2012-03-01

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Recommended Citation

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On Soliton Interactions for a Hierarchy of Generalized Heisenberg Ferromagnetic Models on $SU(3)/S(U(1) \times U(2))$ Symmetric Space

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Abstract

We consider an integrable hierarchy of nonlinear evolution equations (NLEE) related to linear bundle Lax operator L. The Lax representation is $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduced and is naturally associated with the symmetric space $SU(3)/S(U(1) \times U(2))$. The simplest nontrivial equation in the hierarchy is a generalization of Heisenberg ferromagnetic model. We construct the N-soliton solutions for an arbitrary member of the hierarchy by using the Zakharov-Shabat dressing method with an appropriately chosen dressing factor. Two types of soliton solutions: quadruplet and doublet solitons are found. The one-soliton solutions of NLEEs with even and odd dispersion laws have different properties. In particular, the one-soliton solutions for NLEEs with even dispersion laws are *not* traveling waves; their velocities and their amplitudes are time dependent. Calculating the asymptotics of the N-soliton solutions for $t \to \pm\infty$ we analyze the interactions of quadruplet solitons.

1 Introduction

The main object of present paper is the following coupled system of equations

$$
\begin{aligned}
iu_t + u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0 \\
iv_t + v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0
\end{aligned},
\tag{1.1}
$$

where the smooth functions $u : \mathbb{R}^2 \to \mathbb{C}$ and $v : \mathbb{R}^2 \to \mathbb{C}$ satisfy the algebraic constraint $|u|^2 + |v|^2 = 1$. The system (1.1) is a natural candidate to be a multicomponent generalisation of the classical Heisenberg ferromagnetic equation. It is well known [32] that the Heisenberg ferromagnetic model is integrable in the sense of inverse scattering method (ISM). It has a Lax pair related to the algebra $\mathfrak{su}(2)$. Since the time the complete integrability of HF equations was discovered, many attempts for its generalization have been made [20, 21, 22]. A well known method [26, 27, 28, 29, 12, 10, 24, 30, 31] to obtain new integrable nonlinear evolution equations (NLEE) is based on imposing certain algebraic reductions on generic Lax operators. Lax pairs associated to hermitian symmetric spaces represent a special interest in modern theory of integrable systems is study of NLEEs [1, 7, 8, 11] since the NLEEs they produce look relatively simple.

The system (1.1) is also integrable in the sense of ISM. Its Lax operators are associated with the symmetric space $SU(3)/S(U(1) \times U(2))$ with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction imposed on them [15, 16, 13].

The purpose of the present paper is to derive the soliton solutions for the integrable hierarchy of equations related to (1.1) and analyse the interactions between them. In this sense this paper is a natural continuation of our previous papers [15, 16, 13].

In Section 2 we start with some basic facts to be used further in paper. Firstly we describe the hierarchy of nonlinear equations related to (1.1) in terms of recursion operators. Then we outline the spectral properties of the relevant Lax operator and formulate direct scattering problem. The spectrum of scattering operator L consists of a continuous and a discrete parts. As a result of the $\mathbb{Z}_2$ reductions L possesses two configurations of discrete eigenvalues: generic ones, coming in quadruplets $\pm\lambda_k$, $\pm\lambda_k^*$ and purely imaginary ones coming as doublets $\pm i\kappa_j$.

In Section 3 we derive the 1-soliton solutions for the NLEEs of the hierarchy. For this to be done we apply the Zakharov-Shabat dressing method [34, 38, 35, 36] with a rational dressing factor with 2 simple poles. Due to the action of reductions we have two types of 1-soliton solutions: quadruplet solitons to correspond to 4 eigenvalues and doublet ones to correspond to 2 eigenvalues respectively. We present explicit expressions for these two types of one-soliton solutions. In order to construct general multisoliton solutions we discuss two different purely algebraic constructions: by using a multiple pole dressing factor and by applying "one-soliton" dressing factors several times consecutively. It turns out that the properties of the 1-soliton solutions to NLEEs with even and odd dispersion laws differ drastically. For example, the 1-soliton solutions for NLEEs with even dispersion laws are *not* traveling waves. Even the doublet soliton of eq. (1.1) exhibits two maxima (resp. minima) for $|u_1|$ (resp. for $|v_1|$) which first come closer to each other and then move away, one to ∞ and the other to $-\infty$ as time goes to $t \to \infty$. Their velocity, as well as their amplitudes are time dependent. These properties are similar to the ones of the boomerons and trappons discovered by Calogero and Degasperis [2, 3, 4, 5]. At the same time the soliton solutions to the NLEEs with odd dispersion laws (e.g. the solutions of eq. (2.18)) behave as standard solitons, i.e. they are traveling waves.

Section 4 is dedicated to interactions of quadruplet soliton solutions for the NLEE with odd dispersion laws. In order to do this we use the classical method of Zakharov and Shabat, see the monographs [34, 32] for a detailed exposition. Namely, we calculate the limits of the N-soliton solutions for $t \to \pm\infty$ assuming that all solitons move with different velocities. In this way we establish that the solitons preserve their velocities and amplitudes; the only effect of their interaction consists in shifts of the relative mass center and the phase of solitons. We provide explicit expressions for these shifts in terms of the poles μ_k of the dressing factors.

In Section 5 we briefly discuss the conservation laws of the NLEE and finish with some conclusions.

2 Preliminaries

In this section we shall expose in brief some basic facts on Lax operators and direct scattering problem for the integrable hierarchy of the equation (1.1). In doing this we shall use a gauge covariant formulation [14, 17, 18, 19].

2.1 Polynomial Lax Pair Related to $SU(3)/S(U(1) \times U(2))$

The NLEEs under consideration in this paper represent a zero curvature condition $[L, A] = 0$ for Lax operators L and A in the form:

$$L(\lambda) = \mathrm{i}\partial_x + \lambda L_1(x, t) \tag{2.1}$$

$$A(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{N} \lambda^k A_k(x, t), \tag{2.2}$$

where $\lambda \in \mathbb{C}$ is the so-called spectral parameter and the functions L_1 and A_k, $k = 1, \ldots, N$ take values in $\mathfrak{sl}(3, \mathbb{C})$. The Lax operators are subject to the following $\mathbb{Z}_2$ reductions:

$$L^\dagger(\lambda^*) = -\breve{L}(\lambda), \qquad A^\dagger(\lambda^*) = -\breve{A}(\lambda) \tag{2.3}$$

$$\mathbf{C}L(-\lambda)\mathbf{C} = L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda), \tag{2.4}$$

where $\mathbf{C} = \mathrm{diag}\,(1, -1, -1)$ and the operation $\breve{}$ is defined as follows

$$\breve{L}(\lambda)\psi(x, t, \lambda) \equiv \mathrm{i}\partial_x\psi(x, t, \lambda) - \lambda\psi(x, t, \lambda)L_1(x, t, \lambda).$$

Due to reduction (2.3) the matrix coefficients of the Lax pair are hermitian matrices. On the other hand reduction (2.4) represents an action of Cartan's involutive automorphism which defines the symmetric space $SU(3)/S(U(1) \times U(2))$, see [23, 25]. It induces a $\mathbb{Z}_2$-grading in the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$

$$\mathfrak{sl}(3) = \mathfrak{sl}^0(3) \oplus \mathfrak{sl}^1(3), \qquad \mathfrak{sl}^\sigma(3) = \{X \in \mathfrak{sl}(3) \mid \mathbf{C}X\mathbf{C} = (-1)^\sigma X\}. \tag{2.5}$$

It is evident that $L_1, A_k \in \mathfrak{sl}^1(3)$ for k being an odd integer and $A_k \in \mathfrak{sl}^0(3)$ otherwise. This means that A_k for even k are block-diagonal matrices of the form

$$A_k = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

while L_1 and A_k for odd k have the complementary block structure. In particular, L_1 is written as:

$$L_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \tag{2.6}$$

The potential L_1 is required to obey the following conditions:

1. The eigenvalues of L_1 are $0, \pm 1$, i.e. the potential satisfies the characteristic equation $L_1^3 = L_1$.

2. The function $L_1(x, t) - L_\pm$ where

$$\lim_{x \to \pm\infty} L_1(x, t) = L_\pm = \begin{pmatrix} 0 & 0 & e^{\mathrm{i}\phi_\pm} \\ 0 & 0 & 0 \\ e^{-\mathrm{i}\phi_\pm} & 0 & 0 \end{pmatrix}, \qquad \phi_\pm \in \mathbb{R}. \tag{2.7}$$

   is a Schwartz type function, i.e. it is infinitely smooth and tends to 0 faster than any polynomial when $|x| \to \infty$.

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The grading (2.5) means that any function X with values in $\mathfrak{sl}(3)$ can be split as follows:

$$X = X^0 + X^1, \qquad X^{0,1} \in \mathfrak{sl}^{0,1}(3). \tag{2.8}$$

Let us define the Killing form for $\mathfrak{sl}(3)$ as follows:

$$\langle X, Y \rangle = \operatorname{tr}(XY), \qquad X, Y \in \mathfrak{sl}(3).$$

Then each component $X^{0,1}$ splits into a term commuting with L_1 and its orthogonal complement with respect to the Killing form

$$X^0 = X^{0,\perp} + \kappa_0 L_2, \qquad L_2 = L_1^2 - \frac{2}{3}\mathbb{1}, \qquad \langle X^{0,\perp}, L_2 \rangle = 0 \tag{2.9}$$

$$X^1 = X^{1,\perp} + \kappa_1 L_1, \qquad \langle X^{1,\perp}, L_1 \rangle = 0. \tag{2.10}$$

As a simple consequence of condition 1 above L_1 and L_2 are normalized as follows:

$$\langle L_1, L_1 \rangle = 2, \qquad \langle L_2, L_2 \rangle = \frac{2}{3}. \tag{2.11}$$

Therefore the coefficients κ_0 and κ_1 are given by the following equalities

$$\kappa_0 = \frac{3}{2}\langle X^0, L_2 \rangle, \qquad \kappa_1 = \frac{1}{2}\langle X^1, L_1 \rangle. \tag{2.12}$$

The zero curvature condition $[L, A] = 0$ for the pair (2.1), (2.2) leads to certain recurrence relations for the matrix coefficients of L and A, see [13]. Resolving them allows one to express A_k in terms of L_1 and its x-derivatives of order up to $N - k$. Since the maximal order term in the operator A must commute with L_1 there exists two options:

$$\text{a) } A_N = c_{2p}L_2, \qquad \text{if} \quad N = 2p$$
$$\text{b) } A_N = c_{2p+1}L_1, \qquad \text{if} \quad N = 2p+1,$$

where c_{2p} and c_{2p+1} are constants. Then a more detailed analysis [13] shows that the NLEEs look as follows:

$$\text{a) } \operatorname{ad}_{L_1}^{-1} L_{1,t} - \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^{q-1}\Lambda_1 \operatorname{ad}_{L_1}^{-1} L_{2,x} - \sum_{q=0}^{p-1} c_{2q+1}(\Lambda_1\Lambda_2)^q \operatorname{ad}_{L_1}^{-1} L_{1,x} = 0,$$

$$\tag{2.13}$$

$$\text{b) } \operatorname{ad}_{L_1}^{-1} L_{1,t} - \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^{q-1}\Lambda_1 \operatorname{ad}_{L_1}^{-1} L_{2,x} - \sum_{q=0}^{p} c_{2q+1}(\Lambda_1\Lambda_2)^q \operatorname{ad}_{L_1}^{-1} L_{1,x} = 0.$$

The integro-differential operators Λ_1 and Λ_2 appeared above are given by

$$\Lambda_1 = -\mathrm{i}\operatorname{ad}_{L_1}^{-1}\left(\pi\partial_x(\cdot) - \frac{1}{2}L_{1,x}\partial_x^{-1}\langle\partial_x(\cdot), L_1\rangle\right)$$

$$\tag{2.14}$$

$$\Lambda_2 = -\mathrm{i}\operatorname{ad}_{L_1}^{-1}\left(\pi\partial_x(\cdot) - \frac{3}{2}L_{2,x}\partial_x^{-1}\langle\partial_x(\cdot), L_2\rangle\right),$$

where projection $\pi := \operatorname{ad}_{L_1}^{-1}\operatorname{ad}_{L_1}$ cuts all L_1-commuting parts off. The operator

$$\Lambda X := \begin{cases} \Lambda_1\Lambda_2 X, & X \in \mathfrak{sl}^0(3) \\ \Lambda_2\Lambda_1 X, & X \in \mathfrak{sl}^1(3) \end{cases}$$

is called recursion operator. It can be viewed as an adjoint representation of the operator L. Its existence manifests the hierarchies associated with NLEE (nonlinear equations, integrals of motion, simplectic forms etc) and thus plays a very important role in theory of solitons.

