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Milena Venkova
Technological University Dublin, milena.venkova@dit.ie

Christopher Boyd
University College Dublin, chris.boyd@ucd.ie

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HOLOMORPHIC BASIS FOR FAMILIES OF SUBSPACES OF A BANACH SPACE

CHRISTOPHER BOYD AND MILENA VENKOVA

ABSTRACT. In this article we investigate the connection between a family of complemented subspaces of a Banach space having a holomorphic basis, and being holomorphically complemented.

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KEY WORDS: holomorphic projection, holomorphic basis.

1. Introduction

Perhaps the most fundamental of all selection results is the Axiom of choice: given any collection of non-empty sets \( \{ X_\alpha \}_{\alpha \in A} \) it is possible to “choose” an element \( x_\alpha \) from each \( X_\alpha \). The choice of an element is realised by a “choice function”. Additional structure on the indexing set \( A \) or on the range spaces \( X_\alpha \) allows to refine the problem of “selection”. The additional structure on the range spaces can be given by assuming they are, for example, rings, spaces of linear operators, or Banach algebras. Additional structure on the domain can be given by assuming \( A \) is an open subset of a topological or even a complex Banach space. This assumption allows us to ask for the choice function to be continuous or holomorphic. The problem now changes from set-theoretic to analytic. The solution typically consists of two stages. The first is finding a local solution about each point; the second is ”patching” these local solutions to obtain a global one. Until recently, all selection problems have assumed that the domain is finite-dimensional, and sometimes that the range is finite-dimensional as well. Typically, the choice function has values in an operator space and the question has often been considered in the context of invertibility properties of the operators, e.g. ([1, 2, 19]). The recent work of Lempert and Patyi however has allowed to extend such results to infinite-dimensional domains, as in [5, 6, 7].

In this paper, we concentrate on a different, although related (see [7]) problem - the case when our operators are projections. We introduce holomorphic Schauder basis and study the relationship between families of subspaces with such basis and holomorphically complemented families of subspaces. The case when \( \Omega \) is a domain in a finite dimensional space was studied by Shubin in [15]. Saphar ([14]) and Bart ([2]), on the other hand, considered finite holomorphic bases over a domain in \( \mathbb{C} \). Here we consider the non-trivial generalizations to both infinite-dimensional domains and infinite holomorphic bases.

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2. Notation and definitions

Let $X$ and $Y$ be Banach spaces over $\mathbb{C}$, $\mathcal{L}(X, Y)$ will denote the space of continuous linear mappings from the Banach space $X$ into the Banach space $Y$, $GL(X, Y)$ will denote the set of all invertible linear operators from $X$ to $Y$. We let $\mathcal{H}(\Omega; X)$ denote the set of all $X$-valued holomorphic mappings defined on an open subset $\Omega$ of a Banach space. We use the standard notation $X' := \mathcal{L}(X, \mathbb{C})$ and $GL(X) := GL(X, X)$.

We remind the reader that a sequence $(x_n)_n$ in a Banach space $X$ is called a Schauder basis of $X$ if for every $x \in X$ there is a unique sequence of scalars $(a_n)_n$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. A sequence $(x_n)_n$ which is a Schauder basis of its closed linear span is called a basic sequence. Two bases, $(x_n)_n$ for a Banach space $X$ and $(y_n)_n$ for a Banach space $Y$, are equivalent if there exists an isomorphism $T : X \to Y$ such that $T(x_n) = y_n$ for all $n \in \mathbb{N}$.

If $(x_n)_n$ is a Schauder basis for the Banach space $X$, the bounded linear functionals

$$x^*_n \left( \sum_{n=1}^{\infty} a_n x_n \right) = a_n$$

for all $n$, are called the biorthogonal functionals associated to this basis. If for every $x^* \in X'$ the norm of $x^*|_{[x_n]_n^\infty}$, the restriction of $x^*$ to the span of $(x_n)_n^\infty$, tends to zero as $n \to \infty$, then $(x_n)_n$ is called a shrinking basis. The biorthogonal functionals $(x^*_n)_n$ form a basis of $X'$ if and only if the basis $(x_n)_n$ is shrinking.

When $(x_n)_n$ is a basic sequence we define $(x^*_n)_n$ by using the relation $x^*_n(x_m) = \delta_{nm}$ and extending by linearity and continuity to all of $X$ (Definition 1.f.1, [11]).


In the reminder of this section we recall the definition of holomorphic Banach vector bundles and of their sub-bundles, and some of their properties.

