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Incompressible and Compressible Boundary Layers on a Fibre in the Melt Spinning Process

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INCOMPRESSIBLE AND COMPRESSIBLE BOUNDARY LAYERS ON A FIBRE IN THE MELT SPINNING PROCESS

BY

JOHN HARDING

A THESIS SUBMITTED TO THE DUBLIN INSTITUTE OF TECHNOLOGY FOR THE AWARD OF

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SUPERVISED BY

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ABSTRACT

This thesis seeks to investigate shear stress on and heat transfer from the surface of a fibre in the industrial process of melt spinning. Melt spinning is the process whereby molten polymer materials are extruded through the holes of a device known as a spinneret to create continuous filaments of polymer. Fibres manufactured in this way are used in applications as diverse as fashion and clothing, to telecommunications. The magnitude of the shear stress on the fibre surface, and also the rate of heat transfer from the fibre to the surrounding environment are of major importance to the overall quality of the finished product, and this thesis examines the shear stress and heat transfer experienced by an idealised fibre in the melt spinning process. Also examined is the effect of compressible fluid flow on the shear stress on the fibre's surface.
Declaration

I certify that this thesis which I now submit for examination for the award of PhD, is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for postgraduate study by research of the Dublin Institute of Technology and has not been submitted in whole or in part for an award in any other Institute or University.

The work reported on in this thesis conforms to the principles and requirements of the Institute's guidelines for ethics in research.

The Institute has permission to keep, to lend or to copy this thesis in whole or in part, on condition that any such use of the material of the thesis be duly acknowledged.

Signature John Harding Date 16-12-2009
Candidate
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CHAPTER 1

FIBRE SPINNING

The manufacture of synthetic fibres represents a huge global industry. In 1990, the value of synthetic fibre manufacture was worth approximately $70 billion [1], and by 2002 production of polyester fibres reached record levels of 20.4 million tonnes [2]. Despite the increasing appeal of natural fibres such as cotton and wool over recent years, there continues to be significant demand for synthetic fibres, because they can be fabricated to include properties that natural fibres do not possess. Fibres created by spinning are used on a daily basis in industries as diverse as clothing, textiles and telecommunications. Fibre spinning represents a sizeable portion of the activities of many large chemical engineering corporations all over the world.

1.1 THE FIBRE SPINNING PROCESS.

Most synthetic manufactured fibres are created by extruding a viscous liquid at high pressure through the very small holes of a device called a spinneret. This process forms continuous filaments of semi-solid polymer. The
Filaments are then wound onto a take up drum. The fibre forming polymers are initially solid, and therefore must first be converted into a fluid state for extrusion through the spinneret. This is usually achieved by melting if the polymers are thermoplastic, that is, if they melt when heated, or by dissolving them in a solvent if they are unstable at high temperatures [3]. The spinnerets used in the production of synthetic fibres are similar, in effect, to a bathroom shower head. A spinneret may have from one to several hundred apertures. These tiny holes are very sensitive to impurities and consequently, to clogging. The polymer melt feeding them must be finely filtered, and this can prove to be difficult with very viscous materials. Regular maintenance and cleaning of the spinneret is essential.

As the fibre is extruded and wound onto a spinning drum, the resulting tension stretches the fibre. For some materials, this initial stretching is quite important in helping to establish physical properties in the fibre, which depend on whether one is examining the properties in the fibre axis direction or the fibre radius direction. This directional dependence of properties is called anisotropy and the usual example is a plank of wood, in which the properties along the grain are quite different to the properties across the grain. With many fibres however, these properties are controlled later in the manufacturing process, where the fibres are reheated, stretched, and cooled again [1]. There are four main methods of spinning filaments of manufactured fibres: dry, wet, gel, and melt spinning. These methods are introduced over the following pages.
1.2 DRY SPINNING.

Polymers that are not stable at high temperatures are dry spun. The polymer is initially dissolved in a solvent and then extruded into a chamber of hot air or inert gas. In the chamber the solvent evaporates leaving the polymer filament to solidify. Dry spinning is one of the more expensive methods of fibre manufacture as it necessitates the use of a solvent to prepare the material for spinning and the recovery of this solvent from the hot air chamber for further use. There are also the costs (and environmental implications) of disposing of these chemicals. Examples of materials that are dry spun are acetate, triacetate, acrylic and spandex.

1.3 WET SPINNING.

Wet spinning is used for polymers that do not melt, and only dissolve in solvents that are unstable at high temperatures. After the polymer is dissolved, it is extruded from a spinneret into a water bath. The bath contains dissolved agents that precipitate the fibre from the solution. After passing through the water bath, the resulting wet filament still contains some solvent and precipitating agent. This makes it necessary for the fibre to be drawn through a second bath where it solidifies and is then wound onto a take up drum. Wet spinning is even more expensive than dry spinning due to the equipment needed for the process, and the expense of the procurement and recycling of solvents and precipitating agents. Wet spinning also has the disadvantage that spinning speeds are comparatively slow. This is due to the large amount of drag exerted on the filaments by the water bath. Acrylic, rayon and aramid are among the materials produced using this process [3].
1.4 GEL SPINNING.

Gel spinning is a special process used to obtain high strength fibres. The melt is not in a fully liquid state during extrusion. The polymer chains are not completely separated, but are bound together loosely in liquid crystal form. This phenomenon produces strong forces between the chains in the resulting filaments that can increase strength of the fibres significantly. In addition, the liquid crystals are aligned along the fibre axis by the shear stresses during extrusion from the spinneret. Therefore, the filaments end up with a high degree of orientation relative to each other, further enhancing the strength of the fibre. The process is sometimes described as dry-wet spinning, as the filaments first pass through air and then are cooled further in a liquid bath. Some high-strength polyethylene and aramid fibres are produced in this way.

1.5 MELT SPINNING.

By far the most common form of fibre manufacture, melt spinning does not require the use of solvents and the equipment used is considerably less complicated than that used in dry or wet spinning processes. As such, this method enjoys a distinct cost advantage. Melt spun fibres can be extruded from the spinneret in different cross-sectional shapes (round, pentagonal, octagonal, and others). Melt spun fibres can also be hollow. Pentagonal shaped and hollow fibres, when used in carpet, show less soil and dirt. Octagonal-shaped fibres offer glitter-free effects in textiles. Hollow fibres trap air, creating insulation.

Spinning speeds are comparatively very high. As the name would suggest, in the melt spinning process, the polymer is melted into a thick viscous liquid suitable for
extrusion. Melting is normally carried out using polymer chips in an electrically heated extruder, and this type of spinning is used for polymers that are not decomposed or degraded by the high temperatures which are required to melt them. Polymers which are stable up to about $300^\circ C$ are spun in this way. The molten polymer is then extruded into a cool air quench in which it solidifies and is wound onto a drum. Polymers suitable for melt spinning include nylon, polyester, PET, and polystyrene. Below and overleaf is a schematic of the melt spinning process, and also a picture of polymer fibres being extruded from a spinneret.

![Schematic of the melt spinning process](image-url)

**Figure 1:** Schematic of the melt spinning process [3].
Figure 2: Polymer extruded from spinneret [3].

Two other factors are also vitally important in the manufacture of melt spun synthetic fibres. These are: shear stress (air resistance) on the surface of the fibre, and the rate of heat loss from the fibre to the surrounding environment. This thesis will examine the effects of shear stress and heat transfer on an idealised fibre in the melt spinning process. Also included is an analysis of the effects of compressibility on the shear stress on the fibre surface.
CHAPTER 2

BOUNDARY LAYER THEORY

Until the early 1900's, scientists were unable to successfully apply the Navier-Stokes equations, the fundamental equations that govern the motion of Newtonian fluids, to the majority of actual viscous flows. In 1904, at the International Mathematical Congress, Ludwig Prandtl gave a lecture entitled "Fluid Motion with Very Small Friction." The talk introduced the idea of a boundary layer [4].

Prandtl believed that flow around a specific body could be looked at in two sections. The first section was a very thin layer surrounding the outer boundaries of the body. The second region was the outer flow, away from the body. He explained that viscous friction played an important role in the closer, boundary layer and could be ignored in the outer layer. This is explained by the tendency of fluids to 'stick' to a body as they flow past it, slowing the flow of the fluid closest to the body [5]. Since then, boundary layer theory has become a fundamental aspect of fluid dynamics. In addition to Prandtl's original theory concerning momentum, it has been found that a similar layer exists for heat transfer. This thermal boundary layer is treated in much
the same way as the momentum boundary layer. The thermal boundary layer exists even if the fluid in the inviscid region is the same temperature as that of the surface. This is because large velocity gradients associated with the velocity boundary layer create large thermal gradients through viscous dissipation. To understand boundary layer theory, it is important to realise that the boundary layer is incredibly thin; orders of magnitude smaller than the characteristic length associated with a given flow.

To understand why boundary layers occur, it is necessary to know a little about different types of fluid flow. Broadly speaking there are two main types, \textit{laminar} flow, and \textit{turbulent} flow. Laminar flow can be thought of as a fluid moving smoothly in layers over each other with no mixing of fluid particles between the layers. Laminar flow generally occurs at lower velocities. Turbulent flow is seen to occur at higher fluid velocities and it involves the random mixing of particles between the fluid layers.

When a fluid flows over a solid surface, it is seen that the fluid very close to the surface has exactly the same velocity as the surface itself, i.e. the fluid sticks to the surface of the solid. Moving radially away from the surface of the solid, the velocity of the fluid is seen to rapidly approach the velocity of the main body of fluid, until it reaches the same velocity. The region in which this rapid transition happens is defined as the boundary layer [6].

Bearing the above in mind, it is easy to visualise how and why a laminar boundary layer is formed. As the fluid flows over the solid body the layer closest to the surface sticks to it and there is no relative motion. This is known as the no-slip condition. This happens because all surfaces, even surfaces that may be smooth on a
As the laminae above the stationary layer in contact with the surface continue to flow, viscous forces are exerted on them, slowing them down. This effect decreases with distance from the solid surface, until at some point the remaining layers move at the same velocity as the main body of the fluid. This constitutes a boundary layer, and its thickness is defined as the distance it takes to go from zero velocity relative to the solid to the velocity of the free stream, $U_\infty$.

![Schematic of a boundary layer formed by fluid flow over a stationary solid surface.](image)

The melt spinning process may be treated as a boundary layer problem and this thesis seeks to investigate the momentum and thermal boundary layers on an idealised fibre formed in this way. The problem is idealised by treating it as flow past a continuous circular cylinder immersed in a laminar fluid flow.
CHAPTER 3

AXIAL FLOW ALONG A CYLINDER

This chapter introduces the idea of axial flow along a cylinder and discusses methods used to calculate the shear stress on the surface of such a cylinder. Several papers of importance in this area are reviewed, with a view to extending and applying some of the ideas and methods involved to the process of fibre spinning.

Axial flow over a stationary cylinder has been studied extensively in the past by Seban and Bond [7], Kelly [8], Curle [9], Stewartson [10] and Cooke [11], to name but a few. However, this review will concentrate on a paper entitled “The axisymmetric boundary layer on a long thin cylinder”, by M.B. Glauert and M. J. Lighthill [12], who use two different methods to determine the skin friction on the surface of a long thin stationary cylinder immersed in a laminar fluid flow. The first is the Pohlhausen method, which is an approximate method, and is valid along the full length of the cylinder. The second method used is an asymptotic series solution that was first introduced by Glauert and Lighthill [12]. Glauert and Lighthill [12]
find good agreement of results for these two different methods. Also reviewed is a paper by B.C. Sakiadis [13], entitled “Boundary layer behaviour on continuous solid surfaces: III. The boundary layer on a continuous cylindrical surface.”. In this paper Sakiadis [13] derives an expression for the shear stress on the surface of a cylinder moving in a stationary fluid. Sakiadis [13] uses the Pohlhausen method to determine shear stress on the cylinder, and this result is compared to the corresponding exact solution obtained by Crane [14], in a paper entitled: “Boundary layer flow on a circular cylinder moving in a fluid at rest.”

3.1 GLAUERT & LIGHTHILL: “THE AXISYMMETRIC BOUNDARY LAYER ON A LONG THIN CYLINDER”

Glauert and Lighthill [12] examine a laminar fluid flow of constant velocity $U_\infty$ over a stationary cylinder of radius $a$.

![Figure 4: Boundary layer over a stationary cylinder.](image)

A coordinate system $(x,r)$ is fixed at the front end of the cylinder, where $x$ measures axial distance from the front of the cylinder and $r$ measures radial distance from the cylinder axis. The boundary conditions for this flow are $u = v = 0$ on $r = a$, and $u \to U_\infty$ as $r \to \infty$, where $u$ and $v$ are the axial and radial velocity.
components within the boundary layer respectively. The first boundary condition states that at the cylinder surface the velocity of the fluid is zero. The second states that the fluid velocity at the edge of the boundary layer is equal to the velocity of the freestream.

3.2 DERIVATION OF LOGARITHMIC VELOCITY PROFILE.

The ratio of the fluid velocity inside the boundary layer to that of the fluid in the free stream is known as a velocity distribution, and it is represented by a velocity profile. The basis of the Pohlhausen method lies in solving the momentum integral equation. However its success depends on the accuracy of the chosen velocity profile in representing the conditions close to the solid surface. This paper by Glauert and Lighthill [12] is an important paper historically as it was the first paper to use a logarithmic velocity profile. The logarithmic profile is found to represent the velocity close to the cylinder surface with a high degree of accuracy. The logarithmic velocity profile is derived in this section by applying the boundary conditions outlined above to the axially symmetric boundary layer equations (3.1) and (3.2), which are:

\[
ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial r} = \nu \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \tag{3.1}
\]

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0. \tag{3.2}
\]

\( \nu \) is the kinematic viscosity of the fluid and is given by the ratio of the dynamic viscosity, \( \mu \), to the fluid density \( \rho \), i.e. \( \nu = \frac{\mu}{\rho} \). Equation (3.1) is the momentum boundary layer equation, which describes the fluid motion within the boundary layer.
Equation (3.2) is the continuity equation which is a statement of conservation of mass. Inserting the boundary conditions \( u = v = 0 \) at \( r = a \) into (3.1) gives:

\[
\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0 \quad \text{on } r = a
\]

and therefore

\[
\left( \frac{\partial^2 u}{\partial r^2} \right)_{r=a} = -\frac{1}{a} \left( \frac{\partial u}{\partial r} \right)_{r=a} .
\]  

(3.3)

Differentiating (3.1) with respect to \( r \) and applying the surface boundary condition yields:

\[
\left( \frac{\partial^3 u}{\partial r^3} \right)_{r=a} + \frac{1}{a^2} \left( \frac{\partial^2 u}{\partial r^2} \right)_{r=a} - \frac{1}{a^3} \left( \frac{\partial u}{\partial r} \right)_{r=a} = 0.
\]  

(3.4)

Substituting (3.3) into (3.4) gives:

\[
\frac{\partial^3 u}{\partial r^3} = \frac{2}{a^2} \left( \frac{\partial u}{\partial r} \right)_{r=a} .
\]  

(3.5)

It can be seen that the velocity of the fluid near the surface of the cylinder, \( u \), can be expressed by a Taylor series of the form

\[
u = (r-a) \left( \frac{\partial u}{\partial r} \right)_{r=a} + \frac{(r-a)^2}{2!} \left( \frac{\partial^2 u}{\partial r^2} \right)_{r=a} + \frac{(r-a)^3}{3!} \left( \frac{\partial^3 u}{\partial r^3} \right)_{r=a} + O(r-a)^4.
\]  

(3.6)

Utilising (3.3) and (3.5) equation (3.6) may be written as,

\[
u = (r-a) \left( \frac{\partial u}{\partial r} \right)_{r=a} + \frac{(r-a)^2}{2!} \left( - \frac{1}{a} \frac{\partial u}{\partial r} \right)_{r=a} + \frac{(r-a)^3}{3!} \left( \frac{2}{a^2} \frac{\partial u}{\partial r} \right)_{r=a} + O(r-a)^4,
\]  

(3.7)

which following some simplification reduces to:

\[
u = a \left( \frac{\partial u}{\partial r} \right)_{r=a} \left( \frac{(r-a)}{a} - \frac{1}{2} \left( \frac{r-a}{a} \right)^2 + \frac{1}{3} \left( \frac{r-a}{a} \right)^3 \right) + O \left( \frac{r-a}{a} \right)^4.
\]  

(3.8)
Note that the right hand side of equation (3.8) is the first three terms of the MacLaurin series expansion of \( \ln\left(\frac{r}{a}\right) \) multiplied by \( a \left( \frac{\partial u}{\partial r}\right)_{r=a} \). Thus (3.8) may be approximated by:

\[
u = a \left( \frac{\partial u}{\partial r}\right)_{r=a} \ln\left(\frac{r}{a}\right). \tag{3.9}\]

So the velocity profile (the ratio of fluid velocity inside the boundary layer to the velocity of the free stream) for this case is given by

\[
\frac{u}{U_\infty} = \frac{a}{U_\infty} \left( \frac{\partial u}{\partial r}\right)_{r=a} \ln\left(\frac{r}{a}\right). \tag{3.10}\]

Defining the variable \( \frac{1}{\alpha(x)} = \frac{a}{U_\infty} \left( \frac{\partial u}{\partial r}\right)_{r=a} \) allows (3.10) to be rewritten as

\[
\frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln\left(\frac{r}{a}\right). \tag{3.11}\]

This is a logarithmic velocity profile that quite accurately represents the flow of the fluid close to the surface of the cylinder. At the edge of the boundary layer the ratio of the internal velocity to the free stream velocity is unity as the velocities are the same at this interface. Thus the complete velocity profile is approximated by:

\[
\frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln\left(\frac{r}{a}\right) \quad 1 \leq \frac{r}{a} \leq e^a, \quad \frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e^a. \tag{3.12}\]

### 3.3 APPLICATION OF THE DERIVED LOGARITHMIC PROFILE.

In section 3.2 it has been shown that the logarithmic velocity profile represents the fluid behaviour close to the solid surface very accurately, with an error of only \( O\left(\frac{r-a}{a}\right)^4 \). In this part of the analysis the shear stress on the surface of the cylinder
is determined using the Pohlhausen method. The shear stress is quantified by the Bingham number, which is a non-dimensional shear stress coefficient. The Bingham number, $Bi$, is defined as

$$Bi = \frac{\alpha \tau}{\mu U_\infty},$$  \hspace{1cm} (3.13)

where $\tau$ is the shear stress on the surface of the cylinder, $\mu$ is the viscosity of the fluid, $\alpha$ is the cylinder radius and $U_\infty$ is the free stream velocity. Now, noting that $\tau = \mu \left(\frac{\partial u}{\partial r}\right)_{r=\alpha}$, (3.12) gives:

$$\tau = \frac{\mu U_\infty}{\alpha},$$  \hspace{1cm} (3.14)

and therefore from (3.13)

$$Bi = \left|\frac{1}{\alpha}\right|.$$  \hspace{1cm} (3.15)

Integration of equation (3.1) over a normal section of the flow and also using (3.2) yields the momentum integral equation:

$$\frac{d}{dx} \int_a^{ae^x} u(U_\infty - u)2\pi r dr = 2\pi \alpha \frac{\tau}{\rho}.$$  \hspace{1cm} (3.16)

Substituting an expression for $u$ obtained from (3.12), equation (3.16) reduces to:

$$\frac{d}{dx} \left[ \int_a^{ae^x} \frac{U_\infty}{\alpha} \ln\left(\frac{r}{a}\right) \left(U_\infty - \frac{U_\infty}{\alpha} \ln\left(\frac{r}{a}\right)\right) r dr \right] = \frac{vU_\infty}{\alpha}.$$  \hspace{1cm} (3.17)

where $v$ is the kinematic viscosity of the fluid and is given by $v = \frac{\mu}{\rho}$.