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Example 1 *Consider the simplest case when $N = 2$. Then the matrix coefficients of the second Lax operator A read:*

$$A_2 = - \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & |u|^2 - 2/3 & u^*v \\ 0 & v^*u & |v|^2 - 2/3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix} \quad (2.15)$$

$$a = iu_x + i(uu_x^* + vv_x^*)u, \quad b = iv_x + i(uu_x^* + vv_x^*)v \quad (2.16)$$

This L-A pair produces the 2-component system

$$\begin{aligned} iu_t + u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0 \\ iv_t + v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0 \end{aligned} \quad (2.17)$$

we started our paper with (see (1.1)). □

For completeness here we present another member of the hierarchy (2.13). It is the simplest NLEE corresponding to an odd dispersion law.

Example 2 *Consider the case when $\mathbf{f}(\lambda) = -8\lambda^3 J$, i.e. $c_3 = -8$, $c_2 = c_1 = 0$. Then the corresponding 2-component system obtains the form:*

$$\begin{aligned} u_t = 8u_{xxx} + 12(uu_x^* + vv_x^*)u_{xx} + r(u, v)u_x + s(u, v)u \\ v_t = 8v_{xxx} + 12(uu_x^* + vv_x^*)v_{xx} + r(u, v)v_x + s(u, v)v \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} r(u, v) = 3 \left[4(|u_x|^2 + |v_x|^2) + 5(uu_x^* + vv_x^*)^2 + 6(uu_x^* + vv_x^*)_x \right] \\ s(u, v) = 3 \left[2(uu_x^* + vv_x^*)_{xx} + 4(|u_x|^2 + |v_x|^2)_x + 5(uu_x^* + vv_x^*)_x^2 \right]. \quad \square \end{aligned}$$

Sometimes it is more convenient to deal with Lax operators written in canonical gauge. In this gauge the operator (2.1) looks as follows:

$$\tilde{L}(\lambda) = g^{-1}Lg = i\partial_x + U_0(x, t) + \lambda J, \qquad J = \text{diag}(1, 0, -1), \quad (2.19)$$

where

$$g = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & -1 \\ u^* & \sqrt{2}v & u^* \\ v^* & -\sqrt{2}u & v^* \end{pmatrix}. \quad (2.20)$$

The second Lax operator (2.2) is given by

$$\begin{aligned} \text{a)} \quad \tilde{A}(\lambda) = i\partial_t + \sum_{k=0}^{N-1} \lambda^k \tilde{A}_k(x, t) + c_N \lambda^N I, \qquad N = 2p \\ \text{b)} \quad \tilde{A}(\lambda) = i\partial_t + \sum_{k=0}^{N-1} \lambda^k \tilde{A}_k(x, t) + c_N \lambda^N J, \qquad N = 2p + 1, \end{aligned} \quad (2.21)$$

where $I = g^{-1}L_2\, g = \text{diag}(1/3, -2/3, 1/3)$.

2.2 Direct Scattering Problem

In order to formulate a direct scattering problem for L, one needs to introduce auxiliary spectral linear system

$$L(\lambda)\psi(x,t,\lambda) = i\partial_x\psi(x,t,\lambda) + \lambda L_1(x,t)\psi(x,t,\lambda) = 0. \qquad (2.22)$$

Here ψ denotes a fundamental set of solutions or a fundamental solution for short. Since the operators (2.1) and (2.2) commute ψ also satisfies

$$A(\lambda)\psi(x,t,\lambda) = \left(i\partial_t + \sum_{k=1}^{N}\lambda^k A_k(x,t)\right)\psi(x,t,\lambda) = \psi(x,t,\lambda)\mathbf{f}(\lambda) \qquad (2.23)$$

as well. The matrix-valued function

$$\mathbf{f}(\lambda) = \lim_{x\to\pm\infty} g_\pm^{-1}\sum_{k=1}^{N}\lambda^k A_k(x,t)g_\pm \qquad (2.24)$$

is called dispersion law of the nonlinear equation (2.13). The unitary matrix

$$g_\pm = \lim_{x\to\pm\infty} g(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2}\,e^{i\phi_\pm} & 0 \\ e^{-i\phi_\pm} & 0 & e^{-i\phi_\pm} \end{pmatrix}$$

involved in the definition of the dispersion law diagonalizes the asymptotics $L_{1,\pm} = \lim_{x\to\pm\infty} L_1(x,t)$. It can be proven that the dispersion law of (2.13) reads

$$\begin{aligned}
\text{a)} \quad & \mathbf{f}(\lambda) = \sum_{q=0}^{p-1} c_{2q+1}\lambda^{2q+1}J + \sum_{q=1}^{p} c_{2q}\lambda^{2q}I, \\
\text{b)} \quad & \mathbf{f}(\lambda) = \sum_{q=0}^{p} c_{2q+1}\lambda^{2q+1}J + \sum_{q=1}^{p} c_{2q}\lambda^{2q}I.
\end{aligned} \qquad (2.25)$$

The dispersion law of the 2-component system (2.17) is $-\lambda^2 I$ and that of ((2.18) is $-8\lambda^3 J$. It is evident from (2.25) that $\mathbf{f}(\lambda)$ obeys the splitting:

$$\mathbf{f}(\lambda) = \mathbf{f}_0(\lambda)I + \mathbf{f}_1(\lambda)J, \qquad (2.26)$$

which is a result of the $\mathbb{Z}_2$ grading (2.5) of the algebra $\mathfrak{sl}(3)$.

A special type of fundamental solutions are the so-called Jost solutions $\psi_\pm$ which are normalized as follows

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)e^{-i\lambda Jx}g_\pm^{-1} = \mathbb{1}. \qquad (2.27)$$

Due to (2.24) one can show that the asymptotic behavior of $\psi_\pm$ do not depend on time and thus the definition is correct. The transition matrix

$$T(t,\lambda) = [\psi_+(x,t,\lambda)]^{-1}\psi_-(x,t,\lambda). \qquad (2.28)$$

is called scattering matrix. It can be easily deduced from relation (2.23) that the scattering matrix evolves with time according to the linear differential equation

$$i\partial_t T + [\mathbf{f}(\lambda), T] = 0, \qquad (2.29)$$

which is integrated straight away to give

$$T(t, \lambda) = e^{\mathbf{if}(\lambda)t} T(0, \lambda) e^{-\mathbf{if}(\lambda)t}. \tag{2.30}$$

From now on the parameter t will be fixed and we shall omit it to simplify our notation. Due to reasons of simplicity we set $\phi_+ = \phi_- = 0$ as well.

The action of $\mathbb{Z}_2$-reductions (2.3), (2.4) imposes the following restrictions

$$\begin{aligned}
\left[\psi_\pm^\dagger(x, \lambda^*)\right]^{-1} &= \psi_\pm(x, \lambda), & \left[T^\dagger(\lambda^*)\right]^{-1} &= T(\lambda) \\
\mathbf{C}\psi_\pm(x, -\lambda)\mathbf{C} &= \psi_\pm(x, \lambda), & \mathbf{C}T(-\lambda)\mathbf{C} &= T(\lambda)
\end{aligned} \tag{2.31}$$

on the Jost solutions and the scattering matrix.

The continuous spectrum of L fills up the real axis in the complex λ-plane. Thus the λ-plane is divided into two regions denoted by $\mathbb{C}_+$ (the upper half plane) and $\mathbb{C}_-$ (the lower half plane). These regions represent domains for fundamental solutions $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ to be analytic functions in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively [16]. The fundamental analytic solutions (FAS) can be constructed by using Gauss factors in the decomposition of the scattering matrix:

$$T(\lambda) = T^\mp(\lambda)D^\pm(\lambda)(S^\pm(\lambda))^{-1}. \tag{2.32}$$

S^+ and T^+ are upper triangular matrices, S^- and T^- are lower triangular matrices and $D^\pm$ are diagonal ones. Then χ^+ and χ^- are expressed as follows

$$\chi^\pm(x, \lambda) = \psi_-(x, \lambda)S^\pm(\lambda) = \psi_+(x, \lambda)T^\mp(\lambda)D^\pm(\lambda). \tag{2.33}$$

Due to relation (2.33) the FAS can be interpreted as solutions to a local Riemann-Hilbert problem

$$\chi^+(x, \lambda) = \chi^-(x, \lambda)G(x, \lambda), \qquad G(\lambda) = (S^-(\lambda))^{-1}S^+(\lambda). \tag{2.34}$$

The established interrelation between the inverse scattering method and Riemann-Hilbert problem plays an important role in constructing solutions to NLEEs through dressing method.

