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Particle trajectories in extreme Stokes waves over infinite depth.

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Abstract

We investigate the velocity field of fluid particles in an extreme water wave over infinite depth. It is shown that the trajectories of particles within the fluid and along the free surface do not form closed paths over the course of one period, but rather undergo a positive drift in the direction of wave propagation. In addition it is shown that the wave crest cannot form a stagnation point despite the velocity of the fluid particles being zero there.

1 Introduction

In the current paper we analyse qualitative properties of the underlying motion for the Stokes wave of greatest height, over a fluid of infinite depth. A Stokes wave is a symmetric wave profile over an irrotational flow, which rises and falls exactly once per period between crest and trough. The wave of greatest height, or extreme wave, shares many features of a regular Stokes wave, but crucially from a mathematical perspective, it displays certain irregularities at the surface. Specifically, at the wave crest a cusp develops, whereby the profile is continuous but it is no longer differentiable. Physically, this is due to the presence of a so-called stagnation point where the horizontal velocity of the particle equals the wave speed.

The existence of the extreme wave was first conjectured by Stokes [30], and has been the subject of an extensive body of research over for a century and was rigorously established in [1] (cf. [31] for an overview of Stokes waves in general). Subsequently, a number of further interesting features of the extreme wave have been established, for example [8, 27], and from both a mathematical and physical point of view the extreme wave remains a subject of great interest.

The aim of this paper is to provide a clear qualitative picture of the particle trajectories throughout the fluid domain for an infinitely deep extreme Stokes wave. It had been assumed for many years that the trajectories followed by fluid particles in regular waves should be closed, either in the form of ellipses for water of finite depth, or approaching circular paths at infinite depth [26]. However in contrast it has been shown that in a periodic surface gravity wave there are no closed particle trajectories in the linear approximation, see [10, 16]. In recent years there have been several papers which dealt with various aspects of the flow beneath a regular travelling water wave, in irrotational flow or in a flow with vorticity, see discussions in [5, 11, 12, 13, 14, 15, 17, 24, 32]. The analysis employed in these papers is not transferable to the case of the extreme wave, since the presence of a stagnation point at the crest generates a number of insurmountable mathematical difficulties.

Rather, a different approach must be employed, and in [8] an analysis of the particle trajectories for the finite depth extreme Stokes wave was undertaken. In this paper we extend the analysis of particle trajectories for extreme Stokes waves to the setting of infinitely deep fluid, thereby completing the work which was initiated in [21, 24]. Despite the cusp at the wave crest, the velocity field is shown to be continuous throughout the closure of the fluid domain. We then use techniques from conformal mapping theory, together with the approach developed for regular Stokes waves, to prove that the particle trajectories in deep-water extreme Stokes are not closed, but rather undergo a positive drift in the direction of wave propagation.

2 Preliminaries

2.1 The governing equations

We consider a two-dimensional flow, periodic in the horizontal direction, which moves with a constant speed $c > 0$. The flow is assumed to be irrotational and propagates in an infinitely deep body of water. The fluid body is given by $\Omega = \{(X, Y) \in \mathbb{R} \times (-\infty, \eta(X, t))\}$, where $\eta(X, t)$ is the surface profile of the fluid. In addition, the fluid is considered to be inviscid, incompressible and of constant density $\rho = 1$, with the flow being subject to a gravitational acceleration g. The Euler equations are:

$$\begin{aligned} &u_t + uu_X + vu_Y = -P_X, \\ &v_t + uv_X + vv_Y = -P_Y - g \qquad \text{for } (X, Y) \in \Omega. \end{aligned} \tag{1}$$

The incompressibility of the fluid flow is expressed by,

$$u_X + v_Y = 0, \quad \text{for } (X, Y) \in \Omega, \tag{2}$$

while the irrotational character of the flow is described by,

$$u_Y - v_X = 0, \quad \text{for } (X, Y) \in \Omega. \tag{3}$$

In the absence of surface tension, the boundary conditions for the flow are given by:

$$\begin{aligned} v &= \eta_t + u\eta_X, \\ P &= P_0 \qquad \text{on } Y = \eta(X,t), \end{aligned} \tag{4}$$

$$(u,v) \to (0,0) \qquad \text{uniformly in } X \text{ as } \; Y \to -\infty. \tag{5}$$

In the boundary condition (4) above, P_0 is the constant pressure exerted by the atmosphere on the free surface. Each of the unknown functions u, v, P and η is required to be periodic in the variable X, with a period $L > 0$. Moreover each function assumes the form of a travelling wave profile and so depends on the (X,t)-variables via the combination $X - ct$. Without loss of generality we assume the solutions have period $L = 2\pi$. The decay of the velocity field deep down as in (5), is interpreted as the statement that at great depth there is very little fluid motion.

