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Abstract – **In control system theory, a performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system parameters. In this paper, the authors demonstrate two methods to determine analytically the Integral of the Square of the Error (ISE) performance index value for a first-order-plus-dead-time (FOPDT) process model under PI control. The ability of proportional/integral (PI) and proportional/integral/derivative (PID) controllers to compensate most practical industrial processes has led to their wide acceptance in industrial applications. The most direct way to set up PI/PID controller parameters is the use of tuning rules. The second part of this paper examines the performance of ten tuning rules used to compensate six representative processes.**

Keywords – **Performance Index, Integral of Absolute Error, ISE, Optimum Control System.**

1 Introduction

Increasing emphasis on the mathematical formulation and measurement of control system performance can be found in the recent literature on automatic control. Modern control theory assumes that the systems engineer can specify quantitatively the required system performance. Then a performance index can be calculated or measured and used to evaluate the system's performance. A quantitative measure of the performance of a system is also necessary for the operation of modern adaptive control systems, for automatic parameter optimisation of a control system, and for the design of optimum systems [1]. A system is considered an optimum control system when the system parameters are adjusted so that the index reaches an extremum value, commonly a minimum value. A performance index, to be useful, must be a number that is always positive or zero. Then the best system is defined as the system that minimises this index. Two suitable performance indices examined in this paper are the integral of the square of the error, ISE, and the integral of the absolute magnitude of the error, IAE.

The second part of the paper examines the performance of ten PI or PID tuning rules used to compensate six representative processes. The tuning rules are taken from a book by A. O'Dwyer [2] which comprehensively compiles, using a unified notation, the tuning rules to control processes with time delay, proposed over six decades (1942 – 2002).

II.a Analytical Calculation of ISE Using Contour Integration and The Method of Residues

The basic problem that will be considered is that of the evaluation of the integral

$$J = \int_0^{\infty} e^2(t)dt \qquad (1)$$

in which *e(t)* has Laplace transform *E(s)* given by

$$E(s) = \frac{B(s) + D(s)Exp(-s\tau)}{A(s) + C(s)Exp(-s\tau)} \qquad (2)$$

and *A(s), B(s), C(s)* and *D(s)* are polynomials in *s* of finite degree and with real coefficients; τ is the time delay. It will be assumed that the above integral (1) exists, or equivalently that the system is stable. A necessary, but not sufficient, condition for stability is that the poles of *E(s)* lie in the open left-half plane, a fact of which much use will be made.

From Parseval's theorem it follows that

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} E(s)E(-s)ds$$

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left(\frac{B(s) + D(s)\exp(-s\tau)}{A(s) + C(s)\exp(-s\tau)}\right)\left(\frac{B(-s) + D(-s)\exp(s\tau)}{A(-s) + C(-s)\exp(s\tau)}\right)ds \qquad (3)$$

For delay free systems (i.e. those for which *C(s) = D(s) = 0*) it is possible to evaluate such integrals by closing the contour on either the left or the right and using the theory of residues. Such an approach applied to the above integral in its present form offers little hope of success. This is because there are, in general, an infinite number of poles in both the left and the right half-planes and moreover it is not possible to obtain a closed form solution for these. However, it can be shown that it is indeed possible to evaluate such integrals using contour integration and the theory of residues *provided that* the integrand is first suitably rearranged in such a way that there are only a finite number of relevant poles.

The basic idea is to split the integrand into two parts, the first of which contains all the poles arising from the zeros of *(A(s) + C(s) exp(-sτ))* and the second all those arising from the zeros of *(A(-s) + C(-s) exp(sτ))*. This is achieved by first obtaining an equivalent form for *E(-s)* at the poles of *E(s)* [3, page 1066]. If the two parts are treated in different ways, as suggested by Walton and Marshall, the poles arising from the roots of

$$A(-s)A(s) - C(-s)C(s) = 0 \qquad (4)$$

must also be considered. It is now possible to close the contour in the left half-plane and in the right half-plane. In both cases, the only enclosed poles arise from the enclosed zeros of equation 4. Assuming that the integrals round the semicircles at infinity are zero (as will be the case in most situations of practical interest), it follows that [4]

$$J = -\sum_k \underset{s=s_k}{res}\left(\frac{B(s)+D(s)\exp(-s\tau)}{A(s)+C(s)\exp(-s\tau)}\right)\left(\frac{B(-s)A(s)-D(-s)C(s)}{A(-s)A(s)-C(-s)C(s)}\right) \qquad (5)$$

where s_k are the roots of equation 4.

