-\frac{2\pi n}{a}\right)t}\right) dt}{(t+1)^\gamma}$$

$$- \sum_{n=-\infty}^{+\infty} \frac{c\hat{b}_n(\lambda)e^{-i\delta-2i\omega+2\pi in\frac{x}{a}}}{2iW(\psi_+(\lambda), \psi_-(\lambda))}e^{i\left(\frac{2k(\lambda)}{a}+2\omega-\frac{2\pi n}{a}\right)x}$$

$$\times \int_0^x \frac{\left(e^{-i\left(\frac{2k(\lambda)}{a}+2\omega-\frac{2\pi n}{a}\right)t} - e^{-i\left(\frac{2k(\mu)}{a}+2\omega-\frac{2\pi n}{a}\right)t}\right) dt}{(t+1)^\gamma}.$$

Applying Lemma 2 we get

$$(24) \quad |Q_{21}^I(x, \lambda, \mu)| \leq \frac{|c|c_2(\varepsilon(\mu), \beta, \gamma)}{|W(\psi_+(\lambda), \psi_-(\lambda))|} \left(\sum_{n=-\infty}^{+\infty} |\hat{b}_n(\lambda)|\right) \frac{1}{(x+1)^\gamma}.$$

Therefore combining the estimates (22), (24) and (23) the matrix Q can be estimated as follows

$$\|Q(x, \lambda, \mu)\| \leq \frac{1}{(x+1)^\gamma} \frac{|c|}{|W(\psi_+(\lambda), \psi_-(\lambda))|}$$

$$\times \sqrt{\frac{4}{\varepsilon^2(\mu)}\left(\sum_{n=-\infty}^{+\infty} |b_n(\lambda)|\right)^2 + \left(\frac{2}{\varepsilon(\mu)} + c_2(\varepsilon(\mu), \beta, \gamma)\right)^2 \left(\sum_{n=-\infty}^{+\infty} |\hat{b}_n(\lambda)|\right)^2}.$$

Let us estimate now the Fourier coefficients. For $n \neq 0$ we have:

$$b_n(\lambda) = \frac{1}{a} \int_0^a p_+^2(x, \lambda)e^{-2\pi in\frac{x}{a}}dx = -\frac{a}{4\pi^2 n^2} \int_0^a (p_+^2(x, \lambda))'' e^{-2\pi in\frac{x}{a}}dx.$$

Thus

$$(25) \qquad |b_n(\lambda)| \leq \frac{a}{4\pi^2 n^2} \int_0^a |(p_+^2(x, \lambda))''|dx.$$

In terms of the corresponding Bloch solution the second derivative of p_+^2 is

$$(p_+^2(x, \lambda))'' = 2e^{-\frac{2ik(\lambda)x}{a}} \left(\psi_+''(x, \lambda)\psi_+(x, \lambda) - \frac{2k^2(\lambda)}{a^2}\psi_+^2(x, \lambda)\right.$$

$$\left. + (\psi_+'(x, \lambda))^2 - \frac{4ik(\lambda)}{a}\psi_+'(x, \lambda)\psi_+(x, \lambda)\right).$$

Let us estimate the norm in $L_1(0; a)$ of the function $\psi_+''(\cdot, \lambda)$. From the equation

$$\psi_+''(x, \lambda) = (q(x) - \lambda)\psi_+(x, \lambda),$$

we see that

$$\|\psi_+''(\cdot, \lambda)\|_{L_1(0;a)} \leq (\|q\|_{L_1(0;a)} + |\lambda|a) \max_{x \in [0;a]} |\psi_+(x, \lambda)|.$$