Using the change of variable $r = a e^z$ (3.17) becomes:
Integrating (3.18) by parts yields:

$$\frac{d}{dx} \left[ \frac{U_\infty^2 a^2}{\alpha^2} \int_0^1 z(\alpha(x) - z) e^{2z} dz \right] = \frac{\nu U_\infty}{\alpha}. \quad (3.18)$$

Differentiating and simplifying gives:

$$\frac{d}{dx} \frac{U_\infty^2 a^2}{\alpha^2} \left( \frac{1}{4} e^{2\alpha} (\alpha - 1) + \frac{1}{4} (\alpha + 1) \right) = \frac{\nu U_\infty}{\alpha}. \quad (3.19)$$

Integrating (3.21) and simplifying results in:

$$\int_0^\alpha \left( \frac{2a^2 - 3\alpha + 2}{\alpha^2} e^{2\alpha} - (\alpha + 2) \right) d\alpha = \frac{4\nu}{U_\infty a^2}, \quad (3.20)$$

and therefore:

$$\lim_{\xi \to 0} \int_\xi^\alpha \left( \frac{2a^2 - 3\alpha + 2}{\alpha^2} e^{2\alpha} - (\alpha + 2) \right) d\alpha = \frac{4\nu}{U_\infty a^2}. \quad (3.21)$$

Integrating (3.21) and simplifying results in:

$$\left[ e^{2\alpha} + Ei(2\alpha) - \ln(\alpha) - \frac{2}{\alpha} (e^{2\alpha} - 1) \right]_\xi^\alpha = e^\beta, \quad (3.22)$$

where $Ei(z) = \int_{-\infty}^z \frac{e^t}{t} dt$ and $\beta = \ln \left( \frac{4\nu}{U_\infty a^2} \right)$. $\beta$ is a non-dimensional axial coordinate. The thickness of a laminar boundary layer, $\delta$, is proportional to $\sqrt{\frac{4\nu}{U_\infty}}$.

Thus $\frac{4\nu}{U_\infty a^2} \sim \delta^2 \sim \frac{\delta^2}{a^2}$. When $\frac{\delta^2}{a^2}$ is small, the thickness of the boundary layer compared to the radius of the cylinder is small. This is towards the leading edge of the cylinder. When $\frac{\delta^2}{a^2}$ is large, the boundary layer thickness is large in comparison with the radius of the cylinder. This occurs at large distances from the leading edge of the cylinder.
The asymptotic expansion of the exponential integral function, \( E_i(z) \), shows that \( E_i(2\xi) \approx \gamma + \ln(2\xi) \) as \( \xi \to 0 \), where \( \gamma \) is Euler's constant. Thus equation (3.22) may be written as:

\[
e^{2\alpha} + E_i(2\alpha) - \ln(2\alpha) - \frac{2}{\alpha} \left(e^{2\alpha} - 1\right) - \gamma - e^{2\xi} + \frac{2}{\xi} \left(e^{2\xi} - 1\right) = e^{-\beta}.
\] (3.23)

The final two terms on the left hand side of (3.23) may be expanded using the MacLaurin series expansion of \( e^{2\xi} \), such that: \( e^{2\xi} \approx 1 + 2\xi + 2\xi^2 + \ldots \) giving

\[
e^{2\alpha} + E_i(2\alpha) - \ln(2\alpha) - \frac{2}{\alpha} \left(e^{2\alpha} - 1\right) - \gamma + 3 = e^{-\beta},
\] (3.24)

as \( \xi \to 0 \). Equation (3.24) may be solved numerically for \( \alpha \), and the Bingham number calculated using the formula \( Bi = \frac{1}{|\alpha|} \). Solving equation (3.24) in this manner yields valid results for all values of \( \beta \). Glauert and Lighthill [12] also derive a solution valid at only large axial distances from the origin of coordinates.

In this part of the analysis, the region of interest is where the radius of the cylinder is small in comparison with the thickness of the boundary layer. In this region, \( \alpha(x) \) is large. This occurs at large axial distances from the leading edge of the cylinder.

Recall \( \alpha(x) = \frac{U_{\infty}}{a} \left( \frac{\partial u}{\partial r} \right)^{-1}_{rad} \). When the boundary layer thickness is large, the transition of the velocity from the surface to the free stream velocity takes place over a greater distance so \( \left( \frac{\partial u}{\partial r} \right)^{-1}_{rad} \) is small, and thus in this region the magnitude of \( \alpha(x) \) is large. Therefore the exponential terms in (3.21) become very much larger than the other terms, which may be neglected. Also, at sufficiently large
values of $\alpha$ terms of the order of $\frac{e^{2\alpha}}{\alpha^2}$ are also small in comparison with other terms. Thus for large $\alpha$, after integration (3.21) reduces to:

$$e^{2\alpha} - \frac{3e^{2\alpha}}{2\alpha} \approx e^\beta.$$ (3.25)

where in the region of the lower limit, $\alpha = 0$, the integrand in equation (3.21) may be expanded using a MacLaurin series such that:

$$e^{2\alpha} = 1 + 2\alpha + 2\alpha^2 + \ldots.$$ (3.26)

This gives

$$\left(\frac{2\alpha^2 - 3\alpha + 2}{\alpha^2}\right)^{2\alpha} - (\alpha + 2) \approx \frac{2}{3}\alpha + \frac{4}{3}\alpha^2 + \frac{6}{5}\alpha^3 + \ldots$$ (3.27)

Thus at the limit $\alpha = 0$, $\int \left(\frac{2\alpha^2 - 3\alpha + 2}{\alpha^2}\right)^{2\alpha} - (\alpha + 2) d\alpha = 0$.

Taking the natural logarithm of both sides of (3.25), and noting that for large $\alpha$

$$\ln\left(1 - \frac{3}{2\alpha}\right) \approx -\frac{3}{2\alpha},$$

equation (3.25) yields:

$$2\alpha - \frac{3}{2\alpha} \approx \beta,$$ (3.28)

Rearranging (3.28) gives

$$2\alpha^2 - \beta\alpha - \frac{3}{2} \approx 0.$$ (3.29)

Solving this for $\alpha$ yields:

$$\alpha \approx \pm \frac{\sqrt{\beta^2 - 4(2)(-\frac{3}{2})}}{4}.$$ (3.30)

Only the positive value of this is taken, because a negative Bingham number has no physical meaning. This results in:
\[
\alpha \approx \frac{\beta + (\beta^2 + 12)^{1/2}}{4} \approx \frac{\beta + \beta \left(1 + \frac{12}{\beta^2}\right)^{1/2}}{4}. \tag{3.31}
\]

Using the fact that \(\left(1 + \frac{12}{\beta^2}\right)^{1/2} \approx 1 + \frac{6}{\beta^2}\) gives

\[
\alpha \approx \frac{\beta}{2} \left(1 + \frac{3}{\beta^2}\right). \tag{3.32}
\]

Now that an expression for \(\alpha\) has been found, the Bingham number may be determined from (3.15).

\[
Bi = \left|\frac{1}{\alpha}\right| = \frac{2}{\beta} \left(1 + \frac{3}{\beta^2}\right)^{-1}. \tag{3.33}
\]

as \(\beta\) is a positive quantity. Thus:

\[
Bi = \frac{2}{\beta} - \frac{6}{\beta^3} + O\left(\frac{1}{\beta^4}\right). \tag{3.34}
\]

This is Glauert and Lighthill's expression for shear stress on a stationary cylinder at large axial distances using the Pohlhausen method.

### 3.4 COMPARISON OF POHLHAUSEN AND ASYMPTOTIC SERIES SOLUTIONS.

A brief comparison is now drawn between Glauert and Lighthill's [12] Pohlhausen solutions detailed in the previous section, and the asymptotic series expansion of the same problem. Recall that the analytical Pohlhausen solution is:

\[
Bi = \frac{2}{\beta} - \frac{6}{\beta^3} + O\left(\frac{1}{\beta^4}\right). \tag{3.35}
\]

and the asymptotic series solution given by [12] is:

\[
Bi = \left|\frac{\alpha r}{\mu U_\infty}\right| = \frac{2\gamma^2}{\beta} + \frac{2\gamma^2 - \frac{1}{2} \pi^2 - 4\ln 2}{\beta^3} + O\left(\frac{1}{\beta^4}\right). \tag{3.36}
\]
Figure 5 shows a plot of the Bingham number, $Bi$ vs $\beta = \ln\left(\frac{41x}{U_\infty a^2}\right)$ in the region $8 \leq \beta \leq 12$.

It can be seen that as $\beta$ increases, i.e. as axial distance along the cylinder is increased, the Bingham number decreases. It is also clear from the graph that the Pohlhausen method shows good agreement with the asymptotic series solution, with the analytical solution giving surprisingly good results, given the approximations made in its derivation. The Pohlhausen solutions underestimate the Bingham number by an average of just 5.2% over the region of interest. It is also evident that the Pohlhausen and asymptotic series solutions converge as $\beta \to \infty$. 

![Plot of Bi vs $\beta$](image-url)

Figure 5
Sakiadis [13] deals with the boundary layer on a cylinder of radius $a$ moving through a fluid at rest. A Pohlhausen method similar to that used by Glauert and Lighthill [12] is employed to solve the problem. The same coordinate system $(x, r)$ is chosen, where $x$ measures axial distance from the leading edge of the cylinder and $r$ measures radial distance from the axis of the cylinder. Therefore, the boundary layer equations have the same form as (3.1) and (3.2). The boundary conditions are:

$$u = U_f \text{ on } r = a$$

and

$$u \to 0 \text{ as } r \to \infty,$$

(i.e. $u = U_f$, the constant cylinder velocity on the cylinder surface, and $u$ tends to zero at large radial distances from the fibre. A logarithmic velocity profile, similar to that used by Glauert and Lighthill [12] is used. It has the form:

$$\frac{u}{U_f} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right) \quad 1 \leq \frac{r}{a} \leq e^a,$$

$$\frac{u}{U_f} = 0 \quad \frac{r}{a} \geq e^a. \quad (3.38)$$

Again, integration of (3.1) over a normal section of the flow and use of (3.2) yields the momentum integral equation:

$$\frac{d}{dx} \int_{a}^{r} u^2 r dr = -a \nu \left( \frac{\partial u}{\partial r} \right)_{r=a}. \quad (3.39)$$

Substituting an expression for $u$ obtained from the first part of (3.38) and making the change of variable $r = ae^z$ gives:
\[
\frac{d}{dx} \left[ \frac{U_f^2 a^2}{\alpha^2} \int_0^\alpha (\alpha-z)^2 e^{2z} \, dz \right] = \frac{vU_f}{\alpha}. \tag{3.40}
\]

Integrating the left hand side of (3.40) and simplifying gives:

\[
\alpha \frac{d}{dx} \left( \frac{e^{2a}}{\alpha^2} - 2 - \frac{2}{\alpha} - \frac{1}{\alpha^2} \right) = \frac{4v}{U_f a^2}. \tag{3.41}
\]

Differentiating the left hand side of (3.41) gives, after some simplification:

\[
\left( \frac{2e^{2a}(\alpha-1) + 2\alpha + 2}{\alpha^2} \right) \frac{d\alpha}{dx} = \frac{4v}{U_f a^2}. \tag{3.42}
\]

Hence (3.42) becomes:

\[
\lim_{z \to 0} \int_z^\alpha \left( \frac{2e^{2a}(\alpha-1) + \alpha + 1}{\alpha^2} \right) d\alpha = \frac{4vz}{U_f a^2}. \tag{3.43}
\]

Sakiadis [13] evaluates (3.43) by a numerical method, but it may be expressed in terms of the exponential integral function \( Ei(z) \), where \( Ei(z) = \int_{-\infty}^t e^t \, dt \). Integration of (3.43) gives:

\[
\left[ \frac{2e^{2a}}{\alpha} - 2Ei(2\alpha) + 2\ln(\alpha) - \frac{2}{\alpha} \right]_z^\alpha = e^\beta. \tag{3.44}
\]

Using the series expansions for \( Ei(z) \) and \( e^{2z} \) yields:

\[
\frac{2(e^{2a} - 1)}{\alpha} - 2Ei(2\alpha) + 2\ln(2\alpha) + 2\gamma - 4 = e^\beta. \tag{3.45}
\]

Equation (3.45) may be solved for \( \alpha \) numerically to calculate the Bingham number.

As in section 3.3 it is possible to derive a simple analytical solution to equation (3.43) which is valid at large axial distances from the leading edge by integrating the left hand side by parts and including terms up to \( O\left(\frac{e^{2a}}{\alpha^2}\right) \). Equation (3.46) shows the results.
\[
\frac{e^{2\alpha}}{\alpha} \left(1 - \frac{1}{2\alpha}\right) \approx \frac{4\nu x}{U_J a^2}
\]  
(3.46)

as \(\alpha \to \infty\). As in section 3.3, close to the lower limit of \(\alpha = 0\) the integrand of equation (3.43) may be expanded using the Maclaurin series (3.26). This results in:

\[
2 \left(\frac{e^{2\alpha}(\alpha - 1 + \alpha + 1)}{\alpha^2}\right) \approx \frac{4}{3} \alpha + \frac{4}{3} \alpha^2 + \frac{4}{5} \alpha^3 + ...
\]

(3.47)

Therefore at the limit \(\alpha = 0\),

\[
2 \int \left(\frac{e^{2\alpha}(\alpha - 1 + \alpha + 1)}{\alpha^2}\right) d\alpha = 0.
\]

Taking the natural logarithm of (3.46) yields

\[
\beta \approx 2\alpha - \ln \alpha - \frac{1}{2\alpha},
\]

(3.48)

where \(\beta = \ln \left(\frac{4\nu x}{U_J a^2}\right)\). In this case the Bingham number \(Bi\) is defined as:

\[
Bi = \left|\frac{a\tau}{\mu U_J}\right| = \left|\frac{1}{\alpha}\right|
\]

(3.49)

Solving (3.48) for \(\alpha\) gives:

\[
\alpha \approx \frac{\beta}{2} \left[1 + \frac{1}{\beta} \ln \left(\frac{\beta}{2}\right) - \frac{1}{\beta^2}\right].
\]

(3.50)

Inverting (3.50) and simplifying yields:

\[
Bi \approx \frac{1}{\alpha} \approx \frac{2}{\beta} \left[1 - \frac{1}{\beta} \ln \left(\frac{\beta}{2}\right) - \frac{1}{\beta^2}\right].
\]

(3.51)

In a paper entitled: “Boundary layer flow on a circular cylinder moving in a fluid at rest” Crane [14] sought to evaluate the shear stress on the surface of a cylinder moving at constant velocity through still air (the same case as Sakiadis's [13] above). Crane [14] solves the problem using an asymptotic series similar to that used by
Glauert and Lighthill [12]. Figure 6 compares Crane's [14] and Sakiadis's [13] results.

![Plot of Bi vs $\beta$](image)

Figure 6

It can be seen that Sakiadis's [13] Pohlhausen solution underestimates the Bingham number by an average of 6.9% over the region of interest.

### 3.6 EXPERIMENTAL DATA.

There is little in the way of experimental data on the shear stress on the surface of synthetic fibres. This is in part due to the difficulties of measuring such forces on fibres with diameters of the order of microns, and also that fibre manufacturers do not publish their results for commercial reasons. The data that does exist however, such as that obtained by Gould and Smith [15], and Koldenhof [16], shows good agreement with the theoretical predictions of Glauert and Lighthill [12], Sakiadis [13] and Crane [14] respectively. It is clear that expressions for the shear stress on a stationary cylinder in a moving fluid, or a moving cylinder in a fluid at rest can be
derived using the Pohlhausen integral method, a method that shows good agreement with exact solutions to the same problems. The next chapter discusses the thermal boundary layer.
CHAPTER 4
THE THERMAL BOUNDARY LAYER

In chapter 2 it was noted that in addition to the momentum boundary layer, it has been found that a similar thermal boundary layer exists. This is true even if the fluid in the inviscid region is the same temperature as that of the bounding surface. This occurs because large velocity gradients associated with the velocity boundary layer create large thermal gradients through viscous dissipation. As with the study of momentum boundary layers, solutions of the thermal boundary layer generated over a stationary cylinder in laminar flow have been extensively studied by Bourne et al [17], Hieber and Gebhart [18], Khan et al [19] and Richelle et al [20] among others. Of more interest to the fibre spinning process however, is modelling a fibre by treating it as a circular cylinder moving through a fluid at rest, and this problem has been tackled by Bourne and Dixon [21], Kuiken [22] and Glicksman [23]. This chapter however, focuses on a paper written by Bourne and Elliston [24] on the thermal boundary layer created by a fibre moving through still air entitled “Heat transfer through the axially symmetric boundary layer on a moving circular fibre”.

The fibre is extruded from the spinneret aperture into a cool still air quench in which it solidifies. In practice anywhere up to several hundred fibres are extruded from the spinneret at once, with consequent overlapping of the boundary layers. However, Bourne and Elliston [24] idealise the problem by reducing it to one of a single continuous infinite circular cylinder moving with constant velocity $U_f$ at a constant temperature $T_f$. A Pohlhausen method is used to obtain the Nusselt number, $Nu$, which characterises the rate of heat loss from the fibre to the surrounding environment. This quantity is of considerable practical interest in that it is a significant factor in the overall quality of the finished fibre.

4.1 Bourne and Elliston: “Heat Transfer Through the Axially Symmetric Boundary Layer on a Moving Circular Fibre”

The coordinates $(x, r)$ measure axial distance from the front of the fibre and radial distance from the fibre axis respectively. The axial and radial components of the fluid velocity are denoted by $u$ and $v$ respectively, and the temperature of the surrounding air in the thermal boundary layer is $T(x, r)$. The boundary layer equations are given below.