It can be shown that the reduction conditions (2.31) and equation (2.32) lead to the following demands on the Gauss factors

$$\begin{aligned}
\left[S^+(\lambda^*)\right]^\dagger &= [S^-(\lambda)]^{-1}, & \tilde{\mathbf{C}}S^\pm(-\lambda)\tilde{\mathbf{C}} &= S^\mp(\lambda) \\
\left[T^+(\lambda^*)\right]^\dagger &= [T^-(\lambda)]^{-1}, & \tilde{\mathbf{C}}T^\pm(-\lambda)\tilde{\mathbf{C}} &= T^\mp(\lambda) \\
\left[D^+(\lambda^*)\right]^\dagger &= [D^-(\lambda)]^{-1}, & \tilde{\mathbf{C}}D^\pm(-\lambda)\tilde{\mathbf{C}} &= D^\pm(\lambda),
\end{aligned} \tag{2.35}$$

where

$$\tilde{\mathbf{C}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Finally, combining all this information we see that the FAS obey the symmetry conditions

$$\left[\chi^+(x, \lambda^*)\right] = [\chi^-(x, \lambda)]^{-1} \qquad \mathbf{C}\chi^+(x, -\lambda)\mathbf{C} = \chi^-(x, \lambda). \tag{2.36}$$

3 Dressing Method and Soliton Solutions

As we mentioned in the previous section the inverse scattering method is tightly related to Riemann-Hilbert problem. The Riemann-Hilbert problem possesses two types of solutions: regular ones (without singularities) and singular ones. Singular solutions can be generated by dressing regular solutions with a factor which has prescribed singularities. The simplest types of singularities are first order poles and zeroes. It can be proven that they correspond to poles of the resolvent of L. Hence they are discrete eigenvalues of the Lax operator (2.1). The discrete eigenvalues of L form orbits of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$. There exist two types of orbits: generic orbits containing quadruplets of eigenvalues $\{\pm\mu, \pm\mu^*\}$ and degenerate orbits consisting of two imaginary eigenvalues $\pm i\kappa$ (doublets).

There is a very deep connection between singular solutions to Riemann-Hilbert problem and soliton solutions to the corresponding nonlinear problem. In the present section we are going to analyze the soliton solutions to the system (2.13). For this to be done, we are going to apply the dressing method proposed in [38] and developed in [35, 36, 28, 29]. We demonstrate that the NLEE (2.13) has two types of 1-soliton solutions: doublet soliton to be connected with two imaginary discrete eigenvalues of L and quadruplet soliton connected to 4 eigenvalues.

3.1 Rational Dressing

The dressing method is an indirect method for solving a NLEE possessing a Lax representation. This means that it allows one to generate a solution to the NLEE starting from a known one. Let us assume we know a solution

$$L_1^{(0)} = \begin{pmatrix} 0 & u_0 & v_0 \\ u_0^* & 0 & 0 \\ v_0^* & 0 & 0 \end{pmatrix}$$

of (2.13) and a fundamental solution $\psi_0(x, t, \lambda)$ of the auxiliary linear problems

$$L^{(0)}(\lambda)\psi_0 = i\partial_x \psi_0 + \lambda L_1^{(0)} \psi_0 = 0$$

$$A^{(0)}(\lambda)\psi_0 = i\partial_t \psi_0 + \sum_{k=1}^{N} \lambda^k A_k^{(0)} \psi_0 = 0. \tag{3.1}$$

Then one constructs another function $\psi_1(x, t, \lambda) = \Phi(x, t, \lambda)\psi_0(x, t, \lambda)\Phi_-^\dagger(\lambda)$, where $\Phi_-(\lambda) = \lim_{x \to -\infty} \Phi(x, t, \lambda)$. This function is a common solution to

$$L^{(1)}(\lambda)\psi_1 = i\partial_x \psi_1 + \lambda L_1^{(1)} \psi_1 = 0$$

$$A^{(1)}(\lambda)\psi_1 = i\partial_t \psi_1 + \sum_{k=1}^{N} \lambda^k A_k^{(1)} \psi_1 = 0, \tag{3.2}$$

where the potential

$$L_1^{(1)} = \begin{pmatrix} 0 & u_1 & v_1 \\ u_1^* & 0 & 0 \\ v_1^* & 0 & 0 \end{pmatrix}$$

is to be found. From (3.1) and (3.2) it follows that the dressing factor $\Phi(x, t, \lambda)$ satisfies the following equations:

$$i\partial_x \Phi + \lambda L_1^{(1)}\Phi - \lambda\Phi L_1^{(0)} = 0 \tag{3.3}$$

$$i\partial_t \Phi + \sum_{k=1}^{N} \lambda^k A_k^{(1)}\Phi - \Phi \sum_{k=1}^{N} \lambda^k A_k^{(0)} = 0. \tag{3.4}$$

We also assume that the dressing factor is regular at $|\lambda| \to 0, \infty$. Then from (3.3) one can derive the following relation between $L_1^{(1)}$ and $L_1^{(0)}$:

$$L_1^{(1)}(x, t) = \Phi(x, t, \infty)L_1^{(0)}(x, t)\Phi^\dagger(x, t, \infty). \tag{3.5}$$

This equation will play a central role in our further considerations since it allows one to generate a new solution to (2.13) from the given one $L_1^{(0)}$.

Due to the reduction conditions (2.3), (2.4) the dressing factor obeys the symmetries:

$$\mathbf{C}\Phi(x, t, -\lambda)\mathbf{C} = \Phi(x, t, \lambda) \tag{3.6}$$

$$\Phi(x, t, \lambda)\Phi^\dagger(x, t, \lambda^*) = \mathbb{1}. \tag{3.7}$$

In order to obtain a nontrivial dressing we choose $\Phi(x, t, \lambda)$ as a rational function[1] of λ with a minimal number of simple poles. At first we shall consider the case when these poles are generic complex numbers. Hence the dressing factor looks as follows:

$$\Phi(x, t, \lambda) = \mathbb{1} + \frac{\lambda M(x, t)}{\lambda - \mu} + \frac{\lambda \mathbf{C}M(x, t)\mathbf{C}}{\lambda + \mu}, \tag{3.8}$$

where $\operatorname{Re}\mu \neq 0$, $\operatorname{Im}\mu \neq 0$. It is evident that the reduction condition (3.6) is fulfilled. On the other hand (3.7) leads to the conclusion that

$$\Phi^{-1}(x, t, \lambda) = \mathbb{1} + \frac{\lambda M^\dagger(x, t)}{\lambda - \mu^*} + \frac{\lambda \mathbf{C}M^\dagger(x, t)\mathbf{C}}{\lambda + \mu^*}. \tag{3.9}$$

The identity $\Phi(\lambda)\Phi^{-1}(\lambda) = \mathbb{1}$ must hold for any λ. Therefore after equating the residue at $\lambda = \mu^*$ to 0 one gets the equation:

$$\left(\mathbb{1} + \frac{\mu^* M(x, t)}{\mu^* - \mu} + \frac{\mu^* \mathbf{C}M(x, t)\mathbf{C}}{\mu^* + \mu}\right) M^\dagger(x, t) = 0. \tag{3.10}$$

The rest of algebraic relations can be reduced to (3.10) due to the symmetry conditions (2.3), (2.4).

The residue M ought to be singular since otherwise it should be proportional to $\mathbb{1}$ and the dressing becomes trivial. It suffices to consider the case $\operatorname{rank}M = 1$. Then M can be decomposed in the following manner:

$$M = |n\rangle\langle m|, \qquad |n\rangle = (n_1, n_2, n_3)^T, \quad \langle m| = (m_1^*, m_2^*, m_3^*). \tag{3.11}$$

After substituting this representation into (3.10) one derives a linear system for the 3-vector $|n\rangle$:

$$|m\rangle - \frac{\mu^*|n\rangle\langle m|m\rangle}{2i\kappa} + \frac{\mu^*\mathbf{C}|n\rangle\langle m|\mathbf{C}|m\rangle}{2\omega} = 0. \tag{3.12}$$

---

[1] If Φ is λ-independent then it does not depend on x and t either. Thus (3.5) produces simply a unitary transformation of $L_1^{(0)}$ which is not essential because of $U(2)$ gauge symmetry of the model.

where we have used the notation $\omega = \operatorname{Re} \mu$, $\kappa = \operatorname{Im} \mu$. The solution of (3.12) reads:

$$|n\rangle = \frac{1}{\mu^*} \left(\frac{\langle m|m\rangle}{2\mathrm{i}\kappa} - \frac{\langle m|\mathbf{C}|m\rangle}{2\omega} \mathbf{C} \right)^{-1} |m\rangle. \tag{3.13}$$

The vector $|m\rangle$ is an element of the projective space $\mathbb{C}P^2$. Indeed, it is evident that a rescaling $|m\rangle \to h|m\rangle$ with any complex $h \neq 0$ does not change the matrix M.

Taking into account the ansatz (3.8) one can rewrite (3.5) as:

$$L_1^{(1)} = (\mathbb{1} + M + \mathbf{C}M\mathbf{C}) L_1^{(0)} (\mathbb{1} + M + \mathbf{C}M\mathbf{C})^\dagger. \tag{3.14}$$

Notice that the dressing procedure preserves the matrix structure of L since the factor $\mathbb{1} + M + \mathbf{C}M\mathbf{C}$ is a block-diagonal matrix.

We have expressed all quantities needed in terms of $|m\rangle$ and now it remains to find $|m\rangle$ itself. For that purpose we rewrite equations (3.3), (3.4) in the form:

$$\Phi(x,t,\lambda) \left(\mathrm{i}\partial_x + \lambda L_1^{(0)} \right) \Phi^{-1}(x,t,\lambda) = \lambda L_1^{(1)}$$

$$\Phi(x,t,\lambda) \left(\mathrm{i}\partial_t + \sum_{k=1}^N \lambda^k A_k^{(0)} \right) \Phi^{-1}(x,t,\lambda) = \sum_{k=1}^N \lambda^k A_k^{(1)}. \tag{3.15}$$

It is obviously satisfied at $\lambda = 0$. After equating the residues of (3.15) at $\lambda = \mu^*$ to 0 we obtain a set of differential equations

$$\left(\mathbb{1} + \frac{\mu^* M}{\mu^* - \mu} + \frac{\mu^* \mathbf{C}M\mathbf{C}}{\mu^* + \mu} \right) \left(\mathrm{i}\partial_x + \mu^* L_1^{(0)} \right) |m\rangle = 0$$

$$\left(\mathbb{1} + \frac{\mu^* M}{\mu^* - \mu} + \frac{\mu^* \mathbf{C}M\mathbf{C}}{\mu^* + \mu} \right) \left(\mathrm{i}\partial_t + \sum_{k=1}^N (\mu^*)^k A_k^{(0)} \right) |m\rangle = 0. \tag{3.16}$$

Taking into account (3.10) the equations above can be reduced to

$$\left(\mathrm{i}\partial_x + \mu^* L_1^{(0)}(x,t) \right) |m(x,t)\rangle = h(x,t)|m(x,t)\rangle$$

$$\left(\mathrm{i}\partial_t + \sum_{k=1}^N (\mu^*)^k A_k^{(0)}(x,t) \right) |m(x,t)\rangle = h(x,t)|m(x,t)\rangle \tag{3.17}$$

for some arbitrary function h. At this point we recall that the vectors in the decomposition (3.11) are not uniquely determined. Indeed, the operation $|n\rangle \to B^{-1}|n\rangle$ and $|m\rangle \to B^\dagger|m\rangle$ for any nondegenerate 3×3 matrix B produces another decomposition of M. It is not hard to see that it is always possible to choose B in such a way that $h \equiv 0$ is fulfilled. Thus from (3.17) it follows that $|m(x,t)\rangle$ is proportional to some fundamental solution $\psi_0(x,t,\lambda)$ of the bare linear problem, namely

$$|m(x,t)\rangle = \psi_0(x,t,\mu^*)|m_0\rangle, \tag{3.18}$$

where $|m_0\rangle \in \mathbb{C}^3 \backslash \{0\}$ is a constant vector of integration. The new solution $L_1^{(1)}$ of (2.13) and the solution $\psi_1(x,t,\lambda)$ of the corresponding linear system are parameterized by a complex number μ and a complex 3-vector $|m_0\rangle$.