2.2 The moving frame

It will be convenient for us to analyse the system in the moving frame, namely the frame of reference in which the free surface assumes a stationary wave form. The transformation to this frame of reference is induced by the following change of coordinates:

$$x = X - ct, \qquad y = Y, \qquad t = t. \tag{6}$$

Under this change of coordinates the Euler equations in (1) transform as:

$$\begin{aligned} (u-c)u_x + vu_y &= -P_x, \\ (u-c)v_x + vv_y &= -P_y - g \qquad \text{for } (x,y) \in \mathbb{R} \times (-\infty, \eta(x)), \end{aligned} \tag{7}$$

while the incompressibility and irrotationality conditions (2)-(3)become:

$$\begin{aligned} u_x + v_y &= 0, \\ v_x - u_y &= 0 \qquad \text{for } (x,y) \in \mathbb{R} \times (-\infty, \eta(x)). \end{aligned} \tag{8}$$

The boundary conditions (4)-(5) in the moving frame are:

$$\begin{aligned} v &= (u-c) \cdot \eta', \\ P &= P_0 \qquad \text{on } y = \eta(x), \end{aligned} \tag{9}$$

$$(u-c, v) \to (-c, 0) \quad \text{uniformly in } x \text{ as } y \to -\infty. \tag{10}$$

The system of equations (7)-(8) along with the boundary conditions (9)-(10) serve to define the free boundary problem in the moving frame. In addition to the boundary conditions (9), there are several symmetries of the functions that apply in the moving frame: η is symmetric with respect to the crest line $x = 0$, while u is even and v is odd in the x-variable. The advantage of transferring the free boundary problem to the moving frame is that the system of equations now has no explicit time dependence.

In the case of a smooth Stokes wave we always have the condition

$$u(x,y) < c, \tag{11}$$

for all points in the fluid domain and its boundary. In the case of an extreme Stokes wave we have,

$$u(x,y) \leq c, \tag{12}$$

where equality is achieved at the wave crest. Equations (7) and (8) ensure that u, v, η, and P are analytic in the interior of the fluid domain, while (9) and (12) ensure that these functions are merely continuous on the free surface. In the moving frame, we may say that the wave crest is located at $x = 0$, $y = \eta(0)$, while the tangent lines to the profile at the cusp create an opening of 120°. For a discussion of these facts, see [3].

The irrotationality condition in (8) implies,

$$\int_{-\pi}^{\pi}\int_{y_0}^{y_1}[u_y - v_x]\mathrm{d}y\mathrm{d}x = 0,$$

for any fixed depths $y = y_0$ and $y = y_1$ below the trough level. Integrating, we find,

$$\int_{-\pi}^{\pi} u(x,y_1)\mathrm{d}x - \int_{-\pi}^{\pi} u(x,y_0)\mathrm{d}x = 0,$$

where we have reversed the order of integration in the second integral and used $v = 0$ along the trough lines $((\pm\pi,-\infty),(\pm\pi,\eta(\pm\pi))]$. The relation,

$$\kappa = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,y_0)\mathrm{dx} < \mathrm{c}, \tag{13}$$

follows at once, with the situation being similar to that encountered in the case of a flat bed cf. [9]. It is clear that κ may be interpreted as the average horizontal current of the fluid body. In this paper it will be assumed that $\kappa = 0$.

3 The hodograph transform

It follows from (8) that we may introduce a stream function $\psi(x,y)$ defined by:

$$\psi_y = u - c, \tag{14}$$

$$\psi_x = -v. \tag{15}$$

Differentiating (14)-(15) with respect to y and x respectively and applying (8), we deduce $\psi(x,y)$ is harmonic throughout the fluid domain. Integrating (14)-(15) we obtain,

$$\psi(x,y) = \psi(0,y_0) + \int_{y_0}^{y}[u(x,\zeta) - c]\mathrm{d}\zeta - \int_{0}^{x} v(\xi,y_0)\mathrm{d}\xi \quad y \in (-\infty,\eta(x)], \tag{16}$$

where y_0 is some fixed depth below the trough level. The 2π-periodicity in x of $u(x,y)$, along with the fact $v(x,y)$ is odd in the x-variable, together ensure that $\psi(x,y)$ also has a period of 2π in the x-variable at any fixed depth below the free surface.

The first boundary condition in (9) gives us,

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x,\eta(x)) = -v(x,\eta(x)) + (u(x,\eta(x)) - c)\cdot\eta' = 0, \tag{17}$$

and so the stream function is constant along the free surface. Since $\psi(x,y)$ is harmonic in the fluid domain, the stream function must attain its maximum and minimum on the boundary. Furthermore, by (11) & (12) we have $\psi_y \leq 0$ along the free surface $y = \eta(x)$, in which case the minimum of $\psi(x,y)$ must occur on the surface, since $\psi_y \to -c$ as $y \to -\infty$.