Example:
An example is used to demonstrate the method. Refer to the block diagram in figure 1.

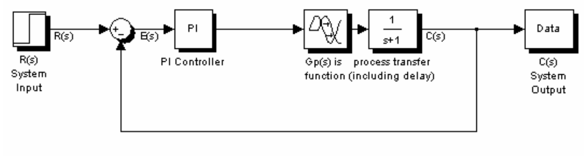

Figure 1. Note that $E = 1/1 + G_{ol}$.

The error of the **ideal PI** controller in series with a first-order-plus-dead-time (**FOPDT**) process (**servo**) is:

$$E(s) = \frac{T_i s + T_p T_i s^2}{T_i s^2 + T_p T_i s^3 + \left(K_c K_p s + K_c K_p T_i s^2\right)e^{-\tau_p s}} \qquad (6)$$

with $G_c(s) = K_c\left(1 + \frac{1}{T_i s}\right)$

and $G_p(s) = \frac{K_p}{1 + T_p s} e^{-\tau_p s}$

The general form of *E(s)* can be expressed as follows:

$$E(s) = \frac{B(s) + D(s)e^{-\tau s}}{A(s) + C(s)e^{-\tau s}} \qquad (7)$$

Hence,

$$B(s) = T_i + T_p T_i s \qquad (8)$$

$$D(s) = 0 \qquad (9)$$

$$A(s) = T_i s + T_p T_i s^2 \qquad (10)$$

$$C(s) = K_c K_p + K_c K_p T_i s \qquad (11)$$

Then the roots have to be calculated from equation 4. The FOPDT process model parameters and the PI controller parameters are as shown in figure 2. The four roots are calculated as follows:

$S_1 = 0.1214975$
$S_2 = 0 + 0.184065i$
$S_3 = -0.1214975$
$S_4 = 0 - 0.184065i$

Each of the four roots are now inserted in turn into equation 5 and summed together to give the cost function value equal to 5.3966.

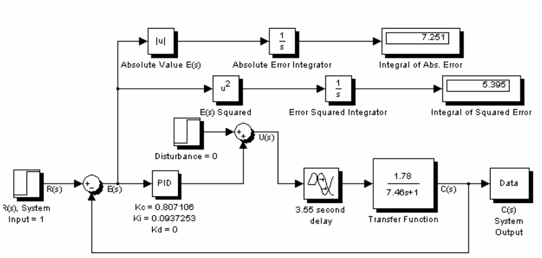

Figure 2. Simulink file to check ISE value.

The file in figure 2 demonstrates the ISE value obtained using simulation techniques. The simulated value of 5.395 compares favourably with the analytical result of 5.3966.

II.b ANALYTICAL CALCULATION OF ISE USING PARSEVAL'S THEOREM AND CONTOUR INTEGRATION

A second method to determine the analytical ISE value for a servo response of a first-order-plus-dead-time process model under PI control is described by Thomas Heeg [5] with reference to Marshall *et al.* [6] as follows:

In order to express the Laplace transform of the error signal *E(s)* for the control system shown in figure 1, we denote