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0, \tag{4.1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{v}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \tag{4.2}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \kappa \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right). \tag{4.3}
\]

where $\nu$ and $\kappa$ are respectively, the kinematic viscosity and thermal diffusivity of the fluid. The boundary conditions are
\[ u = U_f, \quad v = 0, \quad T = T_f, \quad \text{at} \quad r = a, \quad (4.4) \]
\[ u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad \text{as} \quad r \to \infty, \quad (4.5) \]

where \( T_f \) is the fibre temperature and \( T_\infty \) is the temperature of the air outside the thermal boundary layer. It can be seen from (4.2) and (4.3) that the velocity and temperature satisfy similar differential equations, so the velocity and temperature profiles will possess similar forms. A logarithmic temperature profile is chosen, which is given by (4.6):

\[
\frac{T - T_\infty}{T_f - T_\infty} = 1 - \frac{1}{\alpha_f(x)} \ln \left( \frac{r}{a} \right), \quad 1 \leq \frac{r}{a} \leq e^{a_r},
\]
\[
\frac{T - T_\infty}{T_f - T_\infty} = 0, \quad \frac{r}{a} \geq e^{a_r}.
\]  

(4.6)

The velocity profile is:

\[
\frac{u}{U_f} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right), \quad 1 \leq \frac{r}{a} \leq e^a,
\]
\[
\frac{u}{U_f} = 0 \quad \frac{r}{a} \geq e^a.
\]  

(4.7)

Integrating equation (4.3) from the surface of the cylinder to infinity yields the energy integral equation for this case:

\[
\frac{d}{dx} \int_0^\infty u(T - T_\infty) r dr = -a \kappa \left( \frac{\partial T}{\partial r} \right)_{r=a}.
\]  

(4.8)

An expression for \( u \) is obtained from the first part of (4.7), \( (T - T_\infty) \) is obtained from the first equation in (4.6) and these are substituted into (4.8). Also, putting \( r = ae^z \), (4.8) becomes:

\[
\frac{d}{dx} \left( \frac{U_f a^2}{\alpha \alpha_f} \int_0^a (\alpha \alpha_r - \alpha z - \alpha_r z + z^2) e^{2z} dz \right) = \frac{\kappa}{\alpha_f(x)}.
\]  

(4.9)

Integrating the left hand side of (4.9) by parts and simplifying gives:
Differentiating (4.10) and simplifying gives

\[
\frac{d}{dx}\left( e^{2\alpha}\left( \frac{1}{\alpha} - \frac{1}{\alpha_{T}} + \frac{1}{\alpha \alpha_{T}} \right) - \left( 2 + \frac{1}{\alpha} + \frac{1}{\alpha_{T}} + \frac{1}{\alpha \alpha_{T}} \right) \right) = \frac{4\kappa}{U_{f}a^{2}a_{T}}.
\]  

(4.10)

where the term \( \frac{d\alpha}{dx} \) which occurs in the differentiation of (4.10) is eliminated by means of (4.12), obtained from Sakiadis [13], equation (3.42) in chapter 3. Also,

\[
\frac{\nu}{\kappa} = \sigma, \text{ where } \sigma \text{ is the Prandtl number.}
\]

From (3.42)

\[
\frac{dx}{d\alpha} = \frac{U_{f}a^{2}}{2\nu a_{T}} \left[ e^{2a}(\alpha-1) + \alpha + 1 \right].
\]  

(4.12)

The Nusselt number, \( Nu \), is defined as:

\[
Nu = \frac{Q}{\kappa(T_{f} - T_{w})},
\]  

(4.13)

where \( Q(x) \) is the local rate of heat transfer per unit length of the cylinder and is given by:

\[
Q(x) = -2\pi a_{T} \kappa \left( \frac{\partial T}{\partial r} \right)_{r=a}.
\]  

(4.14)

Hence from the first part of (4.6)

\[
Nu = \frac{2\pi}{\alpha_{T}}.
\]  

(4.15)

Thus to calculate the Nusselt number it is necessary to solve (4.11) for \( \alpha_{T} \). At

\[ \alpha = 0, \alpha_{T} = 0 \] and therefore the expression for \( \frac{d\alpha_{T}}{d\alpha} \) given by equation (4.11) is in
indeterminate form. To determine \( \alpha_r \) in the neighbourhood of this point it is necessary to expand (4.11) using the power series given by (4.16), i.e.

\[
\alpha_r = a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^3 + \ldots \ldots
\]  

(4.16)

Substituting (4.16) into (4.11), expanding \( e^{2\alpha} \) by means of a MacLaurin series and comparing coefficients of like powers of \( \alpha \) gives:

\[
a_1 = \frac{1}{3} \left( \frac{\sigma + 2}{\sigma} \right)
\]

(4.17)

\[
a_2 = \frac{a_1 (\sigma - 2a_1 \sigma + 1)}{3a_1 \sigma - 1}
\]

(4.18)

\[
a_3 = \frac{2a_2 \left[ (3\sigma - 2)^2 - 10 \right]}{45(3\sigma + 2)(\sigma + 1)}
\]

(4.19)

In the range \( 0 \leq \alpha \leq 0.15 \), \( \alpha_r \) was determined using the first three terms of the power series. Equation (4.11) is then forward integrated numerically, starting at \( \alpha = 0.15 \). Integration of (4.11) gives \( \alpha_r \) in terms of \( \alpha \), and Bourne and Elliston [24] use equation (3.45) to define \( \alpha \) as a function of \( \beta = \ln \left( \frac{4\nu x}{U_f a^2} \right) \). Thus it is possible to determine the Nusselt number as a function of \( \beta = \ln \left( \frac{4\nu x}{U_f a^2} \right) \). Figure 7 shows the Nusselt number for various values of \( \beta \). A Prandtl number of \( \sigma = 0.72 \) was used in the calculation, which is the Prandtl number of air.
It can be seen from figure 7 that as $\beta$ increases, i.e. as axial distance from the spinneret increases, the Nusselt number which characterises the rate at which the fibre loses heat to the surrounding air, decreases. Thus the Pohlhausen integral method can be used to determine the Nusselt number for an idealised fibre in the melt spinning process. The methods discussed in chapters 3 and 4 will now be utilised to investigate possible different experimental configurations of the fibre spinning process.
CHAPTER 5

INVESTIGATION OF SHEAR STRESS ON A FIBRE IN A COUNTERFLOW AIR STREAM WHEN THE FIBRE VELOCITY IS MUCH GREATER THAN THE FREE STREAM VELOCITY

This chapter details one of the research problems completed. The problem which is investigated uses a similar Pohlhausen method to that used by Glauert and Lighthill [12] and Sakiadis [13]. This work seeks to model the melt spinning process in which a polymer fibre issues from a spinneret at high velocity into a counterflow air stream. It is assumed in this section that the fibre velocity is much greater than that of the free stream and thus the boundary layer forms at the spinneret end of the fibre as opposed to its front end at the take up drum. This is because the fibre velocity has a stronger influence on the flow. In the limiting case where $U_\infty \to 0$, this problem reduces to that considered by Sakiadis [13], where a moving cylinder issues into a stationary fluid, which proves to be a useful check on the results. A coordinate system $(x,r)$ is fixed at the spinneret and the boundary layer builds up from $x = 0$. 

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The boundary layer equations are:

\[
ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial r} = \nu \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right),
\]

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0.
\]

The velocity profile used is a logarithmic profile which satisfies the boundary conditions for this particular problem, and is given by:

\[
\frac{u + U_\infty}{U_f + U_\infty} = \left[ 1 - \frac{1}{\alpha(x)} \ln \frac{r}{a} \right], \quad 1 \leq \frac{r}{a} \leq e^\alpha,
\]

\[
\frac{u}{U_\infty} = -1, \quad \frac{r}{a} \geq e^\alpha.
\]

This means that at \( r = a \), i.e. at the surface of the fibre, \( u = U_f \). This indicates that the velocity of the fluid along the surface of the fibre is the same as the velocity of the fibre. Thus the no-slip condition is satisfied. At \( r = ae^\alpha \), i.e. at the edge of the boundary layer, \( u = -U_\infty \) so that at the edge of the boundary layer, the fluid velocity matches that of the outer flow, with the minus sign indicating that the fluid flow is in
the opposite direction to the fibre motion. Now, putting \( \lambda = \frac{U_\infty}{U_f} \) and \( r = ae^z \), the profile inside the boundary layer given by the first part of (5.3) may be written as:

\[
u = U_f \left[ (\lambda + 1) \left( 1 - \frac{z}{\alpha(x)} \right) - \lambda \right]. \tag{5.4}\]

The Bingham number, \( Bi \), which is a non-dimensional shear stress coefficient, is defined for this case as:

\[ Bi = \frac{\tau}{\mu U_f} \]

where the shear stress \( \tau \), is given by

\[ \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \].

Using (5.4) to evaluate \( \tau \) it is found that:

\[ Bi = \frac{(\lambda + 1)}{\alpha}. \tag{5.5} \]

The inner part of the velocity profile (5.3) is now substituted into the momentum integral equation, which is: (This and subsequent momentum integral equations are derived in appendix A)

\[ \frac{d}{dx} \left[ \int_{a}^{r=a} u(u + U_\infty) 2\pi r dr \right] = -2\pi a \frac{\tau}{\rho}. \tag{5.6} \]

Integration of this equation is from the surface of the fibre at \( r = a \) to the edge of the boundary layer at \( r = ae^a \). Using the expression for \( u \) given by (5.4), the expression for \( \tau \) and putting \( r = ae^z \), the momentum integral equation (5.6) becomes, after some simplification:

\[ \frac{d}{dx} \left[ \frac{U_f^2 a^2}{\alpha^2} \int_{0}^{a} \left[ \lambda z^2 - \lambda \alpha z + \alpha^2 - 2\alpha z + z^2 \right] e^{2z} dz \right] = \frac{vU_f}{\alpha}. \tag{5.7} \]

Following integration by parts this reduces to:

\[ \alpha \frac{d}{dx} \left[ \frac{e^{2z}}{\alpha^2} \left( 1 + \lambda - \lambda \alpha \right) - 2 \frac{2}{\alpha} - \frac{1}{\alpha^2} - \frac{\lambda}{\alpha} - \frac{\lambda}{\alpha^2} \right] = \frac{4v}{U_f a^2}. \tag{5.8} \]
Differentiating and simplifying gives:

\[
\left[ \frac{(2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2)e^{2\alpha} + (2\alpha + 2 + \lambda \alpha + 2\lambda)}{\alpha^2} \right] \frac{d\alpha}{dx} = \frac{4\nu}{U_f a^2},
\]

and therefore

\[
\lim_{\xi \to 0} \int \left[ \frac{(2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2)e^{2\alpha} + (2\alpha + 2 + \lambda \alpha + 2\lambda)}{\alpha^2} \right] d\alpha = \frac{4\nu \alpha}{U_f a^2}.
\]

Equation (5.10) can be seen to agree with equation (3.43) of the Sakiadis [13] solution when \(\lambda = 0\), i.e. when \(U_\infty = 0\). Integrating (5.10) gives

\[
\left[ (2 + 3\lambda)Ei(2\alpha) - (2 + 2\lambda)\left( 2Ei(2\alpha) - \frac{e^{2\alpha}}{\alpha} \right) - \lambda e^{2\alpha} + (2 + \lambda)\ln \alpha - \frac{(2 + 2\lambda)}{\alpha} \right]_{\xi}^{\alpha} = e^{\beta},
\]

where \(Ei(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt\) and \(\beta = \ln \left( \frac{4\nu \alpha}{U_f a^2} \right)\). Asymptotically expanding the exponential integral function, \(Ei(z)\), shows that \(Ei(2\xi) \approx \gamma + \ln(2\xi)\) as \(\xi \to 0\).

Thus (5.11) may be written as:

\[
\frac{(2 + 2\lambda)}{\alpha} (e^{2\alpha} - 1) + (2 + \lambda)(\ln(2\alpha) - Ei(2\alpha) + \gamma) - \lambda e^{2\alpha} + \lambda e^{2\xi} - \frac{(2 + 2\lambda)}{\xi} (e^{2\xi} - 1) = e^{\beta}
\]

The final two terms on the left hand side of (5.12) may be expanded using the MacLaurin series expansion of \(e^{2\xi}\), such that: \(e^{2\xi} \approx 1 + 2\xi + 2\xi^2 + \ldots\) giving

\[
\frac{(2 + 2\lambda)}{\alpha} (e^{2\alpha} - 1) + (2 + \lambda)(\ln(2\alpha) - Ei(2\alpha) + \gamma) - \lambda e^{2\alpha} - 4 - 3\lambda = e^{\beta},
\]

as \(\xi \to 0\). Equation (5.13) may be solved numerically for \(\alpha\) and the Bingham number calculated using (5.5), which is: \(Bi = \left[ \frac{(\lambda + 1)}{\alpha} \right]\). Alternatively, an analytical
solution valid at large axial distances may be obtained using the following method.

Equation (5.10) is integrated by parts, including terms up to \( O\left(\frac{e^{2\alpha}}{\alpha^6}\right) \), which gives:

\[
e^{2\alpha} \left[ \frac{(2+3\lambda)}{2\alpha} - \frac{(2+\lambda)}{4\alpha^2} - \frac{(2+\lambda)}{4\alpha^3} - \frac{(6+3\lambda)}{8\alpha^4} - \frac{(12+6\lambda)}{8\alpha^5} - \frac{(30+15\lambda)}{8\alpha^6} - \lambda \right] \approx e^\beta,
\]

as \( \alpha \to \infty \). Thus

\[
2\alpha + \ln \left[ \frac{(2+3\lambda)}{2\alpha} - \frac{(2+\lambda)}{4\alpha^2} - \frac{(2+\lambda)}{4\alpha^3} - \frac{(6+3\lambda)}{8\alpha^4} - \frac{(12+6\lambda)}{8\alpha^5} - \frac{(30+15\lambda)}{8\alpha^6} - \lambda \right] \approx \beta.
\]

Using equation (5.5) yields:

\[
Bi \approx \left[ \frac{\beta - \frac{1}{2} \ln \left( \frac{(2+3\lambda)}{2\alpha} - \frac{(2+\lambda)}{4\alpha^2} - \frac{(2+\lambda)}{4\alpha^3} - \frac{(6+3\lambda)}{8\alpha^4} - \frac{(12+6\lambda)}{8\alpha^5} - \frac{(30+15\lambda)}{8\alpha^6} - \lambda \right)}{\beta - \frac{1}{2} \ln \left( \frac{(2+3\lambda)}{2\alpha} - \frac{(2+\lambda)}{4\alpha^2} - \frac{(2+\lambda)}{4\alpha^3} - \frac{(6+3\lambda)}{8\alpha^4} - \frac{(12+6\lambda)}{8\alpha^5} - \frac{(30+15\lambda)}{8\alpha^6} - \lambda \right)} \right].
\]

(5.16)

It is desirable to have the solution for the Bingham number solely in terms of the non-dimensional axial coordinate, \( \beta \), and the velocity ratio \( \lambda \). Taking \( \alpha \approx \frac{\beta}{2} \) from equation (5.15) and substituting into (5.16) gives:

\[
Bi \approx \left[ \frac{\beta - \frac{1}{2} \ln \left( \frac{(2+3\lambda)}{\beta} - \frac{(2+\lambda)}{\beta^2} - \frac{(2+\lambda)}{\beta^3} - \frac{(6+3\lambda)}{\beta^4} - \frac{4(12+6\lambda)}{\beta^5} - \frac{8(30+15\lambda)}{\beta^6} - \lambda \right)}{\beta - \frac{1}{2} \ln \left( \frac{(2+3\lambda)}{\beta} - \frac{(2+\lambda)}{\beta^2} - \frac{(2+\lambda)}{\beta^3} - \frac{(6+3\lambda)}{\beta^4} - \frac{4(12+6\lambda)}{\beta^5} - \frac{8(30+15\lambda)}{\beta^6} - \lambda \right)} \right].
\]

(5.17)

Figure 9 overleaf details the plot of the numerical solution of equation (5.13) for values of \( \lambda = \frac{1}{10} \), \( \lambda = \frac{1}{15} \), \( \lambda = \frac{1}{20} \), \( \lambda = \frac{1}{30} \), and \( \lambda = 0 \) (which agrees exactly with Sakiadis [13]), in the region \( 2 \leq \beta \leq 12 \). Also included is the plot of the analytical solution (5.17) for a value of \( \lambda = \frac{1}{10} \) in the region \( 8 \leq \beta \leq 12 \).
ANALYSIS.

Figure 9 plots the Bingham number versus $\beta = \ln \left( \frac{4x}{U_f a^2} \right)$. The effect of fixing the value of $U_f$, and fixing the value of $x$, while varying the velocity ratio $\lambda = \frac{U_\infty}{U_f}$ is now examined. It can be seen that increasing the relative velocities has only a small effect on the shear stress on the fibre surface. For example, trebling $\lambda = \frac{U_\infty}{U_f}$ from $\frac{1}{30}$ to $\frac{1}{10}$ results in an increase of just 6.4% at $\beta = 2$. At larger axial distances from the spinneret this increase in the Bingham number is even smaller and the plots converge as $\beta \to \infty$.

Also evident is that the analytical solution shows good agreement with the numerical solution at large axial distances.
The purpose of this chapter is to derive expressions for the Bingham number when the ratio of fibre to free stream velocities, $\lambda = \frac{U_f}{U_\infty}$, is small, so as to show what would happen to the shear stress on the surface of the fibre if the free stream velocity relative to the fibre velocity was increased, or if the fibre velocity relative to the free stream velocity was decreased. The velocity profile used in chapter 5 is modified, because the boundary layer forms at the take up drum in this case, and a similar method to that used in chapter 5 is used to derive expressions for the Bingham number. A coordinate system is fixed at the take up drum (see figure 10). A boundary layer builds up from $x = 0$, while the motion of the fibre tries to create a boundary layer from the spinneret that expands in the opposite direction. For $U_\infty > U_f$, i.e. $\lambda < 1$, it is reasonable to assume that the free stream velocity has a
stronger influence on the flow and that the boundary layer builds up from the take up drum. In the limit where \( U_f \to 0 \), i.e. where \( \lambda_\infty \to 0 \), the problem reduces to that considered by Glauert and Lighthill [12], where an expression for shear stress on a stationary cylinder in a moving fluid was derived. Thus the results in this section can be compared with those of Glauert and Lighthill [12], which were discussed in chapter 3.

![Modified coordinate system](image)

The velocity profile is modified accordingly and it is detailed below:

\[
\frac{u-U_\infty}{-U_f-U_\infty} = \left[ 1 - \frac{1}{\alpha(x)} \ln \frac{r}{a} \right] \quad 1 \leq \frac{r}{a} \leq e^\alpha, \\
\frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e^\alpha.
\]

(6.1)

So that now, at \( r = a \), on the surface of the fibre, \( u = -U_f \), with the minus sign indicating the fibre motion is in the opposite direction to the fluid flow, and at \( r = ae^\alpha \), i.e. at the edge of the boundary layer, \( u = U_\infty \). It can be seen that the above velocity profile reduces to the corresponding Glauert and Lighthill [12] profile given by (3.12) when \( U_f = 0 \).
Putting \( z = \ln \frac{r}{a} \) and \( \lambda = \frac{U}{U_\infty} \), the profile inside the boundary layer may be written as:

\[
 u = U_\infty \left[ 1 - \left( \lambda + 1 \right) \left( 1 - \frac{z}{\alpha(x)} \right) \right].
\]  

(6.2)

The Bingham number for this case is defined as:

\[
 Bi = \frac{\alpha \tau}{\mu U_\infty},
\]

(6.3)

where \( \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \). Using (6.2) to evaluate \( \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \) it is found that:

\[
 Bi = \frac{\left( \lambda + 1 \right)}{\alpha}
\]

(6.4)

The inner part of the velocity profile (6.1) is substituted into the momentum integral equation, which in this case: (Equation (6.5) is derived in appendix A)

\[
 \frac{d}{dx} \int_a^{ae^{-z}} u(U_\infty - u)2\pi r dr = 2\pi a \frac{\tau}{\rho}.
\]

(6.5)

Integration of this equation is from the surface of the fibre at \( r = a \) to the edge of the boundary layer at \( r = ae^{-z} \). Using the expression for \( u \) given by (6.2), the expression for \( \tau \) given by \( \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \) and letting \( r = ae^z \), the momentum integral equation (6.5) becomes, after simplification:

\[
 \frac{d}{dx} \left[ \frac{U_\infty^2 a^2}{\alpha^2} \int_0^{\alpha - z} \left( \lambda z - \lambda + z \right) e^{2z} dz \right] = \frac{vU_\infty}{\alpha}.
\]

(6.6)

Integrating by parts gives:

\[
 \alpha \frac{d}{dx} \left[ \frac{e^{2z} \left( \alpha - 1 - \lambda \right) + \alpha + 1 + \lambda + 2 \lambda \alpha + 2 \lambda \alpha^2}{\alpha^2} \right] = \frac{4v}{U_\infty a^2}.
\]

(6.7)
Differentiating with respect to $x$ and simplifying yields:

\[
\left[ \frac{(2\alpha^2 - 3\alpha + 2 - 2\lambda_\alpha\alpha + 2\lambda_\alpha\alpha)}{\alpha^2} - \frac{\alpha + 2 + 2\lambda_\alpha\alpha + 2\lambda_\alpha\alpha}{\alpha^2} \right] \frac{d\alpha}{dx} = \frac{4\nu}{U_\omega a^2}.
\]

Hence

\[
\lim_{\xi \to 0} \int_{\xi}^{\alpha} \left[ \frac{(2\alpha^2 - 3\alpha + 2 - 2\lambda_\alpha\alpha + 2\lambda_\alpha\alpha)}{\alpha^2} - \frac{\alpha + 2 + 2\lambda_\alpha\alpha + 2\lambda_\alpha\alpha}{\alpha^2} \right] d\alpha = \frac{4\nu}{U_\omega a^2}
\]

(6.8)

As expected, this can be seen to correspond equation (3.21) in the Glauert and Lighthill [12] case when $\lambda_\alpha \to 0$. Integrating (6.9) results in:

\[
\left[ e^{2\alpha} - (3 + 2\lambda_\alpha) \ln e^{2\alpha} \right] - (2 + 2\lambda_\alpha) \ln \alpha + \frac{2 + 2\lambda_\alpha}{\alpha} = \beta,
\]

(6.10)

where $\text{Ei}(z) = \int_0^z \frac{e^t}{t} dt$ and $\beta = \ln \left( \frac{4\nu}{U_\omega a^2} \right)$. The asymptotic expansion of the exponential integral function, $\text{Ei}(z)$, shows that $\text{Ei}(2\xi) \approx \gamma + \ln(2\xi)$ as $\xi \to 0$, and (6.10) becomes:

\[
e^{2\alpha} + (1 + 2\lambda_\alpha) \ln e^{2\alpha} - \frac{2 + 2\lambda_\alpha}{\alpha} (e^{2\alpha} - 1) - e^{2\xi} + \frac{2 + 2\lambda_\alpha}{\xi} (e^{2\xi} - 1) = \beta,
\]

(6.11)

The final two terms on the left hand side of (6.11) may now be expanded using the MacLaurin series expansion of $e^{2\xi}$, giving:

\[
e^{2\alpha} + (1 + 2\lambda_\alpha) \ln e^{2\alpha} - \frac{2 + 2\lambda_\alpha}{\alpha} (e^{2\alpha} - 1) + 3 + 4\lambda = e^\beta,
\]

(6.12)

as $\xi \to 0$. Equation (6.12) can be solved numerically for $\alpha$ and the Bingham number calculated using (6.3). The next part of this analysis seeks to derive an analytical solution valid at large axial distances from the leading edge of the fibre.

