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ON THE QUADRATIC BUNDLES RELATED TO HERMITIAN SYMMETRIC SPACES

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Abstract. We develop the direct scattering problem for quadratic bundles associated to Hermitian symmetric spaces. We adapt the dressing method for quadratic bundles which allows us to find special solutions to multicomponent derivative Schrödinger equation for instance. The latter is an infinite dimensional Hamiltonian system possessing infinite number of integrals of motion. We demonstrate how one can derive them by block diagonalization of the corresponding Lax pair.

1. Introduction

The modern period in the history of integrable systems started with the discovery of the inverse scattering transform (IST) by Gardner, Greene, Kruskal and Miura [7] who solved the Cauchy problem for the Korteweg-de Vries equation. Ever since that time the applications of IST increased tremendously — from purely discrete equations to multidimensional partial differential equations [1, 22].

Historically the first nonlinear evolution equations (NEEs) solved by means of IST were associated with the scattering operator

$$L(\lambda) = \mathrm{i}\partial_x + Q(x,t) - \lambda\sigma_3 \tag{1}$$

where $\lambda \in \mathbb{C}$ is an external parameter called spectral and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \pm q^*(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

This operator is known as a linear bundle due to its dependence on λ. Ever since that time the scheme of IST has been extended to matrix Lax operators with a polynomial [6, 21] and even rational λ-dependence [28, 40]. The first step in this direction was done by Mikhailov and Kuznetsov [23,26] who proved the integrability of the 2-dimensional Thirring model. This problem can be effectively reduced to the following quadratic bundle Lax operator

$$L(\lambda) = \mathrm{i}\partial_x - \frac{1}{2}|q|^2\sigma_3 + \lambda Q(x,t) - \lambda^2\sigma_3 \tag{3}$$

where

$$Q = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}. \tag{4}$$

Another equation with a physical application [29, 30] was considered by Kaup and Newell [20] who introduced the Lax operator

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2\sigma_3 \tag{5}$$

where $Q(x,t)$ is again in the form (4). This allowed them solve the derivative nonlinear Schrödinger equation (DNSE)

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0. \tag{6}$$

and find integrals of motion for it. The study of DNSE was continued by Gerdjikov, Kulish and Ivanov [15] who developed the generalised Fourier interpretation for DNSE in terms of generating operators, squared solutions etc., found action-angle variables for it and thus proved its complete integrability. Later Gerdjikov and Ivanov [10, 11] carried out an exhaustive study of the generic quadratic bundle

$$L(\lambda) = \mathrm{i}\partial_x + U_0(x,t) + \lambda U_1(x,t) - \lambda^2\sigma_3 \tag{7}$$

where $U_0(x,t)$ is an arbitrary 2×2 matrix while $U_1(x,t)$ has zero diagonal elements. In the latter papers the existence of Riemann-Hilbert problem with canonical normalization was exploited and was clarified its importance.

Another fruitful idea in the soliton theory is to search for multicomponent equations integrable by means of IST. This trend was pioneered by Zakharov and Manakov [25, 38] who derived a 3-wave system and a 2-component counterpart of the nonlinear Schrödinger equation. For that purpose they used a 3×3-matrix analogue of the Lax operator (2). Soon it became clear that Lax pairs can be related to homogeneous and symmetric spaces in a very natural way [2,4,5]. In [4] Fordy derived multicomponent versions of DNSE related to different Hermitian symmetric spaces amongst which is the following one

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \frac{2\mathrm{i}}{m+1}\left(\left(\mathbf{q}^T\mathbf{q}^*\right)\mathbf{q}\right)_x = 0 \tag{8}$$

where $\mathbf{q} : \mathbb{R}^2 \to \mathbb{C}^m$, $m \geq 2$ is an infinitely smooth function. The multicomponent NEEs related to symmetric spaces attracted attention again [12, 13] as a result of recent studies on Bose-Einstein condensates [18, 24, 33].

The aim of the current paper is to build the foundations of the theory of quadratic bundles associated with Hermitian symmetric spaces. In order to do this we are going to use a gauge covariant approach [14]. This will allow us to treat in a uniform manner any quadratic bundle regardless of the structure of the underlying symmetric space.

The paper is organised as follows. In Section 2 we give some basic preliminary facts on quadratic bundles associated with Hermitian symmetric spaces. After introducing the main object of study we are going to develop the direct scattering problem and discuss the spectral properties of the scattering operator. In Section 3 we intend to adapt Zakharov-Shabat dressing method for the case quadratic bundles of the mentioned type. This method will allow us to derive particular solutions of multicomponent DNSE. As we shall see the form of dressing factor depends crucially on the structure of symmetric space. Section 4 is dedicated to Hamiltonian interpretation of DNSE. We are going to prove that there exist infinite number of integrals of motion and a general recursion formula to generate them will be presented. In doing this we shall use the method of (block) diagonalization of Lax pair proposed in [3]. Section 5 contains a summary of our results and some additional remarks.

2. Quadratic Bundles Related to Hermitian Symmetric Spaces

The current section is preliminary in nature. Its purpose is to provide an introduction to the direct scattering theory of quadratic bundles related to Hermitian symmetric spaces. In doing this we shall follow some well-known ideas from soliton theory [14, 39].

Firstly we are going to shed light on the relation that exists between Hermitian symmetric spaces and quadratic bundles. Let G/H be a Hermitian symmetric space, i.e. G is assumed to be a connected simple Lie group[1] and $\mathrm{H} \subset \mathrm{G}$ is a stabilizer of a typical point $p \in \mathrm{G}$, see [16] for more detailed explanations. The Lie algebra $\mathfrak{g}$ corresponding to the Lie group G obeys the splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{9}$$

where $\mathfrak{h}$ is a subalgebra corresponding to the Lie subgroup H and the subspace $\mathfrak{m}$ represents its complement in $\mathfrak{g}$. Since the homogeneous space G/H is symmetric the following relations

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{10}$$

hold true as well. In other words $\mathfrak{g}$ is $\mathbb{Z}_2$-graded and the subspaces $\mathfrak{h}$ and $\mathfrak{m}$ are eigensubspaces

$$\mathfrak{h} = \{X \in \mathfrak{g};\, \mathbf{C}X\mathbf{C}X = X\}, \qquad \mathfrak{m} = \{X \in \mathfrak{g};\, \mathbf{C}X\mathbf{C} = -X\}.$$

[1]We shall deal with matrix Lie groups and Lie algebras only.

of the adjoint action of an involutive automorphism (Cartan's involutive automorphism).

Let us now consider the Lax pair

$$L(\lambda) \;=\; \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J \tag{11}$$

$$A(\lambda) \;=\; \mathrm{i}\partial_t + \sum_{k=1}^{2N} \lambda^k A_k(x,t) \tag{12}$$

where $\lambda \in \mathbb{C}$ is spectral parameter while $Q(x,t)$, J and A_k belong to the Lie algebra $\mathfrak{g}$. Let L and A be subjects to the $\mathbb{Z}_2$ reduction conditions [27, 28]

$$L^\dagger(\lambda^*) \;=\; \tilde{L}(\lambda), \qquad A^\dagger(\lambda^*) = \tilde{A}(\lambda) \tag{13}$$

$$\mathbf{C}L(-\lambda)\mathbf{C} \;=\; L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda) \tag{14}$$

where tilde operation is defined as follows

$$\tilde{L}(\lambda)\psi = -\mathrm{i}\partial_x\psi + \lambda\psi(Q - \lambda J)$$

for $\psi : \mathbb{R}^2 \to \mathbb{C}^n$ being a smooth function. As a result of (13) all coefficients above become Hermitian matrices while the latter reduction implies that $J, A_{2k}(x,t) \in \mathfrak{h}$ while $Q(x,t), A_{2k-1}(x,t) \in \mathfrak{m}$. This way L and A become compatible with $\mathbb{Z}_2$-grading of $\mathfrak{g}$ and thus following [4] we say that the Lax operators are related to the symmetric space $\mathrm{G/H}$.