$\alpha = K_pK_c,\ \beta = K_pT_i,\ \gamma = K_pT_d$ (12)

where K_p is the process gain. The asymptotic stability of the closed-loop system is a basic requirement when searching for optimal controller settings. This requirement constitutes a constraint, which determines the set of admissible values of K_c and/or T_i and/or T_d, depending on the regulator type. The conditions of asymptotic stability for our system can be obtained in an explicit form (see Gorecki *et al.* [7]). We are dealing with the integral square error in equation 1 for the closed-loop control system of figure 1. The system is driven by a step input. In order to calculate the integral performance criterion *J* we use Parseval's theorem. To this end the Laplace transform of the error signal is needed. For the PI controller, the parameter γ in equation 12 is set to zero. The ISE value is now analytically calculated from equation 13:

$$J = \frac{1}{2\Delta}\left[\left(q^2-\rho^2\right)\frac{\beta-\alpha q+\left(q^2-\rho^2\right)\cosh(\rho\tau)}{\alpha\rho^2-q\beta-\rho\left(q^2-\rho^2\right)\sinh(\rho\tau)}+\left(q^2+\sigma^2\right)\frac{\beta-\alpha q+\left(q^2+\sigma^2\right)\cos(\sigma\tau)}{\alpha\sigma^2+q\beta-\sigma\left(q^2+\sigma^2\right)\sin(\sigma\tau)}\right] \quad (13)$$

where

$$\Delta = \sqrt{\left(\alpha^2 - q^2\right)^2 + 4\beta^2} \quad (14)$$

$$\rho = \sqrt{\frac{\Delta + q^2 - \alpha^2}{2}} \quad (15)$$

$$\sigma = \sqrt{\frac{\Delta - q^2 + \alpha^2}{2}} \quad (16)$$

$$q = \frac{1}{T_m} \quad (17)$$

The asymptotic stability conditions are given as

$$\alpha + q > 0, \beta > 0, \tau\sigma < \arccos\frac{\sigma^2 - \alpha^2}{q\alpha + \beta} \quad (18)$$

The software package used to determine the *J* value in equation 13 is Mathematica [8]. The equation for *J* is excessively long for reproduction here, but it will be presented in full at the conference.

The FOPDT process model parameters and the PI controller parameters shown in figure 2 are used in the calculation of *J* using the equation. This results in an ISE value equal to 5.3967 that again compares favourably with the experimental result of 5.395.

The same procedure can be carried out to analytically determine the ISE for the servo/regulator response of a process using different PI/PID controller structures.

III. CONTROL SYSTEMS DESIGN USING PERFORMANCE INDEX MINIMISATION.

Many tuning rules have been defined for performance index minimisation (O'Dwyer [2]). The following eleven representative tuning rules are examined:

- Murrill (1967) [Regulator - PI]
- Edgar *et al.* (1997) [Regulator - PI]
- Smith & Corripio (1997) [Servo - PI]
- Murrill (1967) [Regulator - PID]
- Wang *et al.* (1995) [Servo - PID]
- Kaya & Scheib (1988) [Regulator - PID]
- Shinskey (1988) [Regulator - PID]
- Kaya & Scheib (1988) [Servo - PID]
- Smith & Corripio (1997) N = 10 [Servo - PID]
- Kaya & Scheib (1988) [Regulator - PID]
- Kaya & Scheib (1988) [Servo - PID]

Some of these tuning rules are optimised by their authors for regulator response, while others optimised for servo response, as indicated. In addition, a number of the PID controller tuning rules are associated with PID controller structures other than the ideal PID controller architecture.

The six processes examined are

- $G_P(s) = \dfrac{2e^{-s}}{1 + 8.5s + 22.5s^2 + 18s^3}$
- $G_p(s) = \dfrac{2e^{-s}}{1 + 4.5s + 4.5s^2}$
- $G_p(s) = \dfrac{2e^{-s}}{1 + 18s + 137s^2 + 567s^3 + 1403s^4 + 2103s^5 + 1846s^6 + 856s^7 + 158s^8}$
- $G_p(s) = \dfrac{2e^{-s}}{1 + s + s^2}$
- $G_p(s) = \dfrac{2(1 + 2.25s)e^{-s}}{1 + 8.5s + 22.5s^2 + 18s^3}$
- $G_p(s) = \dfrac{2(1 - 2.25s)e^{-s}}{1 + 8.5s + 22.5s^2 + 18s^3}$

Each process is modelled by a first-order-plus-dead-time model using two different identification techniques. These are 1: Two-point algorithm modelling, in the time domain 2: Analytical and gradient based frequency domain modelling [9].