Thus we have proved the following Proposition:

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Proposition 1 *Let $L_1^{(0)}$ be a solution of (2.13) and $\psi_0(x, t, \lambda)$ be a common solution to (3.1). Let also μ be a complex number to fulfill $\operatorname{Re} \mu \neq 0$, $\operatorname{Im} \mu > 0$ and $|m_0\rangle \in \mathbb{C}^3 \backslash \{0\}$. Then the matrix-valued function $L_1^{(1)}(x, t)$ defined by (3.14) where $M = |n\rangle\langle m|$ is determined by (3.13) and (3.18) is a solution to (2.13) as well. The corresponding fundamental solution $\psi_1(x, t, \lambda)$ of (3.2) is given by $\psi_1 = \Phi\psi_0$ where $\Phi(x, t, \lambda)$ is determined by (3.8), (3.11), (3.13) and (3.18).* $\square$

Let us now consider the case when the poles of the dressing factor are imaginary, i.e. we have:

$$\Phi(x, t, \lambda) = \mathbb{1} + \lambda \left(\frac{M(x, t)}{\lambda - i\kappa} + \frac{\mathbf{C}M(x, t)\mathbf{C}}{\lambda + i\kappa} \right), \qquad \kappa \neq 0. \tag{3.19}$$

Then Φ^{-1} has the same poles as Φ and therefore the equality $\Phi\Phi^{-1} = \mathbb{1}$ already contains second order poles. In this case the natural requirement of vanishing of the matrix coefficients before $(\lambda - i\kappa)^{-2}$ and $(\lambda - i\kappa)^{-1}$ leads to the algebraic relations:

$$M\mathbf{C}M^\dagger = 0 \tag{3.20}$$

$$\left(\mathbb{1} + M + \frac{\mathbf{C}M\mathbf{C}}{2} \right) \mathbf{C}M^\dagger \mathbf{C} + M \left(\mathbb{1} + \mathbf{C}M^\dagger \mathbf{C} + \frac{M^\dagger}{2} \right) = 0. \tag{3.21}$$

As before in order to obtain a nontrivial result M is required to be a degenerate matrix, i.e. decomposition (3.11) holds true. Then relation (3.20) is rewritten as

$$\langle m|\mathbf{C}|m\rangle = 0. \tag{3.22}$$

Relation (3.21) in its turn can be easily reduced to the following linear system for 3-vector $|n\rangle$

$$\left(\mathbb{1} + \frac{\mathbf{C}|n\rangle\langle m|\mathbf{C}}{2} \right) \mathbf{C}|m\rangle = i\sigma|n\rangle. \tag{3.23}$$

by introducing some auxiliary real function σ. That linear system allows one to express $|n\rangle$ through $\langle m|$ and σ, namely:

$$|n\rangle = \left(i\sigma - \frac{\langle m|m\rangle}{2}\mathbf{C} \right)^{-1} \mathbf{C}|m\rangle. \tag{3.24}$$

In order to find $|m\rangle$ and σ we turn back to the equations (3.15). Vanishing of the second order poles in (3.15) leads to the conclusion that

$$|m(x, t)\rangle = \psi_0(x, t, -i\kappa)|m_0\rangle, \tag{3.25}$$

where $|m_0\rangle$ is a constant nonzero 3-vector. After substituting (3.25) into (3.22) and taking into account (2.3) one convinces himself that the components of the polarization vector $|m_0\rangle$ are no longer independent but satisfy the constraint:

$$\langle m_0|\mathbf{C}|m_0\rangle = 0 \qquad \Leftrightarrow \qquad |m_{0,1}|^2 = |m_{0,2}|^2 + |m_{0,3}|^2. \tag{3.26}$$

The vanishing condition of the first order poles leads to some differential constraint on $\sigma(x, t)$ which is integrated to give:

$$\sigma(x, t) = -\kappa\langle m_0|\psi^{-1}(x, t, i\kappa)\dot{\psi}_0(x, t, i\kappa)\mathbf{C}|m_0\rangle + \sigma_0, \tag{3.27}$$

where $\sigma_0 \in \mathbb{R}$ is a costant of integration.

Thus to calculate the soliton solution itself one just substitutes the result for $|n\rangle$ and $|m\rangle$ into M and uses formula (3.14). As it is seen the new solution is parametrized by the polarization vector $|m_0\rangle$, the real number σ_0 and the pole $i\kappa$. All this can be formulated in the following manner:

Proposition 2 *Let there be given a solution $L^{(0)}(x,t)$ to (2.13), a common solution $\psi_0(x,t,\lambda)$ to (3.1), real numbers $\kappa > 0$, σ_0 and a complex nonzero vector $|m_0\rangle$ satisfying (3.26). Then the function $L_1^{(1)}(x,t)$ determined by (3.14), (3.11), (3.24), (3.25) and (3.27) is a solution of the system (2.13) too. The solution $\psi_1(x,t,\lambda)$ of the dressed linear system (3.2) is given by $\psi_1 = \Phi\psi_0$ where Φ is defined by (3.19), (3.11), (3.24), (3.25) and (3.27).*

One can apply the dressing procedure repeatedly to build a sequence of exact solutions

$$L_1^{(0)} \xrightarrow{\Phi_1} L_1^{(1)} \xrightarrow{\Phi_2} \ldots \xrightarrow{\Phi_N} L_1^{(N)}. \tag{3.28}$$

More precisely this alternative procedure will be explained in Section 4.

3.2   Soliton Solutions

Let us apply the dressing procedure to the following seed solution

$$L_1^{(0)}(x,t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.29}$$

of equation (2.13). In this case a fundamental solution to (3.1) reads:

$$\psi_0(x,t,\lambda) = \begin{pmatrix} \cos(\lambda x + \mathbf{f}_1(\lambda)t)e^{\frac{i\mathbf{f}_0(\lambda)t}{3}} & 0 & i\sin(\lambda x + \mathbf{f}_1(\lambda)t)e^{\frac{i\mathbf{f}_0(\lambda)t}{3}} \\ 0 & e^{\frac{-2i\mathbf{f}_0(\lambda)t}{3}} & 0 \\ i\sin(\lambda x + \mathbf{f}_1(\lambda)t)e^{\frac{i\mathbf{f}_0(\lambda)t}{3}} & 0 & \cos(\lambda x + \mathbf{f}_1(\lambda)t)e^{\frac{i\mathbf{f}_0(\lambda)t}{3}} \end{pmatrix}. \tag{3.30}$$

We recall that $\mathbf{f}_0(\lambda)$ and $\mathbf{f}_1(\lambda)$ are even and odd part of the dispersion law induced by the $\mathbb{Z}_2$ grading of $\mathfrak{sl}(3)$, see (2.26).

We are going to consider the generation of a quadruplet soliton first. In this case one uses factor (3.8). It is convenient to decompose the polarization vector $|m_0\rangle$ according to the eigensubspaces of the endomorphism ψ_0 (3.30):

$$|m_0\rangle = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{3.31}$$

where α, β, γ are arbitrary complex constants.

If the vector $|m_0\rangle$ is proportional to one of the eigenvectors of the endomorphism ψ_0, then the corresponding matrix M does not depend on the variables x and t (due to the projective nature of the vector $|m\rangle$) and the corresponding solution (3.14) is a simple unitary rotation of the constant solution $L_1^{(0)}$.

Thus elementary solitons correspond to vectors $|m_0\rangle$, belonging to essentially two-dimensional invariant subspaces of ψ_0, i.e. they correspond to polarization vectors with only one zero coefficient in the expansion (3.31). Let us consider each of these three cases in more detail.
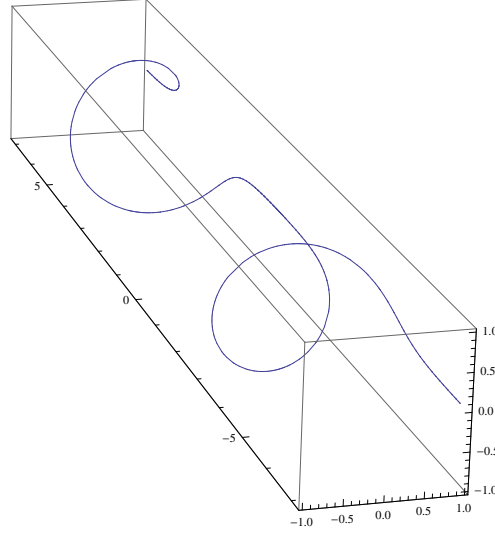
Figure 1: Real and imaginary parts of the component v_1 in 3.33 as a function of x. Here $\kappa = 1, \omega = 10^{-3}, \alpha = 1, \beta = 1 + i$.

Case (i): $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$

The 1-soliton solution is given by:

$$u_1(x, t) = 0$$
$$v_1(x, t) = \exp\left\{4i \arctan\left(\frac{\kappa \cos(2\omega x + 2\mathbf{f}_1^R(\mu)t + \phi_\alpha - \phi_\beta)}{\omega \cosh(2\kappa x + 2\mathbf{f}_1^I(\mu)t + \ln|\alpha/\beta|)}\right)\right\}, \qquad (3.32)$$

where $\phi_\alpha = \arg\alpha$, $\phi_\beta = \arg\beta$. $\mathbf{f}_1^R(\lambda)$ and $\mathbf{f}_1^I(\lambda)$ are the real and the imaginary part of the polynomial $\mathbf{f}_1(\lambda)$ (resp. $\mathbf{f}_0^R(\lambda)$ and $\mathbf{f}_0^I(\lambda)$ stand for the real and imaginary part of $\mathbf{f}_0(\lambda)$ to be used later on). If the dispersion law of NLEE is an even polynomial ($\mathbf{f}_1(\lambda) \equiv 0$) then the solution (3.32) becomes stationary:

$$u_1(x, t) = 0$$
$$v_1(x, t) = \exp\left\{4i \arctan\left(\frac{\kappa \cos(2\omega x + \phi_\alpha - \phi_\beta)}{\omega \cosh(2\kappa x + \ln|\alpha/\beta|)}\right)\right\}. \qquad (3.33)$$

A plot of that solution is presented on Fig.1. It is easy to check that $u = 0$, $v = \exp(if(x))$ is an exact solution of (2.17) for any differentiable function $f(x)$ tending to 0 when $x \to \pm\infty$. This resembles the case of the three-wave equation [33] where one wave of an arbitrary shape is an exact solution of the system and the two other waves are identically zero. The solution (3.33) has a simple spectral characterisation and an explicitly given analytic fundamental solution of the corresponding linear problem.