**Definition 2.1.** Let $\pi : \mathcal{E} \to \Omega$ be a surjective holomorphic map of complex Banach manifolds. We assume that the fibre above $z \in \Omega$, $\mathcal{E}_z := \pi^{-1}(z)$, has been given a Banach space structure whose topology coincides with the topology induced from $\mathcal{E}$. A collection $(U_\alpha, \tau_\alpha)_{\alpha \in \Gamma}$ is called a trivializing cover for $\pi$ if $(U_\alpha)_{\alpha \in \Gamma}$ is an open cover of $\Omega$ and for each $\alpha \in \Gamma$ there is a Banach space $X_\alpha$ such that $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha$ is a biholomorphic mapping and conditions (1), (2) and (3) below are satisfied.

1. $\tau_{\alpha z} := \tau_\alpha|_{\mathcal{E}_z}$ is a linear isomorphism from $\mathcal{E}_z$ onto $X_\alpha$, modulo identifying $\{z\} \times X_\alpha$ and $X_\alpha$, for each $z \in U_\alpha$.
2. $\pi = \pi_\alpha \circ \tau_\alpha$, where $\pi_\alpha$ is the canonical projection from $U_\alpha \times X_\alpha$ onto $U_\alpha$.
3. If $\alpha, \beta \in \Gamma$ and $U_\alpha \cap U_\beta \neq \emptyset$ then the map $z \mapsto g_{\alpha\beta}(z)$ from $U_\alpha \cap U_\beta$ into $\mathcal{L}(X_\beta, X_\alpha)$ is holomorphic.

Two trivializing covers are said to be equivalent if their union is also a trivializing cover.
Definition 2.2. A holomorphic Banach vector bundle is a triple \((\mathcal{E}, \pi, \Omega)\), where \(\pi : \mathcal{E} \to \Omega\) is a surjective holomorphic map of complex Banach manifolds, together with a class of equivalent trivializing covers.

The bundle structure is specified by any one trivializing cover. If \(\Omega\) is connected, then all the \(X_U\) are isomorphic to a common Banach space \(X\), called the fibre type of the bundle. We call \(\mathcal{E}\) the bundle space, \(\pi\) the projection of the bundle, \(\Omega\) the base of the bundle, \(\tau_\alpha\) a trivialization of \(\pi^{-1}(U_\alpha)\) and \(g_{\alpha\beta}\) a transition map. If \(X\) is a Banach space and \(\Omega\) is a complex manifold, the triple \((\Omega \times X, \pi, \Omega)\), where \(\pi\) is the canonical projection from \(\Omega \times X\) onto \(\Omega\), together with the trivializing covers equivalent to the trivializing cover \(\{((\Omega, \pi))\}\), is called the trivial bundle. For convenience, we often write \(\mathcal{E}\) in place of \((\mathcal{E}, \pi, \Omega)\).

A holomorphic section of the holomorphic vector bundle \((\mathcal{E}, \pi, \Omega)\) is a holomorphic mapping \(f : \Omega \to \mathcal{E}\) such that \(\pi \circ f = 1_\Omega\). The set of all holomorphic sections is denoted by \(\mathcal{H}(\Omega; \mathcal{E})\). When \((\mathcal{E}, \pi, \Omega)\) is the trivial bundle \((\Omega \times X, \pi, \Omega)\) we write \(\mathcal{H}(\Omega; X)\) in place of \(\mathcal{H}(\Omega; \mathcal{E})\). Under the restriction maps, the collections \(\mathcal{H}(U; \mathcal{E})\), \(U \subset \Omega\) open, make up a sheaf \(\mathcal{O}\mathcal{E}\) over \(\Omega\).

An endomorphism of the holomorphic vector bundle \((\mathcal{E}, \pi, \Omega)\) is a holomorphic mapping \(f : \mathcal{E} \to \mathcal{E}\) such that \(\pi \circ f = \pi\), \(f_z := f|_{\mathcal{E}_z}\) is a continuous linear mapping for all \(z \in \Omega\), and the mapping

\[
z \in U \mapsto \tau_z \circ f_z \circ \tau_z^{-1} \in \mathcal{L}(X)
\]

is holomorphic for any trivialising map \(\tau : \pi^{-1}(U) \to U \times X\). We denote by \(\mathcal{M}(\mathcal{E})\) the set of all endomorphisms of \(\mathcal{E}\). If \(f_z^2 = f_z\) for all \(z \in \Omega\) we call \(f\) a projection.