The system is examined in the MatLab/Simulink computer environment. The following example demonstrates how the method is applied. A step is applied to the system and the results recorded as shown.

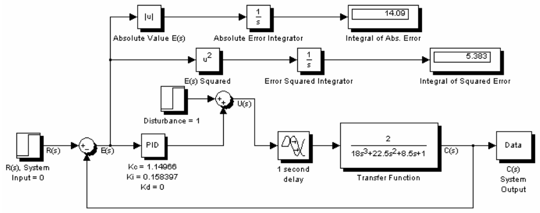

Figure 3. MatLab/Simulink file to determine IAE/ISE value (regulator).

$$\mathrm{Process} = G_p(s) = \frac{2e^{-s}}{1 + 8.5s + 22.5s^2 + 18s^3}$$

Model – frequency domain modelling -

$$G_m(s) = \frac{1.78e^{-3.55s}}{1 + 7.46s}$$

Controller: $G_c(s) = K_c\left(1 + \frac{1}{T_i s}\right)$

Regulator tuning. Tuning rule – Minimum IAE – Murrill (1967) – pages 358 – 363 (see [2]).

The tuning rule is appropriate for: $0.1 \le \frac{\tau_m}{T_m} \le 1.0$.

For our model, $\frac{\tau_m}{T_m} = \frac{3.55}{7.46} = 0.476$

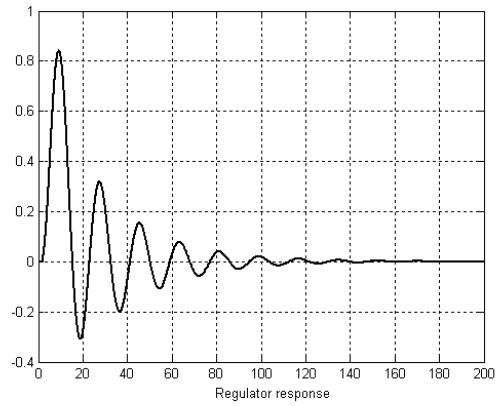

Figure 4. Regulator response using Murrill's rule.

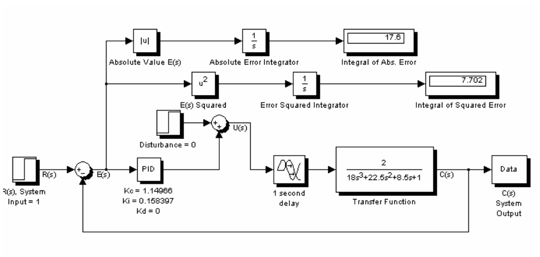

Figure 5. MatLab/Simulink file to determine IAE/ISE value (servo).

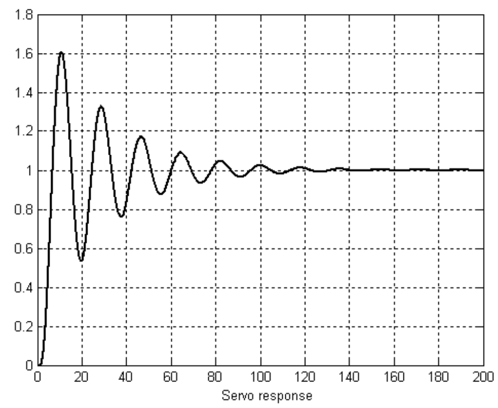

Figure 6. Servo response using Murrill's rule.

The eleven tuning rules mentioned previously, compensating the six processes using the two separate identification methods are examined and the results recorded in a worksheet. The complete worksheet can be obtained from the authors but some sample results are demonstrated in tables 1, 2, 3 and 4.

	Process 1	Process 1
Tuning rule	2-Point	Freq-Dom
Murrill (1967) [Regulator]	11.25	14.10
Edgar *et al.* (1997) [Regulator]	27.04	15.71
Smith & Corripio (1997) [Servo]	13.91	11.70
Murrill (1967) [Regulator]	6.44	5.20
Wang *et al.* (1995) [Servo]	11.94	9.64
Kaya & Scheib (1988) [Regulator]	9.81	7.74
Shinskey (1988) [Regulator]	9.93	7.31
Kaya & Scheib (1988) [Servo]	12.53	9.09
Smith & Corripio (1997) N = 10 [Servo]	10.36	7.61
Kaya & Scheib (1988) [Regulator]	10.68	8.73
Kaya & Scheib (1988) [Servo]	10.60	7.83

Table 1. Process 1 regulator response IAE values.