Since the functions

$$e^{-\frac{ik(\lambda)x}{a}}, \psi_+(x, \lambda), \psi_+'(x, \lambda)$$

are continuous in both variables on the set

$$[0; a] \times U(\beta, \mu)$$

and hence attain their maximums, we have: the integral

$$\int_0^a |(p_+^2(x, \lambda))''| dx$$

is bounded on $U(\beta, \mu)$. For n=0,

$$b_0(\lambda) = \frac{1}{a} \int_0^a p_+^2(x, \lambda) dx = \frac{1}{a} \int_0^a e^{-2ik(\lambda)\frac{x}{a}} \psi_+^2(x, \lambda) dx$$

and is also bounded. The same argument is valid for $\hat{b}_n$. Finally we see that there exists $c_5(\beta, \mu)$ such that for every $\lambda \in U(\beta, \mu)$

$$|b_n(\lambda)|, |\hat{b}_n(\lambda)| \leq \frac{c_5(\beta, \mu)}{n^2 + 1}.$$

The Wronskian $W(\psi_+(\lambda), \psi_-(\lambda))$ does not have zeros in $U(\beta, \mu)$. Also in the formula for the derivative of Q,

$$Q'(x, \lambda, \mu) = \frac{c \sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} \begin{pmatrix} 0 & -p_+^2(x, \lambda) \\ p_-^2(x, \lambda) & 0 \end{pmatrix}$$
$$- \left[Q(x, \lambda, \mu), \begin{pmatrix} -\frac{ik(\lambda)}{a} & 0 \\ 0 & \frac{ik(\lambda)}{a} \end{pmatrix} \right],$$

the functions $\pm\frac{k(\lambda)}{a}, \pm c \sin(2\omega x + \delta)p_\pm^2(x, \lambda)$ are bounded for $(x; \lambda) \in [0; +\infty) \times U(\beta, \mu)$. Hence there exists $c_1(\beta, \mu, \gamma)$ such that for every $\lambda \in U(\beta, \mu)$ and $x \geq 0$ the following estimates hold

$$\|Q(x, \lambda, \mu)\|, \|Q'(x, \lambda, \mu)\| \leq \frac{c_1(\beta, \mu, \gamma)}{(x+1)^\gamma}.$$

This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\square$

Let us study the properties of the remainder

$$
\begin{aligned}
(26) \quad R^{(2)}(x, \lambda, \mu) := e^{-Q(x,\lambda,\mu)} & \left[\begin{pmatrix} -\frac{ik(\lambda)}{a} & 0 \\ 0 & \frac{ik(\lambda)}{a} \end{pmatrix} \right. \\
+ \frac{c \sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} & \begin{pmatrix} -p_+(x, \lambda)p_-(x, \lambda) & -p_+^2(x, \lambda) \\ p_-^2(x, \lambda) & p_+(x, \lambda)p_-(x, \lambda) \end{pmatrix} \\
\left. + R^{(1)}(x, \lambda) \right] e^{Q(x,\lambda,\mu)} & - \begin{pmatrix} -\frac{ik(\lambda)}{a} & 0 \\ 0 & \frac{ik(\lambda)}{a} \end{pmatrix} \\
- \frac{c \sin(2\omega x + \delta)}{(x+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} & \begin{pmatrix} -p_+(x, \lambda)p_-(x, \lambda) & 0 \\ 0 & p_+(x, \lambda)p_-(x, \lambda) \end{pmatrix} \\
& - e^{-Q(x,\lambda,\mu)} \left(e^{Q(x,\lambda,\mu)} \right)' .
\end{aligned}
$$

Lemma 3. *The remainder $R^{(2)}$ given by (26) possesses the following properties:*

(1) $R^{(2)} \in L_1(0; \infty)$ *and the integral*

$$
\int_0^\infty \| R^{(2)}(x, \lambda, \mu) \| dx
$$

converges uniformly with respect to $\lambda \in U(\beta, \mu)$.

(2)

$$
(27) \qquad \begin{aligned}
(R^{(2)}(x, \mu, \mu))_{21} &= \overline{(R^{(2)}(x, \mu, \mu))_{12}}, \\
(R^{(2)}(x, \mu, \mu))_{22} &= \overline{(R^{(2)}(x, \mu, \mu))_{11}}.
\end{aligned}
$$

PROOF. The first assertion follows directly from Theorem 2.