If the dispersion law contains odd powers of λ as well then the elementary soliton is no more stationary. For example in the case of equation (2.18) it reads:

$$u_1(x, t) = 0, \qquad v_1(x, t) = \exp(4i \arctan \zeta_{\text{cub}}(x, t))$$
$$\zeta_{\text{cub}}(x, t) = \left[\frac{\kappa \cos 2\omega[x + 8(3\kappa^2 - \omega^2)t + (\phi_\alpha - \phi_\beta)/2\omega]}{\omega \cosh 2\kappa[x + 8(\kappa^2 - 3\omega^2)t + \ln|\alpha/\beta|/2\kappa]}\right]. \qquad (3.34)$$

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Case (ii): $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$

In this case the solution looks as follows:

$$u_1(x,t) = \frac{4i\omega\kappa Q_{\mathrm{gen}}^* \exp i\{\omega x + (\mathbf{f}_0^R(\mu) + \mathbf{f}_1^R(\mu))t + \phi_\alpha - \phi_\gamma\}}{(\omega - i\kappa)Q_{\mathrm{gen}}^2}$$

$$v_1(x,t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q_{\mathrm{gen}}^2} \quad, \tag{3.35}$$

where $\phi_\alpha = \arg\alpha$, $\phi_\gamma = \arg\gamma$ and

$$Q_{\mathrm{gen}} = 2\omega e^{\kappa x + (\mathbf{f}_0^I(\mu) + \mathbf{f}_1^I(\mu))t + \ln|\alpha/\gamma|} + (\omega + i\kappa)e^{-\kappa x - (\mathbf{f}_0^I(\mu) + \mathbf{f}_1^I(\mu))t - \ln|\alpha/\gamma|}.$$

In particular, when $\mathbf{f}(\lambda) = -\lambda^2 I$, i.e. $\mathbf{f}_0(\lambda) = -\lambda^2$ and $\mathbf{f}_1(\lambda) = 0$ hold, we obtain a solution to (2.17):

$$u_1(x,t) = \frac{4i\omega\kappa Q^* \exp i\{\omega x + (\kappa^2 - \omega^2)t + \phi_\alpha - \phi_\gamma\}}{(\omega - i\kappa)Q^2}$$

$$v_1(x,t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q^2}, \tag{3.36}$$

where

$$Q = 2\omega e^{\kappa(x - 2\omega t) + \ln|\alpha/\gamma|} + (\omega + i\kappa)e^{-\kappa(x - 2\omega t) - \ln|\alpha/\gamma|}.$$

Contour plots of $|u_1|^2$ and $|v_1|^2$ of the solutions (3.36) are shown on Figure 2.

When the dispersion law is odd, say $\mathbf{f}_1(\lambda) = -8\lambda^3$, the quadruplet solution represents a traveling wave of the form:

$$u_1(x,t) = \frac{4i\omega\kappa Q^* \exp i\omega[x + 8(3\kappa^2 - \omega^2)t + (\phi_\alpha - \phi_\gamma)/\omega]}{(\omega - i\kappa)Q^2}$$

$$v_1(x,t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q^2}, \quad, \tag{3.37}$$

where

$$Q = 2\omega e^{\kappa(x + 8(\kappa^2 - 3\omega^2)t + \ln|\alpha/\gamma|/\kappa)} + (\omega + i\kappa)e^{-\kappa(x + 8(\kappa^2 - 3\omega^2)t + \ln|\alpha/\gamma|/\kappa)}.$$

This is an elementary soliton for the cubic flow NLEE (2.18).

Case (iii): $\alpha = 0, \beta \neq 0, \gamma \neq 0$

The solution now can be obtained from the solution in the case (ii), by changing $\alpha \to \beta$ and $x \to -x$.

In the cases (ii) the solution (3.36) is a soliton of width $1/\kappa$ moving with velocity 2ω. The corresponding soliton in the case (iii) moves with a velocity -2ω.

In the generic case, when all three constants are non-zero, the solution represents a nonlinear deformation of the above described solitons. For $\kappa > 0$ it may be viewed as a decay of unstable time independent soliton from the case (i) into two solitons, corresponding to the cases (ii) and (iii) (see fig 2). For $\kappa < 0$, the solution is a fusion of two colliding solitons into a stationary one.

Let us now consider dressing by a factor with two imaginary poles (doublet case), i.e. $\mu = i\kappa$. There are two essentially different cases.
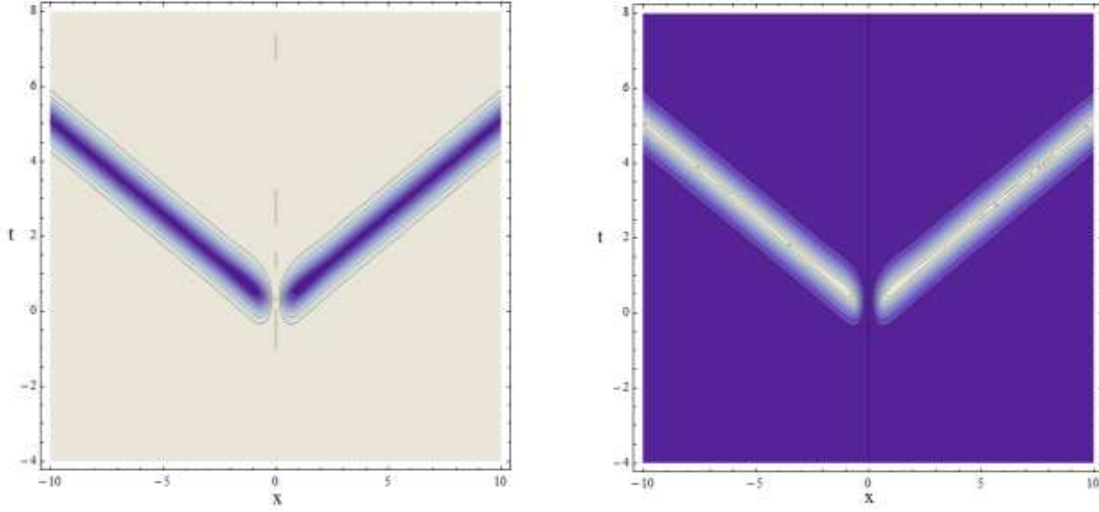
Figure 2: Contour plot of $|u_1|^2$ (left panel) and $|v_1|^2$ (right panel) for a generic soliton solution (3.36) as a function of x and t where $\alpha = \gamma = \kappa = \omega = 1$.

Case (i): $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$

From (3.26) it follows that $|m_{0,1}| = |m_{0,3}|$. It suffices to pick up $m_{0,1} = 1$ and the third component is $m_{0,3} = \exp(i\varphi)$, $\varphi \in \mathbb{R}$. The doublet solution reads:

$$u_1(x,t) = 0, \qquad v_1(x,t) = \exp(4i\arctan\Xi_{\text{gen}}(x,t))$$
$$\Xi_{\text{gen}}(x,t) = \frac{\sigma_0 - 2\kappa(x + \dot{\mathbf{f}}_1(i\kappa)t)\sin\varphi}{\cosh 2(\kappa x + \mathbf{f}_1^I(i\kappa)t) + \sinh 2(\kappa x + \mathbf{f}_1^I(i\kappa)t)\cos\varphi}. \tag{3.38}$$

If the dispersion law of NLEE is even polynomial, i.e. $\mathbf{f}_1(\lambda) \equiv 0$, the 1-soliton solution becomes stationary:

$$u_1(x,t) = 0$$
$$v_1(x,t) = \exp\left\{4i\arctan\left(\frac{\sigma_0 - 2\kappa x\sin\varphi}{\cosh 2\kappa x + \sinh 2\kappa x\cos\varphi}\right)\right\}. \tag{3.39}$$

Figure 3 presents the argument and the imaginary part of $v_1(x)$ in the stationary case as functions of x and the phase φ.

As in the quadruplet case if the dispersion law is an odd polynomial the doublet solution is time-depending. Let us consider the simplest example $\mathbf{f}_1(\lambda) = -8\lambda^3$ corresponding to equation (2.18). Now (3.38) obtains the form:

$$u_1(x,t) = 0, \qquad v_1(x,t) = \exp(4i\arctan\Xi_{\text{cub}}(x,t))$$
$$\Xi_{\text{cub}}(x,t) = \frac{\sigma_0 - 2\kappa(x + 24\kappa^2 t)\sin\varphi}{\cosh 2\kappa(x + 8\kappa^2 t) + \sinh 2\kappa(x + 8\kappa^2 t)\cos\varphi}. \tag{3.40}$$

Case ii. Generic doublet

Now let us assume $m_{0,2} \neq 0$. For simplicity we fix $m_{0,2} = 1$. Then the norms of $m_{0,1}$ and $m_{0,3}$ are interrelated through
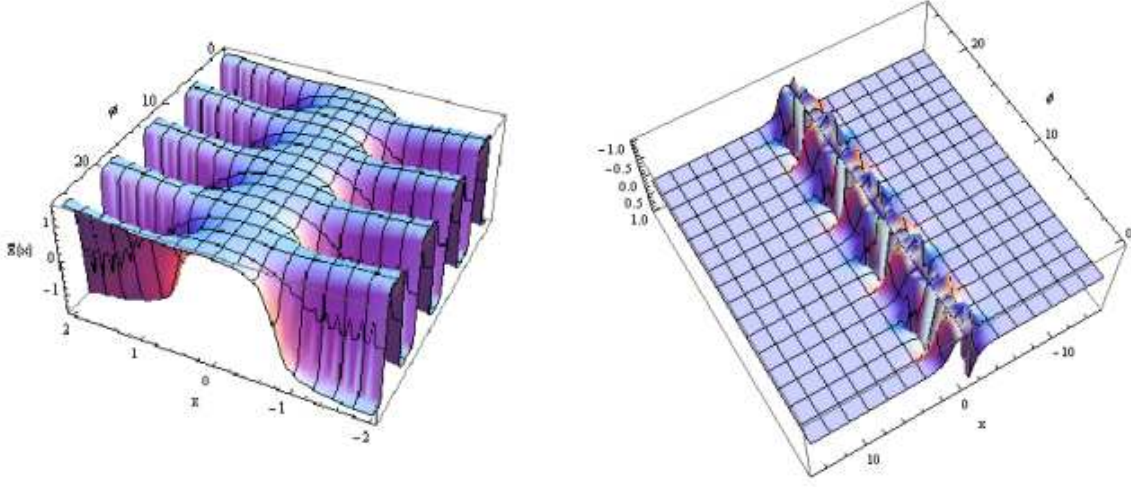$$|m_{0,1}|^2 - |m_{0,3}|^2 = 1.$$

Figure 3: Plots of the argument (left panel) and $\mathrm{Im}\, v_1(x)$ (right panel) for the stationary solution (3.39) as a function of x and φ; $\kappa = \sigma = 1$.