	Process 4	Process 4
Tuning rule	2-Point	Freq-Dom
Murrill (1967) [Regulator]	9.30	9.62
Edgar *et al.* (1997) [Regulator]	21.43	24.24
Smith & Corripio (1997) [Servo]	10.66	10.20
Murrill (1967) [Regulator]	4.27	4.34
Wang *et al.* (1995) [Servo]	6.25	6.31
Kaya & Scheib (1988) [Regulator]	5.32	5.78
Shinskey (1988) [Regulator]	6.45	6.92
Kaya & Scheib (1988) [Servo]	7.53	7.57
Smith & Corripio (1997) N = 10 [Servo]	6.64	6.50
Kaya & Scheib (1988) [Regulator]	5.81	6.16
Kaya & Scheib (1988) [Servo]	6.81	7.22

Table 2. Process 4 regulator response ISE values.

Tuning rule	Process 2 2-Point	Process 2 Freq-Dom
Murrill (1967) [Regulator]	4.98	8.14
Edgar *et al.* (1997) [Regulator]	5.78	4.26
Smith & Corripio (1997) [Servo]	4.46	4.36
Murrill (1967) [Regulator]	4.59	10.53
Wang *et al.* (1995) [Servo]	3.72	3.26
Kaya & Scheib (1988) [Regulator]	5.90	6.62
Shinskey (1988) [Regulator]	5.45	6.31
Kaya & Scheib (1988) [Servo]	4.55	3.70
Smith & Corripio (1997) N = 10 [Servo]	4.24	3.54
Kaya & Scheib (1988) [Regulator]	6.40	7.29
Kaya & Scheib (1988) [Servo]	4.15	3.53

Table 3. Process 2 servo response IAE values.

Tuning rule	Process 6 2-Point	Process 6 Freq-Dom
Murrill (1967) [Regulator]	9.15	9.32
Edgar *et al.* (1997) [Regulator]	15.26	13.33
Smith & Corripio (1997) [Servo]	10.05	9.79
Murrill (1967) [Regulator]	8.77	9.35
Wang *et al.* (1995) [Servo]	8.56	8.51
Kaya & Scheib (1988) [Regulator]	8.73	8.83
Shinskey (1988) [Regulator]	8.77	8.79
Kaya & Scheib (1988) [Servo]	9.86	9.50
Smith & Corripio (1997) N = 10 [Servo]	9.29	8.98
Kaya & Scheib (1988) [Regulator]	9.04	9.23
Kaya & Scheib (1988) [Servo]	9.13	8.89

Table 4. Process 6 servo response ISE values.

Figures 7 and 8 show the average IAE values, obtained over all the controller tuning rules, for each of the process modelling methods. Figures 9 to 12 how the average IAE and ISE values, obtained over all the process modelling methods, for each controller tuning rule.

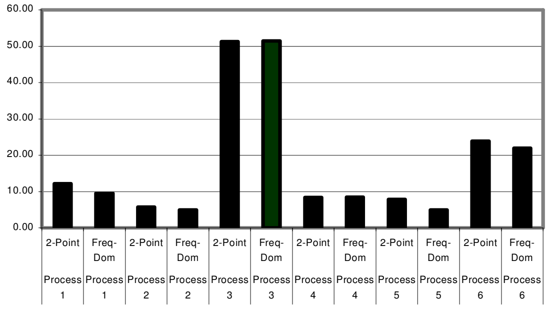

Figure 7. Regulator response, average IAE value.