Remark 1 *It is always possible to pick up J in such a way that $\mathfrak{h}$ coincide with the centralizer $C_J = \{X \in \mathfrak{g};\, [X, J] = 0\}$ of J. This will simplify significantly some of our further considerations.*

Example 2 *Let us consider as a simple illustration a quadratic bundle related to the symmetric space $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$, $m \geq 2$. In this case $\mathbf{C} = \mathrm{diag}(1, -1\ldots, -1)$ and the subspace $\mathfrak{h}$ consists of all $(m+1)\times(m+1)$ block diagonal Hermitian traceless matrices of the form*

$$\begin{pmatrix} * & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \ldots & * \end{pmatrix}$$

while $\mathfrak{m}$ consists of all Hermitian matrices with complementary block structure, namely

$$\begin{pmatrix} 0 & * & \ldots & * \\ * & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \ldots & 0 \end{pmatrix}.$$

In particular, the potential Q is given by

$$Q(x,t) = \begin{pmatrix} 0 & q_1(x,t) & \dots & q_m(x,t) \\ q_1^*(x,t) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_m^*(x,t) & 0 & \dots & 0 \end{pmatrix}. \tag{15}$$

The subalgebra $\mathfrak{h}$ coincides with C_J if $J = \mathrm{diag}(m, -1, \dots, -1)$. The compatibility condition of the operators (11) and (12) for $N = 2$ (i.e the quadratic flow) produces exactly the vector DNSE we mentioned in the previous section (see formula (8)). $\square$

Example 3 *Another example worthy to mention here is given by a quadratic bundle related to the symmetric space $\mathrm{SO}(2r+1)/\mathrm{SO}(2) \times \mathrm{SO}(2r-1)$, $r \geq 2$. Now Cartan's involution is given by $\mathbf{C} = \mathrm{diag}(-1, 1 \dots, 1, -1)$. The subalgebra $\mathfrak{h}$ therefore consists of all Hermitian matrices of the form*

$$\begin{pmatrix} * & 0 & \dots & 0 & 0 \\ 0 & * & \dots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \dots & * & 0 \\ 0 & 0 & \dots & 0 & * \end{pmatrix}$$

while $\mathfrak{m}$ contains all block off-diagonal matrices

$$\begin{pmatrix} 0 & * & \dots & * & 0 \\ * & 0 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \dots & 0 & * \\ 0 & * & \dots & * & 0 \end{pmatrix}.$$

The constant element $J \in \mathfrak{h}$ now is chosen as follows $J = \mathrm{diag}(1, 0, \dots, 0, -1)$ while the potential Q is given by

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^T(x,t) & 0 \\ \mathbf{q}^*(x,t) & 0 & s_0\mathbf{q}(x,t) \\ 0 & \mathbf{q}^\dagger(x,t)s_0 & 0 \end{pmatrix} \tag{16}$$

for some smooth function $\mathbf{q} : \mathbb{R}^2 \to \mathbb{C}^{2r-1}$. The presence of the $2r-1 \times 2r-1$ matrix $(s_0)_{ij} = (-1)^{i-1}\delta_{i\,2r-j}$ takes into account that Q is an element of orthogonal algebra $\mathfrak{so}(2r+1)$.

The compatibility condition of the Lax pair (11), (12) in the quadratic flow case ($N = 2$) is equivalent to the multicomponent DNSE

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \mathrm{i}[2(\mathbf{q}^T\mathbf{q}^*)\mathbf{q} - (\mathbf{q}^T s_0\mathbf{q}^*)s_0\mathbf{q}^*]_x = 0. \quad \square \tag{17}$$

In order to obtain definite results we must impose certain boundary conditions on the potential Q. We shall restrict ourselves with the simplest case of zero boundary conditions

$$\lim_{x\to\pm\infty} Q(x,t) = 0. \tag{18}$$

To be more specific we require that each matrix element of Q is a function of the Schwartz type for $x \in \mathbb{R}$. Moreover, we assume that Q is such that the corresponding Lax operator has a **finite number** of discrete eigenvalues.

The spectrum of the scattering operator L is determined by its resolvent $R(\lambda)$ defined by the equality

$$L(\lambda) \circ R(\lambda) = \mathbb{1} \tag{19}$$

where $\circ$ stands for operator composition. It follows from (19) that the resolvent is an integral operator of the form

$$\left(R(\lambda)\, F\right)(x,t) = \int_{-\infty}^{\infty} \mathcal{R}(x,y,t,\lambda)F(y)\mathrm{d}y \tag{20}$$

for $F : \mathbb{R} \to \mathbb{C}^n$ being any continuous function. The kernel $\mathcal{R}(x,y,t,\lambda)$ is assumed to be continuous with respect to variables x and y and its domain in the spectral λ-plane is complementary to the spectrum of L. More specifically the pole singularities of $\mathcal{R}$ if exist correspond to discrete eigenvalues of L while the locus of points in the λ-plane for which the boundary $\lim_{x,\,y\to\pm\infty}\mathcal{R}(x,y,t,\lambda)$ does not exist determines the continuous part of the spectrum [8, 14]. We shall convince ourselves that the latter requirement is reduced to the following one by the condition

$$\mathrm{im}\lambda^2 J = 0.$$

In all examples we shall encounter later on in text J is a real matrix hence the continuous spectrum is simply the real and imaginary axis of the Cartesian frame in $\mathbb{C}$. On the other hand due to reductions (13) and (14) the discrete eigenvalues of L are sorted into certain discrete orbits of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [14,27,28]. Indeed, the resolvent obeys the symmetries

$$\begin{aligned} R^\dagger(\lambda^*) &= \tilde{R}(\lambda) \qquad \Rightarrow \qquad \mathcal{R}^\dagger(x,y,t,\lambda^*) = \mathcal{R}(y,x,t,\lambda) && (21)\\ \mathbf{C}R(-\lambda)\mathbf{C} &= R(\lambda) \qquad \Rightarrow \qquad \mathbf{C}\mathcal{R}(x,y,t,-\lambda)\mathbf{C} = \mathcal{R}(x,y,t,\lambda) && (22)\end{aligned}$$

where

$$(\tilde{R}(\lambda)F)(x,t) = \int_{-\infty}^{\infty} F(y)\mathcal{R}(y,x,t,\lambda)\mathrm{d}y.$$

It is immediately seen from the above relations that if μ is a pole then $-\mu$ and $\pm\mu^*$ are poles as well. Therefore the eigenvalues of L go into quadruples (each quadrant in $\mathbb{C}$ contains the same number of eigenvalues.

Let us now consider the auxilary linear problem

$$\mathrm{i}\partial_x\psi(x,t,\lambda) + \lambda(Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \tag{23}$$

The function ψ is viewed as a fundamental set of solutions to (23) called fundamental solution for short, i.e. ψ takes values in the Lie group G. Since L and A commute any fundamental solution ψ satisfies

$$A(\lambda)\psi(x,t,\lambda) = \psi(x,t,\lambda)f(\lambda) \tag{24}$$

as well. The quantity

$$f(\lambda) = \lim_{x\to\pm\infty} \sum_{k=1}^{N} \lambda^k A_k(x,t) \tag{25}$$

is called dispersion law. It labels the NEE amongst the integrable hierarchy, i.e. all equations to share the same Lax operator L and it is therefore an essential feature of integrable $1+1$ dimensional equations.

Next we introduce a very special type of fundamental solutions, namely Jost solutions $\psi_\pm$. They are defined through the following equality

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda^2 Jx} = 1\!\!1.$$

As any two fundamental solutions are linearly related so do the Jost solutions. The transition matrix

$$T(t,\lambda) = (\psi_+(x,t,\lambda))^{-1}\,\psi_-(x,t,\lambda) \tag{26}$$

is called scattering matrix. It is not hard to see that the time evolution of T is driven by the dispersion law through the linear equation

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \tag{27}$$

which is easily integrated to give

$$T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t}T(0,\lambda)\mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

Equation (27) is a linearized version of the corresponding NEE. This significant fact underlies the method of IST for integratation of NEEs [1, 14, 39]. From now on shall fix time and thus skip it in all further formulae in order to simplify our notation.