Integration of equation (6.9), including terms up to $O\left( \frac{e^{2\alpha}}{\alpha^6} \right)$ yields:
\[ e^{2\alpha} \left[ 1 - \frac{(3 + 2\lambda) + (1 + 2\lambda)}{2\alpha} + \frac{(1 + 2\lambda)}{4\alpha^2} + \frac{(3 + 6\lambda)}{4\alpha^3} + \frac{(6 + 12\lambda)}{8\alpha^4} + \frac{(15 + 30\lambda)}{8\alpha^5} \right] \approx e^\beta, \]

(6.13)

as \( \alpha \to \infty \). Therefore,

\[ 2\alpha + \ln \left[ 1 - \frac{(3 + 2\lambda) + (1 + 2\lambda)}{2\alpha} + \frac{(1 + 2\lambda)}{4\alpha^2} + \frac{(3 + 6\lambda)}{4\alpha^3} + \frac{(6 + 12\lambda)}{8\alpha^4} + \frac{(15 + 30\lambda)}{8\alpha^5} \right] = \beta. \]

(6.14)

The Bingham number, \( Bi \), for this case is \( Bi = \left( \frac{\lambda_{\infty} + 1}{\alpha} \right) \), and thus from (6.14):

\[ Bi \approx \left[ \frac{\beta}{2} - \frac{1}{2} \ln \left( 1 - \frac{(3 + 2\lambda)}{2\alpha} + \frac{(1 + 2\lambda)}{4\alpha^2} + \frac{(3 + 6\lambda)}{4\alpha^3} + \frac{(6 + 12\lambda)}{8\alpha^4} + \frac{(15 + 30\lambda)}{8\alpha^5} \right) \right]. \]

(6.15)

Taking \( \alpha \approx \frac{\beta}{2} \) from equation (6.14) and substituting into (6.15) gives:

\[ Bi \approx \left[ \frac{\beta}{2} - \frac{1}{2} \ln \left( 1 - \frac{(3 + 2\lambda)}{\beta} + \frac{(1 + 2\lambda)}{\beta^2} + \frac{2(1 + 2\lambda)}{\beta^3} + \frac{2(3 + 6\lambda)}{\beta^4} + \frac{4(6 + 12\lambda)}{\beta^5} + \frac{8(15 + 30\lambda)}{\beta^6} \right) \right]. \]

(6.16)

Figure 11 overleaf illustrates the plot of the numerical solution of equation (6.12) for values of \( \lambda_{\infty} = \frac{1}{2}, \lambda_{\infty} = \frac{1}{3}, \lambda_{\infty} = \frac{1}{5}, \lambda_{\infty} = \frac{1}{10}, \) and \( \lambda_{\infty} = 0 \) (which agrees with Glauert and Lighthill [12]), in the region \( 2 \leq \beta \leq 12 \). Also included is the plot of the analytical solution (6.16) for values of \( \lambda = \frac{1}{2}, \) and \( \lambda = \frac{1}{5} \) in the region \( 8 \leq \beta \leq 12 \).
ANALYSIS.

Figure 11 plots the Bingham number versus $\beta = \ln \left( \frac{4 \lambda x}{U_\infty a^2} \right)$. That is, the effect of fixing the value of $U_\infty$, and fixing the value of $x$, while varying the velocity ratio $\lambda = \frac{U_f}{U_\infty}$ is examined. It can be seen that as $\beta$ increases, that is, as axial distance along the fibre increases so that the boundary layer thickness increases, the Bingham number is reduced. As expected, the solution tends to that of the stationary cylinder given by Glauert and Lighthill's [12] solution when $\lambda \rightarrow 0$, i.e. when $U_f \rightarrow 0$.

Also evident is that as the velocity ratio $\lambda = \frac{U_f}{U_\infty}$ increases, the shear stress on the fibre surface is increased for a fixed value of $\beta$. The analytical solution is seen to give excellent agreement at large axial distances with its numerical counterpart.
CHAPTER 7

INVESTIGATION OF HEAT TRANSFER FROM A FIBRE IN A COUNTERFLOW AIRSTREAM

In this chapter the thermal boundary layer created by an idealised fibre formed in the melt spinning process is examined with a view to finding the Nusselt number, which as stated in chapter 4, quantifies the rate of heat loss from the fibre to the surrounding environment. The fibre is extruded from the spinneret into a counterflow air stream in which it solidifies. The fibre is then wound onto a take up drum. The problem is idealised by treating it as a continuous circular cylinder moving with constant velocity $U_f$, which is at a constant temperature $T_f$, issuing into a steady laminar counterflow of velocity $U_\infty$ which is at a constant temperature $T_\infty$. Section 7.1 deals with the case where $U_\infty > U_f$, and section 7.2 considers the situation where $U_f \gg U_\infty$. 

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7.1 HEAT TRANSFER THROUGH THE BOUNDARY LAYER ON A FIBRE IN A COUNTERFLOW AIR STREAM WHEN THE FREE STREAM VELOCITY IS GREATER THAN THE FIBRE VELOCITY.

In this section, the thermal boundary layer on a fibre moving in a counterflow air stream is examined. The fibre to free stream velocity ratio is assumed to be small.

The same coordinate system \((x,r)\) and velocity profile used in the analysis of surface shear stress in chapter 6 are employed, where again the origin of coordinates is at the take up drum. The temperature profile takes a similar logarithmic form to that of the velocity profile. This is because the differential equations for temperature and velocity have the same form. In the limit where \(U_f \to 0\), i.e. where \(\lambda_\infty = \frac{U_f}{U_\infty} \to 0\), the problem reduces to that considered by Bourne et al [17], where heat transfer from the surface of a stationary cylinder in a moving fluid was calculated using a similar Pohlhausen technique. The boundary layer equations for this case are given by the following.

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0, \tag{7.1}
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial r} = \frac{\nu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \tag{7.2}
\]

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial r} = \kappa \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right). \tag{7.3}
\]

where \(\nu\) is the kinematic viscosity of the fluid, and \(\kappa\) is its thermal diffusivity. The temperature profile is given by:

\[
\frac{T - T_\infty}{T_f - T_\infty} = 1 - \frac{1}{\alpha_r} \ln \frac{r}{a} \quad 1 \leq \frac{r}{a} \leq e^{\alpha_r}, \tag{7.4}
\]

\[
\frac{T - T_\infty}{T_f - T_\infty} = 0 \quad \frac{r}{a} \geq e^{\alpha_r}. \tag{7.4}
\]
and the velocity profile is:

\[ \frac{u-U_\infty}{-U_f - U_\infty} = 1 - \frac{1}{\alpha(x) \ln \left( \frac{r}{a} \right)} \quad 1 \leq \frac{r}{a} \leq e^a, \]

\[ \frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e^a. \]  

(7.5)

Which is consistent with the boundary conditions:

\[ u = -U_f, \; T = T_f, \; \text{ at } \; r = a, \]  

(7.6)

\[ u \to U_\infty, \; T \to T_\infty, \; \text{ as } \; r \to \infty. \]  

(7.7)

The energy integral equation given by (7.8) for this case has two parts on the left hand side.

\[ \frac{d}{dx} \int_a^r u(T-T_\infty)2\pi r dr + \frac{d}{dx} \int_{ae^a}^r \left[ U_\infty(T-T_\infty)2\pi r dr \right] = -2\pi \alpha \kappa \left( \frac{\partial T}{\partial r} \right)_{r=a}. \]  

(7.8)

This is because the thermal boundary layer is thicker than the momentum boundary layer, which is the case for air under normal conditions [4] [5] [6] [27]. Therefore, heat transfer occurs from the fibre surface across the momentum boundary layer, in which the velocity of the fluid is governed by the first equation in (7.5), and also from the outer edge of the momentum boundary layer to the edge of the thermal boundary layer. In this region the velocity of the fluid is the constant free stream velocity given by the second part of (7.5). The first term in the left hand side of (7.8) is obtained by integrating the left hand side of (7.3) from the surface of the cylinder at \( r = a \), to the edge of the momentum boundary layer at \( r = ae^a \), with velocity \( u \) that obeys the logarithmic velocity profile given by the first part of (7.5). The second term on the left hand side is obtained by integrating (7.3) from the edge of the momentum boundary layer at \( r = ae^a \), to the edge of the thermal boundary layer at \( r = ae^{2a} \). This region of the flow moves with velocity \( U_\infty \). This derivation can be
seen in Appendix A. Substituting an expression for \( u \) obtained from the first expression in (7.5) and expressions for \((T - T_s)\) and \(\left(\frac{\partial T}{\partial r}\right)_{r=a}\) from the first part of (7.4) into (7.8) yields, after some simplification:

\[
\frac{d}{dx} U_\alpha a^2 \int_0^\alpha \left[1 - \left(\lambda_\alpha + 1\right) \left(1 - \frac{z}{\alpha}\right) \right] \left[1 - \frac{z}{\alpha_T}\right] e^z \, dz + \frac{d}{dx} U_\alpha a^2 \int_\alpha^\infty \left[1 - \frac{z}{\alpha_T}\right] e^z \, dz = \frac{\kappa}{\alpha_T}.
\]

(7.9)

where the substitution \( r = \alpha e^z \) has again been used. This gives after further manipulation:

\[
\frac{d}{dx} \left[ \frac{U_\alpha a^2}{\alpha \alpha_T} \int_0^\alpha \left(\lambda_\alpha \alpha z - \lambda_\alpha \alpha_T z - \lambda_\alpha \alpha_T z^2 + \alpha_T z - z^2\right) e^z \, dz \right]
+ \frac{d}{dx} \left[ \frac{U_\alpha a^2}{\alpha \alpha_T} \left((\alpha \alpha_T - \alpha) e^z \, dz \right) \right] = \frac{\kappa}{\alpha_T}.
\]

(7.10)

Integrating this expression by parts gives:

\[
\frac{d}{dx} \left[ \left(\lambda_\alpha + 1\right) \left(\frac{1}{\alpha_T} - \frac{1}{\alpha \alpha_T}\right) \right] e^{2a} + \left(\frac{1}{\alpha_T}\right) e^{2a_T}
+ \left(2\lambda_\alpha + \frac{\lambda_\alpha}{\alpha_T} + \frac{\lambda_\alpha}{\alpha_T} + \frac{1}{\alpha} + \frac{1}{\alpha \alpha_T}\right) = \frac{4\kappa}{U_\alpha a^2 \alpha_T}
\]

(7.11)

Equation (7.11) is now differentiated with respect to \( x \) giving:

\[
\frac{d \alpha_T}{dx} \left[ \left(\lambda_\alpha + 1\right) \left(\frac{1}{\alpha \alpha_T} - \frac{1}{\alpha_T^2}\right) e^{2a} - \left(\frac{1}{\alpha_T} - \frac{2}{\alpha_T^2}\right) e^{2a_T} - \frac{\lambda_\alpha}{\alpha_T^2} - \frac{\lambda_\alpha}{\alpha_T^2} - \frac{1}{\alpha \alpha_T^2}\right]
+ \frac{d \alpha}{dx} \left[ \left(\lambda_\alpha + 1\right) \left(\frac{2}{\alpha_T} + \frac{1}{\alpha^2} - \frac{2}{\alpha \alpha_T} - \frac{2}{\alpha \alpha_T}\right) e^{2a} - \left(\lambda_\alpha + 1\right) \left(\frac{1}{\alpha^2} + \frac{1}{\alpha^2 \alpha_T}\right) \right]
= \frac{4\kappa}{U_\alpha a^2 \alpha_T}.
\]

(7.12)
Now, to get this equation in terms of $\frac{d\alpha_T}{d\alpha}$ and to eliminate the term which occurs in (7.12) an expression for $\frac{dx}{d\alpha}$ must be found. This is obtained from equation (6.8), which is:

$$
\left[ \frac{(2\alpha^2 - 3\alpha + 2 - 2\lambda_\infty \alpha - 2\lambda_\infty) e^{2\alpha}}{\alpha^2} - (\alpha + 2 + 2\lambda_\infty \alpha + 2\lambda_\infty) \right] \frac{d\alpha}{dx} = \frac{4\nu}{U_\infty a^2}.
$$

Thus

$$
\frac{dx}{d\alpha} = \frac{U_\infty a^2}{4\nu a^2} \left[ e^{2\alpha} (2\alpha^2 - 3\alpha + 2 - 2\lambda_\infty \alpha + 2\lambda_\infty) - (2 + \alpha + 2\lambda_\infty + 2\lambda_\infty \alpha) \right].
$$

(7.13)

Multiplying (7.12) by $\frac{dx}{d\alpha}$ and simplifying results in:

$$
\frac{d\alpha_T}{d\alpha} \left[ (\lambda_\infty + 1)(1 - \alpha) e^{2\alpha} + (2\alpha \alpha_T - \alpha) e^{2\alpha_T} - \lambda_\infty \alpha - \lambda_\infty - 1 \right] + \frac{\alpha_T}{\alpha} \left[ (\lambda_\infty + 1)(2\alpha^2 + \alpha_T - 2\alpha \alpha_T + 1 - 2\alpha) e^{2\alpha_T} - (\lambda_\infty + 1)(\alpha_T + 1) \right] = \frac{\alpha_T}{\alpha_T} \left[ e^{2\alpha} (2\alpha^2 - 3\alpha + 2 - 2\lambda_\infty \alpha + 2\lambda_\infty) - 2 - \alpha - 2\lambda_\infty - 2\lambda_\infty \alpha \right],
$$

(7.14)

where the Prandtl number is given by $\sigma = \frac{\nu}{\kappa}$. The Nusselt number, $Nu$, is defined as

$$
Nu = \frac{Q(x)}{\kappa(T_f - T_\infty)}.
$$

(7.15)

where $Q(x)$ is the local rate of heat transfer per unit length of the cylinder and is given by: $Q(x) = -2\pi \alpha \kappa \left( \frac{\partial T}{\partial r} \right)_{r=a}$. Hence:

$$
Nu = \frac{2\pi}{\alpha_T}.
$$

(7.16)

So, solution of equation (7.14) for $\alpha_T$ will allow the Nusselt number to be found. In solving (7.14) it should be noted that the condition of $\alpha_T = 0$ when $\alpha = 0$ exists. As
with equation (4.11) in chapter 4, the expression for $\frac{da_T}{d\alpha}$ given by equation (7.14) is in indeterminate form at the point $\alpha_T = 0, \alpha = 0$. To determine $\alpha_T$ in the neighbourhood of this point it is necessary to expand (7.14) using the following series:

$$\alpha_T \approx a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^3 + \ldots$$  \hspace{1cm} (7.17)

$$e^{2\alpha} \approx 1 + 2\alpha + 2\alpha^2 + \frac{4\alpha^3}{3} + \frac{2\alpha^4}{3} + \ldots$$  \hspace{1cm} (7.18)

$$e^{2\alpha_T} \approx 1 + 2\alpha_T + 2\alpha_T^2 + \frac{4\alpha_T^3}{3} + \frac{2\alpha_T^4}{3} + \ldots$$  \hspace{1cm} (7.19)

Expanding (7.14) and comparing coefficients of $\alpha^0$, $\alpha^1$ and $\alpha^2$, the constants $a_1$, $a_2$ and $a_3$ are found for a given value of fibre to free stream velocity ratio $\lambda_\infty = \frac{U_f}{U_\infty}$. (This process is detailed in the accompanying disc). Using the constants $a_1$, $a_2$ and $a_3$, the initial value of $\alpha_T$ is calculated for $\alpha = 0.15$, and used as the starting point for the integration of (7.14).

Having obtained the starting points for the integration of (7.14), under the condition outlined above, the equation was solved in Maple 11 using a Fehlberg fourth-fifth order Runge–Kutta method. The solution output gives a value of $\alpha_T$ for inputted values of $\alpha(x)$. For the range $2 \leq \beta \leq 12$ the appropriate values of $\alpha(x)$ are determined from equation (6.12). The Nusselt number, $Nu = \frac{2\pi}{\alpha_T}$, is then plotted versus the non-dimensional axial coordinate $\beta = \ln \left( \frac{4\ell x}{U_\infty \alpha^2} \right)$.
Figure 12 shows the plot of the Nusselt number, \( Nu \), versus \( \beta = \ln \left( \frac{4\alpha}{U_\infty a^2} \right) \), for values of fibre to free stream velocity ratios of \( \lambda_\infty = \frac{1}{2} \), \( \lambda_\infty = \frac{1}{3} \), \( \lambda_\infty = \frac{1}{5} \), \( \lambda_\infty = \frac{1}{10} \), and \( \lambda_\infty = 0 \). The plot of the solution for \( \lambda_\infty = 0 \) is seen to agree exactly with the solution given by Bourne et al [17], for heat transfer from a stationary cylinder.

\[ \text{Plot of } Nu \text{ vs } \beta \]

**ANALYSIS.**
Looking at figure 12 it can be seen that the effect on heat transfer due to a non-zero fibre velocity is very small, with the graphs of the various solutions being virtually indistinguishable from each other. However, from the data it can be seen that for a fixed value of \( \beta \), heat transfer decreases marginally with increasing fibre velocity. This increase is most pronounced at smaller values of \( \beta \), i.e. close to the take up drum. Also apparent is that as \( \beta \to \infty \), \( Nu \to 0 \). This is expected behaviour because with increasing axial distance from the take up drum where the boundary
layer is formed, the boundary layer thickness increases. Thus the magnitude of 

\[ \left( \frac{\partial T}{\partial r} \right)_{r=a} \]

decreases. Remembering that the local rate of heat transfer per unit length of the fibre is given by \( Q(x) = -2\pi \alpha k \left( \frac{\partial T}{\partial r} \right)_{r=a} \), it is evident that heat transfer will be reduced for increasing \( \beta \).