The property of matrices from the second assertion is preserved under summation and multiplication of such matrices as well as taking the inverse. We can see from (11) and (18) that $R^{(1)}(x, \mu)$ and $Q(x, \mu, \mu)$ possess this conjugation property. Therefore

$$
Q'(x, \mu, \mu), e^{Q(x,\mu,\mu)}, \left(e^{Q(x,\mu,\mu)} \right)'
$$

and finally $R^{(2)}(x, \mu, \mu)$ (from (26)) also possess this property. It should be taken into account that $W(\psi_+(\mu), \psi_-(\mu))$ is pure imaginary.   □

Let us denote

$$
\nu(x, \lambda) := -\frac{ik(\lambda)}{a} - \frac{c \sin(2\omega x + \delta)p_+(x, \lambda)p_-(x, \lambda)}{(x+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))}.
$$

Finally we obtain a system of the Levinson form:

$$
(28) \qquad \widetilde{v}'(x) = \left[\begin{pmatrix} \nu(x, \lambda) & 0 \\ 0 & -\nu(x, \lambda) \end{pmatrix} + R^{(2)}(x, \lambda, \mu) \right] \widetilde{v}(x).
$$

4. A Levinson-type theorem for 2×2 systems

In this section, we prove two statements that give a uniform estimate and asymptotics of solutions to certain 2×2 differential systems. The approach is the same as for the Levinson theorem [8], but the difference is that we are interested in properties of solution with a given initial condition.

Consider the system

$$(29) \qquad u_1'(x) = \left[\begin{pmatrix} \lambda(x) & 0 \\ 0 & -\lambda(x) \end{pmatrix} + R(x) \right] u_1(x)$$

for $x \geq 0$, where $u_1(x)$ is a two-dimensional vector function and $R(x)$ is a 2×2 matrix with complex entries.

Lemma 4. *Assume that*

$$(30) \qquad \int_0^\infty \|R(t)\| dt < \infty.$$

and that there exists a constant M such that for every $x \leq y$ it holds

$$\int_x^y Re \ \lambda(t) dt \geq -M.$$

Then every solution u_1 to (29) satisfies the estimate

$$(31) \qquad \|u_1(x)\| \leq \|u_1(0)\| e^{\int_0^x Re \ \lambda(t) dt} \sqrt{1 + e^{4M}} e^{\sqrt{1+e^{4M}} \int_0^\infty \|R(t)\| dt}.$$

PROOF. First transform the system (29) by variation of parameters. Denote

$$\Lambda(x) := \begin{pmatrix} \lambda(x) & 0 \\ 0 & -\lambda(x) \end{pmatrix}$$

and take

$$(32) \qquad u_1(x) = e^{\int_0^x \Lambda(t) dt} u_2(x), \ \text{or} \ u_2(x) := e^{-\int_0^x \Lambda(t) dt} u_1(x).$$

After substitution (29) becomes

$$(33) \qquad u_2'(x) = e^{-\int_0^x \Lambda(t) dt} R(x) u_1(x).$$

Integrating this from 0 to x and returning back to the function u_1 on the left-hand side we get

$$(34) \qquad u_1(x) = e^{\int_0^x \Lambda(t) dt} u_1(0) + \int_0^x e^{\int_t^x \Lambda(s) ds} R(t) u_1(t) dt.$$

Now multiply this expression by $e^{-\int_0^x \Lambda(s) ds}$ and denote

$$(35) \qquad u_3(x) := e^{-\int_0^x \Lambda(t) dt} u_1(x).$$

We get the following equation for u_3 considered in $L_\infty(0, \infty); \mathbb{C}^2)$

(36)
$$u_3(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_0^x \lambda(s)ds} \end{pmatrix} u_1(0) + \int_0^x \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_t^x \lambda(s)ds} \end{pmatrix} R(t)u_3(t)dt.$$

The norm of the operator V,
$$V : u \mapsto \int_0^x \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_t^x \lambda(s)ds} \end{pmatrix} R(t)u(t)dt$$

is bounded by
$$\|V\| \le \sqrt{1 + e^{4M}} \int_0^\infty \|R(t)\|dt$$

and the norm of the j-th power is bounded by
$$\|V^j\| \le \frac{(\sqrt{1 + e^{4M}} \int_0^\infty \|R(t)\|dt)^j}{j!}.$$

Hence
$$u_3(x) = (I - V)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_0^x \lambda(s)ds} \end{pmatrix} u_1(0)$$

and

$$\|u_3\|_{L_\infty((0;\infty),\mathbb{C}^2)} \le \exp\left(\sqrt{1 + e^{4M}} \int_0^\infty \|R(t)\|dt\right) \sqrt{1 + e^{4M}} \|u_1(0)\|.$$

Returning to u_1, we arrive at the estimate (31). $\qquad\square$

The second lemma states the asymptotics of the solution.