A sub-bundle of \((\mathcal{E}, \pi, \Omega)\) is a bundle \((\mathcal{E}', \pi', \Omega)\) where \(\mathcal{E}'\) is a subset of \(\mathcal{E}\), \(\pi' = \pi|_{\mathcal{E}'}\), \(\mathcal{E}'_z\) is a closed subspace of \(\mathcal{E}_z\) for all \(z \in \Omega\) and the following condition holds:

for each \(z\) in \(\Omega\) there exists an open neighbourhood \(U\) of \(z\) in \(\Omega\), a subspace \(Y_U\) of \(X_U\) and trivializations \(\tau : \pi^{-1}(U) \to U \times X_U\) and \(\sigma : (\pi')^{-1}(U) \to U \times Y_U\) such that

\[
\tau_z \circ (\sigma^{-1})_z = 1_{U \times Y_U}.
\]

A sub-bundle \((\mathcal{E}', \pi', \Omega)\) is direct if its fibres are complemented subspaces of the corresponding fibres of \((\mathcal{E}, \pi, \Omega)\). We say that there is a projection from \((\mathcal{E}, \pi, \Omega)\) onto \((\mathcal{E}', \pi', \Omega)\) if there is an endomorphism of \(\mathcal{E}\) which on each fibre is a continuous projection onto the corresponding fibre of \(\mathcal{E}'\).

In [7] Dineen and the second author proved the following proposition:

Proposition 2.3. Let \(\Omega\) be a pseudo-convex open subset of a Banach space with an unconditional basis and \((\mathcal{E}, \pi, \Omega)\) be a holomorphic Banach vector bundle over \(\Omega\). If \((\mathcal{F}, \pi', \Omega)\) is sub-bundle of the holomorphic vector bundle \((\mathcal{E}, \pi, \Omega)\), then \((\mathcal{F}, \pi', \Omega)\) is a direct sub-bundle if and only if there exists a holomorphic projection \(p \in \mathcal{M}(\mathcal{E})\) such that \(p(\mathcal{E}) = \mathcal{F}\).

This result relied upon the following important theorem of Lempert ([9, 10]):

Theorem 2.4. Let \(Z\) be a Banach space with a Schauder basis, \(\Omega \subset Z\) pseudo-convex open, \(\mathcal{E} \to \Omega\) a holomorphic Banach vector bundle. If plurisubharmonic domination holds in every pseudo-convex open subset of \(\Omega\), then the sheaf cohomology groups \(H^q(\Omega, \mathcal{O}\mathcal{E})\) vanish for all \(q \geq 1\).
Theorem 2.4 implies the solvability of the additive Cousin problem. We will not go into details about plurisubharmonic domination - let us just say that in his recent paper [18] Patyi showed that plurisubharmonic domination holds on a pseudo-convex open set $\Omega$ of a space with a Schauder (not necessarily unconditional) basis, and on a convex set $\Omega$ in a separable space. For the rest of this article we will assume the former case, i.e. that $\Omega$ is a pseudo-convex open subset of a space with a Schauder basis, but it is worth remembering that the same results will hold when $\Omega$ is a convex open subset of a separable space.

Let $\{M(z)\}_{z \in \Omega}$ be a family of complemented subspaces of $E$. If there exists $P \in \mathcal{H}(\Omega, \mathcal{L}(E))$ such that $P(z)$ is a projection mapping of $E$ onto $M(z)$ for all $z \in \Omega$, we will call $\{M(z)\}_{z \in \Omega}$ a holomorphically complemented family of subspaces of $E$. This definition, together with Patyi’s result, allows us to re-state and generalize Proposition 2.3 in the following form:

**Proposition 2.5.** Let $\Omega$ be a pseudo-convex open subset of a Banach space with a Schauder basis and $E$ be a Banach space. Suppose $\{M(z)\}_{z \in \Omega}$ is a family of complemented subspaces of $E$. The following are equivalent:

1. $\{M(z)\}_{z \in \Omega}$ a holomorphically complemented family of subspaces of $E$.
2. $(z \in \Omega, M(z))$ is a direct holomorphic sub-bundle of $\Omega \times E$.

A multiplicative Cousin data for $(U_\alpha)_{\alpha \in \Gamma}$, an open covering of $\Omega$, is a collection of functions $(f_{\alpha\beta})_{\alpha,\beta \in \Gamma} \subset \mathcal{H}(U_{\alpha\beta}, \mathcal{G}(E))$ on $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, satisfying

$$f_{\alpha\beta} \circ f_{\beta\alpha} = 1$$

on $U_{\alpha\beta}$, and

$$f_{\alpha\beta} \circ f_{\beta\gamma} \circ f_{\gamma\alpha} = 1$$

on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ whenever $U_{\alpha\beta\gamma} \neq \emptyset$.

The multiplicative Cousin problem consists in finding a collection of holomorphic mappings $(f_\alpha)_{\alpha \in \Gamma} \subset \mathcal{H}(U_\alpha, \mathcal{G}(E))$ such that

$$f_\alpha|_{U_{\alpha\beta}} \circ f_\beta^{-1}|_{U_{\alpha\beta}} = f_{\alpha\beta}$$

whenever $U_{\alpha\beta} \neq \emptyset$.