7.2 HEAT TRANSFER THROUGH THE BOUNDARY LAYER ON A FIBRE IN A COUNTERFLOW AIR STREAM WHEN THE FIBRE VELOCITY IS MUCH GREATER THAN THE FREE STREAM VELOCITY.

In this section the heat transfer from a fibre is examined when the fibre velocity is much greater than that of the free stream. In this case, the boundary layer is seen to form at the spinneret. In chapter 5, the shear stress on a fibre in a counterflow air stream at large velocity ratios was examined, and the same velocity profile is used to represent the flow. In the limit where \( U_\infty \to 0 \), i.e. where \( \lambda = \frac{U_\infty}{U_f} \to 0 \), this problem reduces to that considered by Bourne and Elliston [24], where heat transfer from the surface of a fibre moving into a stationary fluid was examined. As in section 7.1, the temperature of the fluid \( T(x,r) \) is given by:

\[
\frac{T - T_\infty}{T_f - T_\infty} = 1 - \frac{1}{\alpha_r} \ln \frac{r}{a}, \quad 1 \leq \frac{r}{a} \leq e^{\alpha_r},
\]

\[
\frac{T - T_\infty}{T_f - T_\infty} = 0, \quad \frac{r}{a} \geq e^{\alpha_r}.
\]  

(7.20)
The velocity profile is:

\[
\frac{u + U_\infty}{U_f + U_\infty} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right) \quad 1 \leq \frac{r}{a} \leq e^\alpha, \\
\frac{u}{U_\infty} = -1 \quad \frac{r}{a} \geq e^\alpha. 
\] (7.21)

Thus the boundary conditions are:

\[
u = U_f, \quad T = T_f, \quad \text{at} \quad r = a, \quad (7.22) \\
u \to -U_\infty, \quad T \to T_\infty, \quad \text{as} \quad r \to \infty. \quad (7.23)
\]

As in section 7.1, the left hand side of energy integral equation (7.24) consists of two terms, because the thermal boundary layer is thicker than the momentum boundary layer. It is:

\[
\frac{d}{dx} \int_0^{\alpha r} [u(T - T_\infty)] 2\pi r dr - \frac{d}{dx} \int_0^{\alpha r} [U_\infty(T - T_\infty)] 2\pi r dr = -2\pi \alpha k \left( \frac{\partial T}{\partial r} \right)_{r=a}. \quad (7.24)
\]

Substituting an expression for \(u\) inside the boundary layer obtained from the first expression in (7.21) and expressions for \((T - T_\infty)\) and \(\frac{\partial T}{\partial r}\) from the first part of (7.20) into (7.24) gives, after simplification:

\[
\frac{d}{dx} U_f a^2 \int_0^{\alpha r} \left[ (\lambda + 1) \left(1 - \frac{z}{\alpha} \right) - \lambda \left[ 1 - \frac{z}{\alpha_T} \right] e^{z^2} dz - \frac{d}{dx} \lambda U_f a^2 \int_0^{\alpha r} \left[ 1 - \frac{z}{\alpha_T} \right] e^{z^2} dz = \frac{k}{\alpha_T}. \right. \quad (7.25)
\]

where the substitution \(r = ae^z\) is made. Further manipulation of (7.25) leads to:

\[
\frac{d}{dx} \left[ \frac{U_f a^2 \alpha_T}{\alpha \alpha_T} \int_0^{\alpha r} \left( \lambda \alpha_z - \lambda \alpha_T z + \alpha \alpha_T - \alpha z + z^2 \right) e^{z^2} dz \right] \\
- \frac{d}{dx} \left[ \frac{U_f a^2 \alpha_T}{\alpha \alpha_T} \int_0^{\alpha r} \left( \lambda \alpha_T - \lambda \alpha z \right) e^{z^2} dz \right] = \frac{k}{\alpha_T}. \quad (7.26)
\]
Integrating (7.26) by parts gives:

\[
\frac{d}{dx} \left[ \left( \frac{1}{\alpha} - \frac{1}{\alpha + \alpha_r} \right) \right] e^{2\alpha} - \left( \frac{\alpha_r}{\alpha} \right) e^{2\alpha_r} \\
- \left( 2 + \frac{1}{\alpha + \alpha_r} + \frac{\lambda}{\alpha} + \frac{1}{\alpha + \alpha_r} + \frac{\lambda}{\alpha_r} \right) = \frac{4\kappa}{U_f a^2 \alpha_r} 
\]  

(7.27)

Differentiating with respect to \( x \) results in:

\[
\frac{d\alpha_r}{dx} \left[ \left( \frac{1}{\alpha_r^2} - \frac{1}{\alpha_r + \alpha} \right) e^{2\alpha} - \left( \frac{\alpha_r}{\alpha} \right) e^{2\alpha_r} + \frac{1}{\alpha_r^2} + \frac{1}{\alpha_r^2 + \alpha} \right] \\
+ \frac{d\alpha}{dx} \left[ \left( \frac{2}{\alpha} - \frac{2}{\alpha_r} - \frac{1}{\alpha_r^2} - \frac{1}{\alpha^2} \right) e^{2\alpha} + (\lambda + 1) \left( \frac{1}{\alpha_r^2} + \frac{1}{\alpha_r^2 + \alpha} \right) \right] \\
= \frac{4\kappa}{U_f a^2 \alpha_r}. 
\]

(7.28)

The expression \( \frac{dx}{d\alpha} \) required to get this equation in terms of \( \frac{d\alpha_r}{d\alpha} \) and to eliminate the \( \frac{d\alpha}{dx} \) term which is implicit in (7.28) is obtained from equation (5.9), which is:

\[
\left[ \left( 2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2 \right) e^{2\alpha} + \left( 2\alpha + 2 + \lambda \alpha + 2\lambda \right) \right] \frac{d\alpha}{dx} = \frac{4\nu}{U_f a^2}. 
\]

Therefore:

\[
\frac{dx}{d\alpha} = \frac{U_f a^2}{4\nu \alpha^2} \left[ e^{2\alpha} \left( 2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2 \right) + \left( 2\alpha + 2 + \lambda \alpha + 2\lambda \right) \right]. 
\]

(7.29)

Multiplying (7.29) by \( \frac{d\alpha_r}{d\alpha} \) and simplifying results in:

\[
\frac{d\alpha_r}{d\alpha} \left[ \left( \alpha + 1 \right) (\alpha_r - 1) e^{2\alpha} - \left( 2\lambda \alpha + \lambda \right) e^{2\alpha} + \alpha + 1 + \lambda \right] \\
+ \frac{\alpha_r}{\alpha} \left[ \left( \alpha + 1 \right) (2\alpha + \alpha_r - 2\alpha^2 + 2\alpha - \alpha_r - 1) e^{2\alpha} + \left( \alpha + 1 \right) (\alpha_r + 1) \right] \\
= \frac{\alpha_r}{\sigma \alpha} \left[ e^{2\alpha} \left( 2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2 \right) + \left( 2\alpha + 2 + \lambda \alpha + 2\lambda \right) \right]. 
\]

(7.30)
where the Prandtl number is given by $\sigma = \frac{\nu}{\kappa}$. Equation (7.30) can be seen to agree with Bourne and Elliston [24], when $\lambda = \frac{U_m}{U_f} = 0$. In section 7.1 the Nusselt number, $Nu$, was shown to be given by $Nu = \frac{2\pi}{\alpha T}$, so solution of (7.30) for $\alpha T$ will yield the rate of heat transfer from the fibre surface. Like equation (7.14), the condition $\alpha T = 0$ when $\alpha = 0$ exists, and (7.30) is expanded using the series (7.17), (7.18) and (7.19) to find the initial conditions under which to solve the differential equation. In the range $2 \leq \beta \leq 12$ the appropriate values of $\alpha(x)$ are calculated using equation (5.13) and used in the attached computer program. Figure 13 shows the plot of the Nusselt number versus $\beta = \ln \left( \frac{4\sqrt{x}}{U_f a^2} \right)$.

![Plot of $Nu$ vs $\beta$](image)

Figure 13
ANALYSIS.

Examining figure 13 it can be seen that the introduction of a non-zero free stream velocity results in a marginal reduction in heat transfer from the surface of a fibre. Also of interest is that the larger the value of \( \lambda \) examined, it is found that the solution breaks down at smaller axial distances from the spinneret. The error report from various maple solvers states that the problem is probably singular at the point of breakdown. So increasing \( \lambda = \frac{U_\infty}{U_f} \), that is, increasing \( U_\infty \) (for a fixed value of \( U_f \)) induces the breakdown of the solution. Recall equation (7.24)

\[
\frac{d}{dx} \int_a u(T - T_\infty) 2\pi rdr - \frac{d}{dx} \int_a U_\infty (T - T_\infty) 2\pi rdr = -a\kappa \left( \frac{\partial T}{\partial r} \right)_{r=a}.
\]

(7.31)

It can be seen that the second term on the left hand side is a negative quantity. This is due to the boundary conditions of the problem, and the magnitude of this term increases with increasing \( U_\infty \). It was decided to examine whether at the point of breakdown

\[
\frac{d}{dx} \int_a u(T - T_\infty) 2\pi rdr = \frac{d}{dx} \int_a U_\infty (T - T_\infty) 2\pi rdr \quad \text{and} \quad -a\kappa \left( \frac{\partial T}{\partial r} \right)_{r=a} = 0.
\]

From equation (7.15), \( Nu = \frac{Q(x)}{\kappa(T_f - T_\infty)} \), where \( Q(x) \) is the local rate of heat transfer per unit length of the cylinder and is given by: \( Q(x) = -2\pi\kappa \left( \frac{\partial T}{\partial r} \right)_{r=a} \). Thus when

\[
\left( \frac{\partial T}{\partial r} \right)_{r=a} = 0, \quad Nu = \frac{Q(x)}{\kappa(T_f - T_\infty)} = 0,
\]

and the solution breaks down. Putting

\[
\phi_1 = \frac{d}{dx} \int_a u(T - T_\infty) 2\pi rdr \quad \text{and} \quad \phi_2 = \frac{d}{dx} \int_a U_\infty (T - T_\infty) 2\pi rdr,
\]

it can be seen that for
this hypothesis to be validated the ratio of $\frac{\phi_2}{\phi_1}$ must tend to unity approaching the point at which the solution breaks down. $\phi_1$ and $\phi_2$ may be written as:

$$
\phi_1 = \left\{ \frac{d\alpha_r}{d\alpha} \left[ (2\lambda \alpha - \lambda + \alpha - 1 - 2\lambda \alpha^2) e^{2\alpha} + \alpha + 1 + \lambda \right] + \frac{\alpha_r}{\alpha} \left[ ((\lambda + 1)(2\alpha \alpha_r - 2\alpha^2 + 2\alpha - \alpha_r - 1) - 4\lambda \alpha^2 \alpha_r + 4\lambda \alpha^3) e^{2\alpha} + (\lambda + 1)(\alpha_r + 1) \right] \right\},
$$

(7.32)

and

$$
\phi_2 = \left\{ \frac{d\alpha_r}{d\alpha} \left[ (\lambda \alpha - 2\lambda \alpha^2) e^{2\alpha} - (\lambda \alpha - 2\lambda \alpha_r) e^{2\alpha_r} \right] + \frac{\alpha_r}{\alpha} \left[ (4\lambda \alpha^3 - 4\lambda \alpha^2 \alpha_r) e^{2\alpha_r} \right] \right\},
$$

(7.33)

Equation (7.30) may be solved for $\frac{d\alpha_r}{d\alpha}$ explicitly in terms of $\alpha$, $\alpha_r$, $\lambda$, and $\sigma$, giving:

$$
\frac{d\alpha_r}{d\alpha} = \frac{\alpha_r^{\alpha} \left[ e^{2\alpha} (2\alpha - 2 + 3\lambda \alpha - 2\lambda - 2\lambda \alpha^2) + (2\alpha + 2 + \lambda \alpha + 2\lambda) \right]}{\left[ (\lambda + 1)(\alpha - 1)e^{2\alpha} - (2\lambda \alpha \alpha_r - \lambda \alpha) e^{2\alpha_r} + \alpha + 1 + \lambda \right] - \frac{\alpha_r}{\alpha} \left[ ((\lambda + 1)(2\alpha \alpha_r - 2\alpha^2 + 2\alpha - \alpha_r - 1) e^{2\alpha} + (\lambda + 1)(\alpha_r + 1) \right] \left[ (\lambda + 1)(\alpha - 1)e^{2\alpha} - (2\lambda \alpha \alpha_r - \lambda \alpha) e^{2\alpha_r} + \alpha + 1 + \lambda \right]}.
$$

(7.34)

This expression for $\frac{d\alpha_r}{d\alpha}$ can now be used in equations (7.32) and (7.33). The numerical values of $\alpha_r$ (for a given value of $\alpha$) returned by the Maple solver are now utilised to investigate whether $\frac{\phi_2}{\phi_1} \rightarrow 1$ when the solution breaks down. Figure 14 details the results for values of $\lambda = \frac{U_s}{U_f} = \frac{1}{20}$ and $\lambda = \frac{U_s}{U_f} = \frac{1}{30}$.
The graph clearly shows that \( \frac{\phi_2}{\phi_1} \to 1 \) at the point of breakdown of the solution of (7.30). The spikes above the critical value of 1 can be attributed to numerical anomalies in the Maple solver. So when the solution breaks down \( \left( \frac{\partial T}{\partial r} \right)_{r=a} \to 0 \).

Solving the velocity profile (7.4) for \( T \), differentiating with respect to \( r \), and then solving for \( \alpha_T \) it can be seen that \( \alpha_T = \frac{-\left(T_r - T_a\right)}{a} \left( \frac{\partial T}{\partial r} \right)_{r=a}^{-1} \). This indicates that \( \alpha_T \to \infty \) as \( \left( \frac{\partial T}{\partial r} \right)_{r=a} \to 0 \). \( \alpha_T \) is essentially a measure of the thermal boundary layer thickness, and at the point of breakdown, it becomes very large. However, the assumptions used to derive the boundary layer equations state that the boundary layer must be sufficiently thin for the equations to be valid. Clearly this is not the case here and the solution breaks down.
CHAPTER 8

INVESTIGATION OF THE EFFECTS OF COMPRESSIBILITY ON THE BINGHAM NUMBER

In this chapter the effects of compressibility on shear stress on the surface of a stationary fibre immersed in a laminar fluid flow, and also a moving fibre in a counterflow free stream are investigated. The temperature of the fibre, $T_f$, is assumed to be considerably larger than the temperature of the surrounding environment, $T_\infty$, and this temperature difference gives rise to large variations in the density and viscosity of the surrounding fluid. In effect the fluid is compressible. It is noted that the density of a fluid is inversely proportional to its temperature, and Crocco’s relation, which defines the relationship between a fluid’s temperature and velocity, is used to obtain an expression for $\rho$, the density of the fluid. Crocco’s relation is valid for fluids with a Prandtl number of unity, and this is a good approximation for air [27]. A Pohlhausen method is used to evaluate the Bingham number at large axial distances from the leading edge of the fibre. The results are
compared to the corresponding incompressible results obtained by Glauert and Lighthill [12], who obtained a solution for surface shear stress on a stationary fibre immersed in a laminar fluid flow, and also the analytical results from chapter 6 of this thesis, which details the corresponding incompressible counterflow problem.

8.1 INVESTIGATION OF THE EFFECTS OF A COMPRESSIBLE FLUID FLOW ON SURFACE SHEAR STRESS ON A STATIONARY FIBRE.

Figure 15 below shows the coordinate system for the stationary fibre.

\[ \text{Figure 15} \]

The compressible boundary layer equations for a fluid of Prandtl number of unity are given by:

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial r} (\rho v) = 0, \]  
\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mu \frac{\partial u}{\partial r} \right), \]  
\[ \rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mu \frac{\partial T}{\partial r} \right) + \frac{\mu}{C_p} \left( \frac{\partial u}{\partial r} \right)^2, \]

where \( C_p \) is specific heat capacity for an isobaric process. The boundary conditions are:

\[ u = 0 \text{ on } r = a, \quad u \to U_\infty \text{ as } r \to \infty, \]
\[ T = T_f \text{ on } r = a, \quad T \to T_\infty \text{ as } r \to \infty. \]

(8.4)
As for the corresponding incompressible case dealt with in section 3.3, the velocity profile for this case is given by:

\[
\frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right) \quad 1 \leq \frac{r}{a} \leq e\alpha, \\
\frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e\alpha. 
\]  

Integration of (8.1) and (8.2) over a normal section of the flow yields the compressible momentum integral equation (see appendix A for derivation), which is:

\[
\frac{d}{dx} \int_a^{ae\alpha} \rho u (U_\infty - u) 2\pi r dr = 2\pi \alpha \mu_f \left( \frac{\partial u}{\partial r} \right)_{r=a}, 
\]

where \( \mu_f \) is the fluid viscosity at the fibre surface. Noting that \( \frac{\rho}{\rho_f} = \frac{T_f}{T} \) applies in this case it is seen that \( \rho = \frac{\rho_f T_f}{T} \), where \( \rho_f \) and \( T_f \) are respectively, the density and temperature of the fluid at the fibre surface. An expression for \( T \) is obtained from Crocco’s relation shown below. Crocco’s relation defines the relationship between a fluid’s velocity and temperature, and it is:

\[
C_p T = A - \frac{1}{2} u^2. 
\]

where \( C_p \) is the specific heat capacity of air at constant pressure.