Lemma 5. *Let that all conditions of Lemma 4 be satisfied, then the following asymptotics hold:*

(1) *If*

(37)
$$\int_0^\infty Re\ \lambda(t)dt < +\infty,$$

then every solution u_1 of (29) has the following asymptotics:

$$u_1(x) = \begin{pmatrix} e^{\int_0^x \lambda(s)ds} & 0 \\ 0 & e^{-\int_0^x \lambda(s)ds} \end{pmatrix} [u_1(0)$$

$$+ \int_0^\infty \begin{pmatrix} e^{-\int_0^t \lambda(s)ds} & 0 \\ 0 & e^{\int_0^t \lambda(s)ds} \end{pmatrix} R(t)u_1(t)dt + o(1) \Bigg] \quad as\ x \to \infty.$$

(2) *If*

(38)
$$\int_0^\infty Re\ \lambda(t)dt = +\infty,$$

then every solution u_1 of (29) *has the following asymptotics:*

$$u_1(x) = e^{\int_0^x \lambda(s)ds} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (u_1(0) \right.$$

$$\left. + \int_0^\infty e^{-\int_0^t \lambda(s)ds} R(t)u_1(t)dt \right) + o(1) \right] \quad as\ x \to \infty.$$

PROOF. Asymptotics 1. Consider the function u_2 given by (32) and integrate (33):

$$(39) \qquad u_2(x) = u_1(0) + \int_0^x e^{-\int_0^t \Lambda(s)ds} R(t)u_1(t)dt.$$

Since for every $x \le y$ we have the estimate

$$\int_x^y \mathrm{Re}\ \lambda(s)ds \le \int_x^y |\mathrm{Re}\ \lambda(s)|ds$$

$$\le \int_0^\infty |\mathrm{Re}\ \lambda(s)|ds \le \int_0^\infty \mathrm{Re}\ \lambda(s)ds + 2M,$$

the exponent under the integral in (39) is bounded. The solution $u_1(t)$ is also bounded in this case due to Lemma 4. Hence the integral in (39) converges as $x \to \infty$ and there exists

$$\lim_{x \to \infty} u_2(x) = u_1(0) + \int_0^\infty e^{-\int_0^t \Lambda(s)ds} R(t)u_1(t)dt.$$

Returning to u_1 we obtain the answer.

Asymptotics 2. Consider the function u_3 given by (35) and the corresponding equation (36). It follows from Lemma 4 that $u_3(t)$ is bounded, and hence Lebesgue's dominated convergence theorem implies that the following limit exists

$$\lim_{x \to \infty} u_3(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[u_1(0) + \int_0^\infty R(t)u_3(t)dt \right].$$

This is equivalent to the announced asymptotics for u_1. $\qquad\square$

5. Asymptotics for the solution φ and Weyl-Titchmarsh type formula

In this section, we obtain the asymptotics for the solution $\varphi_\alpha(x, \lambda)$ and prove the Weyl-Titchmarsh type formula for the operator $\mathcal{L}_\alpha$. Consider the set

$$U(\beta) := \bigcup_{\mu \in \sigma(\mathcal{L}_{per}) \setminus \{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \ge 0\}} U(\beta, \mu)$$

that belongs to $\overline{\mathbb{C}_+}$ and contains

$$\sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}$$

as a part of its boundary. The number β is arbitrary here.