In the fluid domain, the Euler equations are alternatively stated using Bernoulli's law,

$$\frac{(u-c)^2 + v^2}{2} + gy + P = Q, \tag{18}$$

where Q is constant. Using Bernoulli's law we may reconstruct Euler's equations in (7), along with the boundary conditions (9), in terms of the stream function $\psi(x,y)$ and height function $\eta(x)$, both of which are periodic and even in the x-variable. The free boundary problem (7) and (9), when expressed in terms of $\eta(x)$ and $\psi(x,y)$ using (18), may be written as:

$$\begin{aligned}
&\Delta\psi = 0 \quad \text{in } -\infty < y < \eta(x),\\
&\frac{1}{2}|\nabla\psi|^2 + gy + P_0 = Q \quad \text{on } y = \eta(x),\\
&\psi = 0 \quad \text{on } y = \eta(x),\\
&\nabla\psi \to (0,-c) \quad \text{uniformly in } x \text{ as } y \to -\infty.
\end{aligned} \tag{19}$$

The level sets of the steam function provide a foliation of the closure of the fluid domain $\bar{\Omega}$, with $\psi = 0$ on the surface.

We divide the fluid domain Ω into distinct regions as follows:

$$\begin{aligned}
\Omega_+ &= \left\{(x,y) \in \mathbb{R}^2 : \ x \in (0,\pi),\ -\infty < y < \eta(x)\right\},\\
\Omega_- &= \left\{(x,y) \in \mathbb{R}^2 : \ x \in (-\pi,0),\ -\infty < y < \eta(x)\right\},
\end{aligned} \tag{20}$$

which are two halves of the fluid domain in one period. In addition we have the two halves of the surface over one period given by:

$$\begin{aligned}
S_+ &= \{x \in (0,\pi),\ y = \eta(x)\},\\
S_- &= \{x \in (-\pi,0),\ y = \eta(x)\}.
\end{aligned} \tag{21}$$

The regions $\Omega_\pm$ are separated by the crest line $((0,-\infty),(0,\eta(0))]$ and are bounded laterally by the trough lines $((\pm\pi,-\infty),(\pm\pi,\eta(\pm\pi))]$. The height function η is even in x, in which case $\eta(-\pi) = \eta(\pi)$, while $\eta' \leq 0$ on S_+ and $\eta' \geq 0$ on S_-.

The irrotationality condition in (8) allows us to introduce a velocity potential $\phi(x, y)$ such that;

$$\phi_x = u - c, \qquad \phi_y = v, \tag{22}$$

with $\phi = 0$ along the crest line. Integrating (22) we find,

$$\phi(x, y) = \int_0^x [u(\xi, y_0) - c]\mathrm{d}\xi + \int_{y_0}^y v(x, \zeta)\mathrm{d}\zeta, \tag{23}$$

where y_0 is a fixed depth below the trough level. It is clear from this integral representation that $\phi(x, y) + cx$ is 2π-periodic in x. In particular we have,

$$\phi(2\pi, y) = \int_0^{2\pi} [u(\xi, y) - c]\mathrm{d}\xi = -2c\pi,$$

for all y below the trough level.

We can use the steam function and the potential function to perform a conformal hodograph transform induced by the change of variables:

$$\begin{aligned} q &= -\phi(x, y), \\ p &= -\psi(x, y). \end{aligned} \tag{24}$$

The image of the semi-infinite strip $\Omega_\pm$ is:

$$\begin{aligned} \hat{\Omega}_+ &= \left\{(p, q) \in \mathbb{R}^2 : \ q \in (0, c\pi), \ p \in (-\infty, 0)\right\}, \\ \hat{\Omega}_- &= \left\{(p, q) \in \mathbb{R}^2 : \ q \in (-c\pi, 0), \ p \in (-\infty, 0)\right\}, \end{aligned} \tag{25}$$

while the free surface region $S_\pm$ transforms as:

$$\begin{aligned} \hat{S}_+ &= \{q \in (0, c\pi), p = 0\}, \\ \hat{S}_- &= \{q \in (-c\pi, 0), \ p = 0\}, \end{aligned} \tag{26}$$

under the conformal mapping in (24).