This is why it proves to be convenient to parametrize them as follows:

$$m_{0,1} = \cosh\theta_0 \mathrm{e}^{\mathrm{i}(\varphi_0 + \tilde{\varphi})}, \qquad m_{0,3} = |\sinh\theta_0| \mathrm{e}^{\mathrm{i}(\varphi_0 - \tilde{\varphi})}, \tag{3.41}$$

where θ_0, φ_0 and $\tilde{\varphi}$ are arbitrary real numbers. Then the doublet soliton solution reads:

$$
\begin{aligned}
u_1(x,t) &= \frac{2\Delta^*}{\Delta^2}\mathrm{e}^{\mathrm{i}(\mathbf{f}_0(\mathrm{i}\kappa)t + \varphi_0)} \left[\sinh\theta_+ \cos\tilde{\varphi} + \mathrm{i}\sinh\theta_- \sin\tilde{\varphi}\right] \\
v_1(x,t) &= 1 + \frac{2(2\mathrm{i}\sigma - 1)}{\Delta} + \frac{4\mathrm{i}\sigma(\mathrm{i}\sigma - 1)}{\Delta^2},
\end{aligned}
\tag{3.42}
$$

where

$$
\begin{aligned}
\Delta(x,t) &= \cosh^2\theta_+ \cos^2\tilde{\varphi} + \cosh^2\theta_- \sin^2\tilde{\varphi} - \mathrm{i}\sigma \\
\sigma(x,t) &= \sigma_0 + \kappa\dot{\mathbf{f}}_0^I(\mathrm{i}\kappa)t + \kappa\left(x + \dot{\mathbf{f}}_1(\mathrm{i}\kappa)t\right)\sinh 2\theta_0 \sin 2\tilde{\varphi} \\
\theta_\pm(x,t) &= \kappa x + \mathbf{f}_1^I(\mathrm{i}\kappa)t \pm \theta_0.
\end{aligned}
$$

Let us consider the special case when the dispersion law is $-\lambda^2 I$. The solution (3.42) is significantly simplified if in addition one assumes that $m_{0,3}/m_{0,1} > 0$ ($\tilde{\varphi} = 0$). The result reads:

$$
\begin{aligned}
u_1 &= \frac{2\left(\cosh^2(\kappa x + \theta_0) + \mathrm{i}(\sigma_0 - 2\kappa^2 t)\right)}{\left(\cosh^2(\kappa x + \theta_0) - \mathrm{i}(\sigma_0 - 2\kappa^2 t)\right)^2}\, \mathrm{e}^{\mathrm{i}\left(\kappa^2 t + \varphi_0\right)} \sinh(\kappa x + \theta_0) \\
v_1 &= \left(\frac{\cosh^2(\kappa x + \theta_0) + \mathrm{i}(\sigma_0 - 2\kappa^2 t)}{\cosh^2(\kappa x + \theta_0) - \mathrm{i}(\sigma_0 - 2\kappa^2 t)}\right)^2 - \frac{2}{\left(\cosh^2(\kappa x + \theta_0) - \mathrm{i}(\sigma_0 - 2\kappa^2 t)\right)^2}.
\end{aligned}
\tag{3.43}
$$

A plot of $\mathrm{Re}\, u_1(x,t)$ and $\mathrm{Re}\, v_1(x,t)$ is shown on Fig. 4

It proves to be of some interest to consider the odd dispersion case as well. In the simplest nontrivial situation when $\mathbf{f}_1(\lambda) = -8\lambda^3$ (equation (2.18)) we have

$$
\begin{aligned}
u_1 &= \frac{2\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0 + \mathrm{i}\sigma_0\right)}{\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - \mathrm{i}\sigma_0\right)^2}\, \mathrm{e}^{\mathrm{i}\varphi_0} \sinh(\kappa x + 8\kappa^3 t + \theta_0) \\
v_1 &= \left(\frac{\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) + \mathrm{i}\sigma_0}{\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - \mathrm{i}\sigma_0}\right)^2 - \frac{2}{\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - \mathrm{i}\sigma_0\right)^2}.
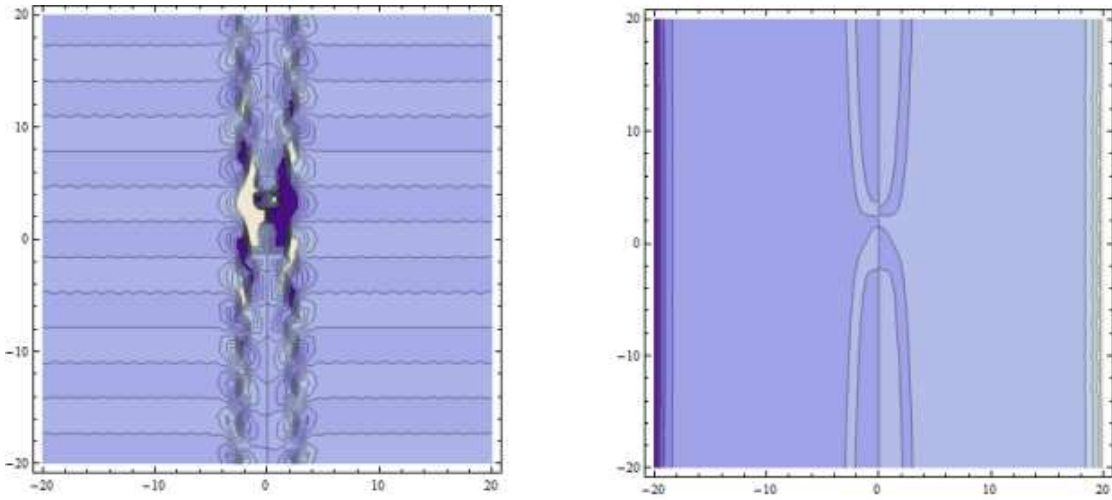\end{aligned}
\tag{3.44}
$$

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Figure 4: Contour plot of Re $u_1(x,t)$ (left panel) and Re $v_1(x,t)$ (right panel) for doublet soliton (3.43) as functions of x and t. Here $\kappa = 0$, $\sigma_0 = 5$ and $\theta_0 = 0$.

We have assumed above that $\tilde{\varphi} = 0$.

Remark 1 *Let us make a few short remarks on the behaviour of doublet soliton (3.43). First of all it is evident that this* **is not** *a travelling wave solution. Moreover, as it is seen from Fig. 5 the component $|u_1(x,t)|^2$ has two symmetric maxima and one minimum at the origin (resp. $|v_1(x,t)|^2$ has two symmetric minima and one maximum at the origin). The value of the maximum of $|u_1(x,t)|^2$ (resp. the minimum of $|v_1(x,t)|^2$) first increases with time ($\sigma(t) > 0$) and then decreases ($\sigma(t) < 0$). The maxima positions of u_1 depend on t according to:*

$$\xi_0(t) = -\frac{\theta_0}{\kappa} + \frac{1}{\kappa}\ln\left(\sqrt{1 + \sqrt{1 + \sigma^2(t)}} + \sqrt[4]{1 + \sigma^2(t)}\right), \qquad (3.45)$$

where $\sigma(t) = \sigma_0 - 2\kappa^2 t$. The soliton velocity $v := d\xi_0/dt$ **is not constant** *but changes with t as given by:*

$$v(t) = -\frac{2\kappa^2 t \sigma(t)}{1 + \sigma^2(t)} \frac{\sqrt[4]{1 + \sigma^2(t)}}{\sqrt{1 + \sqrt{1 + \sigma^2(t)}}}. \qquad (3.46)$$

Such behavior resembles the boomerons and the trappons [4, 5]. On Fig. 6 it is plotted the t-dependence of the soliton velocity.

3.3 Multisoliton Solutions

As we have already mentioned the dressing procedure can be applied several times consequently. Thus after dressing the 1-soliton solution one derives a 2-soliton solution, after dressing the 2-soliton solution one obtains a 3-soliton solution and so on. Of course, in doing this one is allowed to apply either of dressing factors (3.8) and (3.19). Therefore the multisoliton obtained will be a certain combination of quadruplet and doublet solitons. Another way of derivation the multisoliton solution consists in using a dressing factor
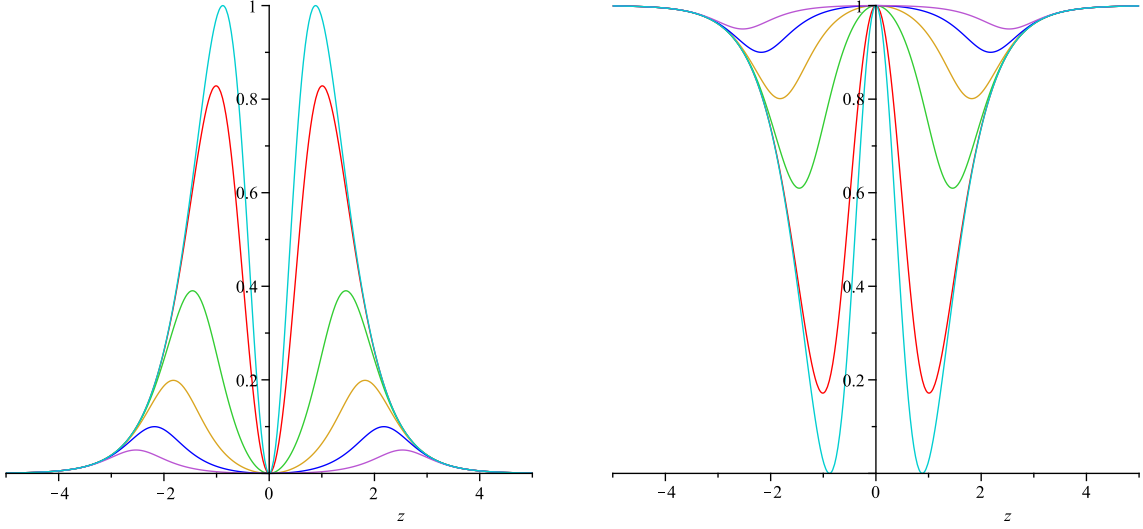
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Figure 5: Contour plot of $|u_1(x,t)|^2$ (left panel) and $|v_1(x,t)|^2$ (right panel) for doublet soliton solution (3.43) as a function of x for several values of t: $t = 0, 1, 5, 10, 20, 40$.

with a proper number of poles:

$$\Phi = \mathbb{1} + \lambda \sum_{k=1}^{N_1} \left(\frac{M_k}{\lambda - \mu_k} + \frac{\mathbf{C}M_k\mathbf{C}}{\lambda + \mu_k} \right) + \lambda \sum_{l=1}^{N_2} \left(\frac{P_l}{\lambda - \mathrm{i}\kappa_l} + \frac{\mathbf{C}P_l\mathbf{C}}{\lambda + \mathrm{i}\kappa_l} \right). \tag{3.47}$$

As it follows from (3.47) the multisoliton solution obtained will be a mixture of N_1 quadruplet solitons and N_2 doublet ones. In order to determine the residues of Φ one follows basically the same steps as in the case of a 2-poles dressing factor. Firstly, the identity $\Phi\Phi^{-1} = \mathbb{1}$ implies that the residues of Φ and Φ^{-1} fulfill some algebraic restrictions. For example, the condition

$$\lim_{\lambda \to \mu_k} (\lambda - \mu_k)\Phi\Phi^{-1} = 0, \qquad k = 1, \ldots, N_1 \tag{3.48}$$

for vanishing of the residue of $\Phi\Phi^{-1}$ at $\lambda = \mu_k$ leads to the following algebraic restrictions:

$$M_k \left[\mathbb{1} + \mu_k \sum_{r=1}^{N_1} \left(\frac{M_r^\dagger}{\mu_k - \mu_r^*} + \frac{\mathbf{C}M_r^\dagger\mathbf{C}}{\mu_k + \mu_r^*} \right) + \mu_k \sum_{l=1}^{N_2} \left(\frac{P_l^\dagger}{\mu_k + \mathrm{i}\kappa_l} + \frac{\mathbf{C}P_l^\dagger\mathbf{C}}{\mu_k - \mathrm{i}\kappa_l} \right) \right] = 0. \tag{3.49}$$