Theorem 3. *Let $\frac{2a\omega}{\pi} \notin \mathbb{Z}$ and $q_1 \in L_1(\mathbb{R}_+)$, then the solution φ_α of the Cauchy problem*

$$-\varphi_\alpha''(x, \lambda) + \left(q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x)\right)\varphi_\alpha(x, \lambda) = \lambda\varphi_\alpha(x, \lambda),$$
$$\varphi_\alpha(0, \lambda) = \sin\alpha,$$
$$\varphi_\alpha'(0, \lambda) = \cos\alpha$$

has the following asymptotics. For every $\lambda \in U(\beta)$ there exists $A_\alpha(\lambda)$ such that

   (1) *If $\lambda \in \mathbb{C}_+ \cap U(\beta)$, then*

$$\varphi_\alpha(x, \lambda) = A_\alpha(\lambda)\psi_-(x, \lambda) + o\left(e^{Im\ k(\lambda)\frac{x}{a}}\right),$$
$$\varphi_\alpha'(x, \lambda) = A_\alpha(\lambda)\psi_-'(x, \lambda) + o\left(e^{Im\ k(\lambda)\frac{x}{a}}\right)$$

      as $x \to \infty$.

   (2) *If $\lambda \in \sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}$, then*

$$\varphi_\alpha(x, \lambda) = A_\alpha(\lambda)\psi_-(x, \lambda) + \overline{A_\alpha(\lambda)}\psi_+(x, \lambda) + o(1),$$
$$\varphi_\alpha'(x, \lambda) = A_\alpha(\lambda)\psi_-'(x, \lambda) + \overline{A_\alpha(\lambda)}\psi_+'(x, \lambda) + o(1)$$

      as $x \to \infty$.

The function A_α is analytic in the interior of $U(\beta)$ and has boundary values on

$$\sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}.$$

PROOF. We are going to omit the index α since the value of the boundary parameter is fixed throughout this proof. According to (8) and (10) we write
(40)
$$\begin{pmatrix} \varphi(x) \\ \varphi'(x) \end{pmatrix} = \begin{pmatrix} \psi_-(x, \lambda) & \psi_+(x, \lambda) \\ \psi_-'(x, \lambda) & \psi_+'(x, \lambda) \end{pmatrix} \begin{pmatrix} e^{ik(\lambda)\frac{x}{a}} & 0 \\ 0 & e^{-ik(\lambda)\frac{x}{a}} \end{pmatrix} v_\varphi(x, \lambda).$$

This is the definition of v_φ, a solution of (12) corresponding to φ. Let us fix the point

$$\mu \in \sigma(\mathcal{L}_{per})\backslash\{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}$$

and consider $\lambda \in U(\beta, \mu)$. The function

$$\widetilde{v}_\varphi(x, \lambda, \mu) := e^{-Q(x, \lambda, \mu)}v_\varphi(x, \lambda),$$

is a solution to (28) corresponding to φ. Let us see that conditions of Lemma 4 are satisfied for the system (28) uniformly with respect to $\lambda \in U(\beta, \mu)$. First of all we have estimate (30) from Lemma 3 and

$$\text{Re } \nu(x, \lambda) = \frac{\text{Im } k(\lambda)}{a} - \text{Re } \left(\frac{c \sin(2\omega x + \delta) p_+(x, \lambda) p_-(x, \lambda)}{(x+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} \right).$$

Estimating the second term in the same way as in Theorem 2 we have:

$$\left| \int_x^y \text{Re } \frac{c \sin(2\omega t + \delta) p_+(t, \lambda) p_-(t, \lambda)}{(t+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} dt \right| \leq \frac{|c|a}{\pi |W(\psi_+(\lambda), \psi_-(\lambda))|}$$

$$\times \left(\sum_{n=-\infty}^{\infty} |\widetilde{b}_n(\lambda)| \left(\frac{1}{\left| \frac{2a\omega}{\pi} + 2n \right|} + \frac{1}{\left| \frac{2a\omega}{\pi} - 2n \right|} \right) \right) \frac{1}{(x+1)^\gamma},$$

where

$$\widetilde{b}_n(\lambda) := \frac{1}{a} \int_0^a p_+(x, \lambda) p_-(x, \lambda) e^{-2\pi i n \frac{x}{a}} dx$$

are Fourier coefficients for $p_+(\cdot, \lambda) p_-(\cdot, \lambda)$. Analogously to (25) we have:

$$|\widetilde{b}_n(\lambda)| \leq \frac{a}{4\pi^2 n^2} \int_0^a |(\psi_+(x, \lambda) \psi_-(x, \lambda))''| dx.$$