We define a new harmonic function $h(q, p)$, such that,

$$h(q, p) = y. \tag{27}$$

In terms of the (q, p)-variables, the free boundary problem in (19), may be equivalently written as a nonlinear fixed boundary problem, given by:

$$\begin{aligned} &\Delta_{q,p} h = 0 \quad \text{for } q \in (-c\pi, c\pi), \ p \in (-\infty, 0), \\ &2(E_0 - gh)(h_p^2 + h_q^2) = 1 \quad \text{on } p = 0, \\ &\nabla h \to \left(0, \frac{1}{c}\right) \quad \text{uniformly in } q \in (-c\pi, c\pi) \text{ as } p \to -\infty, \end{aligned} \tag{28}$$

where $E_0 = Q - P_0$. In addition the function $h(q, p)$ is even and periodic in the q-variable with period $2\pi c$.

We introduce the relations:

$$\begin{aligned} \partial_q &= h_p \partial_x + h_q \partial_y, & \partial_x &= (c-u)\partial_q + v\partial_p, \\ \partial_p &= -h_q \partial_x + h_p \partial_y, & \partial_y &= -v\partial_q + (c-u)\partial_p, \end{aligned} \tag{29}$$

and upon inverting, we find:

$$\begin{aligned} h_q &= -\frac{v}{(c-u)^2+v^2} = -\frac{\partial x}{\partial p} = \frac{\partial y}{\partial q}, \\ h_p &= \frac{c-u}{(c-u)^2+v^2} = \frac{\partial x}{\partial q} = \frac{\partial y}{\partial p}, \end{aligned} \tag{30}$$

which will be used further on.

4 The velocity field of an extreme Stoke's wave

We introduce the function $f : \hat{\Omega}_+ \to \Omega_+$ defined by $f : \xi \mapsto x + iy$, with $\xi = q + ip$, where f is analytic in $\hat{\Omega}_+$ and continuous on the closure $\hat{\Omega}_+ \cup \partial\hat{\Omega}_+$. Furthermore the function has analytic continuation to any point on $\partial\hat{\Omega}_+$ except the point (0,0). However due to the cusp in the wave crest, the behaviour of f' is singular at the point $\xi = 0$.

Nevertheless the fact that the fluid domain Ω has a corner with two Hölder continuous curves issuing from $f(0)$, the tangents of which form an angle of $\frac{2\pi}{3}$ at the cusp, ensures that the function $\xi \mapsto \xi^{1/3} f'(\xi)$ is continuous on the closure of $\hat{\Omega}_+$. Specifically we may say that,

$$\lim_{\xi \to 0} \xi^{1/3} f'(\xi) = \xi_0 \neq 0, \tag{31}$$

where ξ_0 is constant, cf. [28]. Using the results of (30), along with the definition of $f(\xi)$, we see that

$$\frac{1}{f'(\xi)} = \frac{1}{x_q + iy_q} = (c-u) + iv. \tag{32}$$

Meanwhile (31) ensures $\frac{1}{f'(\xi)}$ is continuous in the closure of $\hat{\Omega}_+$, which together with (32) imply $[(c-u)+iv]$ is also continuous in the closure of $\hat{\Omega}_+$.

In fact we may specifically write,

$$\frac{(c-u)+iv}{\xi^{1/3}} \to \frac{1}{\xi_0} \quad \text{as } \xi \to 0, \tag{33}$$

and so it follows that,

$$\lim_{\xi \to 0} \frac{[(u-c)+iv]\xi^{2/3} - 0}{\xi} = \lim_{\xi \to 0} [(u-c)+iv]\xi^{-1/3} = \frac{1}{\xi_0}. \tag{34}$$

The function $[(u-c)+iv]\xi^{2/3}$ is therefore differentiable at $\xi = 0$.

Evaluating the derivative of $\frac{\xi^{2/3}}{f'(\xi)}$ along $\xi = ip$ with $p \uparrow 0$, i.e. where $v = 0$, we see that $\Re[\xi_0] \neq 0$ and $\Im[\xi_0] \neq 0$. It follows from this, and evaluating the same derivative along $\xi = q$, that the limit of $[c - u(q,0)]q^{-1/3}$ as $q \downarrow 0$ is also non-zero. As such the map $q \mapsto u(q,0)$ is continuous and periodic in $\mathbb{R}$ but does not belong to the Sobolev space $W^{1,k}(0,1)$ for $k \geq 3/2$. However, if this were so then $u(0,0) - c = 0$ would yield,

$$\lim_{q \to 0} \left([c - u(q,0)]q^{-1/3} \right) = 0,$$

which we see from,

$$0 \leq c - u(q,0) = \int_0^q |u_q(s,0)| \mathrm{d}s \leq q^{(k-1)/k} \left(\int_0^q |u_q(s,0)|^k \mathrm{d}s \right)^{1/k},$$

as $\lim_{q \downarrow 0} \int_0^q |u_q(s,0)|^k \, \mathrm{d}s = 0$.