The following Theorem ([17, 16]) shows the multiplicative Cousin problem is solvable on certain domains:

**Theorem 2.6.** Let $Z$ be a Banach space with a Schauder basis, $\Omega \subset Z$ be pseudo-convex and open. If plurisubharmonic domination holds in every pseudo-convex open subset of $\Omega$, then for any Banach space $E$ any multiplicative Cousin problem for $\mathcal{O}^{\mathcal{G}(E)}$ is solvable over $\Omega$ as soon as it is continuously solvable.

In particular, since for contractible (i.e. homotopically equivalent to a point) set $\Omega$ the bundle $\mathcal{O}^{\mathcal{G}(E)}$ is continuously trivial, under the constraints of Theorem 2.6 it will be holomorphically trivial.

### 3. Holomorphic bases

**Definition 3.1.** Let $E$ and $X$ be Banach spaces and let $\Omega$ be an open subspace of $X$. The sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathcal{H}(\Omega, E)$ for all $n \in \mathbb{N}$ is said to form a holomorphic basis (resp. holomorphic basic sequence) for $E$ if the following two conditions are satisfied:

1. 
2. 

**Theorem 2.4** implies the solvability of the additive Cousin problem.
(1) \((x_n(z))_{n \in \mathbb{N}}\) is a Schauder basis (resp. basic sequence) for \(E\) for every \(z \in \Omega\);
(2) for every \(z_0 \in \Omega\) there exist a neighbourhood \(V_0\) of \(z_0\) and continuous mappings \(l_0 : V_0 \to \mathbb{R}^+\) and \(L_0 : V_0 \to \mathbb{R}^+\) such that
\[
\sum_{n=1}^{N} a_n x_n(z_0) \leq \sum_{n=1}^{N} a_n x_n(z) \leq L_0(z) \sum_{n=1}^{N} a_n x_n(z_0)
\]
for all \((a_n)_{n \in \mathbb{N}}\), all \(N \in \mathbb{N}\), and all \(z \in V_0\).

If \((x_n)_{n \in \mathbb{N}}\) is a holomorphic basis (basic sequence) than the bases (resp. basic sequences) \((x_n(z_0))_{n \in \mathbb{N}}\) and \((x_n(z))_{n \in \mathbb{N}}\) are equivalent for all \(z \in V_0\). Note that if a Banach space has a Schauder basis, then it has infinitely many non-equivalent bases (see [11]), thus allowing us to ask whether we can 'select' in such a way that we obtain a holomorphic basis over \(\Omega\).

**Example 3.2.**

Let \(E\) be a subspace of a Banach space \(F\). Suppose \(E\) a Schauder basis \((e_n)\) and let \(\Omega\) be an open subset of \(E\). Let \(f \in \mathcal{H}(\Omega, E)\) is such that for each \(z \in \Omega\) the derivative of \(f\) at \(z\), \(\hat{df}(z)\), is an invertible linear mapping from \(E\) into \(E\). For each \(n\) in \(\mathbb{N}\) let \(x_n(z) = \hat{df}(z)e_n\). Then for each \(z \in \Omega\) we have that \((x_n(z))\) is a Schauder basis for \(E\). Let \(z_0\) be a point of \(\Omega\). Since the function \(\hat{df} : z \to \hat{df}(z)\) is holomorphic we can choose a neighbourhood \(V_0\) of \(z_0\) so that \(\|\hat{df}(z) - \hat{df}(z_0)\| < \frac{1}{\|\hat{df}(z_0)^{-1}\|}\) for all \(z \in V_0\). Then for \(z \in V_0\) we have
\[
\hat{df}(z) = \hat{df}(z_0) + (\hat{df}(z) - \hat{df}(z_0)) = \hat{df}(z_0)\left(I + \hat{df}(z_0)^{-1}(\hat{df}(z) - \hat{df}(z_0))\right).
\]
Hence for each \(z \in V_0\), each sequence of complex numbers \((a_n)\) and each \(N \in \mathbb{N}\) we have that
\[
l_0(z) \sum_{n=1}^{N} a_n x_n^*(z_0) \leq \sum_{n=1}^{N} a_n x_n^*(z) \leq L_0(z) \sum_{n=1}^{N} a_n x_n^*(z_0)
\]
where
\[
l_0(z) = \frac{1}{\|\left(I + \hat{df}(z_0)^{-1}(\hat{df}(z) - \hat{df}(z_0))\right)^{-1}\|}
\]
and
\[
L_0(z) = \left\|I + \hat{df}(z_0)^{-1}(\hat{df}(z) - \hat{df}(z_0))\right\|.
\]
Thus we have that \((x_n(z))\) is a holomorphic basis for \(E\) over \(\Omega\). Regarding \(x_n\) as a holomorphic function from \(\Omega\) into \(F\) we get that \((x_n(z))\) is a holomorphic basic sequence for \(F\) over \(\Omega\).