Thus

\[
T = \frac{1}{C_p} \left( A - \frac{1}{2} u^2 \right). 
\]

(A proof of the validity of Crocco’s relation is contained in Appendix B). Applying the boundary conditions to Crocco’s relation the constant \( A \) is obtained, so that:
On \( r = a \),
\[
C_p T_f = A - \frac{1}{2} (0)^2,
\]
giving \( A = C_p T_f \),
and as \( r \to \infty \),
\[
C_p T_\infty = A - \frac{1}{2} U_\infty^2.
\]
Inserting an expression for \( \rho \) obtained from \( \rho = \frac{\rho_f T_f}{T} \) into (8.6) gives:
\[
\frac{d}{dx} \left[ \int_a^r \frac{\rho_f T_f (U_\infty - u) r dr}{C_p \left( A - \frac{1}{2} u^2 \right)} \right] = a \mu_f \left( \frac{\partial u}{\partial r} \right)_{r=a}. \tag{8.10}
\]
Now, in this analysis of the effects of a compressible flow, it is convenient to make the substitution \( \tilde{u} = \frac{u}{U_\infty} \), thus \( u = \tilde{u} U_\infty \). From the boundary conditions (8.4), on the fibre surface \( u = 0 \), so that \( \tilde{u} = 0 \) on \( r = a \). At the edge of the boundary layer, \( u = U_\infty \), so that \( \tilde{u} = 1 \) on \( r = a e^x \). Now, \( \tilde{u} = \frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right) \). Putting \( R = \frac{r}{a} \) gives:
\[
\tilde{u} = \frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln(R), \quad \text{and thus} \quad R = e^{a \tilde{u}}. \tag{8.11}
\]
Thus \( r dr = a^2 \alpha e^{a \tilde{u}} d \tilde{u} \) and (8.10) becomes:
\[
\frac{d}{dx} \left[ \int_0^{1/\alpha(x)} \frac{\tilde{u} (1 - \tilde{u}) \alpha e^{a \tilde{u}} d \tilde{u}}{C_p \left( A - \frac{1}{2} U_\infty^2 \tilde{u}^2 \right)} \right] = U_\infty \mu_f \left( \frac{\partial \tilde{u}}{\partial R} \right)_{\tilde{u}=0}. \tag{8.12}
\]
From (8.11) \( \tilde{u} = \frac{1}{\alpha(x)} \ln(R) \), and \( U_\infty \mu_f \left( \frac{\partial \tilde{u}}{\partial R} \right)_{\tilde{u}=0} = \frac{\mu_f U_\infty}{\alpha} \). Substituting into (8.12) gives, after some simplification:
\[ \alpha \frac{d}{dx} \left[ \int_0^1 \alpha \epsilon^{2\alpha \tilde{u} - \alpha \epsilon^{2\alpha \tilde{u}}} \tilde{u} \, d \tilde{u} \right] = \frac{\nu_f}{U_\infty \alpha^2 C_p T_f}. \] (8.13)

where \( \nu_f = \frac{\mu_f}{\rho_f} \), and \( \nu_f \) is the kinematic viscosity of the fluid at the fibre surface.

Putting \( f(\tilde{u}) = \frac{\tilde{u}(1-\tilde{u})}{A - \frac{1}{2} U_\infty^2 \tilde{u}^2} \), (8.13) reduces to:

\[ \alpha \frac{d}{dx} \left[ \int_0^1 f(\tilde{u}) \alpha \epsilon^{2\alpha \tilde{u}} \tilde{u} \, d \tilde{u} \right] = \frac{\nu_f}{U_\infty \alpha^2 C_p T_f}. \] (8.14)

Equation (8.14) is now integrated by parts, giving:

\[ \left\{ f(\tilde{u}) \alpha \epsilon^{2\alpha \tilde{u}} \tilde{u} \, d \tilde{u} \approx \left[ \left( f(\tilde{u}) \epsilon^{2\alpha \tilde{u}} \right)^{\infty} \right]_0 \right\} \]

as \( \alpha \to \infty \). The suffix "1" notates the first derivative of \( f(\tilde{u}) \) with respect to \( \tilde{u} \), and so on. Terms \( O \left( \frac{\epsilon^{2\alpha \tilde{u}}}{\alpha^7} \right) \) have been neglected. Applying the limits yields:

\[ \int_0^1 f(\tilde{u}) \alpha \epsilon^{2\alpha \tilde{u}} \tilde{u} \, d \tilde{u} \approx \left[ \left( f(\tilde{u}) \epsilon^{2\alpha \tilde{u}} \right)^{\infty} \right]_0 \]

where non-exponential terms, which are small at large values of \( \alpha \), have been disregarded. This expansion is valid for large values of \( \alpha \), but it is also necessary to
see the behaviour of \( \int_0^1 f(u) \alpha e^{2\alpha u} u \, du \) for small axial distances. Let \( t = \alpha \tilde{u} \). Then

\[
\tilde{u} = \frac{t}{\alpha} \quad \text{and} \quad d\tilde{u} = \frac{dt}{\alpha}. \quad f\left(\frac{t}{\alpha}\right) e^{2\alpha \tilde{u}} d\tilde{u} \text{ now becomes:}
\]

\[
f\left(\frac{t}{\alpha}\right) e^{2\alpha t} dt = \frac{(\alpha t - t^2) e^{2\alpha t} dt}{A \alpha^2 - \frac{1}{2} U_\infty^2 t^2}.
\]

(8.17)

For \( 0 < \alpha < 1 \) terms with a coefficient of \( \alpha \) may be neglected and equation (8.17) reduces to:

\[
f\left(\frac{t}{\alpha}\right) e^{2\alpha t} dt \approx \frac{2 e^{2\alpha t} dt}{U_\infty^2}.
\]

(8.18)

Now, given that \( t = \alpha \tilde{u} \), the condition \( \tilde{u} = 0 \) becomes \( t = 0 \), and the condition \( \tilde{u} = 1 \) becomes \( t = \alpha \). Therefore for small axial distances:

\[
\int_0^1 f\left(\frac{t}{\alpha}\right) e^{2\alpha t} dt \approx \frac{2 e^{2\alpha t} dt}{U_\infty^2} \approx \frac{\left(e^{2\alpha} - 1\right)}{U_\infty^2}.
\]

(8.19)

Equation (8.14) now becomes:

\[
\alpha \frac{d}{dx} \left(\frac{4(e^{2\alpha} - 1)}{U_\infty^2}\right) + \alpha \frac{d}{dx} \left(\left[\left(2 f(t) - \frac{f_1(t)}{\alpha} + \frac{f_2(t)}{2\alpha^2} - \frac{f_3(t)}{4\alpha^4} + \frac{f_4(t)}{8\alpha^4} - \frac{f_5(t)}{16\alpha^5} + \frac{f_6(t)}{32\alpha^6}\right) e^{2\alpha}\right]\right)
\]

\[
\approx \frac{4\nu_f}{U_\infty \alpha^2 C_p T_f}.
\]

(8.20)

Differentiating (8.20) with respect to \( x \) gives:

\[
\left[ \frac{8 \alpha e^{2\alpha}}{U_\infty^2} \right] \frac{d\alpha}{dx} + e^{2\alpha} \left[ 4 \alpha f(t) - 2 f_1(t) + \frac{1}{\alpha} (f_2(t) + f_2(t)) - \frac{1}{\alpha^2} \left(f_3(t) + f_3(t)\right) \right]
\]

\[
+ \frac{1}{\alpha^3} \left( \frac{3 f_4(t)}{4} + \frac{f_5(t)}{4}\right) - \frac{1}{\alpha^4} \left( \frac{f_4(t)}{2} + \frac{f_5(t)}{8}\right) + \frac{1}{\alpha^5} \left( \frac{5 f_4(t)}{16} + \frac{f_5(t)}{16}\right) - \frac{1}{\alpha^6} \left( \frac{3 f_6(t)}{16}\right) \cdot \frac{d\alpha}{dx}
\]

\[
\approx \frac{4\nu_f}{U_\infty \alpha^2 C_p T_f},
\]

(8.21)

and:

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\[
\int_0^1 \left[ \frac{8 \alpha e^{2\alpha}}{U_w^2} \right] d\alpha + \int_1^\alpha e^{2\alpha} \left[ 4af(t) - 2f_1(t) + \frac{1}{\alpha} (f_1(t) + f_2(t)) - \frac{1}{\alpha^2} \left( f_2(t) + \frac{f_3(t)}{2} \right) \right] d\alpha \\
+ \frac{1}{\alpha^3} \left( \frac{3f_1(t) + f_3(t)}{4} \right) - \frac{1}{\alpha^4} \left( \frac{f_4(t)}{2} + \frac{f_5(t)}{8} \right) + \frac{1}{\alpha^5} \left( \frac{5f_6(t)}{16} + \frac{f_7(t)}{16} \right) - \frac{1}{\alpha^6} \left( \frac{3f_8(t)}{16} \right) \right] d\alpha \\
\approx \frac{4v_x x}{U_w a^2 C_p T_f},
\]

(8.22)

where the first integral represents the behaviour of the flow at small axial distances.

Integration of (8.22), including terms up to \( O\left( \frac{e^{2\alpha}}{\alpha^6} \right) \), results in:

\[
e^{2\alpha} \left[ 2af(t) - f(t) - f_1(t) + \frac{1}{2\alpha} [f_1(t) + f_2(t)] \right] \\
+ \frac{1}{4\alpha^2} [f_1(t) - f_2(t) - f_3(t)] \\
+ \frac{1}{8\alpha^3} [2f_1(t) - 2f_2(t) + f_3(t) + f_4(t)] \\
+ \frac{1}{16\alpha^4} [6f_1(t) - 6f_2(t) + 3f_3(t) - f_4(t) - f_5(t)] \\
+ \frac{1}{32\alpha^5} [24f_1(t) - 24f_2(t) + 12f_3(t) - 4f_4(t) + f_5(t) + f_6(t)] \\
+ \frac{1}{64\alpha^6} [120f_1(t) - 120f_2(t) + 60f_3(t) - 20f_4(t) + 5f_5(t) - f_6(t)] \right] \\
\approx \frac{4v_x x}{U_w a^2 C_p T_f},
\]

(8.23)

where non-exponential terms and terms \( O(e^2) \) have been neglected. Defining the sums of the derivatives above as:

\[
s_1 = [f_1(t) + f_2(t)], \\
s_2 = [f_1(t) - f_2(t) - f_3(t)], \\
s_3 = [2f_1(t) - 2f_2(t) + f_3(t) + f_4(t)], \\
s_4 = [6f_1(t) - 6f_2(t) + 3f_3(t) - f_4(t) - f_5(t)], \\
s_5 = [24f_1(t) - 24f_2(t) + 12f_3(t) - 4f_4(t) + f_5(t) + f_6(t)], \\
s_6 = [120f_1(t) - 120f_2(t) + 60f_3(t) - 20f_4(t) + 5f_5(t) - f_6(t)].
\]

(8.24)
equation (8.23) may be written as:
\[ e^{2\alpha} \left[ C_p T_\infty \left( 2\alpha \overline{f(t)} - f_1(t) - f_1(1) + \frac{s_1}{2\alpha} + \frac{s_2}{4\alpha^2} + \frac{s_3}{8\alpha^3} + \frac{s_4}{16\alpha^4} + \frac{s_5}{32\alpha^5} + \frac{s_6}{64\alpha^6} \right) \right] \approx \frac{4v_f x T_\infty}{U_\infty a^2 T_f}. \] (8.25)

Now, the Bingham number for this compressible analysis is defined as: 
\[ Bi = \left| \frac{\mu \tau}{U_f U_\infty} \right|, \]
where \( \tau = \frac{U_\infty \mu_f}{a} \left( \frac{\partial u}{\partial R} \right) \bigg|_{R=0} \). (The shear stress, \( \tau \), is measured at the surface of the fibre, and at the fibre surface \( \mu = \mu_f \)). Evaluating \( \tau = \frac{U_\infty \mu_f}{a} \left( \frac{\partial u}{\partial R} \right) \bigg|_{R=0} \) from (8.11), it is seen that as in the incompressible case, the Bingham number, \( Bi = \left| \frac{1}{\alpha} \right| \). So the solution of (8.25) for \( \frac{1}{\alpha} \) will yield an expression for shear stress on the fibre surface.

From (8.25):
\[ 2\alpha + \ln \left[ C_p T_\infty \left( 2\alpha \overline{f(t)} - f_1(t) - f_1(1) + \frac{s_1}{2\alpha} + \frac{s_2}{4\alpha^2} + \frac{s_3}{8\alpha^3} + \frac{s_4}{16\alpha^4} + \frac{s_5}{32\alpha^5} + \frac{s_6}{64\alpha^6} \right) \right] \approx \beta^*. \] (8.26)

where \( \beta^* = \ln \left( \frac{4v_f x T_\infty}{U_\infty a^2 T_f} \right) \). The kinematic viscosity at the surface, \( \nu_f \), may be written in terms of the ambient kinematic viscosity outside the boundary layer, \( \nu \). So \( \nu_f = n \nu \), where the value of the constant \( n \) is determined by the relative temperature, and consequently, relative viscosity of the fluid at the surface and outside the boundary layer respectively. For example, for \( T_f = 398K \),
\[ \nu_f = 2.578 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}. \] For \( T_\infty = 293K \), \( \nu = 1.511 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}. \) Thus \( n = 1.706. \)
Now, \[ \beta^* = \ln \left( \frac{4V_f \cdot x T_{\infty}}{U_{\infty} a^2 T_f} \right) = \ln \left( \frac{4n \cdot V x}{U_{\infty} a^2} \right) = \ln \left( \frac{4n x}{U_{\infty} a^2} \right) + \ln(n) - \ln \left( \frac{T_f}{T_{\infty}} \right), \]
gives:
\[ \beta^* = \beta + \ln(n) - \ln \left( \frac{T_f}{T_{\infty}} \right). \] (8.27)

Solving (8.26) for \( \frac{1}{\alpha} \) yields:

\[ Bi \approx \left( \frac{\beta^*}{2} - \frac{1}{2} \ln \left[ C_{p} T_{\infty} \left( \frac{2 \alpha f(l) - f(t_f) - f_1(l)}{2a} + \sum_{n=1}^{6} \frac{s_n}{8^n \alpha^n} \right) \right] \right)^{-1}. \] (8.28)

Taking \( \alpha \approx \frac{\beta^*}{2} \) gives:

\[ Bi \approx \left( \frac{\beta^*}{2} - \frac{1}{2} \ln \left[ C_{p} T_{\infty} \left( \frac{2 \alpha f(l) - f(t_f) - f_1(l)}{2a} + \sum_{n=1}^{6} \frac{s_n}{8^n \alpha^n} \right) \right] \right)^{-1}. \] (8.29)

Equation (8.27) allows the solution for the Bingham number given by (8.29) to be expressed in terms of the non dimensional axial coordinate \( \beta \).

Finally, the derivatives of \( f(\tilde{u}) \) must be evaluated. Differentiating \( f(\tilde{u}) \) and applying the upper limit of \( \tilde{u} = 1 \) gives:
so that, \( m^2 = \frac{u^2}{\gamma_g \left( \frac{C_T}{C_p} \right) C_p T} \) and therefore \( m^2 = \frac{u^2}{\gamma_g \left( 1 - \frac{1}{\gamma_g} \right) C_p T} \) \( (8.32) \)

Now, in the free stream \( u = U_\infty \), and \( T = T_\infty \), thus:

\[
m^2 = \frac{U_\infty^2}{2 C_p T_\infty} \frac{1}{\gamma_g - 1}.
\] \( (8.33) \)

For air under normal conditions \( \gamma_g = \frac{7}{5} \), which leads to:

\[
m^2 = \frac{5}{2} \frac{U_\infty^2}{C_p T_\infty} \quad \text{and so} \quad \frac{U_\infty^2}{C_p T_\infty} = \frac{2m^2}{5}.
\] \( (8.34) \)

Equation (8.30) now reduces to:

\[
f(\theta) = 0,
\]

\[
f_1(\theta) = -\frac{1}{C_p T_\infty},
\]

\[
f_2(\theta) = -\frac{2}{C_p T_\infty} \left( 1 + \frac{2m^2}{5} \right),
\]

\[
f_3(\theta) = -\frac{6}{C_p T_\infty} \left( \frac{3m^2}{5} + \frac{4m^4}{25} \right),
\]

\[
f_4(\theta) = -\frac{24}{C_p T_\infty} \left( \frac{m^2}{5} + \frac{8m^4}{25} + \frac{8m^6}{125} \right),
\]

\[
f_5(\theta) = -\frac{120}{C_p T_\infty} \left( \frac{5m^4}{25} + \frac{20m^6}{125} + \frac{16m^8}{625} \right),
\]

\[
f_6(\theta) = -\frac{720}{C_p T_\infty} \left( \frac{m^4}{25} + \frac{18m^6}{125} + \frac{48m^8}{625} + \frac{32m^{10}}{3125} \right).
\] \( (8.35) \)

Using equations (8.34) and (8.35) all of the terms in equation (8.29) may be evaluated for a given temperature \( T \) and Mach number \( m \), at a given axial distance \( \beta \). Figure 16 below shows a plot of the solution in the region \( 8 \leq \beta \leq 12 \) for
\[ T_f = 398K, \quad T_f = 373K \quad \text{and} \quad T_f = 348K \] with \( U_\infty = 10ms^{-1} \), and also shown is the incompressible solution.

![Plot of Bi vs \( \beta \)](image)

**ANALYSIS.**

Figure 16 demonstrates that the effects of a compressible flow reduce the shear stress on the surface of the fibre by a small amount. For example, the maximum reduction is seen when \( T_f = 398K \) at \( \beta = 8 \). Here the shear stress is reduced by approximately 4.2%. It is clear also that the compressible solutions tend towards the incompressible case as \( \beta \to \infty \), and for a fixed value of \( \beta \), the compressible solutions tend towards their incompressible counterpart as \( T_f \to T_\infty \).
8.2 INVESTIGATION OF THE EFFECTS OF COMPRESSIBILITY ON SHEAR STRESS ON THE SURFACE OF A FIBRE IN A COUNTERFLOW AIRSTREAM.

In chapter 6 an expression for the shear stress on the surface of a fibre in a counterflow air stream was derived. The fibre to free stream velocity ratio, \( \lambda = \frac{U_f}{U_\infty} \), was assumed to be small, and the boundary layer was seen to form at the take up drum. In this section an expression for shear stress on the fibre surface far from the take up drum is derived, when the flow is compressible. The flow is compressible because, as was the case in the last section, the fibre temperature is much greater than that of the free stream.