Like in the case of a quadratic bundle related to $\mathfrak{sl}(2)$ [11, 20] the Jost solutions are defined on the continuous spectrum of L only. To see this one introduces the auxiliary functions $\xi_\pm = \psi_\pm \mathrm{e}^{\mathrm{i}\lambda^2 Jx}$ to satisfy the linear equation

$$\mathrm{i}\partial_x \xi_\pm + \lambda Q \xi_\pm - \lambda^2 [J, \xi_\pm] = 0$$

with boundary condition

$$\lim_{x\to\pm\infty} \xi_\pm(x,\lambda) = \mathbb{1}.$$

Equivalently $\xi_\pm$ are solutions to the following Voltera type integral equations

$$\xi_\pm(x,\lambda) = \mathbb{1} + \mathrm{i}\lambda \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}\lambda^2 J(x-y)} Q(x) \xi_\pm(y,\lambda) \mathrm{e}^{\mathrm{i}\lambda^2 J(x-y)} \mathrm{d}y. \tag{28}$$

Outside of the continuous spectrum of L (i.e. $\lambda^2 \notin \mathbb{R}$) there always exist at least one increasing exponential factor to make the integral divergent. This is why $\xi_\pm$ as well as $\psi_\pm$ **can not** be analytically extended outside of the continuous spectrum. A more detailed analysis however shows that there exists a solution χ^+ to be analytic in the first and third quadrant in $\mathbb{C}$ and another denoted by χ^- analytic in the second and forth quadrant. The fundamental analytic soluitons are related to the Jost solutions through

$$\chi^\pm(x,\lambda) = \psi_-(x,\lambda) S^\pm(\lambda) = \psi_+(x,\lambda) T^\mp(\lambda) D^\pm(\lambda) \tag{29}$$

where all matrix factors introduced above appear in the LDU decomposition

$$T(\lambda) = T^\mp(\lambda) D^\pm(\lambda) (S^\pm(\lambda))^{-1}$$

of the scattering matrix. In fact this is a generalization of the usual LDU decomposition since all matrices involved have a block structure compatible with the splitting (9) of the Lie algebra $\mathfrak{g}$. For example when dealing with symmetric spaces of the type $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(n))$ we have the following

$$\begin{aligned}
S^+(\lambda) &= \begin{pmatrix} \mathbb{1}_m & \mathbf{s}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{t}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix} \\
S^-(\lambda) &= \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{s}_-(\lambda) & \mathbb{1}_n \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{t}_-(\lambda) & \mathbb{1}_n \end{pmatrix} \\
D^+(\lambda) &= \begin{pmatrix} d_m^+(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^+(\lambda) \end{pmatrix}, \qquad D^-(\lambda) = \begin{pmatrix} d_m^-(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^-(\lambda) \end{pmatrix}
\end{aligned}$$

where $\mathbf{s}_{\pm}(\lambda)$ and $\mathbf{t}_{\pm}(\lambda)$ are $n\times m$ complex matrices; $d_m^{\pm}(\lambda)$ are $m\times m$ and $d_n^{\pm}(\lambda)$ are $n \times n$ complex matrices respectively. All these quantities can be expressed algorithmically in terms of matrix elements of T, see [9] for instance.

It is clear from (29) that the fundamental analytic solutions are interrelated through

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G(\lambda), \qquad \lambda \in \mathbb{R}\cup \mathrm{i}\mathbb{R}$$

for $G(\lambda) = (S^{-}(\lambda))^{-1}S^{+}(\lambda)$. Thus $\chi^{+}(x,\lambda)$ and $\chi^{-}(x,\lambda)$ can be viewed as solutions[2] to a local Riemann-Hilbert factorization problem in λ-plane for continuous spectrum of L being the boundary contour. As we shall see in next section this fact is important for elaboration of dressing method.

The reductions (13) and (14) impose certain symmetry conditions on the Jost solutions, the scattering matrix and fundamental analytic solutions. Here is a list of these:

$$\begin{aligned}
&\left[\psi^{\dagger}_{\pm}(x,\lambda^*)\right]^{-1} = \psi_{\pm}(x,\lambda), \qquad \left[T^{\dagger}(\lambda^*)\right]^{-1} = T(\lambda)\\
&\mathbf{C}\psi_{\pm}(x,-\lambda)\mathbf{C} = \psi_{\pm}(x,\lambda), \qquad \mathbf{C}T(-\lambda)\mathbf{C} = T(\lambda)\\
&\left[\chi^{+}(x,\lambda^*)\right]^{\dagger} = [\chi^{-}(x,\lambda)]^{-1}, \qquad \mathbf{C}\chi^{\pm}(x,-\lambda)\mathbf{C} = \chi^{\pm}(x,\lambda).
\end{aligned} \tag{30}$$

One important application of the fundamental analytic solutions is in the spectral theory of the scattering operator. To see this let us define the function:

$$\mathcal{R}(x,y,\lambda) = \begin{cases} \mathcal{R}^{+}(x,y,\lambda), & \mathrm{im}\lambda^2 > 0\\ \mathcal{R}^{-}(x,y,\lambda), & \mathrm{im}\lambda^2 < 0\end{cases} \tag{31}$$

where

$$\mathcal{R}^{\pm}(x,y,\lambda) \;=\; \pm\mathrm{i}\chi^{\pm}(x,\lambda)\Theta^{\pm}(x-y)\left(\chi^{\pm}(y,\lambda)\right)^{-1}. \tag{32}$$

Above $\Theta^{\pm}$ is a matrix-valued function expressed in terms of Heavyside's step function and certain constant projectors. For instance, for the symmetric space $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$ we have

$$\Theta^{\pm}(x-y) = \theta(\pm(y-x))P - \theta(\pm(x-y))\,(\mathbb{1}-P) \tag{33}$$

for P being a constant projector of the $P = \mathrm{diag}(1,0,\ldots,0)$.

It directly follows from (29) that the asymptotic behaviour of the fundamental analytic solutions is given by

$$\begin{aligned}
\chi^{\pm}(x,\lambda) &\xrightarrow[x\to\infty]{} \mathrm{e}^{-\mathrm{i}\lambda^2 Jx}S^{\pm}(\lambda)\\
\chi^{\pm}(x,\lambda) &\xrightarrow[x\to-\infty]{} \mathrm{e}^{-\mathrm{i}\lambda^2 Jx}T^{\mp}(\lambda)D^{\pm}(\lambda).
\end{aligned} \tag{34}$$

[2] Strictly speaking solutions to a Riemann-Hilbert problem are the functions $\chi^{\pm}\mathrm{e}^{\mathrm{i}\lambda^2 Jx}$ since they have the proper normalization as $|\lambda|\to\infty$.

Taking into account (34) we see that for $\lambda^2 \in \mathbb{R}$ the kernel $\mathcal{R}$ becomes asymptotically unbounded. Hence the integral (19) does not converge and $R(\lambda)$ is not defined. As we mentioned this condition determines the continuous part of the spectrum of the scattering operator L. On the other hand when $\lambda^2 \notin \mathbb{R}$ it is the role of the projector P to ensure that $\mathcal{R}^{\pm}$ decreases exponentially as $x, y \to \pm\infty$. P is therefore implicitly related to the structure of the underlying symmetric space, i.e. the $\mathbb{Z}_2$-grading of the corresponding Lie algebra.

We shall state without proof the following important theorem:

Theorem 4 *The function $\mathcal{R}$ defined through (31) and (32) represents an integral kernel of the resolvent of L, i.e. the equality*

$$L(\lambda)\mathcal{R}(x, y, \lambda) = \delta(x - y) \tag{35}$$

holds true. The kernel $\mathcal{R}$ is a meromorphic function in $\mathbb{C}$ with a finite number of poles $\{\pm\mu_k, \pm\mu_k^*\}_{k=1}^{l}$ to form the discrete spectrum of the scattering operator L. The continuous part of the spectrum coincides with the real and the imaginary axis in the spectral λ-plane.

3. Special Solutions to DNSE

In this section we are going to integrate multicomponent DNSE related to symmetric spaces, i.e. find their particular solutions. There are different approaches for integration of nonlinear evolution equations, see [14, 17, 31]. It is our belief that the dressing technique proposed by Zakharov-Shabat [41] and developed further in [19, 40] provides a very convenient and powerful tool to solve multicomponent evolution equations associated with homogeneous or symmetric spaces. Being purely algebraic in nature the dressing method takes into account the algebraic structures (if present) underlying the Lax pairs and that way offers a uniform approach to a variety of integrable nonlinear problems. This is why our main purpose here is to adapt the dressing method for quadratic bundles. This will allow us to easily find the soliton solutions to DNSE.

3.1. Dressing Method

As we saw in the previous section the inverse scattering method can be reduced to a matrix Riemann problem on the λ-plane. This remarkable fact underlies one of the formulations of the dressing method [14, 39] in terms of 1-parameter families of gauge transforms of Lax pair. The dressing method allows one to integrate a given NEE indirectly, i.e. starting from a known solution one obtains another.