In particular, if \(f : \Omega \to f(\Omega)\) is bi-holomorphic then \((x_n(z))\) is a holomorphic basic sequence for \(F\) over \(\Omega\).

We will need the following lemma, proven in [6]:

**Lemma 3.3.** If \(P\) and \(P'\) are projections in \(\mathcal{L}(X)\) and \(\|P - P'\| < 1\) then \((1_X - P + P') \in GL(X)\) and \((1_X - P + P')(P(X)) = P'(X)\). In particular, \(P(X) \simeq P'(X)\).
Lemma 3.3, (space with a Schauder basis and
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V
Let
the space
holomorphically complemented family of subspaces of E and that for some z₀ ∈ Ω
for all z ∈ Vα. In this way we obtain an open cover Γ := {Vα}z₀∈Ω for Ω. Let
Qα(z) := 1E − P(zα) + P(z). As Ω is connected and open it is path-connected,
thus there exists a continuous path in Ω connecting z₀ and zα. This path is a
compact set, hence it can be covered by a finite number of sets {Vi}i=0,...,k 1. Let (x₀)i=1 denote the Schauder basis of M(z₀).
If z₀ ∈ V₀ ∩ V₁, then
\[ x₀^n(z₀) = (1E − P(z₀) + P(z))(x₀^n) = Q₀(z)(x₀^n) \]
is a basis for M(z₀). Since z₀ ∈ V₁, the mapping Q₁⁻¹(z₀) = (1E − P(z₁) + P(z₀))⁻¹
is well defined, and
\[ x₁^n(z₁) = Q₁⁻¹(z₀)(x₀^n(z₀)) \]
is a basis for M(z₁). Moreover,
\[ x₁^n(z) = Q₁(z)x₁^n(z₁) = Q₁(z)Q₁⁻¹(z₀)(x₀^n(z₀)) \]
will form holomorphic bases for {M(z)}z∈V₁.
Next we choose z₁ ∈ V₁ ∩ V₂, and by repeating the steps above we obtain
\[ x₂^n(z) = Q₂(z)Q₂⁻¹(z₁)(x₁^n(z₁)) \]
a holomorphic family for {M(z)}z∈V₂. Let at each step A₁ := Q₁⁻¹(z₁)Q₁⁻¹(z₁) where
i = 1,...,k. After a finite number of steps we will get
\[ x_k^n := A_k \ldots A₁(x₀^n), \]
and
\[ x_k^n(z) := Q_k(z)A_k \ldots A₁(x₀^n) \]
is a holomorphic basis for {M(z)}z∈Vₖ. Clearly the bases (x₁(z))z∈V₁ are equivalent
for all i = 0,...,k.
Suppose Vα and Vβ belong to Γ and Vα ∩ Vβ ̸= Ø. Let z ∈ Vα ∩ Vβ, then as before we
can construct Aα ∈ GL(E) such that x₀^n(z) = Qα(z)Aα(x₀^n) and Aβ ∈ GL(E) such
that x₀^n(z) = Qβ(z)Aβ(x₀^n). The mapping AβAα⁻¹ is a linear isomorphism mapping
x₀^n onto x₀^n for all n. Then
\[ x₀^n(z) = Qβ(z)AβAα⁻¹(Qα(z))⁻¹(x₀^n(z)) \]
for all n. Let Tβα(z) := Qβ(z)AβAα⁻¹(Qα(z))⁻¹, then Tβα is holomorphic on Vα ∩ Vβ
and Tβα(z) ∈ GL(M(z)) for every z ∈ Vα ∩ Vβ. Let Vα ∩ Vβ ∩ Vγ ̸= Ø, then if
z ∈ Vα ∩ Vβ ∩ Vγ we have
\[ Tαβ(z) ◦ Tβγ(z) ◦ Tγα(z)(x₀^n(z)) = x₀^n(z) \]
for all n. Hence Tαβ(z) ◦ Tβγ(z) ◦ Tγα(z) = 1M(z), so we can consider a holomor-
phic vector bundle S with base Ω, open cover {Vα}α∈Γ, fibre M(z) and transition
mappings {Tαβ}. Clearly the mappings {Tαβ} form a (multiplicative) Cousin data
for {Vα}α∈Γ. By Theorem 2.6 the multiplicative Cousin problem is solvable over Ω.
The solution \( (x_n^a)_{n \in \mathbb{N}} \) gives us the desired holomorphic basis. Indeed, part (1) of Definition 3.1 is clearly satisfied. To show that part (2) of Definition 3.1 is satisfied, take a fixed \( z_\alpha \) in \( \Omega \), on the neighbourhood \( V_\alpha \) the bounded and invertible linear operator \( Q_\alpha(z) \) maps \( (x_n(z_\alpha))_n \) to \( (x_n(z))_n \). Hence for each \( z \in V_\alpha \), each sequence of complex numbers \( (a_n)_n \) and each \( N \in \mathbb{N} \) we have that