So there exists $c_6(\beta, \mu)$ such that for every $\lambda \in U(\beta, \mu)$ and $n \neq 0$

$$|\widetilde{b}_n(\lambda)| \leq \frac{c_6(\beta, \mu)}{n^2},$$

while

$$|\widetilde{b}_0(\lambda)| \leq c_6(\beta, \mu).$$

Eventually there exists $c_7(\beta, \mu)$ such that

$$\left| \int_x^y \text{Re } \frac{c \sin(2\omega t + \delta) p_+(t, \lambda) p_-(t, \lambda)}{(t+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))} dt \right| \leq c_7(\beta, \mu)$$

for every $0 \leq x \leq y$ and $\lambda \in U(\beta, \mu)$. Thus we can take

$$M(\lambda) \equiv c_7(\beta, \mu)$$

for these values of λ. Lemma 4 gives the estimate

(41) $$\|\widetilde{v}_\varphi(x, \lambda, \mu)\| \leq \|\widetilde{v}_\varphi(0, \lambda, \mu)\| e^{\text{Im } k(\lambda) \frac{x}{a}} c_8(\beta, \mu),$$

where

$$c_8(\beta, \mu) := \sqrt{1 + e^{4c_7(\beta, \mu)}}$$

$$\times \exp \left(\sqrt{1 + e^{4c_7(\beta, \mu)}} \max_{\lambda \in U(\beta, \mu)} \int_0^\infty \|R^{(2)}(t, \lambda)\| dt \right).$$

Conditions of Lemma 5 are also satisfied: (37) holds for $\lambda \in \mathbb{R} \cap U(\beta, \mu)$ and (38) holds for $\lambda \in \mathbb{C}_+ \cap U(\beta, \mu)$. So Lemma 5 gives the following asymptotics:

- for $\lambda \in \mathbb{C}_+ \cap U(\beta, \mu)$,

$$
\widetilde{v}_\varphi(x, \lambda, \mu) = e^{-ik(\lambda)\frac{x}{a} - \int_0^x \frac{c \sin(2\omega t + \delta) p_+(t,\lambda) p_-(t,\lambda) dt}{(t+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))}} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\widetilde{v}_\varphi(0, \lambda, \mu) \right. \right.
$$
$$
\left. \left. + \int_0^\infty e^{ik(\lambda)\frac{t}{a} + \int_0^t \frac{c \sin(2\omega s + \delta) p_+(s,\lambda) p_-(s,\lambda) ds}{(s+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))}} R^{(2)}(t, \lambda, \mu) \widetilde{v}_\varphi(t, \lambda, \mu) dt \right) + o(1) \right],
$$

- for $\lambda = \mu$,

$$
(42) \quad \widetilde{v}_\varphi(x, \mu, \mu) = \begin{pmatrix} e^{-ik(\mu)\frac{x}{a} - \int_0^x \frac{c \sin(2\omega t + \delta) p_+(t,\mu) p_-(t,\mu) dt}{(t+1)^\gamma W(\psi_+(\mu), \psi_-(\mu))}} & 0 \\ 0 & e^{ik(\mu)\frac{x}{a} + \int_0^x \frac{c \sin(2\omega t + \delta) p_+(t,\mu) p_-(t,\mu) dt}{(t+1)^\gamma W(\psi_+(\mu), \psi_-(\mu))}} \end{pmatrix}
$$
$$
\times \left[\widetilde{v}_\varphi(0, \mu, \mu) + \int_0^\infty \begin{pmatrix} e^{ik(\mu)\frac{t}{a} + \int_0^t \frac{c \sin(2\omega s + \delta) p_+(s,\mu) p_-(s,\mu) ds}{(s+1)^\gamma W(\psi_+(\mu), \psi_-(\mu))}} & 0 \\ 0 & e^{-ik(\mu)\frac{t}{a} - \int_0^t \frac{c \sin(2\omega s + \delta) p_+(s,\mu) p_-(s,\mu) ds}{(s+1)^\gamma W(\psi_+(\mu), \psi_-(\mu))}} \end{pmatrix} \right.
$$
$$
\left. \times R^{(2)}(t, \mu, \mu) \widetilde{v}_\varphi(t, \mu, \mu) dt + o(1) \right].
$$