Let $Q_0(x,t)$ be a known solution to a DNSE related to some Hermitian symmetric space. It plays the role of a potential for the linear problem

$$L_0\psi_0 = \mathrm{i}\partial_x\psi_0 + \lambda(Q_0 - \lambda J)\psi_0 = 0. \tag{36}$$

Let us apply the gauge transform

$$g : \psi_0(x,t,\lambda) \to \psi_1(x,t,\lambda) = g(x,t,\lambda)\psi_0(x,t,\lambda). \tag{37}$$

to the fundamental solution ψ_0. Under the assumption of g-covariance of the linear problem, i.e. the dressed function ψ_1 is a fundamental solution to the linear problem

$$L_1\psi_1 = \mathrm{i}\partial_x\psi_1 + \lambda(Q_1 - \lambda J)\psi_1 = 0 \tag{38}$$

where $Q_1(x,t)$ is some other potential to be found, we deduce that the dressing factor g satisfies

$$\mathrm{i}\partial_x g + \lambda Q_1\, g - \lambda g Q_0 - \lambda^2[J,g] = 0. \tag{39}$$

Similarly, by comparing the two linear problems

$$\begin{aligned}
A_0(\lambda)\psi_0 &= \mathrm{i}\partial_t\psi_0 + \sum_{k=1}^{2N}\lambda^k A_k^{(0)}\psi_0 = \psi_0 f(\lambda)\\
A_1(\lambda)\psi_1 &= \mathrm{i}\partial_t\psi_1 + \sum_{k=1}^{2N}\lambda^k A_k^{(1)}\psi_1 = \psi_1 f(\lambda)
\end{aligned}$$

we obtain another differential equation for g, namely

$$\mathrm{i}\partial_t g + \sum_{k=1}^{2N}\lambda^k A_k^{(1)} g - g\sum_{k=1}^{2N}\lambda^k A_k^{(0)} = 0. \tag{40}$$

The gauge transform (37) acts on any fundamental solution including the Jost solutions. To ensure that the dressing procedure leads to Jost solutions to (38) one has to modify (37) into

$$\psi_{0,\pm} \to \psi_{1,\pm} = g\psi_{0,\pm}\, g_{\pm}^{-1}, \qquad g_{\pm} = \lim_{x\to\pm\infty} g. \tag{41}$$

This results in the following transformation law for the scattering matrix

$$T_0 \to T_1 = g_+\, T_0\, g_-^{-1}. \tag{42}$$

The fundamental analytic solutions in their turn are dressed through the formula below

$$\chi_1^{\pm} = g\chi_0^{\pm} g_-^{-1}. \tag{43}$$

Using (43) it is seen that the resolvent kernel $\mathcal{R}_0$ for the bare operator L_0 is transformed into

$$\mathcal{R}_1(x,y,t,\lambda) = g(x,t,\lambda)\mathcal{R}_0(x,y,t,\lambda)[g(y,t,\lambda)]^{-1}. \tag{44}$$

Formula (44) shows that even though the bare kernel $\mathcal{R}_0$ might not have any singular points at all the dressed one could — these are singularities introduced by the dressing factor and/or its inverse. The new singular points contribute to the discrete spectrum of the dressed operator L_1. As we discussed in the previous section these points could not be arbitrary. Another way to see this is to write down the symmetry conditions fulfilled by the dressing factor. Indeed, due to the $\mathbb{Z}_2$ reductions (30) we have

$$\begin{aligned} \left[g^{\dagger}(x,t,\lambda^*)\right]^{-1} &= g(x,t,\lambda) && (45)\\ \mathbf{C}g(x,t,-\lambda)\mathbf{C}^{-1} &= g(x,t,\lambda). && (46) \end{aligned}$$

Relation (46) implies that if μ is a singularity point of g so is $-\mu$ while from (45) we deduce that $\pm\mu^*$ are singularities for g^{-1}. This proves that the singularities of the resolvent go in quadruples which resonates to our statement from the previous section.

In order to proceed further we need to make some additional assumptions for the structure of the dressing factor. It is evident from (39) and (40) that if g does not depend on λ then it is simply a constant. On the other hand the connection between the inverse scattering method and Riemann-Hilbert problem implies that the dressing factor has to be divergent as $|\lambda| \to \infty$ to ensure that dressed solutions $\chi_1^{\pm}$ have the proper λ-asymptotics. So to obtain nontrivial results we should pick up a dressing factor possessing certain number of singularities. For the sake of simplicity we shall restrict ourselves with dressing factors having simple poles only. Such a factor can be presented as follows

$$g(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k}\left(\frac{B_k(x,t)}{\lambda-\mu_k} + \frac{\mathbf{C}B_k(x,t)\mathbf{C}}{\lambda+\mu_k}\right), \qquad \mu_k^2 \notin \mathbb{R}. \tag{47}$$

According to (46) its inverse looks as follows

$$g^{-1} = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k^*}\left(\frac{B_k^{\dagger}}{\lambda-\mu_k^*} + \frac{\mathbf{C}B_k^{\dagger}\mathbf{C}}{\lambda+\mu_k^*}\right). \tag{48}$$

After multiplying (39) by g^{-1}/λ and then taking the limit as $|\lambda| \to \infty$ we get the following interrelation between the seed solution Q_0 and the dressed one

$$Q_1 = AQ_0A^{\dagger} + \sum_{k=1}^{l}[J, B_k - \mathbf{C}B_k\mathbf{C}]A^{\dagger} \tag{49}$$

where

$$A = \mathbb{1} + \sum_{k=1}^{l} \frac{1}{\mu_k}(B_k + \mathbf{C}B_k\mathbf{C}). \tag{50}$$

So Q_1 is completely determined if we know the residues B_k. The power of the dressing method consists in the fact that B_k can be expressed in terms of fundamental solutions to (36) (and its λ derivatives) only. To see this we shall analyse the identity

$$gg^{-1} = \mathbb{1}. \tag{51}$$

Since (51) holds identically with respect to λ it gives rise to certain algebraic relations for the residues of g. The form of these relations depends crucially on whether a part of the poles of g and its inverse coincide or not. This is why we shall consider two examples which are more or less representative ones.

Example 5 *Let us consider the case of a quadratic bundle associated with symmetric space* $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$. *Then the simplest choice for* g *is*

$$g(x,t,\lambda) = \mathbb{1} + \frac{\lambda B(x,t)}{\mu(\lambda-\mu)} + \frac{\lambda \mathbf{C}B(x,t)\mathbf{C}}{\mu(\lambda+\mu)}, \qquad \mu^2 \notin \mathbb{R} \tag{52}$$

and formula (49) simplifies into

$$Q_1 = AQ_0A^\dagger + [J, B - \mathbf{C}B\mathbf{C}]A^\dagger \tag{53}$$

where

$$A = \mathbb{1} + \frac{1}{\mu}(B + \mathbf{C}B\mathbf{C})\cdot$$

After calculating the residue at $\lambda = \mu$ *in (51) we obtain the algebraic relation*

$$B\left(\mathbb{1} + \frac{\mu B^\dagger}{\mu^*(\mu-\mu^*)} + \frac{\mu \mathbf{C}B^\dagger\mathbf{C}}{\mu^*(\mu+\mu^*)}\right) = 0\cdot \tag{54}$$

If B *is invertible then (54) implies that it is proportional to* $\mathbb{1}$. *In order to obtain nontrivial dressing we assume* B *is degenerate. Hence there exist two rectangular* $(m+1)\times k$ *matrices* $X(x,t)$ *and* $F(x,t)$ *such that* $B = XF^T$. *Then (54) is reduced to an algebraic equation for* X *whose solution reads*

$$X = \frac{\mu}{\mu^*}\left(\frac{F^TF^*}{\mu-\mu^*} - \frac{F^T\mathbf{C}F^*}{\mu+\mu^*}\mathbf{C}\right)^{-1} F^*\cdot \tag{55}$$

The factor F *can be found from differential equation (39). Evaluating the residue at* $\lambda = \mu$ *and taking into account (54) leads to the differential equation*

$$\mathrm{i}\partial_x F^T - F^T(\mu Q_0 - \mu^2 J) = 0. \tag{56}$$

Therefore we have

$$F^T(x) = F_0^T [\psi_0(x,\mu)]^{-1} \tag{57}$$

where ψ_0 is any fundamental solution to (36) defined in a vicinity of μ and F_0 is a constant matrix. What remains is to recover the time evolution. For this to be done we analyse equation (40) in the same way we did with (39). The residue of (40) at the point μ gives rise to a differential equation for F in the form

$$\mathrm{i}\partial_t F^T - F^T \sum_{k=1}^{2N} \lambda^k A_k^{(0)} = 0. \tag{58}$$

After taking into account (57) and (40) we deduce that the matrix F_0 evolves with time according to

$$\mathrm{i}\partial_t F_0^T - F_0^T f(\mu) = 0 \tag{59}$$

where $f(\lambda)$ is the dispersion law of the nonlinear equation. Thus in order to derive the time dependence for the dressed potential one does the following substitution

$$F_0^T \to F_0^T \mathrm{e}^{-\mathrm{i}f(\mu)t}. \quad \square \tag{60}$$

In the previous example the poles of the dressing factor and its inverse were distinct. As we shall see in next example this is not always possible to achieve. This results in a more complicated procedure to find the residues of g.