\[
\frac{1}{\|Q_\alpha(z)\|} \left\| \sum_{n=1}^{N} a_n x_n(z_\alpha) \right\| \leq \left\| \sum_{n=1}^{N} a_n x_n(z) \right\| \leq \|Q_\alpha(z)\| \left\| \sum_{n=1}^{N} a_n x_n(z_\alpha) \right\|
\]
on \( V_\alpha \).

### 4. Applications and properties of holomorphic bases

The following proposition is a partial converse to Proposition 3.4:

**Proposition 4.1.** Let \( \Omega \) be a pseudo-convex open subset of a Banach space with a basis and \( E \) be a Banach space. Suppose \( (x_n)_{n \in \mathbb{N}} \) is a holomorphic basic sequence such that the closed linear span of \( (x_n(z))_{n \in \mathbb{N}} \), \( M(z) \), is a complemented subspace of \( E \) for every \( z \in \Omega \). Then \( \{M(z)\}_{z \in \Omega} \) is holomorphically complemented in \( E \).

**Proof.** Suppose \( z_0 \in \Omega \) is fixed, and let \( P_0 \) denote a continuous projection from \( E \) onto \( M(z_0) \). Let \( x \in E \). By part (2) of Definition 3.1 there exist a neighbourhood \( V_0 \) of \( z_0 \) and a continuous mapping \( L_0 : V_0 \to \mathbb{R}^+ \) such that

\[
\left\| \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z) \right\| \leq L_0(z) \left\| \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z_0) \right\|
\]

for all \( z \in V_0 \). Hence

\[
\left\| x - P_0x + \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z) \right\|
\]

\[
\leq \|1 - P_0\| \|x\| + \left\| \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z) \right\|
\]

\[
\leq \|1 - P_0\| \|x\| + L_0(z) \left\| \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z_0) \right\|
\]

\[
= \|1 - P_0\| \|x\| + L_0(z) \|P_0\| \|x\| .
\]

Thus the mapping defined by

\[
A(z)x = x - P_0x + \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z),
\]

is continuous on \( V_0 \). For each \( k \in \mathbb{N} \), the function \( \sum_{n=1}^{k} [x_n^*(z_0)(P_0x)]x_n(z_0) \) is holomorphic. Since \( \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z_0) \) converges uniformly on \( V_0 \), it is the uniform limit of a series of holomorphic functions, hence \( A \in \mathcal{H}(\Omega, \mathcal{L}(E)) \). Moreover,

\[
A(z_0)x = x - P_0x + \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0x)]x_n(z_0) = x,
\]
i.e. $A(z_0) = I_E$. Hence there exists a neighbourhood of $z_0$ such that $A \in \mathcal{H}(V_0, GL(E))$. Without loss of generality we will assume this neighbourhood is $V_0$. Clearly,

$$A(z) x_n(z_0) = \sum_{n=1}^{\infty} [x_n^*(z_0)(P_0 x_n(z_0))] x_n(z) = x_n(z).$$

The mapping $Q(z) = A(z) P_0 A^{-1}(z)$ is a projection onto $M_z$ and $Q \in \mathcal{H}(V_0, \mathcal{L}(E))$. The mappings

$$V_0 \times E \to V_0 \times E, (z_0, x) \to (z_0, A^{-1}(z)x)$$

provide a trivialization so that $V_0 \times M_z$ is a direct holomorphic sub-bundle of the trivial bundle $V_0 \times E$. By Proposition 2.5, the family of subspaces $\{M(z)\}_{z \in \Omega}$ is holomorphically complemented in $E$. \hfill \Box

**Lemma 4.2.** Let $\Omega$ be an open subset of a Banach space $X$, $E$ and $F$ be Banach spaces. Suppose that $f \in \mathcal{H}(\Omega, \mathcal{L}(E, F))$ is holomorphic. Then $f^t$ given by $f^t(z) = f(z)^t$ for $z \in \Omega$, is holomorphic.