Apart of this type of constraints we have another one originating from vanishing of the coefficients before the imaginary poles:

$$\lim_{\lambda \to \mathrm{i}\kappa_l} (\lambda - \mathrm{i}\kappa_l)^2 \Phi\Phi^{-1} = (\mathrm{i}\kappa_l)^2 P_l\mathbf{C}P_l^\dagger = 0, \quad l = 1, \ldots, N_2, \tag{3.50}$$

$$\lim_{\lambda \to \mathrm{i}\kappa_l} \partial_\lambda[(\lambda - \mathrm{i}\kappa_l)^2 \Phi\Phi^{-1}] = \mathrm{i}\kappa_l\Theta_l\mathbf{C}P_l^\dagger\mathbf{C} + \mathrm{i}\kappa_l P_l\mathbf{C}\Theta_l^\dagger\mathbf{C} = 0, \tag{3.51}$$

where

$$\Theta_l = \mathbb{1} + \mathrm{i}\kappa_l \sum_{k=1}^{N_1} \left(\frac{M_k}{\mathrm{i}\kappa_l - \mu_k} + \frac{\mathbf{C}M_k\mathbf{C}}{\mathrm{i}\kappa_l + \mu_k} \right) + P_l + \frac{\mathbf{C}P_l\mathbf{C}}{2}$$

$$+ \mathrm{i}\kappa_l \sum_{s \neq l}^{N_2} \left(\frac{P_s}{\mathrm{i}(\kappa_l - \kappa_s)} + \frac{\mathbf{C}P_s\mathbf{C}}{\mathrm{i}(\kappa_l + \kappa_s)} \right).$$
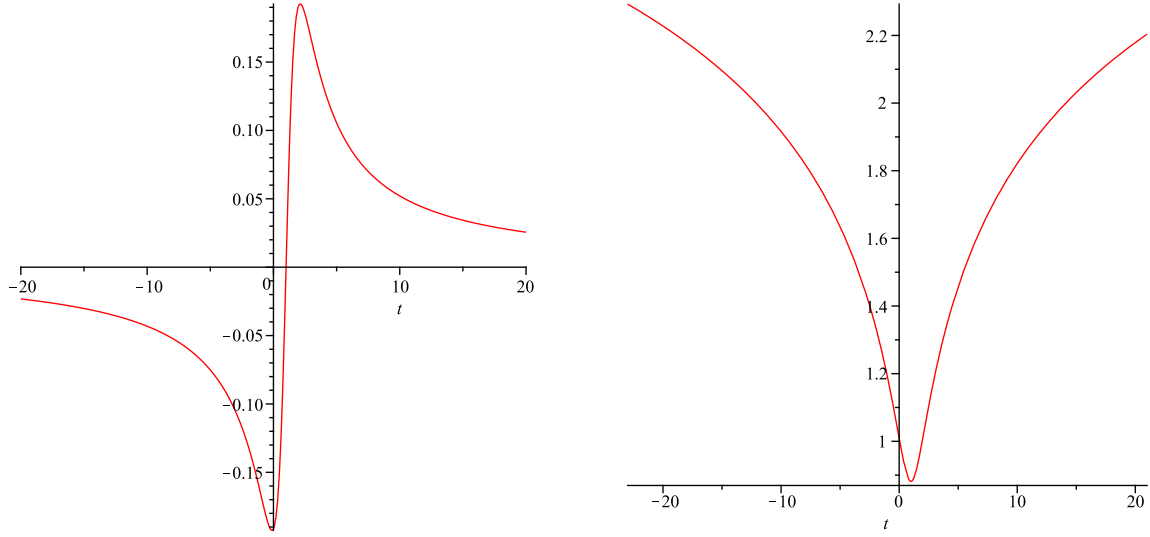
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Figure 6: The soliton velocity $v(t)$ and position of the maxima $\xi_0(t)$ of solution (3.43) as a functions of t. Here $\kappa = \sigma_0 = 1$, $\theta_0 = 0$ and $\tilde{\varphi} = 0$.

Vanishing of the rest of poles of $\Phi\Phi^{-1}$ leads to algebraic constraints which coincide with (3.49)–(3.51) due to the action of $\mathbb{Z}_2$ reductions.

Since $M_k(x,t)$ and $P_l(x,t)$ must be degenerate matrices one introduces their factorizations $M_k = |n_k\rangle\langle m_k|$ and $P_l = |q_l\rangle\langle p_l|$. Substituting it into (3.49)–(3.51) we reduce the first and the third constraint to linear systems for $|n_k\rangle$ and $|q_l\rangle$

$$
\begin{aligned}
|m_k\rangle &= \sum_{r=1}^{N_1} \mathcal{B}_{rk}|n_r\rangle + \sum_{l=1}^{N_2} \mathcal{D}_{sk}|q_s\rangle \\
\mathbf{C}|p_l\rangle &= \sum_{r=1}^{N_1} \mathcal{E}_{rl}|n_r\rangle + \sum_{s=1}^{N_2} \mathcal{F}_{sl}|q_s\rangle,
\end{aligned}
\tag{3.52}
$$

where the matrix coefficients read

$$
\begin{aligned}
\mathcal{B}_{rk} &:= \mu_k^* \left(\frac{\langle m_r|m_k\rangle}{\mu_r - \mu_k^*} - \frac{\langle m_r|\mathbf{C}|m_k\rangle}{\mu_r + \mu_k^*}\mathbf{C} \right) \\
\mathcal{D}_{sk} &:= \mu_k^* \left(\frac{\langle p_s|m_k\rangle}{i\kappa_s - \mu_k^*} - \frac{\langle p_s|\mathbf{C}|m_k\rangle}{i\kappa_s + \mu_k^*}\mathbf{C} \right) \\
\mathcal{E}_{rl} &:= -i\kappa_l \left(\frac{\langle m_r|\mathbf{C}|p_l\rangle}{i\kappa_l - \mu_k} + \frac{\langle m_r|p_l\rangle}{i\kappa_l + \mu_k}\mathbf{C} \right), \quad \mathcal{F}_{ss} := i\sigma_s - \frac{\langle p_s|p_s\rangle}{2}\mathbf{C} \\
\mathcal{F}_{sl} &:= \kappa_l \left(\frac{\langle p_s|\mathbf{C}|p_l\rangle}{\kappa_s - \kappa_l} - \frac{\langle p_s|p_l\rangle}{\kappa_s + \kappa_l}\mathbf{C} \right), \qquad s \neq l
\end{aligned}
$$

By inverting the linear system (3.52) we can express $|n_r\rangle$ and $|q_s\rangle$ through all $|m_k\rangle$, $|p_l\rangle$ and σ_l and that way determine the dressing factor in terms of the latter. The vectors $|m_k\rangle$ and $|p_l\rangle$ as well as the functions σ_l can be found from the natural requirement of vanishing of the poles in (3.15). The result reads

$$
\begin{aligned}
|m_k(x,t)\rangle &= \psi_0(x,t,\mu_k^*)|m_{k,0}\rangle \\
|p_l(x,t)\rangle &= \psi_0(x,t,-i\kappa_l)|p_{l,0}\rangle \\
\sigma_l(x,t) &= -\kappa_l \langle p_{l,0}|\psi^{-1}(x,t,i\kappa_l)\dot{\psi}_0(x,t,i\kappa_l)\mathbf{C}|p_{l,0}\rangle + \sigma_{l,0}.
\end{aligned}
\tag{3.53}
$$

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Analogously to the 2-poles case the components of $|p_l\rangle$ are not independent. As a result of (3.50) that the following relations holds true:

$$\langle p_l(x,t)|\mathbf{C}|p_l(x,t)\rangle = \langle p_{l,0}|\mathbf{C}|p_{l,0}\rangle = 0. \tag{3.54}$$

Thus we have proved that the dressing factor in the multiple poles case is determined if one knows the initial fundamental solution $\psi_0(x,t,\lambda)$. The multisoliton solution itself can be derived through the following formula

$$L_1^{(1)}(x,t) = \Phi(x,t,\infty)L_1^{(0)}(x,t)\Phi^\dagger(x,t,\infty), \tag{3.55}$$

where

$$\Phi(x,t,\infty) = \mathbb{1} + \sum_{k=1}^{N_1}(M_k + \mathbf{C}M_k\mathbf{C}) + \sum_{l=1}^{N_2}(P_l + \mathbf{C}P_l\mathbf{C}).$$

From all said above it follows that the algorithm for obtaining the multisoliton solution can be presented symbolically as follows

$$L_1^{(0)} \longrightarrow \{|m_k\rangle\}_{k=1}^{N_1}, \ \{|p_l\rangle\}_{l=1}^{N_2}, \ \{\sigma_l\}_{l=1}^{N_2} \longrightarrow$$
$$\{|n_k\rangle\}_{k=1}^{N_1}, \{|q_l\rangle\}_{l=1}^{N_2} \longrightarrow \{M_k\}_{k=1}^{N_1}, \{P_l\}_{l=1}^{N_2} \longrightarrow L_1^{(1)}.$$

4 Interactions of Quadruplet Solitons

In this section we aim to study the interactions of solitons we have derived. We shall restrict ourselves with quadruplet solitons for NLEEs with odd dispersion laws. This is the simplest case since the solitons are travelling wave-type solutions. The interactions of the other types of solitons require a special treatment and will be done elsewhere.

Our study will be based on the Zakharov-Shabat scheme [34] applied to the recursive procedure (3.28). Their approach consists in calculating the asymptotics of generic N-soliton solution for $t \to \pm\infty$ and establishing the pure elastic character of the interactions of generic soliton, i.e. solitons travelling at different velocities. The pure elastic character of the soliton interactions is demonstrated by the fact that for $t \to \pm\infty$ the N-soliton solution splits into a sum of N one soliton solutions preserving its amplitudes and velocities. The only effect of the interaction consists in shifting the center of mass and the initial phase of the solitons.

The 1-soliton dressing factor corresponding to the quadruplet case with poles at $\pm\mu_k$ is given by:

$$\Phi_k(x,t,\lambda,\mu_k) = \mathbb{1} + \frac{\lambda}{\lambda - \mu_k}M_k(x,t) + \frac{\lambda}{\lambda + \mu_k}\mathbf{C}M_k(x,t)\mathbf{C}. \tag{4.1}$$

The residues $M_k(x,t) = |n_k\rangle\langle m_k|$ are determined by the following equalities

$$|n_k\rangle = \frac{1}{\mu_k^*}\left(\frac{\langle m_k|m_k\rangle}{2i\kappa_k} - \frac{\langle m_k|\mathbf{C}|m_k\rangle}{2\omega_k}\mathbf{C}\right)^{-1}|m_k\rangle$$

$$|m_k(x,t)\rangle = \psi_0(x,t,\mu_k)|m_{k0}\rangle, \qquad |m_{k0}\rangle = \begin{pmatrix} \alpha_k + \beta_k \\ \gamma_k \\ \alpha_k - \beta_k \end{pmatrix}. \tag{4.2}$$

Let us now outline the alternative procedure for constructing the N-soliton solutions of the NLEE (2.13). The idea is to apply subsequently N times the the one-soliton dressing.