Example 6 *Let us consider now quadratic bundles related to* $\mathbf{BD.I}$ *Hermitian symmetric spaces. Then apart of (45) and (46) the dressing factor must obey the orthogonality condition*

$$g^T S g = S \tag{61}$$

where S is the metric involved in the definition of the orthogonal group. To meet the requirements of all reductions we pick up g in the form:

$$g = 1\!\!1 + \frac{\lambda B}{\mu(\lambda-\mu)} + \frac{\lambda \mathbf{C}B\mathbf{C}}{\mu(\lambda+\mu)} + \frac{\lambda S B^* S}{\mu^*(\lambda-\mu^*)} + \frac{\lambda \mathbf{C}SB^*S\mathbf{C}}{\mu^*(\lambda+\mu^*)}\,, \tag{62}$$

while its inverse looks as follows:

$$g^{-1} = 1\!\!1 + \frac{\lambda B}{\mu(\lambda-\mu)} + \frac{\lambda \mathbf{C}B\mathbf{C}}{\mu(\lambda+\mu)} + \frac{\lambda S B^* S}{\mu^*(\lambda-\mu^*)} + \frac{\lambda \mathbf{C}SBS\mathbf{C}}{\mu^*(\lambda+\mu^*)}\,\cdot \tag{63}$$

Relation (49) now looks as follows

$$Q_1 = AQ_0A^\dagger + [J, B + SB^*S - \mathbf{C}B\mathbf{C} - S\mathbf{C}B^*\mathbf{C}S]A^\dagger \tag{64}$$

for A in the form

$$A = \mathbb{1} + \frac{1}{\mu}(B + \mathbf{C}B\mathbf{C}) + \frac{1}{\mu^*}S(B^* + \mathbf{C}B^*\mathbf{C})S \cdot \tag{65}$$

The identity (51) now leads to a couple of algebraic conditions for B, namely

$$BSB^T = 0 \tag{66}$$

$$BS\Omega^T S + \Omega SB^T S = 0 \tag{67}$$

where

$$\Omega = \mathbb{1} + \frac{\mathbf{C}B\mathbf{C}}{2\mu} + \frac{\mu SB^*S}{\mu^*(\mu - \mu^*)} + \frac{\mu\mathbf{C}SB^*S\mathbf{C}}{\mu^*(\mu + \mu^*)} \cdot$$

The former relation means that B is a degenerate, i.e. it is decomposable into $B = XF^T$ for $X(x,t)$ and $F(x,t)$ being $m \times k$ rectangular matrices. Relations (66) and (67) can be rewritten in terms of F and X to give

$$F^T SF = 0 \tag{68}$$

$$\Omega SF = X\alpha \tag{69}$$

for $\alpha(x,t)$ being some appropriately chosen $k \times k$ skew-symmetric matrix. In the simplest case $k = 1$ it simply vanishes and (69) obtains the form

$$SF = a\mathbf{C}X + bSX^* + c\,\mathbf{C}SX^* \tag{70}$$

where we have introduced

$$a = -\frac{F^T\mathbf{C}SF}{2\mu}\,, \qquad b = -\frac{\mu F^\dagger F}{\mu^*(\mu - \mu^*)}\,, \qquad c = -\frac{\mu F^\dagger \mathbf{C}F}{\mu^*(\mu + \mu^*)} \cdot$$

Due to the $\mathbb{Z}_2$ symmetries the algebraic relations derived at the other 3 poles read

$$\mathbf{C}SF = aX + b\mathbf{C}SX^* + cSX^* \tag{71}$$

$$F^* = a^*\mathbf{C}SX^* + b^*X + c^*\mathbf{C}X \tag{72}$$

$$\mathbf{C}F^* = a^*SX^* + b^*\mathbf{C}X + c^*X. \tag{73}$$

The system (70)–(73) is regarded as a linear system for the factor X (as well as for SX^*, $\mathbf{C}X$ and $\mathbf{C}SX^*$). After performing elementary manipulations for X we get

$$X = \frac{1}{\Delta}\left(\Delta_d SF + \Delta_a \mathbf{C}SF + \Delta_b F^* + \Delta_c \mathbf{C}F^*\right) \tag{74}$$

where

$$\begin{aligned}
\Delta_d &= a^*(bc^* + cb^*), \qquad \Delta_a = a^*(|a|^2 - |b|^2 - |c|^2) \\
\Delta_b &= b|b|^2 - b|a|^2 - b^*c^2, \qquad \Delta_c = c|c|^2 - c|a|^2 - c^*b^2 \\
\Delta &= |a|^4 - 2|ac|^2 - 2|ab|^2 + |b|^4 - (b^*)^2c^2 - b^2(c^*)^2 + |c|^4.
\end{aligned}$$

Thus we have expressed X through F. It is not hard to be verified that the formula (57) holds in this case too.

In order to recover the time evolution one follows the same steps as in the previous example. By doing this one can convince himself that the rule (60) is still valid.

3.2. Soliton Solutions

In the current subsection we shall apply the general results from the previous one to evaluate the simplest class of solutions — the 1-soliton solutions. We shall focus our attention to the vector DNSE related to $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$, see (8). To derive the 1-soliton solution we set $Q_0 = 0$. As a fundamental solution to (36) we can pick up the plane wave $\exp(-\mathrm{i}\lambda^2 Jx)$. Then in the case when $\mathrm{rank} B = 1$ F becomes a column vector of the form

$$F(x,t) = \left(\begin{array}{c} \mathrm{e}^{m\mathrm{i}\mu^2 x} F_{0,1} \\ \mathrm{e}^{-\mathrm{i}\mu^2 x} F_{0,2} \\ \vdots \\ \mathrm{e}^{-\mathrm{i}\mu^2 x} F_{0,m+1} \end{array}\right). \tag{75}$$

After substituting (75) into (55) and then into (53) we get the reflectionless potential to be

$$\begin{aligned} q_1^{j-1}(x) &= (Q_1)_{1\,j}(x) = 2\mathrm{i}(m+1)\sum_{l=2}^{m+1}\frac{\rho\sin(2\varphi)\mathrm{e}^{-\mathrm{i}\sigma_l(x)}\mathrm{e}^{\theta_l(x)}}{\mathrm{e}^{-2\mathrm{i}\varphi}+\sum_{p=2}^{m+1}\mathrm{e}^{2\theta_p(x)}}\times \\ &\left(\delta_{jl} - 2\mathrm{i}\sin(2\varphi)\frac{\mathrm{e}^{\theta_j(x)+\theta_l(x)}\mathrm{e}^{\mathrm{i}(\delta_j-\delta_l-2\varphi)}}{\mathrm{e}^{-2\mathrm{i}\varphi}+\sum_{p=2}^{m+1}\mathrm{e}^{2\theta_p(x)}}\right). \end{aligned} \tag{76}$$

We have used above the notation

$$\begin{aligned} \theta_p(x) &= (m+1)\rho^2\sin(2\varphi)x - \xi_{0,p}, \qquad p = 2,\dots,m+1 \\ \sigma_p(x) &= (m+1)\rho^2\cos(2\varphi)x + \delta_1 - \delta_p - \varphi, \qquad \mu = \rho\exp(\mathrm{i}\varphi) \\ \xi_{0,p} &= \ln|F_{0,1}/F_{0,p}|, \qquad \delta_1 = \arg F_{0,1}, \qquad \delta_p = \arg F_{0,p}. \end{aligned}$$

In order to obtain the 1-soliton solution from (76) one needs to recover the time dependence. Taking into account that for (8) $f(\lambda) = -(m+1)\lambda^4 J$ formula (60) leads to the following correspondence

$$\begin{aligned} &\xi_{0,p} \to \xi_{0,p} - 2(m+1)\rho^4\sin(4\varphi)t \\ &\delta_1 \to \delta_1 + 2m\rho^4\cos(4\varphi)t, \qquad \delta_p \to \delta_p - 2\rho^4\cos(4\varphi)t. \end{aligned} \tag{77}$$

Remark 7 *Let us consider the simplest case possible when* $m = 1$*. Then the dressing factor (52) obtains the form*

$$g = \mathbb{1} + \frac{\lambda B}{\mu(\lambda - \mu)} + \frac{\lambda \sigma_3 B \sigma_3}{\mu(\lambda + \mu)} \cdot \tag{78}$$

According to (53) the reflectionless potential can be written as follows

$$q_1(x) = \frac{4\mathrm{i}\rho \sin(2\varphi) \mathrm{e}^{-\mathrm{i}\sigma(x)} \mathrm{e}^{\theta(x)} \left[\mathrm{e}^{2\mathrm{i}\varphi} + \mathrm{e}^{2\theta(x)}\right]}{\left[\mathrm{e}^{-2\mathrm{i}\varphi} + \mathrm{e}^{2\theta(x)}\right]^2} \tag{79}$$

where

$$\begin{aligned} \theta(x) &= 2\rho^2 \sin(2\varphi)x - \xi_0, \qquad \xi_0 = \ln |F_{0,1}/F_{0,2}| \\ \sigma(x) &= 2\rho^2 \cos(2\varphi)x - \delta_0, \qquad \delta_0 = \delta_2 - \delta_1 - 3\varphi. \end{aligned}$$

To obtain the 1*-soliton solution for DNSE (6) we should recover the time dependence in (79) by using the rule*

$$\xi_0 \to \xi_0 - 4\rho^4 \sin(4\varphi)t, \qquad \delta_0 \to \delta_0 - 2\rho^4 \cos(4\varphi)t.$$

This way we have just reproduced the Kaup-Newell soliton obtained in [20].