**Proof.** Take $z_0$ in $\Omega$. Then we can find a neighbourhood $V_0$ of $z_0$ and $M > 0$ such that $\|f(z)\| < M$ for $z$ in $V_0$. Then we have that $\|f^t(z)\| = \|f(z)^t\| = \|f(z)\| < M$ for all $z$ in $V_0$ and thus we have that $f^t$ is locally bounded. Given $x$ in $E$ and $\varphi$ in $F'$ we have that

$$\langle \varphi \otimes x, f^t(z) \rangle = (f^t(z) \varphi)(x) = \varphi(f(z)x).$$

Hence the function $z \to \langle \varphi \otimes x, f^t(z) \rangle$ if holomorphic for all $x$ in $E$ and $\varphi$ in $F'$. As $\{\varphi \otimes x : x \in E, \varphi \in F'\}$ is a separating subset for $\mathcal{L}(E, F)$, Theorem 3 of [8] implies that $f^t$ is holomorphic. \hfill \Box

**Proposition 4.3.** Let $\Omega$ be a connected pseudo-convex open subset of a Banach space with a basis and $E$ be a Banach space with holomorphic basic sequence $(x_n)_n$ on $\Omega$ such that the closed linear span of $(x_n(z))_n$, $M_z$, is a complemented subspace of $E$ for every $z$ in $\Omega$. Then the associated biorthogonal functionals $x_n^*$ belong to $\mathcal{H}(\Omega, E')$ for all $n$. Moreover, if there is $z_0$ in $\Omega$ such that $(x_n(z_0))_n$ is a shrinking basis for $M_{z_0}$, then $(x_n^*)_n$ is a holomorphic basis sequence on $\Omega$.

**Proof.** Fix $z_0$ in $\Omega$ and let $P_0$ be a continuous projection from $E$ onto $M_{z_0}$. For $x$ in $E$ let

$$A(z)x = x - P_0 x + \sum_{n=1}^{\infty} x_n^*(z_0)(P_0 x)x_n(z).$$

Then as shown in Proposition 4.1 $A$ is bounded, continuous and invertible on some neighbourhood $V_0$ of $z_0$. Moreover, we have that $A(z)x_n(z_0) = x_n(z)$ for all $n$ in $\mathbb{N}$. Let $B(z) = (A(z)^{-1})^t$ for $z$ in $V_0$. It follows from Lemma 4.2 that $B$ is analytic on $V_0$. Also, if $z$ belongs to $V_0$ then for all $n, m$ in $\mathbb{N}$

$$\langle x_n(z), B(z)x_m^*(z_0) \rangle = \langle A(z)x_n(z_0), (A(z)^{-1})^t x_m^*(z_0) \rangle$$

$$= \langle (A(z)^{-1})A(z)x_n(z_0), x_m^*(z_0) \rangle$$

$$= \langle x_n(z_0), x_m^*(z_0) \rangle$$

$$= \delta_{nm}$$

proving that $x_m^*(z) = B(z)x_m^*(z_0)$. It follows that $x_m^*$ is holomorphic on a neighbourhood of $z_0$, and hence on $\Omega$. 
Suppose there is $z_s$ in $\Omega$ such that $(x_n(z_s))_n$ is a shrinking basis for $M_{z_s}$, and let $P_s$ be a continuous projection from $E$ onto $M_{z_s}$. We have already shown that each of the biorthogonal functionals $x_n^*$ belongs to $H(\Omega, E')$ and there is a neighbourhood $V_s$ of $z_s$ so that for each $z$ in $V_s$,

$$A(z)x = x - P_sx + \sum_{n=1}^{\infty} x_n^*(z_s)[(P_sx)]x_n(z)$$

is bounded and invertible linear operator which maps $x_n(z_s)$ to $x_n(z)$ and hence $M_{z_s}$ onto $M_z$. In addition we have that $B(z) := (A(z)^{-1})^t$ maps $x_n^*(z_s)$ to $x_n^*(z)$ for each $z$ in $V_s$. Let $z$ belong to $V_s$ and take $x^*$ in $M'_z$. Then $B(z)^{-1}x^*$ belongs to $M'_{z_s}$. As $(x_n^*(z_s))_n$ is a basis for $M'_{z_s}$, we can find a sequence of complex numbers $(a_n)_n$ such that $B(z)^{-1}x^* = \sum_{n=1}^{\infty} a_n x_n^*(z_s)$. Applying $B(z)$ we get that $x^* = \sum_{n=1}^{\infty} a_n B(z)x_n^*(z) = \sum_{n=1}^{\infty} a_n x_n^*(z)$ and thus $(x_n^*(z))_n$ is a basis for $M'_{z_s}$. Moreover, for each $z$ in $V_s$, each sequence of complex numbers $(a_n)_n$ and each $N$ in $\mathbb{N}$ we have that