Since $Q(x, \lambda, \mu) = O\left(\frac{1}{(x+1)^\gamma} \right)$, we can denote for $\lambda \in U(\beta, \mu)$:

$$
A(\lambda, \mu) := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{-\int_0^\infty \frac{c \sin(2\omega t + \delta) p_+(t,\lambda) p_-(t,\lambda) dt}{(t+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))}} \left[e^{-Q(0,\lambda,\mu)} v_\varphi(0, \lambda) \right. \right.
$$
$$
\left. \left. + \int_0^\infty e^{ik(\lambda)\frac{t}{a} + \int_0^t \frac{c \sin(2\omega s + \delta) p_+(s,\lambda) p_-(s,\lambda) ds}{(s+1)^\gamma W(\psi_+(\lambda), \psi_-(\lambda))}} R^{(2)}(t, \lambda, \mu) e^{Q(t,\lambda,\mu)} v_\varphi(t, \lambda) dt \right] \right\rangle
$$

(where $< \cdot, \cdot >$ stands for the scalar product in $\mathbb{C}^2$) and have:

$$
(43) \qquad \lim_{x \to \infty} v_\varphi(x, \lambda) e^{ik(\lambda)\frac{x}{a}} = \begin{pmatrix} A(\lambda, \mu) \\ 0 \end{pmatrix}.
$$

From this we see that the coefficient $A(\lambda, \mu)$ does not depend on μ, so we will denote it by $A(\lambda)$. Relation (40) can be written as
(44)
$$
v_\varphi(x, \lambda) = \frac{1}{W(\psi_+(\lambda), \psi_-(\lambda))} \begin{pmatrix} \psi'_+(x, \lambda)\varphi(x, \lambda) - \psi_+(x, \lambda)\varphi'(x, \lambda) \\ \varphi'(x, \lambda)\psi_-(x, \lambda) - \varphi(x, \lambda)\psi'_-(x, \lambda) \end{pmatrix},
$$

so $v_\varphi(x, \cdot)$ is analytic in $\mathbb{C}_+$ and continuous up to

$$
\sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, n \geq 0\}.
$$

From the estimate (41) and properties of $Q(x, \lambda, \mu)$ and $R^{(2)}(x, \lambda, \mu)$ given by Theorem 2 and Lemma 3 it follows that $A(\lambda)$ is continuous in

$U(\beta, \mu)$ and analytic in its interior. Thus A is analytic in the interior of $U(\beta)$ having non-tangential boundary limits on

$$\sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}.$$

that coincide with its values on this set.

The solution $\varphi(x, \lambda)$ and its derivative are real if λ is real. Thus (44) shows that the upper and the lower components of the vector $v_\varphi(x, \lambda)$ are complex conjugate for $\lambda \in \sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, n \geq 0\}$. This property is preserved if we multiply the vector by a matrix X such that

$$X_{21} = \overline{X_{12}}, \ X_{22} = \overline{X_{11}},$$

like (27). It follows from Lemma 3 that the upper and the lower components of the vectors in the equality (42) are complex conjugate to each other. Hence for $\lambda = \mu$ we have

(45) $\qquad v_\varphi(x, \mu) = \begin{pmatrix} A(\mu)e^{-ik(\mu)\frac{x}{a}} \\ A(\mu)e^{ik(\mu)\frac{x}{a}} \end{pmatrix} + o(1) \text{ as } x \to \infty.$

The asymptotics of the solution φ and its derivative follows from (40), (43) and (45). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Using the obtained asymptotics both on the spectrum and in $\mathbb{C}_+$ we now prove the Weyl-Titchmarsh type formula.