It is clear that by dressing (76) once again one is able to construct a 2-soliton solution and so on. Proceeding this way one can generate step by step the multisoliton solutions

$$Q_0 \to Q_1 \to \ldots \to Q_l \to \ldots$$

Another way to do this is by using a dressing factor with an appropriate number of simple poles, namely

$$g = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k} \left(\frac{B_k}{\lambda - \mu_k} + \frac{\mathbf{C} B_k \mathbf{C}}{\lambda + \mu_k} \right), \qquad \mu_k^2 \notin \mathbb{R}. \tag{80}$$

Then the multisoliton solution can be derived from the formula (49) by setting $Q_0 = 0$. As before the residues of g can be presented as a product of two rectangular matrices X_k and F_k. A detailed analysis quite similar to what we did before shows that the factor F_k are expressed by a fundamental solution to the bare linear problem as follows

$$F_k^T(x) = F_{0,k}^T [\psi_0(x, \mu_k)]^{-1}. \tag{81}$$

On the other hand the factors X_k are solutions to the linear system

$$F_k^* = \sum_{j=1}^{l} \frac{\mu_k^*}{\mu_j} \left(X_j \frac{F_j^T F_k^*}{\mu_j - \mu_k^*} - \mathbf{C} X_j \frac{F_j^T \mathbf{C} F_k^*}{\mu_j + \mu_k^*} \right) \cdot \tag{82}$$

By solving it one is able to find the residues B_k and then derive the reflectionless potential. To recover the time dependence one should apply the same considerations as in the example 5. The result is given by the rule

$$F^T_{k,0} \to F^T_{k,0} \mathrm{e}^{-\mathrm{i}f(\mu_k)t}$$

which is a natural generalization of correspondence (60).

4. Integrals of Motion

As it was shown in [4] the multicomponent DNSEs related to Hermitian symmetric spaces can be viewed as infinite dimensional Hamiltonian systems whose Hamiltonian is connected to the curvature tensor of the corresponding symmetric space. In this section we aim to describe analytically the conserved densities of integrals of motion for multicomponent DNSEs. For this to be done we are going to use the method of diagonalization of Lax pair proposed by Drinfel'd and Sokolov [3]. This will allow us to derive a general formula generating the conserved quantities in a recursive manner.

We shall start with some general remarks on quadratic bundles related to arbitrary symmetric spaces. Then in order to obtain more concrete results we shall consider two examples referring to symmetric spaces of the type $\mathbf{A}.\mathbf{III}$ and $\mathbf{BD}.\mathbf{I}$, see [16]. Let us consider the quadratic bundle Lax pair

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J && (83)\\ A(\lambda) &= \mathrm{i}\partial_t + \sum_{k=1}^{2N} A_k(x,t)\lambda^k && (84)\end{aligned}$$

which is related to a Hermitian symmetric space G/H. This means that the potential Q as well as A_{2j-1}, $j = 1,\dots,N$ take values in $\mathfrak{m} \in \mathfrak{g}$ while J and A_{2j} take values in the subalgebra $\mathfrak{h}$ (see the beginning of Section 2 for detailed explanations). In accordance with the discussion in Section 2 we pick up J in such a way that its centralizer coincides with $\mathfrak{h}$.

Let

$$\mathcal{P}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{\infty} p_k(x,t)\lambda^{-k} \qquad (85)$$

be a 1-parameter family of gauge transformations[3] acting on the fundamental solutions to the linear problem (23) as follows

$$\psi(x,t,\lambda) \to \tilde{\psi}(x,t,\lambda) = (\mathcal{P}(x,t,\lambda))^{-1}\psi(x,t,\lambda).$$

The Lax pair (83) and (84) is transformed into

$$\tilde{L} \;=\; \mathcal{P}^{-1}L\mathcal{P} = \mathrm{i}\partial_x - \lambda^2 J + \lambda\mathcal{L}_{-1} + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \frac{\mathcal{L}_2}{\lambda^2} + \cdots \tag{86}$$

$$\tilde{A} \;=\; \mathcal{P}^{-1}A\mathcal{P} = \mathrm{i}\partial_t + \sum_{k=1}^{2N}\lambda^k\mathcal{A}_{-k} + \mathcal{A}_0 + \frac{\mathcal{A}_1}{\lambda} + \frac{\mathcal{A}_2}{\lambda^2} + \cdots \tag{87}$$

Let us now assume that $\mathcal{L}_k, \mathcal{A}_k \in \mathfrak{h}$, i.e. they are block diagonal matrices. As we shall see in next examples for certain diagonal matrix elements (or traces of diagonal blocks) of $\mathcal{L}_k$ and $\mathcal{A}_k$ the commutator in the zero curvature representation

$$\mathrm{i}\partial_t\mathcal{L}_k - \mathrm{i}\partial_x\mathcal{A}_k + \sum_j[\mathcal{A}_j, \mathcal{L}_{k-j}] = 0, \qquad k = -1, 0, \ldots \tag{88}$$

vanishes. Thus (88) reduces to continuity equation, i.e. the corresponding elements (or traces of blocks) of $\mathcal{L}_k$ are local conserved densities. Apart of local conserved densities there exist nonlocal ones connected to matrix elements for which the commutator does not vanish.

To find the conserved densities we simply substitute (83) and (85) into (86) and then compare coefficients before the same powers of λ. In result we get the following system of recurrence relations:

$$\mathcal{L}_{-1} = Q - [J, p_1] \tag{89}$$

$$\mathcal{L}_0 + p_1\mathcal{L}_{-1} = Qp_1 - [J, p_2] \tag{90}$$

$$\mathcal{L}_1 + p_1\mathcal{L}_0 + p_2\mathcal{L}_{-1} = \mathrm{i}p_{1,x} + Qp_2 - [J, p_3] \tag{91}$$

$$\ldots$$

$$\mathcal{L}_k + \sum_{j=1}^{k+1} p_j\mathcal{L}_{k-j} = \mathrm{i}p_{k,x} + Qp_{k+1} - [J, p_{k+2}] \tag{92}$$

$$\ldots$$

[3] Strictly speaking the gauge transformation $\mathcal{P}$ takes values in G and one should use an expansion of the form

$$\mathcal{P}(x,t,\lambda) = \exp\left(\sum_{k=1}^{\infty}\mathcal{P}_k(x,t)\lambda^{-k}\right), \qquad \mathcal{P}_k(x,t) \in \mathfrak{g}$$

instead of (85). But since we deal with matrix Lie groups and Lie algebras the expansion (85) is correctly defined. Of course, one should keep in mind that $p_k(x,t)$ are neither group nor algebra elements — they are arbitrary matrices. This choice of expansion parameters although not quite aesthetic from theoretical point of view is very useful from purely practical one, since it will significantly simplify our further calculations.