$$\frac{1}{\|B(z)^{-1}\|} \left| \sum_{n=1}^{N} a_n x_n^*(z_s) \right| \leq \left| \sum_{n=1}^{N} a_n x_n^*(z) \right| \leq \|B(z)\| \left| \sum_{n=1}^{N} a_n x_n^*(z_s) \right|$$

showing that part (2) of Definition 3.1 is satisfied on $V_s$. We now repeat the above procedure with each $z$ in $V_s$ to get a neighbourhood $V_z$ of $z$ such that for each $w$ in $V_z$ we have an invertible continuous linear operator $B_z(w)$ on $E'$ which maps $M'_{z_s}$ onto $M'_{w}$, in the process mapping $x_n^*(z)$ to $x_n^*(w)$ for each $n$ in $N$. As in the above it follows that $(x_n^*(w))_n$ is a holomorphic basis for $M_w$ with

$$\frac{1}{\|B_z(w)^{-1}\|} \left| \sum_{n=1}^{N} a_n x_n^*(z) \right| \leq \left| \sum_{n=1}^{N} a_n x_n^*(w) \right| \leq \|B_z(w)\| \left| \sum_{n=1}^{N} a_n x_n^*(z) \right|$$

for all sequence of complex numbers $(a_n)_n$ and each $N$ in $\mathbb{N}$. Using the same method as in the proof of Proposition 3.4, we will eventually reach each point of $\Omega$. Moreover, as the sequence $(x_n^*(z))_n$ is the dual of the sequence $(x_n(z))_n$, it is uniquely determined and we have that $(x_n^*)_n$ is a holomorphic basis sequence on $\Omega$. 

As an application of holomorphic bases, we will use them to show the existence of holomorphic generalized inverses. To remind the reader: if $T \in \mathcal{L}(X, Y)$ and there exists $S \in \mathcal{L}(Y, X)$ such that $TST = T$ and $STS = S$, we call $S$ a generalized inverse for $T$.

The following definition appears in [7]:

**Definition 4.4.** Let $f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y))$, where $X$ and $Y$ are Banach spaces and $\Omega$ is an open subset of a Banach space. A mapping $g \in \mathcal{H}(\Omega, \mathcal{L}(Y, X))$ is called a holomorphic generalized inverse for $f$ if, for all $z \in \Omega$, $g(z)$ is a generalized inverse for $f(z)$.

Also in [7] it is shown that the existence of holomorphic generalized inverse is equivalent to three other conditions - none of which, unfortunately, is easy to check:

**Theorem 4.5.** Let $\Omega$ be a pseudo-convex open subset of a Banach space with an unconditional basis and $X$ and $Y$ be Banach spaces. Suppose $f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y))$ has a generalized inverse for each $z \in \Omega$. Then the following conditions are equivalent:

1. $f$ has a holomorphic generalized inverse on $\Omega$. 

There exist holomorphic projections $P \in \mathcal{H}(\Omega, \mathcal{L}(X))$ onto $\ker(f(z))$ and $Q \in \mathcal{H}(\Omega, \mathcal{L}(Y))$ onto $\mathrm{Im}(f(z))$.

3. \{z \in \Omega : (z, \ker f(z))\} and \{z \in \Omega : (z, \mathrm{Im} f(z))\} are holomorphic subbundles of $\Omega \times X$ and $\Omega \times Y$ respectively.

4. For every $w \in \Omega$ there exist a neighbourhood $V_w$ of $w$ and closed subspaces $X_w \subset X$ and $Y_w \subset Y$ such that for all $z \in V_w$, $\ker f(z) \oplus X_w = X$ and $\mathrm{Im} f(z) \oplus Y_w = Y$ are direct sum decompositions.

As a straightforward application of Proposition 4.1 we obtain the following:

**Proposition 4.6.** Let $\Omega$ be a pseudo-convex open subset of a Banach space with a basis, $E$ and $F$ be Banach spaces. Suppose $T \in \mathcal{H}(\Omega, \mathcal{L}(E,F))$ has a generalized inverse for each $z \in \Omega$. Then if $\{\ker T(z)\}_{z \in \Omega}$ and $\{\mathrm{Im} T(z)\}_{z \in \Omega}$ have holomorphic bases, $T$ has a holomorphic generalized inverse.

Note that if $P \in \mathcal{H}(\Omega, \mathcal{L}(E))$ is a projection then $1 - P$ is a holomorphic projection onto its complement, hence in Proposition 4.6 the condition that $\{\ker T(z)\}_{z \in \Omega}$ and $\{\mathrm{Im} T(z)\}_{z \in \Omega}$ have holomorphic bases can be substituted by a condition that their complements can be chosen so that they form families with holomorphic bases.

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**References**