4.1 The vertical velocity

In the case of almost extreme water waves we have that $v > 0$ in $\hat{\Omega}_+$, cf. [14], while for the extreme Stokes wave we have $v \geq 0$ in $\hat{\Omega}_+$ with equality achieved on the boundary, see [8]. In both cases $\hat{\Omega}_+$ is a bounded domain since the fluid domain is of finite depth. In our present case we would like to extend these results to the case when $\hat{\Omega}_+$ is a semi-infinite domain, corresponding to the case of an infinitely deep Stokes wave.

To begin we assume there exists a point $(q_0, p_0) \in \hat{\Omega}_+$ such that $v(q_0, p_0) = 0$ for the extreme Stokes wave. We may choose $\varepsilon \in (0, \sqrt{q_0^2 + p_0^2})$ and consider the modified region $\hat{\Omega}_+^\varepsilon$, which is the region $\hat{\Omega}_+$, with a quarter disc of radius ε and centre $(0,0)$, removed. Clearly (q_0, p_0) is an interior point of $\hat{\Omega}_+^\varepsilon$, in which case the harmonic function v attains its minimum at an interior point $\hat{\Omega}_+^\varepsilon$. Maximum principles then require $v \equiv 0$ throughout this region, cf. [20].

We also see from (7) that,

$$\nabla^2 P = -2(u_x^2 + u_y^2) \leq 0 \quad \text{in } \Omega, \tag{35}$$

in which case P is super harmonic. The weak maximum principle and the periodicity of P then ensure that the minimum must be attained on the free surface $y = \eta(x)$, cf. [20]. Furthermore since $P = P_0$ all along the free surface, it follows that P attains its minimum all along the free surface.

Supposing there exists a point $(q_0, 0)$ with $q_0 \in (0, \pi)$, where the harmonic function $v = 0$, then this corresponds to a point $(x_0, \eta(x_0)) \in \Omega_+$ where $v = 0$. The mapping $x \mapsto \eta(x)$ is strictly decreasing for all $x \in (0, \pi)$, in which case $\eta'(x_0) < 0$. Since v is harmonic in the region Ω_+^ε, where Ω_+^ε is the pre-image of $\hat{\Omega}_+^\varepsilon$ under the conformal transformation (24), it follows that $(x_0, \eta(x_0))$ must be a minimum point of v in Ω_+^ε. Hopf's maximum principle then requires that $v_y(x_0, \eta(x_0)) < 0$, cf. [19].

However the first member of (8) now requires $u_x > 0$ at $(x_0, \eta(x_0))$, which together with (12) gives $(u-c)u_x \leq 0$ at $(x_0, \eta(x_0))$. It follows that,

$$P_x = (c-u)u_x \geq 0 \quad \text{at } (x_0, \eta(x_0)). \tag{36}$$

Since $P = P_0$ on the free surface, when combined with the maximum principle requires,

$$\begin{aligned} P_x + P_y \eta' &= 0, \\ P_y &< 0 \qquad \text{on } y = \eta(x), \end{aligned} \tag{37}$$

and so it follows from $\eta' < 0$ on $x \in (0, \pi)$ that $P_x < 0$ at $(x_0, \eta(x_0))$. Thus (36) is a contradiction. It follows that the velocity component v is strictly positive in Ω_+, except at the wave crest and wave trough.

4.2 The horizontal velocity

Along the lateral sides of the excised region Ω_+^ε we have that $v = 0$, therefore v is also zero along the lateral edges of its image $\hat{\Omega}_+^\varepsilon$. Furthermore we now know $v > 0$ in the interior of $\hat{\Omega}_+$ which together with Hopf's maximum principle requires,

$$v_q(0, p) > 0 \qquad v_q(c\pi, p) < 0,$$

for $p \in (-\infty, 0]$. The incompressibility and irrotationality of the flow in (8) along with the relations in (29) give:

$$u_q + v_p = 0, \qquad u_p - v_q = 0. \tag{38}$$

We see that $u_p(0, p) = v_q(0, p) > 0$ and $u_p(c\pi, p) = v_q(c\pi, p) < 0$, and so u decreases as we descend along the crest line while u increases as we descend the trough line of $\hat{\Omega}_+^\varepsilon$, cf. [8]. Moreover the relations (29)-(30) give us,

$$u_q = \frac{(c-u)u_x - v u_y}{(c-u)^2 + v^2}, \tag{39}$$

which is well defined for all points of the fluid domain and its boundary except at the wave crest. Therefore along the lateral edges of $\hat{\Omega}_+^\varepsilon$, where $v = 0$, it follows that,

$$u_q = \frac{u_x}{c-u} = -\frac{v_y}{c-u} = 0, \tag{40}$$

and so u_q is also zero along the lateral edges of $\hat{\Omega}_+^\varepsilon$. In addition, since the transformation (24) is a conformal mapping, while u is harmonic in the fluid domain Ω_+^ε, it follows that u is also harmonic in the domain $\hat{\Omega}_+^\varepsilon$.