In order to resolve it we need to introduce the following projector

$$\Pi_J = \mathrm{ad}_J^{-1}\mathrm{ad}_J, \qquad (\mathrm{ad}_J^{-1}X)_{rs} = \frac{X_{rs}}{J_r - J_s}, \qquad r \neq s$$

which cuts off the corresponding block diagonal part of matrices. Thus extracting the block diagonal part from the first recurrence relation we see that $\mathcal{L}_{-1}$ does not contribute to the integrals of motion while the off-block diagonal part reads

$$Q = [J, p_1]. \tag{93}$$

To fix the existing ambiguity we assume that the matrices p_j , $j = 1, 2, \ldots$ do not have block diagonal parts. Then (93) allows one to write

$$p_1 = \mathrm{ad}_J^{-1}Q. \tag{94}$$

To obtain a nonzero conserved density one considers relation (90) which splits into

$$\mathcal{L}_0 \;=\; (\mathbb{1} - \Pi_J)\, Q p_1 = (\mathbb{1} - \Pi_J)\left(Q\, \mathrm{ad}_J^{-1}Q\right) \tag{95}$$

$$p_2 \;=\; \mathrm{ad}_J^{-1}\Pi_J(Qp_1) = \mathrm{ad}_J^{-1}\Pi_J\left(Q\, \mathrm{ad}_J^{-1}Q\right). \tag{96}$$

Proceeding in the same way with the general recursion relation (92) we get the following result

$$\mathcal{L}_k \;=\; (\mathbb{1} - \Pi_J)\left(Qp_{k+1} - \sum_{j=1}^{k+1} p_j \mathcal{L}_{k-j}\right), \qquad k = 1, 2, \ldots \tag{97}$$

$$p_{k+2} \;=\; \mathrm{i}\mathrm{ad}_J^{-1}p_{k,x} + \mathrm{ad}_J^{-1}\Pi_J\left(Qp_{k+1} - \sum_{j=1}^{k+1} p_j \mathcal{L}_{k-j}\right). \tag{98}$$

Formula (97) allows us to find the conserved density contained in $\mathcal{L}_k$ in a purely algorithmic manner.

In order to interpret DNSE as a Hamiltonian equation one needs to introduce a Poisson structure. Let

$$F([Q(x,t)]) = \int_{-\infty}^{\infty} \mathcal{F}([Q(x,t)])\mathrm{d}x$$

be a functional of the potential Q and its x-derivatives. The variational derivative $\delta F/\delta Q$ is a matrix whose matrix elements are defined by the equality

$$\left(\frac{\delta F}{\delta Q}\right)_{rs} = \frac{\delta F}{\delta Q_{rs}}\,.$$

For any two functionals F and G the simplest Poisson bracket[4] for DNSE reads

$$\{F,G\} = \int_{-\infty}^{\infty} \mathrm{d}x \,\mathrm{tr}\left(\frac{\delta F}{\delta Q} \frac{\partial}{\partial x} \frac{\delta G}{\delta Q^T} \right) . \tag{99}$$

In order to be more specific let us illustrate our results with two examples.

Example 8 *Consider the symmetric space* $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$. *Then taking into account formula (15) for* p_1 *we get*

$$p_1(x,t) = \frac{1}{m+1} \left(\begin{array}{cc} 0 & \mathbf{q}^T(x,t) \\ -\mathbf{q}^*(x,t) & 0 \end{array} \right) \tag{100}$$

where $\mathbf{q}(x,t)$ *is a complex* m*-vector. According to (95) and (96) the coefficient* $\mathcal{L}_0$ *is given by*

$$\mathcal{L}_0 = \frac{1}{m+1} \left(\begin{array}{cc} -\mathbf{q}^T\mathbf{q}^* & 0 \\ 0 & \mathbf{q}^*\mathbf{q}^T \end{array} \right) \tag{101}$$

while p_2 *vanishes. Thus as an integral density one can choose* $\mathcal{I}_1 = \mathbf{q}^\dagger\mathbf{q}$. *The general recursion formula (97) in its turn simplifies into*

$$\mathcal{L}_k = Q p_{k+1} \tag{102}$$

where p_k *can be found from the equality*

$$p_k = \mathrm{ad}_J^{-1} \left(\mathrm{i} p_{k-2,x} - \sum_{j=1}^{k-2} p_j \mathcal{L}_{k-2-j} \right) . \tag{103}$$

Taking into account (102) and (103) it is evident that $\mathcal{L}_1 = 0$. *Thus next nonzero integral density* $\mathcal{I}$ *is connected to the matrix* $\mathcal{L}_2(x,t)$. *The result reads*

$$\mathcal{I}_2 = \mathrm{i}\mathbf{q}^\dagger\mathbf{q}_x - \frac{1}{m+1}(\mathbf{q}^\dagger\mathbf{q})^2. \tag{104}$$

It is not hard to be checked that it represents the Hamiltonian density $\mathcal{H}$ *for the multicomponent DNSE (8) provided the Poisson bracket is defined as in (99). The DNSE can be written down in a Hamiltonian form as follows*

$$q_{k,\,t} = \partial_x \frac{\partial \mathcal{H}}{\partial q_k^*}, \qquad k = 1,\ldots,m. \tag{105}$$

The results we have just obtained can be summarized in the following theorem:

[4] In fact, there is a whole infinite hierarchy of Poisson brackets introduced by appropriate recursion operator.

Theorem 9 *All matrices $\mathcal{L}_k$ corresponding to odd indices vanish while the rest are generated by formulae (102), (103).*

Proof: *We already saw that $\mathcal{L}_{-1} = \mathcal{L}_1 = 0$. So the statement of the theorem follows immediately from (102) and (103) after performing elementary induction.* ∎

Example 10 *Let us now examine the case when the Hermitian symmetric space is of the type $\mathrm{SO}(2r+1)/\mathrm{SO}(2)\times\mathrm{SO}(2r-1)$. The potential in this case is given by (16) and the coefficient p_1 reads*

$$p_1 = \begin{pmatrix} 0 & \mathbf{q}^T & 0 \\ -\mathbf{q}^* & 0 & s_0\mathbf{q} \\ 0 & -\mathbf{q}^\dagger s_0 & 0 \end{pmatrix}. \tag{106}$$

According to formulae (95) and (96) we have

$$\mathcal{L}_0 = \begin{pmatrix} -\mathbf{q}^T\mathbf{q}^* & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbf{q}^*\mathbf{q}^T - s_0\mathbf{q}\mathbf{q}^\dagger s_0 & \mathbf{0} \\ 0 & \mathbf{0}^T & \mathbf{q}^\dagger\mathbf{q} \end{pmatrix} \tag{107}$$

$$p_2 = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{0}^T & \mathbf{q}^T s_0\mathbf{q} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{q}^\dagger s_0\mathbf{q} & \mathbf{0}^T & 0 \end{pmatrix}. \tag{108}$$

Hence the first conserved density $\mathcal{I}_1 = \mathbf{q}^\dagger\mathbf{q}$ formally coincides with that in the previous case. It is not hard to be verified that $\mathcal{L}_1 = 0$ so next conserved density is obtained from $\mathcal{L}_2$. Substituting all quantities needed in (97) leads to the following result

$$\mathcal{I}_2 = \mathrm{i}\mathbf{q}^\dagger\mathbf{q}_x - \left(\mathbf{q}^\dagger\mathbf{q}\right)^2 + \frac{1}{2}\left|\mathbf{q}^T s_0\mathbf{q}\right|^2. \tag{109}$$

This is the Hamiltonian density of DNSE (17) provided the Poisson structure is picked up as in (99).

5. Conclusions

In the present paper we have studied some general properties of quadratic bundles related to arbitrary Hermitian symmetric spaces. In particular, we have introduced all basic notions like Jost solutions, scattering matrix, fundamental analytic solutions etc., required to formulate direct scattering problem. Using the fundamental analytic solutions we have constructed the resolvent of the scattering operator and discussed its properties which determine the spectrum of the scattering operator L.

We have adapted the Zakharov-Shabat dressing technique to quadratic bundles of the afore-mentioned type. Though the method itself is not sensitive to the symmetric space type (more precisely to its structure), the form of the dressing factor may vary from one symmetric space to another. For example in the case of $\mathbf{A.III}$ symmetric spaces one can use the 2-pole dressing factor (52) while for $\mathbf{BD.I}$ this is not possible any more — one needs to use a 4 poles factor, see formula (62). By applying the dressing method we have derived the 1-soliton solution to the multi-component DNSE related to $\mathbf{A.III}$ and discussed how one can construct multisoliton solutions. These results generalize the classical ones by Kaup and Newell [20] for the scalar DNSE — the latter can be obtained by using a dressing factor of the form (78). Similarly, one can derive soliton solutions for DNSE related to other symmetric spaces, say symmetric spaces of the series $\mathbf{BD.I}$. However, this requires much more technical efforts due to the complicated form of the dressing factor (62) .

Since multicomponent DNSE are infinite dimensional Hamiltonian systems there exist at least one integral of motion for them — the Hamiltonian itself. We have proved in the previous section that in fact there are infinite number of conserved quantities associated with multicomponent DNSE and we have derived a general recursion formula which allows one to generate them. For that purpose we have applied the method of block-diagonalization of Lax pair. As a simple illustration we have evaluated the first two integrals of motion in the case of the symmetric spaces $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(m))$ and $\mathrm{SO}(2r+1)/\mathrm{SO}(2)\times\mathrm{SO}(2r-1))$. The second integrals of motion represent the Hamiltonian of the multicomponent DNSE (8) and (17) respectively provided the Poisson bracket is defined as in (99). All this underlies the proof of the complete integrability of the multicomponent DNSE in the sense of Liouville-Arnol'd, i.e. the construction of symplectic basis and action-angle variables. To do this one needs to develop the generalized Fourier transform interpretation of IST by introducing squared solutions (adjoint solutions) and recursion operator [14, 34–36]. All this is to be done elsewhere.