Theorem 4. *Let* $\frac{2a\omega}{\pi} \notin \mathbb{Z}$ *and* $q_1 \in L_1(\mathbb{R}_+)$, *then for almost all* $\lambda \in \sigma(\mathcal{L}_{per})$ *the spectral density of the operator* $\mathcal{L}_\alpha$, *defined by* (1), *is given by*

$$\rho_\alpha'(\lambda) = \frac{1}{2\pi |W(\psi_+(\lambda), \psi_-(\lambda))| \, |A_\alpha(\lambda)|^2},$$

where A_α *is the same as in Theorem 3.*

PROOF. In addition to φ_α consider another one solution of (7), to be denoted by $\theta_\alpha := \varphi_{\alpha + \frac{\pi}{2}}$, satisfying the initial conditions

$$\theta_\alpha(0, \lambda) = \cos \alpha, \ \theta_\alpha'(0, \lambda) = -\sin \alpha.$$

The Wronskian of φ_α and θ_α is equal to one. Theorem 3 yields for $\lambda \in U(\beta) \cap \mathbb{C}_+$,

$$\theta_\alpha(x, \lambda) = A_{\alpha+\frac{\pi}{2}}(\lambda)\psi_-(x, \lambda) + o\left(e^{\text{Im } k(\lambda)\frac{x}{a}}\right) \text{ as } x \to \infty.$$

Since the operator $\mathcal{L}_\alpha$ is in the limit point case, the combination

$$\theta_\alpha + m_\alpha \varphi_\alpha$$

belongs to $L_2(0, \infty)$ (where m_α is the Weyl function for $\mathcal{L}_\alpha$). It has the asymptotics

$$\theta_\alpha(x, \lambda) + m_\alpha(\lambda)\varphi_\alpha(x, \lambda) = (A_{\alpha+\frac{\pi}{2}}(\lambda) + m_\alpha A_\alpha(\lambda))\psi_-(x, \lambda) + o(e^{\text{Im } k(\lambda)\frac{x}{a}}).$$

Therefore

$$m_\alpha(\lambda) = -\frac{A_{\alpha+\frac{\pi}{2}}(\lambda)}{A_\alpha(\lambda)}$$

for $\lambda \in U(\beta) \cap \mathbb{C}_+$ and

$$m_\alpha(\lambda + i0) = -\frac{A_{\alpha+\frac{\pi}{2}}(\lambda)}{A_\alpha(\lambda)}$$

for $\lambda \in \sigma(\mathcal{L}_{per}) \backslash \{\lambda_n, \mu_n, \lambda_n^+, \lambda_n^-, n \geq 0\}$. It follows from the subordinacy theory [10] that the spectrum of $\mathcal{L}_\alpha$ on this set is purely absolutely continuous and

$$(46) \quad \rho_\alpha'(\lambda) = \frac{1}{\pi}\mathrm{Im}\; m_\alpha(\lambda+i0) = \frac{A_\alpha(\lambda)\overline{A_{\alpha+\frac{\pi}{2}}(\lambda)} - \overline{A_\alpha(\lambda)}A_{\alpha+\frac{\pi}{2}}(\lambda)}{2\pi i|A_\alpha(\lambda)|^2}.$$

Theorem 3 yields for these values of λ:

$$\theta_\alpha(x,\lambda) = A_{\alpha+\frac{\pi}{2}}(\lambda)\psi_-(x,\lambda) + \overline{A_{\alpha+\frac{\pi}{2}}(\lambda)}\psi_+(x,\lambda) + o(1),$$
$$\theta_\alpha'(x,\lambda) = A_{\alpha+\frac{\pi}{2}}(\lambda)\psi_-'(x,\lambda) + \overline{A_{\alpha+\frac{\pi}{2}}(\lambda)}\psi_+'(x,\lambda) + o(1),$$

as $x \to \infty$. Substituting these asymptotics and the asymptotics of φ_α and φ_α' into the expression for the Wronskian we get:

$$1 = (\overline{A_\alpha(\lambda)}A_{\alpha+\frac{\pi}{2}}(\lambda) - A_\alpha(\lambda)\overline{A_{\alpha+\frac{\pi}{2}}(\lambda)})W(\psi_+(\lambda), \psi_-(\lambda))$$

(the term $o(1)$ cancels, since both sides are independent of x). Combining with (46) we have

$$\rho_\alpha'(\lambda) = \frac{1}{-2\pi i W(\psi_+(\lambda), \psi_-(\lambda))|A_\alpha(\lambda)|^2}$$
$$= \frac{1}{2\pi|W(\psi_+(\lambda), \psi_-(\lambda))||A_\alpha(\lambda)|^2},$$

which completes the proof. $\qquad\square$

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