The first of the Euler equations gives us,

$$P_x = (c-u)u_x - v u_y,$$

which together with the relations (29)-(30) give,

$$u_q = \frac{P_x}{h_q^2 + h_p^2}, \tag{41}$$

which holds at all points in the fluid domain and along the free boundary, except at the wave crest. Along the free surface it was previously found that,

$$P_x < 0 \quad \text{on } y = \eta(x),$$

except at the wave crest and wave trough, where $P_x = 0$. It follows that,

$$u_q(q, 0) < 0 \qquad \text{for } q \in (0, c\pi), \tag{42}$$

in which case u is strictly decreasing along the free surface, except at the wave crest and wave trough cf. [8].

Since u is harmonic in $\hat{\Omega}_+^\varepsilon$, then u_q is also harmonic in the same region and so attains its maximum and minimum values along the boundary $\partial\hat{\Omega}_+^\varepsilon$. If we suppose that $u_q(q_0, p_0) = m > 0$, where (q_0, p_0) is an interior point of $\hat{\Omega}_+^\varepsilon$, then we have a contradiction of the maximum principle. To see this, we note that since $(u, v) \to (0, 0)$ as $p \to -\infty$, we may choose $p_1 < p_0$ such that $u_q(q, p_1) < m$. On the bounded region consisting of $\hat{\Omega}_+^\varepsilon$ truncated by the line $p = p_1$, u_q is larger at an interior point than on the boundary, which is a contradiction, and so $u_q \le 0$ in the interior of $\hat{\Omega}_+^\varepsilon$. The strong maximum principle also requires that if $u_q = 0$ at some interior point, then $u_q \equiv 0$ throughout $\hat{\Omega}_+^\varepsilon$, contradicting (42). Thus we conclude $u_q < 0$ in the interior of $\hat{\Omega}_+^\varepsilon$, cf. [24, 25].

Along the streamline defined via $\psi(x, y(x)) = \psi_0 < 0$, by definition we have $\psi_x(x, y(x)) + \psi_y(x, y(x))y'(x) = 0$, and so using (30) we find $y'(x) = -\frac{v}{c-u} = \frac{h_q}{h_p}$. Differentiating u along the streamline and using (29) we find,

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x, y(x)) = \frac{u_q}{h_p} < 0, \tag{43}$$

since $h_p > 0$ and $u_q < 0$ in the interior of Ω_+^ε. As such u is strictly decreasing along any streamline in the interior of Ω_+^ε. By a limiting process we therefore deduce that $u_q \le 0$ in the interior of $\hat{\Omega}_+$ for the wave of greatest height, cf. [8]. Maximum principles now ensure a strict inequality in the interior, since u_q is harmonic. Consequently, along any streamline u is strictly decreasing between the crest line and a successive trough line.

5 Particle trajectories

Along a stream line $(x(t), y(t))$ we have $\psi(x(t), y(t)) = \psi_0$, in which case we also have $\partial_t \psi(x(t), y(t)) = 0$. Along the image of the stream line p is constant and so we may write $p(t) = p(0) = p_0$. Furthermore, since we always have the strict inequality $u - c < 0$ in the fluid domain except at the wave crest, it follows that $x(t)$ goes from $+\infty$ to $-\infty$ as t goes from $-\infty$ to $+\infty$. It follows that there exist a time t_0 such that $x(t_0) = 0$ if we are beneath the free surface. Moreover, there also exists a time, say $t = 0$, when $x(0) = \pi$, and a later time $t = \theta$ such that $x(\theta) = -\pi$. Since $x(0) = \pi$ and $x(\theta) = -\pi$ are the endpoints of one period, it follows that θ is the time required for a particle to traverse one period in the moving frame. That is to say θ is the elapsed time per period along a stream line beneath the surface.