The results presented in the paper could be extended in several ways. Firstly, one can can study complete quadratic bundles

$$L(\lambda) = \mathrm{i}\partial_x + U_0(x,t) + \lambda U_1(x,t) - \lambda^2 J \tag{110}$$

where $U_0(x,t)$ splits into a diagonal and off-diagonal part, $U_1(x,t)$ is strictly off-diagonal and J is a diagonal matrix. In general the bundle (110) can not be associated with symmetric spaces unless U_0 contains block diagonal part only and U_1 has a block structure complementary to U_0 (otherwise symmetry conditions (14) will be violated). As it is expected the theory of such bundles becomes more complicated than that of bundles related to symmetric spaces.

We have been dealing in this paper with solutions satisfying zero boundary conditions (the so-called trivial background solutions). These represent the simplest class of solutions to a NEE. On the other hand finding nontrivial background solutions is of current interest even for classical integrable equations like the scalar nonlinear Schrödinger equation [32, 37]. Hence extending the results presented here for potentials satisfying more complicated boundary conditions is another meaningful direction of further development.

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References

[1] Ablowitz M. and Clarkson P., *Soliton, Nonlinear Equations and Inverse Scattering*, London Math. Society Lecture Notes Series **149**, Cambridge Univ. Press, 1991.

[2] Athorne C. and Fordy A., *Generalised KdV and MKdV Equations Associated with Symmetric Spaces*, J. Phys. A **20** (1987) 1377–1386.

[3] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg-de Vries Type*, Sov. J. Math. **30** (1985) 1975–2036.

[4] Fordy A., *Derivative Nonlinear Schrödinger Equations and Hermitian Symmetric Spaces*, J. Phys. A: Math. Gen. **17** (1984) 1235–1245.

[5] Fordy A. and Kulish P., *Nonlinear Schrödinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[6] Gadzhiev I., Gerdzhikov V. and Ivanov M. *Hamiltonian Structures of Nonlinear Evolution Equations Associated with a Polynomial Bundle*, J. Sov. Math. **37** (1987) 1186–1194.

[7] Gardner C., Greene J., Kruskal M. and Miura R., *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19** (1967) 1095–1097.

[8] Gerdjikov V., *On Spectral Theory of Lax Operators on Symmetric Spaces: Vanishing Versus Constant Boundary Conditions*, JGSP **15** (2009) 1–41.

[9] Gerdjikov V., *Algebraic and Analytic Aspects of N-wave Type Equations*, Contemp. Math. **301** (2002) 35–68, nlin.SI/0206014.

[10] Gerdjikov V. and Ivanov M., *The Quadratic Bundle of General Form and the Nonlinear Evolution Equations. I. Expansions over the "Squared" Solutions — Generalized Fourier Transform*, Bulg. J. Phys. **10** (1983) 13–26.

[11] Gerdjikov V. and Ivanov M., *The Quadratic Bundle of General Form and the Nonlinear Evolution Equations. II. Hierarchies of Hamiltonian Structures*, Bulg. J. Physics **10** (1983) 130–143.

[12] Gerdjikov V., Grahovski G., Mikhailov A. and Valchev T., *Polynomial Bundles and Generalised Fourier Transforms for Integrable Equations on* **A.III**-*type Symmetric Spaces*, SIGMA **7** (2011) 096 (48 pages).

[13] Gerdjikov V., Kostov N. and Valchev T., *Solutions of Multi-component NLS Models and Spinor Bose-Einstein Condensates*, Physica D **238** (2009) 1306–1310.

[14] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics **748** Springer Verlag, Berlin, Heidelberg, New York, 2008.

[15] Gerdzhikov V., Kulish P. and Ivanov M., *Quadratic Bundle and Nonlinear Equations*, TMF **44** (1980) 342–357.

[16] Helgasson S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[17] Hirota R., *The Direct Method in Soliton Theory*, Cambridge Univ. Press, New York, 2004.

[18] Ieda J., Miyakawa T. and Wadati M., *Exact Analysis of Soliton Dynamics in Spinor Bose-Einstein Condensates*, Phys. Rev Lett. **93** (2004) 194102.

[19] Ivanov R., *On the Dressing Method for the Generalised Zakharov-Shabat System*, Nuc. Phys. B **694** (2004) 509–524.

[20] Kaup D. and Newell A., *An Exact Solution for a Derivative Nonlinear Schrödinger Equation*, J. Math. Phys. **19** (1978) 798–801.

[21] Konopelchenko B., *The Polynomial Spectral Problem of Arbitrary Order: A General Form of the Integrable Equations and Bäcklund Transformations*, J. Phys. A: Math. Gen. **14** (1981) 3125–3141.

[22] Konopelchenko B.,*Introduction to Multidimensional Integrable Equations: The Inverse Spectral Transform in* $2+1$ *Dimensions*, Series: Plenum Monographs in Nonlinear Physics, Spinger, 1993.

[23] Kuznetsov E. and Mikhailov A. *On the Complete Integrability of the Two-dimensional Classical Thirring Model*, TMF **30** (1977) 303–314.

[24] Li L., Li Z., Malomed B., Mihalache D. and Liu W., *Exact Soliton Solutions and Nonlinear Modulation Instability in Spinor Bose-Einstein Condensates*, Phys. Rev. A **72** (2005) 033611.

[25] Manakov S., *Contribution to the theory of Two-dimensional Stationary Self-focusing of Electromagnetic Waves*, JETP **56** (1973) 505–516.

[26] Mikhailov A. *Integrability of the Two-dimensional Thirring Model*, JETP Letts. **23** (1976) 356–358.

[27] Mikhailov A., *Reductions in Integrable Systems. The Reduction Groups*, Lett. in Jour. of Exper. and Theor. Phys. **32** (1980) 187–192.

[28] Mikhailov A., *The Reduction Problem and the Inverse Scattering Method*, Physica D **3** (1981) 73–117.

[29] Mio K., Ogino T., Minami K. and Takeda S., *Modified Nonlinear Schrödinger Equation for Alfvén Waves Propagating along the Magnetic Field in Cold Plasmas*, J. Phys. Soc. Japan **41** (1976) 265–271.

[30] Mj ̸olhus E., *On the Modulational Instability of Hydromagnetic Waves Parallel to the Magnetic Field*, J. Plasma Physics **16** (1976) 321–334.

[31] Polyanin A., Zaitsev V. and Zhurov A., *Method for Solving Nonlinear Equations of Mathematical Physics and Mechanics*, FizMatLit, Moscow 2005 (In Russian).

[32] Tajiri M. and Watanabe Y., *Breather Solutions to the Focusing Nonlinear Schrödinger equation*, Phys. Rev. E **57** (1998) 3510–3519.

[33] Uchiyama M., Ieda J. and Wadati M., *Multicomponent Bright Solitons in* $F = 2$ *Spinor Bose-Einstein Condensates*, J. Phys. Soc. Japan **76** (2007) 74005–74010.

[34] Yanovski A., *Recursion Operators and Bi-Hamiltonian Formulations of the Landau-Lifshitz Equation Hierarchies*, J. Phys. A: Math. Gen. **39** (2006) 2409Ű-2433.

[35] Yanovski A., *On the Recursion Operators for the Gerdjikov, Mikhailov and Valchev System*, JMP **52** (2011) 082703, 14 pages.

[36] Yanovski A., *Geometry of the Recursion Operators for Caudrey-Beals- Coifman System in the Presence of Mikhailov Type* Z_p *Reductions*, JGSP **25** (2012) 77–97.

[37] Zakharov V. and Gelash A., *Soliton on Unstable Condensate*, 4 pages (2011) nlin.SI:1109.0620.

[38] Zakharov V. and Manakov S. *Resonant Interaction of Wave Packets in Nonlinear Media*, JETP Lett. **18** (1973) 413–417.

[39] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Plenum, New York 1984.

[40] Zakharov V. and Mikhailov A., *On the Integrability of Classical Spinor Models in Two-dimensional Space-time*, Commun. Math. Phys. **74** (1980) 21–40.

[41] Zakharov V. and Shabat A., *Integration of Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform II*, Funct. Anal. Appl. **13** (1979) 13–22 (in Russian).

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