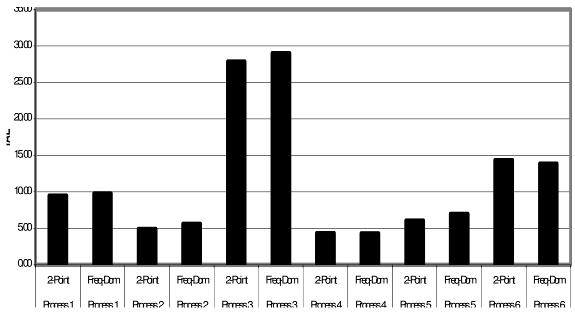

Figure 8. Servo response, average IAE value.

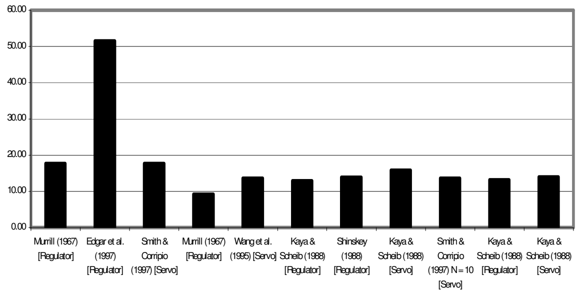

Figure 9. Regulator response, average IAE value.

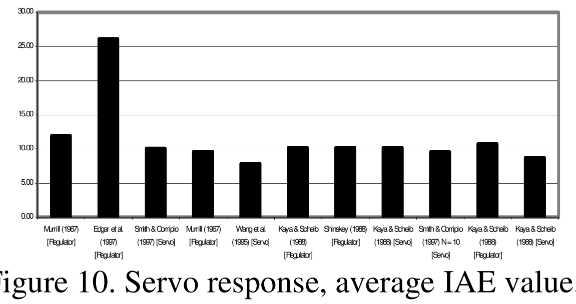

Figure 10. Servo response, average IAE value.

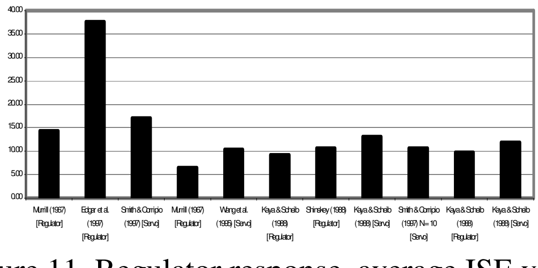

Figure 11. Regulator response, average ISE value.

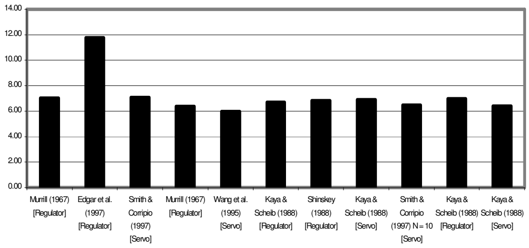

Figure 12. Servo response, average ISE value.

IV. Conclusions.

From the bar-charts in figure 7 and 8, it is concluded that the largest IAE value is obtained for the control of process 3. This is an 8th order process, modelled using a first-order-plus-dead-time model. With the exception of process 4, the lowest IAE obtained from the regulator response is achieved using the frequency-domain modelling method. The opposite is true for the servo response. In this case, most of the controlled systems give a lower IAE value when using the 2-point process modelling method.

From the bar-chart results in figures 9, 10, 11 and 12, it is concluded that the lowest regulator response average IAE value for all the processes is obtained when the Murrill (1967) [Regulator] tuning rule is used. The lowest servo response average is obtained when using the Wang *et al.* (1995) [Servo] tuning rule. Two of the other good performing rules are the Kaya & Scheib (1988) [Servo] and the Smith & Corripio (1997) N = 10 [Servo] rules.

A feature of the charts is the observation that the Murrill (1967) [Regulator] tuning rule has low IAE values for both the regulator and servo responses.

V. Present work.

The work carried out on the six processes using the MatLab/Simulink software is being extended by applying the tuning rules to a real process. This work is presently being carried out on the Process Trainer, PT326, from Feedback Instruments Limited. Preliminary results show that the results obtained from the real process are compatible with the results obtained from the simulated processes. More implementation information will be available in the final paper.

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