It follows that in the region $\hat{\Omega}_+$ with $p < 0$ we have,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q} = -\phi_x \dot{x} - \phi_y \dot{y} = -(c-u)^2 - v^2 < 0, \tag{44}$$

while at the endpoints we have,

$$q(0) = c\pi > 0, \qquad q(\theta) = -c\pi < 0. \tag{45}$$

From (44) we see that $\mathrm{d}t = -\frac{1}{(u-c)^2+v^2}\mathrm{d}q$. Along the particle path, the vertical displacement over time θ is given by,

$$y(\theta) - y(0) = \int_0^\theta v(x(t), y(t)) dt = \int_{-c\pi}^{c\pi} \frac{v}{(u-c)^2 + v^2} dq = 0, \tag{46}$$

since v is odd in q while the denominator in the integrand is even in q. In addition the period along a streamline p may be written as,

$$\theta(p) = \int_0^\theta \frac{\dot{x}}{u(x(t), y(t)) - c} \mathrm{d}t = \int_{-\pi}^{\pi} \frac{1}{c - u(x, y(x))} \mathrm{d}x, \tag{47}$$

which we now use to demonstrate $\theta(p) > \frac{2\pi}{c}$, when $p \in (-\infty, 0)$.

To begin, we note that the Cauchy-Schwarz inequality gives,

$$\left[\int_{-\pi}^{\pi} \frac{1}{c - u(x, y(x))} \mathrm{d}x\right] \left[\int_{-\pi}^{\pi} (c - u(x, y(x))) \mathrm{d}x\right] \geq \left[\int_{-\pi}^{\pi} \mathrm{d}x\right]^2 = 4\pi^2. \tag{48}$$

We now consider the region $D \subset \Omega_+$, which is bounded above by the stream line $\{(x, y(x)) : x \in (-\pi, \pi),\ \psi(x, y(x)) = \psi_0 < 0\}$ and below by the line segment $\{(x, y_0) : x \in (-\pi, \pi)\}$, while it is laterally bounded by the line segments $\{(-\pi, y) : y \in (y_0, y(-\pi))\}$ and $\{(\pi, y) : y \in (y_0, y(\pi))\}$. The divergence theorem applied to the vector field $(v, u - c)$ in the region D gives,

$$2\pi c = \int_{-\pi}^{\pi} [c - u(x, y(x))][1 + y'(x)^2] \mathrm{d}x > \int_{-\pi}^{\pi} [c - u(y(x))] \mathrm{d}x. \tag{49}$$

Taken together the relations (47), (48) and (49) imply,

$$\theta(p) \geq \frac{4\pi^2}{\int_{-\pi}^{\pi} [c - u(x, y(x))] \mathrm{d}x} > \frac{2\pi}{c}, \tag{50}$$

the inequality being strict since $[c - u(x, y(x))]$ only achieves a constant value as $y \to -\infty$.

The horizontal drift of a particle is defined as the net horizontal distance moved by the particle between two consecutive trough lines. That is to say,

$$X(\theta) - X(0) = c\theta - 2\pi = X(t+\theta) - X(t), \quad t \in \mathbb{R}, \tag{51}$$

which corresponds to the motion of a particle over one period in the stationary frame. In the (X,Y)-frame the particle trajectory is governed by the system:

$$\begin{aligned} \dot{X}(t) &= u(X(t), Y(t)), \\ \dot{Y}(t) &= v(X(t), Y(t)), \end{aligned} \tag{52}$$

in which case a solution of (52) of period $\theta = \frac{2\pi}{c}$ corresponds to a closed trajectory in the physical frame (X, Y).

5.1 Particle trajectories beneath the free surface

It is clear from (50) and (52) that the particle drift in the stationary frame is in the positive X-direction. The level set $\{u = 0\}$, consists of a continuous curve $\mathcal{C}_+$ in Ω_+ that intersects each streamline $\psi = p$ exactly once, where $p \in (-\infty, 0]$. The corresponding level set $\mathcal{C}_-$ in Ω_- where $\{u = 0\}$, is the reflection of $\mathcal{C}_+$ about the line $x = 0$. As there is a unique point along each streamline in Ω_+ at which $u = 0$, it follows that $u < 0$ between $\mathcal{C}_+$ and $x = \pi$. Similarly, between $x = -\pi$ and $\mathcal{C}_-$ we have $u < 0$, while between the two level sets C_- and C_+ we have $u > 0$.

We consider a particle initially located at $a = (\pi, y_0)$ when $t = 0$, which moves to the left and intersects the level set $\mathcal{C}_+$ at point b, and later intersects the crest line at point c, before moving on to intersect $\mathcal{C}_-$ at point d, and finally intersecting the trough line at $e = (-\pi, y_0)$ at time $t = \theta(y_0) = \theta$.

Between the trough lines and the level sets we have $u < 0$, in which case the motion of the particle between a & b and d & e must be in the negative X-direction when viewed from the (X,Y)-frame. While moving from b to d in the moving frame we have $u > 0$. Thus between b and d the particle trajectory in the stationary frame is in the positive X-direction.