![Figure 17](image)

The same velocity profile as was used in chapter 6, the corresponding incompressible case is chosen, and is given by:

\[
\frac{u - U_\infty}{-U_f - U_\infty} = \left[ 1 - \frac{1}{\alpha(x)} \ln \frac{r}{a} \right], \quad 1 \leq \frac{r}{a} \leq e^\alpha,
\]

\[
\frac{u}{U_\infty} = 1, \quad \frac{r}{a} \geq e^\alpha.
\]

so that at \( r = a \), i.e. at the surface of the fibre, \( u = -U_f \), and at \( r = ae^\alpha \), the outer
edge of the boundary layer \( u = U_{\infty} \). The momentum integral equation is the same as in section (8.1), namely:

\[
\frac{d}{dx} \int_{r}^{e^{e^{a}}} \rho \mu (U_{\infty} - u) 2 \pi r dr = 2 \pi a u_{f} \left( \frac{\partial u}{\partial r} \right)_{r=a}. \tag{8.37}
\]

As in the previous case, the condition \( \rho = \frac{\rho_{f} T_{f}}{T} \) applies, and by Crocco's relation,

\[
T = \frac{1}{C_{p}} \left( A - \frac{1}{2} u_{f}^2 \right). \tag{8.38}
\]

So (8.37) becomes:

\[
\frac{d}{dx} \left[ \int_{a}^{e^{e^{a}}} \rho_{f} T_{f} (U_{\infty} - u) r dr \right] \left( \frac{1}{C_{p}} \left( A - \frac{1}{2} u_{f}^2 \right) \right) = a u_{f} \left( \frac{\partial u}{\partial r} \right)_{r=a}. \tag{8.39}
\]

The boundary conditions are:

- on \( r = a \), \( u = -U_{f} \) and \( T = T_{f} \),

- as \( r \to \infty \), \( u = U_{\infty} \) and \( T = T_{\infty} \). \tag{8.40}

From the boundary conditions, \( u = -U_{f} \) on the fibre surface at \( r = a \). Putting

\[
\bar{u} = \frac{u - U_{\infty}}{-U_{f} - U_{\infty}} = \left[ 1 - \frac{1}{\alpha(x)} \ln \frac{r}{a} \right] \text{ the condition } r = a \text{ yields } \bar{u} = 1. \text{ At the edge of the boundary layer, } u = U_{\infty}, \text{ and the condition } r = a e^{a} \text{ yields } \bar{u} = 0. \text{ Putting } R = \frac{r}{a} \text{ into the inner condition in (8.36) gives:}
\]

\[
\bar{u} = \left[ 1 - \frac{1}{\alpha(x)} \ln (R) \right] \text{ and thus } R = e^{\frac{a-\bar{u}}{a}}. \tag{8.40}
\]
Now, \( r = aR \) i.e. \( r = ae^{(a-a\tilde{u})} \). Thus \( rdr = -a^2 \alpha e^{2(a-a\tilde{u})} d\tilde{u} \) and therefore

\[
u = U_\infty \left[ 1 - (\lambda_\infty + 1)u \right], \text{ where } \lambda_\infty = \frac{U_f}{U_\infty}\.
\]

Hence equation (8.38) now becomes, after some simplification:

\[
\begin{align*}
-\alpha \frac{d}{dx} \left[ \int_0^1 \left( \frac{\tilde{u} \left( 1 - \tilde{u}[\lambda_\infty + 1] \right) \alpha e^{2(a-a\tilde{u})} d\tilde{u}}{\left( A - \frac{1}{2} U_\infty^2 \left[ 1 - 2\tilde{u}(\lambda_\infty + 1) + \tilde{u}^2 (\lambda_\infty + 1)^2 \right] \right)} \right] &= \frac{\nu_f}{U_\infty a^2 C_p T_f},
\end{align*}
\]

(8.41)

Letting

\[
f(u) = \frac{\tilde{u} \left( 1 - \tilde{u}[\lambda_\infty + 1] \right)}{\left( A - \frac{1}{2} U_\infty^2 \left[ 1 - 2\tilde{u}(\lambda_\infty + 1) + \tilde{u}^2 (\lambda_\infty + 1)^2 \right] \right)},
\]

(8.42)

(8.41) reduces to:

\[
-\alpha \frac{d}{dx} \left[ \int_0^1 f(\tilde{u}) \alpha e^{2(a-a\tilde{u})} d\tilde{u} \right] = \frac{\nu_f}{U_\infty a^2 C_p T_f}.
\]

(8.43)

Equation (8.43) may be written as:

\[
\alpha \frac{d}{dx} \left[ \int_0^1 f(\tilde{u}) \alpha e^{2(a-a\tilde{u})} d\tilde{u} \right] = \frac{\nu_f}{U_\infty a^2 C_p T_f}.
\]

(8.44)

Integrating \( f(\tilde{u}) \alpha e^{2(a-a\tilde{u})} \) by parts yields:

\[
\left( f(\tilde{u}) - \frac{f_1(\tilde{u})}{4\alpha} - \frac{f_2(\tilde{u})}{8\alpha^2} - \frac{f_3(\tilde{u})}{16\alpha^3} - \frac{f_4(\tilde{u})}{32\alpha^4} - \frac{f_5(\tilde{u})}{64\alpha^5} - \frac{f_6(\tilde{u})}{128\alpha^6} \right) e^{2(a-a\tilde{u})} \right)_0^1
\]

(8.45)
as $\alpha \to \infty$. Again the suffix "1" notates the first derivative with respect to $\ddot{u}$ and so

on. Terms $O\left(\frac{e^{2(a-a\ddot{u})}}{\alpha^7}\right)$ have been neglected. Applying the limits yields:

$$\int_0^1 f(\ddot{u}) e^{2(a-a\ddot{u})} d\ddot{u} \approx \left[\left(\frac{f(0)}{2} + \frac{f_1(0)}{4\alpha} + \frac{f_2(0)}{8\alpha^2} + \frac{f_3(0)}{16\alpha^3} + \frac{f_4(0)}{32\alpha^4} + \frac{f_5(0)}{64\alpha^5} + \frac{f_6(0)}{128\alpha^6}\right)e^{2\alpha}\right]$$

(8.46)

where non-exponential terms, which are small at large values of $\alpha$ have been neglected. This expansion is valid for large values of $\alpha$, but as in section 8.1, it is also necessary to see the behaviour of $\int_0^1 f(\ddot{u}) e^{2(a-a\ddot{u})}$ for small axial distances.

The substitution $t = (\alpha - \alpha \ddot{u})$ is made. Then $\ddot{u} = \left(1 - \frac{t}{\alpha}\right)$ and $d\ddot{u} = -\frac{dt}{\alpha}$. $f(\ddot{u})$

now becomes:

$$f\left(1 - \frac{t}{\alpha}\right) = \frac{-\left(\alpha^2 - \alpha t\right) - (\xi_\alpha + 1)(\alpha - t)^2}{A\alpha^2 - \frac{1}{2} U_{\infty}^2 \left[\lambda_\alpha^2 \alpha^2 - 2\lambda_\alpha \alpha t - 2\lambda_\alpha^2 \alpha t^2 + \lambda_\alpha^2 t^2 + 2\lambda_\alpha \alpha t + t^2\right]}$$

(8.47)

Again for $0 < \alpha < 1$ terms with a coefficient of $\alpha$ may be neglected giving:

$$f(\ddot{u}) e^{2(a-a\ddot{u})} \approx \frac{-2e^{2t}}{U_{\infty}^2 (\xi_\alpha + 1)} dt.$$

(8.48)

Now, as $t = (\alpha - \alpha \ddot{u})$, the condition $\ddot{u} = 0$ becomes $t = \alpha$, and the condition $\ddot{u} = 1$ becomes $t = 0$. Therefore for small axial distances:

$$\int_0^1 f(\ddot{u}) e^{2(a-a\ddot{u})} d\ddot{u} \approx \int_0^\alpha \frac{2e^{2t}}{U_{\infty}^2 (\xi_\alpha + 1)} dt \approx \frac{(e^{2\alpha} - 1)}{U_{\infty}^2 (\xi_\alpha + 1)}.$$

(8.49)

Equation (8.44) now becomes:
Differentiating (8.50) with respect to $x$ leads to:

\[
\alpha \frac{d}{dx} \left( \frac{4(e^{2\alpha} - 1)}{U^2 \left( \lambda_x + 1 \right)} \right) + \alpha \frac{d}{dx} \left( \left[ \frac{2 f_i(0)}{\alpha} + \frac{f_i(0)}{2\alpha^2} + \frac{f_i(0)}{4\alpha^3} + \frac{f_i(0)}{8\alpha^4} + \frac{f_i(0)}{16\alpha^5} + \frac{f_i(0)}{32\alpha^6} \right] e^{2\alpha} \right) \approx \frac{4\nu_f}{U_x a^2 C_p T_f},
\]

(8.50)

and therefore,

\[
\int_0^\alpha \left[ \frac{8\alpha e^{2\alpha}}{U_x^2 \left( \lambda_x + 1 \right)} \right] d\alpha + \int e^{2\alpha} \left[ 4\alpha f(0) + 2 f_i(0) + \frac{1}{\alpha} (f_i(0) - f_i(0)) + \frac{1}{\alpha^2} \left( \frac{f_i(0)}{2} - f_i(0) \right) \right]
\]

\[
+ \frac{1}{\alpha^3} \left( \frac{f_i(0)}{4} - \frac{3 f_i(0)}{4} \right) + \frac{1}{\alpha^4} \left( \frac{f_i(0)}{8} - \frac{f_i(0)}{2} \right) + \frac{1}{\alpha^5} \left( \frac{f_i(0)}{16} - \frac{5 f_i(0)}{16} \right) - \frac{1}{\alpha^6} \left( \frac{3 f_i(0)}{16} \right) \right] d\alpha
\]

\[
\approx \frac{4\nu_f x}{U_x a^2 C_p T_f},
\]

(8.51)

where the first integral represents the behaviour of the flow at small axial distances.

Integration of (8.52), including terms up to $O\left( \frac{e^{2\alpha}}{\alpha^5} \right)$, results in:
\[ e^{2\alpha} \left[ 2\alpha f(0) - f(0) + f_1(0) + \frac{1}{2\alpha} \left[ f_2(0) - f_1(0) \right] \right. \]
\[ + \frac{1}{4\alpha^2} \left[ f_3(0) - f_2(0) - f_1(0) \right] \]
\[ + \frac{1}{8\alpha^3} \left[ f_4(0) - f_3(0) - 2f_2(0) - 2f_1(0) \right] \]
\[ + \frac{1}{16\alpha^4} \left[ f_5(0) - f_4(0) - 3f_3(0) - 6f_2(0) - 6f_1(0) \right] \]
\[ + \frac{1}{32\alpha^5} \left[ f_6(0) - f_5(0) - 4f_4(0) - 12f_3(0) - 24f_2(0) - 24f_1(0) \right] \]
\[ + \frac{1}{64\alpha^6} \left[ - f_6(0) - 5f_5(0) - 20f_4(0) - 60f_3(0) - 120f_2(0) - 120f_1(0) \right] \]
\[ \approx \frac{4V_f x}{U_w a^2 C_p T_f}, \] (8.53)

where non-exponential terms and terms \( O(e^2) \) have been neglected. Defining the differences of the derivatives above as:

\[ d1 = [f_2(0) - f_1(0)], \]
\[ d2 = [f_3(0) - f_2(0) - f_1(0)], \]
\[ d3 = [f_4(0) - f_3(0) - 2f_2(0) - 2f_1(0)], \]
\[ d4 = [f_5(0) - f_4(0) - 3f_3(0) - 6f_2(0) - 6f_1(0)], \]
\[ d5 = [f_6(0) - f_5(0) - 4f_4(0) - 12f_3(0) - 24f_2(0) - 24f_1(0)], \]
\[ d6 = [- f_6(0) - 5f_5(0) - 20f_4(0) - 60f_3(0) - 120f_2(0) - 120f_1(0)], \] (8.54)

equation (8.53) becomes:

\[ e^{2\alpha} \left[ C_p T_w \left( 2\alpha f(0) - f(0) + f_1(0) + \frac{d1}{2\alpha} + \frac{d2}{4\alpha^2} + \frac{d3}{8\alpha^3} + \frac{d4}{16\alpha^4} + \frac{d5}{32\alpha^5} + \frac{d6}{64\alpha^6} \right) \right] \approx \frac{4V_f x T_w}{U_w a^2 T_f}. \] (8.55)

Taking the natural logarithm of both sides gives:

\[ 2\alpha + \ln \left[ C_p T_w \left( 2\alpha f(0) - f(0) + f_1(0) + \frac{d1}{2\alpha} + \frac{d2}{4\alpha^2} + \frac{d3}{8\alpha^3} + \frac{d4}{16\alpha^4} + \frac{d5}{32\alpha^5} + \frac{d6}{64\alpha^6} \right) \right] \approx \beta^*, \] (8.56)
where again $\beta^* = \ln \left( \frac{4V_f x T_\infty}{U_\infty a^2 T_f} \right)$. $\beta^*$ is related to the non-dimensional axial coordinate $\beta = \ln \left( \frac{4x}{U_\infty a^2} \right)$ by equation (8.27) which is, $\beta^* = \beta + \ln(n) - \ln \left( \frac{T_f}{T_\infty} \right)$, where $n = \frac{V_f}{V}$. As in the incompressible case, $Bi = \left( \frac{\lambda_\infty + 1}{\alpha} \right)$, so solution of (8.62) yields:

$$Bi \approx (\lambda_\infty + 1) \left( \frac{\beta^*}{2} - \frac{1}{2} \ln \left[ C_p T_\infty \left( 2\alpha f(0) - f(0) + f_1(0) \right) \right] \right)^{-1}. \quad (8.57)$$

Taking $\alpha = \frac{\beta^*}{2}$ gives:

$$Bi \approx (\lambda_\infty + 1) \left( \frac{\beta^*}{2} - \frac{1}{2} \ln \left[ C_p T_\infty \left( 2\alpha f(0) - f(0) + f_1(0) \right) \right] \right)^{-1}. \quad (8.58)$$

Finally the derivatives of $f\left( u \right)$ are evaluated. Differentiating $f\left( u \right)$ and applying the limit of $\tilde{u} = 0$ gives:
\[ f(0) = 0, \]
\[ f_1(0) = \frac{1}{C_p T_\infty}, \]
\[ f_2(0) = -\frac{2(\lambda \infty + 1) U_\infty^2}{C_p^2 T_\infty^2} - \frac{2(\lambda \infty + 1)}{C_p T_\infty}, \]
\[ f_3(0) = \frac{6(\lambda \infty + 1)^2 U_\infty^4}{C_p^3 T_\infty^3} + \frac{9(\lambda \infty + 1)^3 U_\infty^2}{C_p^2 T_\infty^2}, \]
\[ f_4(0) = \frac{24(\lambda \infty + 1)^3 U_\infty^6}{C_p^4 T_\infty^4} - \frac{48(\lambda \infty + 1)^2 U_\infty^4}{C_p^3 T_\infty^3} - \frac{12(\lambda \infty + 1)^3 U_\infty^2}{C_p^2 T_\infty^2}, \]
\[ f_5(0) = \frac{120(\lambda \infty + 1)^4 U_\infty^8}{C_p^5 T_\infty^5} + \frac{300(\lambda \infty + 1)^4 U_\infty^6}{C_p^4 T_\infty^4} + \frac{150(\lambda \infty + 1)^4 U_\infty^4}{C_p^3 T_\infty^3}, \]
\[ f_6(0) = -\frac{720(\lambda \infty + 1)^5 U_\infty^{10}}{C_p^6 T_\infty^6} - \frac{2160(\lambda \infty + 1)^5 U_\infty^8}{C_p^5 T_\infty^5} - \frac{1620(\lambda \infty + 1)^5 U_\infty^6}{C_p^4 T_\infty^4} - \frac{180(\lambda \infty + 1)^5 U_\infty^4}{C_p^3 T_\infty^3}. \]

where from Crocco’s relation (8.7) and the outer boundary condition given by equation (8.39), \( C_p T_\infty = A - \frac{1}{2} U_\infty^2 \). Equation (8.59) may now be written as:

\[ f(1) = 0, \]
\[ f_1(1) = \frac{1}{C_p T_\infty}, \]
\[ f_2(1) = -\frac{2(\lambda + 1)}{C_p T_\infty} \left( 1 + \frac{2m^2}{5} \right), \]
\[ f_3(1) = \frac{6(\lambda + 1)^2}{C_p T_\infty} \left( \frac{3m^2}{5} + \frac{4m^4}{25} \right), \]
\[ f_4(1) = -\frac{24(\lambda + 1)^3}{C_p T_\infty} \left( \frac{m^2}{5} + \frac{8m^4}{25} + \frac{8m^6}{125} \right), \]
\[ f_5(1) = \frac{120(\lambda + 1)^4}{C_p T_\infty} \left( \frac{5m^4}{25} + \frac{20m^6}{125} + \frac{16m^8}{625} \right), \]
\[ f_6(1) = -\frac{720(\lambda + 1)^5}{C_p T_\infty} \left( \frac{m^4}{25} + \frac{18m^6}{125} + \frac{48m^8}{625} + \frac{32m^{10}}{3125} \right). \]
where \( \frac{U_\infty^2}{C_p T_\infty} = \frac{2m^2}{5} \). Using equations (8.54) and (8.60) all terms in equation (8.58) can be evaluated. Figure 18 illustrates the solution with \( T_f = 398K \) and \( T_f = 373K \) with \( U_\infty = 10ms^{-1} \) for various values of \( \lambda_\infty \). Also shown are the incompressible solutions.

**ANALYSIS.**

Examining figure 18 it can be seen that as in the case of the stationary fibre discussed in section 8.1, increasing the fibre temperature \( T_f \) reduces the shear stress on the fibre surface. Again this effect is seen to be small. For example, at \( \beta = 8 \) and \( T_f = 398K \) with \( \lambda_\infty = \frac{1}{2} \), the maximum reduction in shear stress due to compressible effects is just 3.7%. As expected, for a fixed value of \( \beta \), increasing the fibre to free stream velocity ratio \( \lambda_\infty \), increases the shear stress on the fibre surface. The solutions are also seen to tend to the incompressible case as \( T_f \to T_\infty \).
CONCLUSION

This thesis has investigated shear stress on, and heat transfer from the surface of a fibre moving in a counterflow air stream in the melt spinning process. It has also investigated the effects of a compressible flow on shear stress on a stationary fibre in a moving fluid and a fibre in a counterflow air stream. The method of solution used is an approximate Pohlhausen method, which is valid along the full length of the fibre for the treatment of shear stress and heat transfer. The Pohlhausen method was shown to be accurate to within a few percent of an exact asymptotic series solution to the same problem, and therefore a reliable and efficient means of solving the boundary layer equations.

It was demonstrated that in the case where the fibre velocity is much greater than that of the free stream, increasing the velocity ratio $\lambda = \frac{U_\infty}{U_f}$ increases the shear stress on the fibre surface. This is due to the fact that the greater the relative velocity between the fibre and the free stream, the thinner the boundary layer generated by the flow over the fibre becomes. The shear stress on the fibre, $\tau$, is given by $\mu \left( \frac{\partial u}{\partial r} \right)_{r=a}$ and...
clearly as the boundary layer thickness is reduced, the greater the magnitude of the velocity gradient $\left( \frac{\partial u}{\partial r} \right)_{r=a}$. The shear stress on the fibre surface was seen to decrease with increasing axial distance from the spinneret, where the boundary layer is formed.

In the case where the free stream velocity is greater than that of the fibre, the boundary layer is formed at the take up drum, and the greater the relative velocity between the fibre and the free stream, the greater the shear stress on the fibre surface.

In the treatment of heat transfer it was found that the value of the velocity ratio made very little difference to the thermal energy transferred from the surface. The Nusselt number was seen to decrease with increasing axial distance from the take up drum in the case where $U_\infty > U_f$. The boundary layer is formed at the take up drum in this case and it increases in thickness with increasing axial distance. Thus the temperature gradient and consequently, the heat transfer from the fibre surface are reduced.

When $U_f >> U_\infty$ the boundary layer forms at the spinneret and heat transfer was seen to decrease with increasing axial distance from this point. The solution in this case was seen to break down under certain conditions. The thermal boundary layer thickness was shown to be extremely large when this occurred, but the assumptions used to derive the boundary layer equations state that the boundary layer must be sufficiently thin for the equations to be valid. Hence the solution breaks down.
The effects of a compressible flow were examined and this showed that increasing the fibre's temperature reduced shear stress on the surface. This reduction was seen to be very small, with the compressible solutions differing from their incompressible counterparts by just a few percent. Also demonstrated was that considering the effect of a compressible flow greatly increases the complexity of a given problem.
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APPENDIX A

In this appendix the momentum and energy integral equations utilised in this thesis are derived.

DERIVATION OF EQUATION (5.6).