In Section 4.1 it was demonstrated that $v > 0$ in Ω_+ and when combined with the antisymmetry of v about the crest line, it follows that $v > 0$ between a and c, while $v < 0$ between c and e. Therefore we deduce that the particle experiences no net vertical drift over the course of one period. The initial particle position is $(X(0), Y(0)) = (\pi, Y_0)$ while the final location is $(X(\theta), Y(\theta)) = (-\pi + c\theta, Y_0)$, which together with (50) implies $X(\theta) > \pi$. Consequently $X(\theta) > X(0)$, thereby indicating a drift to the right experienced by the particle after one complete period. Thus beneath the free surface the particle trajectory along any streamline is not a closed loop, but rather drifts to the right over the course of one period, when observed in the (X, Y)-frame.

5.2 Particle trajectories on the free surface

In order to demonstrate the existence of non-trivial solutions on the free surface, we want to show that if a particle is initially located at $(x_0, \eta(x_0))$ when $t = 0$

and $x_0 \in (0, \pi)$, then it will always reach the wave crest $(0, \eta(0))$ in finite time. If there exists a time τ, such that for all $t \in (0, \tau)$ we have $x(t) \neq 0$, then we also have,

$$\dot{x}(t) = u(x(t), \eta(x(t))) - c < 0, \quad t \in (0, \tau), \tag{53}$$

as $u \leq c$, except at the wave crest $(0, \eta(0))$. Integrating (53) we obtain,

$$\tau = \int_{x(\tau)}^{x_0} \frac{1}{c - u(\xi, \eta(\xi))} \mathrm{d}\xi. \tag{54}$$

At the wave crest $(0, \eta(0))$ we have a symmetric cusp whose tangents form an angle of $\frac{2\pi}{3}$, and so the magnitude of opening between the horizontal and each tangent is $\frac{\pi}{6}$, in which case,

$$\lim_{x \to 0} [\eta'(x)]^2 = \lim_{x \to 0} \frac{v^2(x, \eta(x))}{[u(x, \eta(x)) - c]^2} = \tan^2\left(\frac{\pi}{6}\right) = \frac{1}{3}. \tag{55}$$

In addition the free boundary problem (19) implemented at the wave crest requires,

$$g\eta(0) + P_0 = Q. \tag{56}$$

Applying Bernoulli's law in (18) then gives us,

$$[u(x, \eta(x)) - c]^2 + v^2(x, \eta(x)) = 2[Q - P_0 - g\eta(x)] = 2g[\eta(0) - \eta(x)], \tag{57}$$

which implies,

$$\lim_{x \to 0} \frac{[u(x, \eta(x)) - c]^2 + v^2(x, \eta(x))}{|x|} = 2g \lim_{x \to 0} \frac{\eta(0) - \eta(x)}{|x|} = \frac{2g}{\sqrt{3}}. \tag{58}$$

The velocity components of a fluid particle approaching the cusp are related through (55) which together with (58) gives,

$$\lim_{x \to 0} \frac{[c - u(x, \eta(x))]^2}{|x|} = \frac{g\sqrt{3}}{2}. \tag{59}$$

From relation (59) we conclude,

$$\int_0^{\pi} \frac{1}{c - u(s, \eta(s))} \mathrm{d}s < \infty, \tag{60}$$

thus the time it takes the particle to travel from the initial location $(x_0, \eta(x_0))$ to the wave crest $(0, \eta(0))$ is finite. Physical considerations require that the particle can only occupy the crest point $(0, \eta(0))$ for an instant before it is replaced by a new particle, since particles resting there for a finite time would accumulate, something which is not observed. It is clear then that the crest is an apparent stagnation point.

With these issues resolved, arguments analogous to those presented in Section 5.1 allow us to define the elapsed time $\theta(0)$ along with the horizontal drift

$c\theta(0) - 2\pi$ over one period, for a particle travelling on the free surface. Evaluating the relations (47) and (51) in the limit $p \uparrow 0$, we find:

$$\theta(0) \geq \frac{2\pi}{c}, \qquad \int_{-\pi}^{\pi} [c - u(\xi, \eta(\xi))] \mathrm{d}\xi \leq 2\pi. \tag{61}$$

On the other hand the Cauchy-Schwarz inequality (48) imposes

$$\left[\int_{-\pi}^{\pi} \frac{1}{c - u(x, \eta(x))} \mathrm{d}x\right] \left[\int_{-\pi}^{\pi} (c - u(x, \eta(x))) \mathrm{d}x\right] \geq 4\pi^2, \tag{62}$$

with equality possible only if $c - u(x, \eta(x))$ is constant over $x \in [-\pi, \pi]$. In contrast to particle trajectories beneath the free surface, there are no horizontal tangents at the wave crest, but rather a pair of tangents which create an opening of $120°$. The same observation remains true upon passing from the moving frame to the (X, Y)-frame, cf. [8].

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