Chapter 5 investigates shear stress on an idealised fibre in a counterflow air stream when $U_f >> U_{\infty}$. The momentum integral equation is derived from the boundary layer equations, which are:

\[
ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial r} = v \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \tag{A.1}
\]

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0. \tag{A.2}
\]

Now, using equation (A.2), (A.1) may be written as:

\[
\frac{\partial}{\partial x} ru (u + U_\infty) + \frac{\partial}{\partial r} rv (u + U_\infty) = v \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right). \tag{A.3}
\]

The velocity profile used to describe the fluid motion close to the fibre surface is:
\[
\frac{u + U_\infty}{U_f + U_\infty} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right)
\]
\[
1 \leq \frac{r}{a} \leq e^a,
\]
\[
\frac{u}{U_\infty} = -1
\]
\[
\frac{r}{a} \geq e^a.
\]

So that on \( r = a \), i.e. at the surface of the fibre, \( u = U_f \). At \( r = ae^a \), the edge of the boundary layer \( u = -U_\infty \). Integrating (A.3) from the fibre surface to the edge of the boundary layer gives:

\[
\int_a^{ae^a} \frac{\partial}{\partial x} u(u + U_\infty) r dr + \int_a^{ae^a} \frac{\partial}{\partial r} u(u + U_\infty) r dr = \nu \int_a^{ae^a} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) dr,
\]

thus

\[
\int_a^{ae^a} \frac{\partial}{\partial x} u(u + U_\infty) r dr + [\nu(u + U_\infty)]_a^{ae^a} = \nu \left( r \frac{\partial u}{\partial r} \right)_{ae^a}.
\]

Now, on \( r = a \), the radial component of the velocity, \( v = 0 \), and at \( r = ae^a \), the edge of the boundary layer \( u = -U_\infty \). So, \( [\nu(u + U_\infty)]_a^{ae^a} = 0 \) and (A.6) reduces to:

\[
\int_a^{ae^a} \frac{\partial}{\partial x} u(u + U_\infty) r dr = \nu \left( r \frac{\partial u}{\partial r} \right)_{ae^a}.
\]

Applying the limits to the right hand side of (A.7) gives

\[
\int_a^{ae^a} \frac{\partial}{\partial x} u(u + U_\infty) r dr = -\nu \left( r \frac{\partial u}{\partial r} \right)_{r=ae^a},
\]

as at \( r = ae^a \), \( u = -U_\infty \) and \( \nu \left( r \frac{\partial u}{\partial r} \right)_{ae^a} = 0 - \nu \left( r \frac{\partial u}{\partial r} \right)_{r=ae^a} \). Now, \( \nu = \frac{\mu}{\rho} \), so

\[
\int_a^{ae^a} \frac{\partial}{\partial x} u(u + U_\infty) r dr = -\frac{\mu}{\rho} \left( r \frac{\partial u}{\partial r} \right)_{r=ae^a}.
\]

Shear stress, \( \tau \), is defined as \( \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=ae^a} \), and (A.9) becomes:
\[ \int_a^{ae^a} \frac{\partial}{\partial x} u(u+U_\infty) dr = -a \frac{\tau}{\rho}. \] (A.10)

Now, by Leibniz's rule,

\[ \int_a^{ae^a} \frac{\partial}{\partial x} u(u+U_\infty) dr = \frac{d}{dx} \left[ u(u+U_\infty) \right]_a^{ae^a} - \left[ ru(u+U_\infty) \right]_{ae^a}^{ae^a} \frac{\partial}{\partial \alpha} (ae^a) \]

\[ + \left[ ru(u+U_\infty) \right]_{ae^a}^{ae^a} \frac{\partial}{\partial \alpha} (a). \] (A.11)

At \( r = ae^a, \ u = -U_\infty \) and \( \left[ ru(u+U_\infty) \right]_{ae^a}^{ae^a} \frac{\partial}{\partial \alpha} (ae^a) = 0 \). Also, \( \frac{\partial}{\partial \alpha} (a) = 0 \) and

\[ \left[ ru(u+U_\infty) \right]_{ae^a}^{ae^a} \frac{\partial}{\partial \alpha} (a) = 0. \] Thus

\[ \int_a^{ae^a} \frac{\partial}{\partial x} u(u+U_\infty) dr = \frac{d}{dx} \left[ u(u+U_\infty) \right]_a^{ae^a} \]

may be written as:

\[ \frac{d}{dx} \int_a^{ae^a} u(u+U_\infty) 2\pi r dr = -2\pi a \frac{\tau}{\rho}. \] (A.12)

This is the momentum integral equation (5.6).

**DERIVATION OF EQUATION (6.5).**

Chapter 6 investigates shear stress on a fibre when \( U_\infty > U_f \), and the derivation of the momentum integral equation (6.5) follows the same approach as the derivation of (5.6). However, the boundary layer is seen to form at the take up drum in this problem, and this necessitates a change in the origin of the coordinate system used.

Once again, we start with the boundary layer equations.

\[ ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial r} = \nu \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \] (A.13)

\[ \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0. \] (A.14)

Now, using equation (A.14), (A.13) may be written as:
\[
\frac{\partial}{\partial x} ru(u-U_f) + \frac{\partial}{\partial r} rv(u-U_f) = \nu \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right). \quad (A.15)
\]

The velocity profile used to describe the fluid motion close to the fibre surface in this case is:

\[
\frac{u-U_f}{-U_f - U_\infty} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right), \quad 1 \leq \frac{r}{a} \leq e^\alpha,
\]

\[
u \frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e^\alpha. \quad (A.16)
\]

So that on \( r = a \), i.e. at the surface of the fibre, \( u = -U_f \). At \( r = ae^\alpha \), the edge of the boundary layer, \( u = U_\infty \). Integrating (A.15) from the fibre surface to the edge of the boundary layer gives:

\[
\int_a^{ae^\alpha} \frac{\partial}{\partial x} u(u-U_\infty) dr + \int_a^{ae^\alpha} \frac{\partial}{\partial r} v(u-U_\infty) dr = \nu \int_a^{ae^\alpha} \left( r \frac{\partial u}{\partial r} \right) dr, \quad (A.17)
\]

thus

\[
\int_a^{ae^\alpha} \frac{\partial}{\partial x} u(u-U_\infty) dr + \left[ rv(u-U_\infty) \right]_a^{ae^\alpha} = \nu \left( r \frac{\partial u}{\partial r} \right)_{ae^\alpha}^{ae^\alpha}. \quad (A.18)
\]

Now, on \( r = a \), the radial component of the velocity, \( v = 0 \), and at \( r = ae^\alpha \), the edge of the boundary layer \( u = U_\infty \). So, \( [ rv(u-U_\infty) ]_{ae^\alpha}^{ae^\alpha} = 0 \) and (A.18) reduces to:

\[
\int_a^{ae^\alpha} \frac{\partial}{\partial x} u(u-U_\infty) dr = \nu \left( r \frac{\partial u}{\partial r} \right)_{ae^\alpha}. \quad (A.19)
\]

Applying the limits to the right hand side of (A.19) gives

\[
\int_a^{ae^\alpha} \frac{\partial}{\partial x} u(u-U_\infty) dr = -a \nu \left( \frac{\partial u}{\partial r} \right)_{r=a}^{ae^\alpha}. \quad (A.20)
\]

because at \( r = ae^\alpha \), \( u = U_\infty \) and \( \nu \left( r \frac{\partial u}{\partial r} \right)_{r=a}^{ae^\alpha} = 0 - a \nu \left( \frac{\partial u}{\partial r} \right)_{r=a}^{ae^\alpha}. \)
Now, \( v = \frac{\mu}{\rho} \), so
\[
\int_a^\infty \frac{\partial}{\partial x} u(u - U_\infty) \, dr = -a \frac{\mu}{\rho} \left( \frac{\partial u}{\partial r} \right)_{r=a}.
\] (A.21)

Shear stress, \( \tau \), is defined as \( \tau = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \), and (A.21) becomes:
\[
\int_a^\infty \frac{\partial}{\partial x} u(u - U_\infty) \, dr = -a \frac{\tau}{\rho}.
\] (A.22)

By Leibniz's rule,
\[
\int_a^\infty \frac{\partial}{\partial x} u(u - U_\infty) \, dr = \frac{d}{dx} \left[ u(u - U_\infty) \right]_a^\infty - \left[ ru(u - U_\infty) \right]_{r=a} \frac{\partial}{\partial \alpha} (ae^\alpha)
\]
\[+ \left[ ru(u - U_\infty) \right]_{r=a} \frac{\partial}{\partial \alpha} (a).
\] (A.23)

At \( r = ae^\alpha \), \( u = U_\infty \) and \( \left[ ru(u - U_\infty) \right]_{r=a} \frac{\partial}{\partial \alpha} (ae^\alpha) = 0 \). Also, \( \frac{\partial}{\partial \alpha} (a) = 0 \) and
\[
\left[ ru(u - U_\infty) \right]_{r=a} \frac{\partial}{\partial \alpha} (a) = 0.
\]
Thus
\[
\int_a^\infty \frac{\partial}{\partial x} u(u - U_\infty) \, dr = \frac{d}{dx} \left[ u(u - U_\infty) \right]_a^\infty
\]
and (A.22) may be written as:
\[
\frac{d}{dx} \int_a^\infty u(U_\infty - u) \, dr = 2\pi a \frac{\tau}{\rho}
\] (A.24)

This is the momentum integral equation (6.5)

**DERIVATION OF EQUATION (7.8).**

Chapter 7, section 7.1 uses the energy integral equation (7.8) to find the Nusselt number, which characterises heat transfer from the fibre to the surrounding environment. Equation (7.8) is derived from the energy boundary layer equation, and the continuity equation is also used. The boundary layer equations are:
\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0, \tag{A.26}
\]

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = \kappa \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right). \tag{A.27}
\]

Now, using (A.26), (A.27) can be written as:

\[
\frac{\partial}{\partial x} ru(T - T_\infty) + \frac{\partial}{\partial r} rv(T - T_\infty) = \kappa \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right). \tag{A.28}
\]

Section 7.1 seeks to derive an expression for heat transfer from the fibre when

\( U_\infty > U_f \). In this case the boundary layer forms at the take up drum, so the velocity profile is that used in chapter 6, where the boundary layer was seen to form at the take up drum.

\[
\frac{u - U_\infty}{-U_f - U_\infty} = 1 - \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right), \quad 1 \leq \frac{r}{a} \leq e^\alpha, \tag{A.29}
\]

\[
\frac{u}{U_\infty} = 1, \quad \frac{r}{a} \geq e^\alpha, \tag{A.29}
\]

and the temperature profile is given by:

\[
\frac{T - T_\infty}{T_f - T_\infty} = 1 - \frac{1}{\alpha_T} \ln \frac{r}{a}, \quad 1 \leq \frac{r}{a} \leq e^{\alpha_T}, \tag{A.30}
\]

\[
\frac{T - T_\infty}{T_f - T_\infty} = 0, \quad \frac{r}{a} \geq e^{\alpha_T}. \tag{A.30}
\]

Thus, on \( r = a \), i.e. at the surface of the fibre, \( u = -U_f \). At \( r = ae^{\alpha} \), the edge of the momentum boundary layer \( u = U_\infty \). At \( r = a \), \( T = T_f \), and at \( r = ae^{\alpha_T} \), the edge of the thermal boundary layer, \( T = T_\infty \).
Now, it is noted that for air, the thermal boundary layer is thicker than the momentum boundary layer, and to fully examine heat transfer from the fibre, the energy integral equation must be derived over two distinct regions of the flow. The first is from the surface of the fibre to the edge of the momentum boundary layer, with velocity \( u \) that obeys (A.29). The second is from the edge of the momentum boundary layer to the edge of the thermal boundary layer with the free stream velocity \( U_\infty \). So (A.28) becomes

\[
\int_a^{ae}\frac{\partial}{\partial x} u(T - T_\infty) \, dr + \int_a^{ae}\frac{\partial}{\partial r} r v(T - T_\infty) \, dr + \int_{ae}^{ae T} U_\infty (T - T_\infty) \, dr + \int_{ae}^{ae T} r v(T - T_\infty) \, dr = \kappa \int_a^{ae} \left( r \frac{\partial T}{\partial r} \right) \, dr,
\]

which reduces to:

\[
\int_a^{ae}\frac{\partial}{\partial x} u(T - T_\infty) \, dr + \left[ rv(T - T_\infty) \right]_{ae} = \kappa \left( r \frac{\partial T}{\partial r} \right)_{ae}.
\]

\[\text{(A.31)}\]

Now, on \( r = a \), the radial component of the velocity, \( v = 0 \), so,

\[\left[ rv(T - T_\infty) \right]_{ae} = \left[ rv(T - T_\infty) \right]_{ae} - 0.\]

On \( r = ae \), \( T = T_\infty \), so

\[\left[ rv(T - T_\infty) \right]_{ae} = 0 - \left[ rv(T - T_\infty) \right]_{ae} .\]

Also,

\[\left( r \frac{\partial T}{\partial r} \right)_{ae} = -a \left( \frac{\partial T}{\partial r} \right)_{ae} .\]

Equation (A.32) now becomes:

\[
\int_a^{ae}\frac{\partial}{\partial x} u(T - T_\infty) \, dr + \int_{ae}^{ae T} U_\infty (T - T_\infty) \, dr = -\kappa a \left( \frac{\partial T}{\partial r} \right)_{ae} .
\]

\[\text{(A.33)}\]

By Leibniz's rule equation (A.33) becomes:

\[
\frac{d}{dx} \left[ u(T - T_\infty) \right]_{ae} - ru(T - T_\infty) \, dr = -ru(T - T_\infty) \, dr + \frac{\partial}{\partial \alpha} (ae) + [ru(T - T_\infty)]_{ae} \frac{\partial}{\partial \alpha} (a)
\]

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and may be written as:

\[ \frac{d}{dx} \left[ \frac{\partial}{\partial \alpha} \left( \frac{a}{\alpha^2} \right) \right] \]

\[ + \left[ \frac{\partial}{\partial \alpha} \left( \frac{b}{\alpha^2} \right) \right] \]

\[ + \left[ \frac{\partial}{\partial \alpha} \left( \frac{c}{\alpha^2} \right) \right], \]

(A.34)

This is the energy integral equation (7.8). Equation (7.24) is derived in the same manner using the boundary conditions for large velocity ratios.

**DERIVATION OF EQUATION (8.6).**

Chapter 8 investigates shear stress on a stationary fibre and shear stress on a fibre moving in a counterflow air stream at small velocity ratios when the flow is compressible. The momentum integral equations (8.6) and (8.39) are solved to find an expression for the compressible Bingham number and these equations are derived here. The compressible boundary layer equations are:

\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mu \frac{\partial u}{\partial r} \right), \]

(A.36)

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial r} (\rho v) = 0, \]

(A.37)

\[ \rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial r} = \mu \frac{\partial}{\partial r} \left( \frac{r \frac{\partial T}{\partial r}}{C_p} \right) + \frac{\mu}{C_p} \left( \frac{\partial u}{\partial r} \right)^2. \]

(A.38)

where \( \rho \) is the fluid density and \( \mu \) the fluid viscosity. Now, using equation (A.37), (A.36) may be written as:

\[ \rho \frac{\partial}{\partial x} ru(u - U_\infty) + \rho \frac{\partial}{\partial r} rv(u - U_\infty) = \frac{\partial}{\partial r} \left( \mu \frac{\partial u}{\partial r} \right). \]

(A.39)
The velocity profile used to describe the fluid motion close to the fibre surface for the stationary fibre case is:

\[
\frac{u}{U_\infty} = \frac{1}{\alpha(x)} \ln \left( \frac{r}{a} \right) \quad 1 \leq \frac{r}{a} \leq e^a,
\]

\[
\frac{u}{U_\infty} = 1 \quad \frac{r}{a} \geq e^a.
\] (A.40)

So that on \( r = a \), i.e. at the surface of the fibre, \( u = 0 \). At \( r = ae^a \), the edge of the boundary layer, \( u = U_\infty \). Integrating (A.39) from the fibre surface to the edge of the boundary layer gives:

\[
\int_a^{ae^a} \rho \frac{\partial}{\partial x} u(u-U_\infty) r dr + \int_a^{ae^a} \rho \frac{\partial}{\partial r} v(u-U_\infty) r dr = \int_a^{ae^a} \frac{\partial}{\partial r} \left( \mu r \frac{\partial u}{\partial r} \right) dr.
\] (A.41)

thus

\[
\int_a^{ae^a} \rho \frac{\partial}{\partial x} u(u-U_\infty) r dr + \left[ \rho r v(u-U_\infty) \right]_a^{ae^a} = \left( \mu r \frac{\partial u}{\partial r} \right)_{rad}^{ae^a}.
\] (A.42)

Now, on \( r = a \), the radial component of the velocity, \( v = 0 \), and at \( r = ae^a \), the edge of the boundary layer \( u = U_\infty \). So, \( \left[ \rho r v(u-U_\infty) \right]_{a}^{ae^a} = 0 \) and (A.42) reduces to:

\[
\int_a^{ae^a} \rho \frac{\partial}{\partial x} u(u-U_\infty) r dr = \left( \mu r \frac{\partial u}{\partial r} \right)_{a}^{ae^a}.
\] (A.43)

Applying the limits to the right hand side of (A.43) gives

\[
\int_a^{ae^a} \rho \frac{\partial}{\partial x} u(u-U_\infty) r dr = -a \mu_f \left( \frac{\partial u}{\partial r} \right)_{r=ae^a},
\] (A.44)

because at \( r = ae^a \), \( u = U_\infty \) and \( \left( \mu r \frac{\partial u}{\partial r} \right)_{a}^{ae^a} = 0 - a \mu_f \left( \frac{\partial u}{\partial r} \right)_{r=ae^a} \), and at \( r = a \) on the
fibre surface $\mu = \mu_f$. Now, by Leibniz's rule (A.44) can be written as:

$$\frac{d}{dx} \int_{a}^{e} \rho u(u - U) 2\pi r dr = -2\pi a \mu_j \left( \frac{\partial u}{\partial r} \right)_{ra}$$

(A.45)

This is the compressible momentum integral equation (8.6). The compressible momentum integral equation (8.39) is derived in the same manner using the boundary conditions for a fibre in a counterflow free stream.
APPENDIX B

Appendix B provides a proof of the validity of Crocco's relation, utilised in Chapter 8 of this thesis.

PROOF OF THE VALIDITY OF CROCCO'S RELATION.

Crocco's relation relates a fluid's temperature to its velocity and is given by:

\[ C_p T = A - \frac{1}{2} u^2, \]  

(B.1)

where \( C_p \) is the specific heat capacity of a gas at constant pressure, \( T \) is the temperature of the fluid, \( u \) is the fluid velocity, and \( A \) is a constant obtained by applying the appropriate boundary conditions. This thesis has used Crocco's relation in Chapter 8 whilst investigating the effects of compressibility on the shear stress experienced on the surface of (i) a stationary fibre in a moving fluid, and (ii) a fibre moving in a counterflow air stream.

To prove that Crocco's relation is valid, it should satisfy the energy equation, the equation which describes the thermal boundary layer around a solid object, i.e. the
energy equation should reduce to the compressible momentum boundary layer equation. The energy equation (for a Prandtl number of 1) is given by:

\[
\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mu r \frac{\partial T}{\partial r} \right) + \frac{\mu}{C_p} \left( \frac{\partial u}{\partial r} \right)^2.
\]

(B.2)

Hence

\[
\frac{\rho u}{C_p} \frac{\partial}{\partial x} \left( A - \frac{1}{2} u^2 \right) + \frac{\rho v}{C_p} \frac{\partial}{\partial y} \left( A - \frac{1}{2} u^2 \right)
\]

\[
= \frac{1}{C_p} \frac{1}{r} \frac{\partial}{\partial r} \left[ \mu r \frac{\partial \left( A - \frac{1}{2} u^2 \right)}{\partial r} \right] + \frac{\mu}{C_p} \left( \frac{\partial u}{\partial r} \right)^2.
\]

(B.3)

Where \( T = \frac{1}{C_p} \left( A - \frac{1}{2} u^2 \right) \). Simplification of (B.3) gives:

\[-u \left[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} \right] = -u \left[ \mu \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{\mu}{r} \right) \right],
\]

(B.4)

i.e.

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{1}{r} \frac{\partial}{\partial r} \left( \mu r \frac{\partial u}{\partial r} \right).
\]

(B.5)

Equation (B.5) is the compressible momentum boundary layer equation.