For simplicity we assume that all N solitons are of quadruplet type. As a result the sequence of mappings (3.28) allows us to constructs a sequence of Lax operators with potentials $L_1^{(k)}$, $k = 1, \ldots, N$ and eigenfunctions:

$$\chi_{(k)}^\pm(x, t, \lambda) = \mathbf{\Phi}_k(x, t, \lambda, \mu_k)\mathbf{\Phi}_{k-1}(x, t, \lambda, \mu_{k-1}) \ldots \mathbf{\Phi}_1(x, t, \lambda, \mu_1)$$
$$\times \psi_0(x, t, \lambda)\mathbf{\Phi}_{1,-}^\dagger(\lambda, \mu_1) \ldots \mathbf{\Phi}_{k-1,-}^\dagger(\lambda, \mu_{k-1})\mathbf{\Phi}_{k,-}^\dagger(\lambda, \mu_k), \tag{4.3}$$

where

$$\mathbf{\Phi}_{k,-}(\lambda, \mu_k) = \lim_{x \to -\infty} \mathbf{\Phi}_k(x, t, \lambda, \mu_k) \tag{4.4}$$

The dressing factors $\mathbf{\Phi}_k(x, t, \lambda, \mu_k)$ are constructed in analogy with (4.1) as follows:

$$\mathbf{\Phi}_k(x, t, \lambda, \mu_k) = \mathbb{1} + \frac{\lambda}{\lambda - \mu_k}\mathbf{M}_k(x, t) + \frac{\lambda}{\lambda + \mu_k}\mathbf{C}\mathbf{M}_k(x, t)\mathbf{C}$$
$$\mathbf{M}_k(x, t) = \frac{1}{\mu_k^*}\left(\frac{\langle \mathbf{m}_k|\mathbf{m}_k\rangle}{2\mathrm{i}\kappa_k} - \frac{\langle \mathbf{m}_k|\mathbf{C}|\mathbf{m}_k\rangle}{2\omega_k}\mathbf{C}\right)^{-1}|\mathbf{m}_k\rangle\langle \mathbf{m}_k| \tag{4.5}$$
$$|\mathbf{m}_k\rangle = \Phi_{k-1}(x, t, \mu_k, \mu_{k=1}) \ldots \Phi_1(x, t, \mu_k, \mu_1)|m_k\rangle.$$

Thus for the N-soliton potential we obtain:

$$L_1^{(N)}(x, t) = \lim_{\lambda \to \infty} \chi_{(N)}^\pm(x, t, \lambda)L_1^{(0)}.\hat{\chi}_{(N)}^\pm(x, t, \lambda) \tag{4.6}$$

Next we recall that we are considering NLEE with odd dispersion laws (2.13b). Their one-soliton solutions are traveling waves and depend on $Z_k = x - V_k t$, where $V_k = 1/\kappa_k \mathrm{Im} f_1(\mu_k)$. In particular, for the eq. (2.18) $\mathbf{f}_1(\lambda) = -8\lambda^3$ and $V_k = 8(3\mu_k^2 - \kappa_k^2)$. Now let us to pick up the trajectory of the N-th soliton: $Z_N \equiv x - 2\omega_N t/3 = $ fixed and evaluate the asymptotics of $L_1^{(N)}(x, t)$ for $t \to \pm\infty$ for fixed Z_N. This will allow us to see what are the effects of the soliton interactions on the N-th soliton.

In what follows we will assume that all solitons move with different velocities, i.e. $V_j \neq V_k$ for $k \neq j$. It is natural to split the solitons in two groups:

$$\mathcal{M}_+ \equiv \{V_k : V_k > V_N\}, \qquad \mathcal{M}_- \equiv \{V_k : V_j < \omega_N\}, \tag{4.7}$$

i.e., the solitons belonging to $\mathcal{M}_+$ are moving faster than the N-th soliton, while the ones belonging to $\mathcal{M}_-$ are slower.

Now we are able to calculate the limits of $\Phi_k(x, t, \lambda)$ for $t \to \pm\infty$ for fixed Z_N. To do this we firstly need to obtain the limits of the one-soliton dressing factor for $x \to \pm\infty$. It can be verified that:

$$\Phi_{k,-}(\lambda, \mu_k) := \lim_{x \to -\infty} \Phi_k(x, t, \lambda) = \begin{pmatrix} c_k(\lambda) & 0 & -c_k'(\lambda), \\ 0 & 1 & 0 \\ -c_k'(\lambda) & 0 & c_k(\lambda) \end{pmatrix},$$
$$\Phi_{k,+}(\lambda, \mu_k) := \lim_{x \to \infty} \Phi_k(x, t, \lambda) = \begin{pmatrix} c_k(\lambda) & 0 & c_k'(\lambda) \\ 0 & 1 & 0 \\ c_k'(\lambda) & 0 & c_k(\lambda) \end{pmatrix}, \tag{4.8}$$

where

$$c_k(\lambda) = \frac{\mu_k}{\mu_k^*}\frac{\lambda^2 - |\mu_k|^2}{\lambda^2 - \mu_k^2}, \qquad c_k'(\lambda) = -\frac{\mu_k}{\mu_k^*}\frac{\lambda(\mu_k - \mu_k^*)}{\lambda^2 - \mu_k^2}.$$

Note that the asymptotics $\Phi_{k,\pm}(\lambda, \mu_k)$ do not depend upon the polarization vectors $|m_{k0}\rangle$ and that they commute for different values of λ. This allows us to describe explicitly the N-soliton interactions of quadruplet solitons.

The action of $\Phi_{k,\pm}(\lambda, \mu_k)$ on the polarization vectors produces the equalities:

$$
\Phi_{k,\pm}(\lambda, \mu_k) \begin{pmatrix} \alpha_k + \beta_k \\ \gamma_k \\ \alpha_k - \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_k^{\pm} + \beta_k^{\pm} \\ \gamma_k \\ \alpha_k^{\pm} - \beta_k^{\pm} \end{pmatrix}
$$
$$
\frac{\alpha_k^{\pm}}{\alpha_k} = \frac{\mu_k}{\mu_k^*} \frac{\lambda \pm \mu_k^*}{\lambda \pm \mu_k}, \qquad \frac{\beta_k^{\pm}}{\beta_k} = \frac{\mu_k}{\mu_k^*} \frac{\lambda \mp \mu_k^*}{\lambda \mp \mu_k}.
$$
(4.9)

Next we have to evaluate the asymptotics of $|\mathbf{m}_k(x,t)\rangle$ when $t \to \pm\infty$ along the trajectory $Z_N(x,t) = \text{const}$. This is done recursively using (4.8). Skipping all technical details here we get:

$$
|\mathbf{m}_N(x,t)\rangle \underset{t\to\infty}{\simeq} \prod_{j\in\mathcal{M}_+} \Phi_+(\mu_N, \mu_j) \prod_{j\in\mathcal{M}_-} \Phi_-(\mu_N, \mu_j)|m_N(x,t)\rangle
$$
$$
|\mathbf{m}_N(x,t)\rangle \underset{t\to-\infty}{\simeq} \prod_{j\in\mathcal{M}_+} \Phi_-(\mu_N, \mu_j) \prod_{j\in\mathcal{M}_-} \Phi_+(\mu_N, \mu_j)|m_N(x,t)\rangle.
$$
(4.10)

Then from (4.9) and (4.10) one deduces that:

$$
\frac{\alpha_N^+}{\alpha} = \prod_{k=1}^{N} \frac{\mu_k}{\mu_k^*} \prod_{j\in\mathcal{M}_+} A_{N,j} \prod_{j\in\mathcal{M}_-} B_{N,j} \qquad \frac{\alpha_N^-}{\alpha} = \prod_{k=1}^{N} \frac{\mu_k}{\mu_k^*} \prod_{j\in\mathcal{M}_+} B_{N,j} \prod_{j\in\mathcal{M}_-} A_{N,j}
$$
$$
\frac{\beta_N^+}{\beta} = \prod_{k=1}^{N} \frac{\mu_k}{\mu_k^*} \prod_{j\in\mathcal{M}_+} B_{N,j} \prod_{j\in\mathcal{M}_-} A_{N,j} \qquad \frac{\beta_N^-}{\beta} = \prod_{k=1}^{N} \frac{\mu_k}{\mu_k^*} \prod_{j\in\mathcal{M}_+} A_{N,j} \prod_{j\in\mathcal{M}_-} B_{N,j}
$$
$$
A_{N,j} = \frac{\mu_N + \mu_j^*}{\mu_N + \mu_j}, \qquad\qquad B_{N,j} = \frac{\mu_N - \mu_j^*}{\mu_N - \mu_j}.
$$
(4.11)

As a result we obtain that: i) the soliton interactions are purely elastic, and ii) their effect is shifts of the relative center of mass and the phase $\delta_N = \arg\alpha - \arg\beta$ of the solitons:

$$
Z_N^{\pm} = Z_N \mp \sum_{j\in\mathcal{M}_+} z_{N,j} \pm \sum_{j\in\mathcal{M}_-} z_{N,j}
$$
$$
\delta_N^{\pm} = \delta_N \pm \sum_{j\in\mathcal{M}_+} \phi_{N,j} \mp \sum_{j\in\mathcal{M}_-} \phi_{N,j}
$$
(4.12)
$$
z_{N,j} = \frac{1}{2\kappa_N}(\ln|A_{N,j}| - \ln|B_{N,j}|), \quad \phi_{N,j} = \arg(A_{N,j}) - \arg(B_{N,j}).
$$

5 Integrals of Motion

Here we will sketch briefly the direct method for finding integrals of motion, introduced by Drinfel'd and Sokolov [6]. We will apply it to the system (2.17). In order to do that it proves to be technically more convenient to deal with the Lax pair (2.19), (2.21). We will

use the transformation $\mathcal{P}(x, t, \lambda)$ that diagonalizes simultaneously the Lax pair $\tilde{L}$ and $\tilde{A}$:

$$\mathcal{L} = \mathcal{P}^{-1}\tilde{L}\mathcal{P} = \mathrm{i}\partial_x + \lambda J + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \cdots$$

$$\mathcal{A} = \mathcal{P}^{-1}\tilde{A}\mathcal{P} = \mathrm{i}\partial_t + \lambda^2 I + \lambda \mathcal{A}_{-1} + \mathcal{A}_0 + \frac{\mathcal{A}_1}{\lambda} + \cdots .$$

(5.1)

Here all matrix coefficients $\mathcal{L}_k$, $\mathcal{A}_{-1}$ and $\mathcal{A}_k$, $k = 0, 1, \ldots$ are diagonal. Using the asymptotic expansion for $\mathcal{P}(x, t, \lambda)$:

$$\mathcal{P}(x, t, \lambda) = \mathbb{1} + \frac{p_1(x, t)}{\lambda} + \frac{p_2(x, t)}{\lambda^2} + \cdots$$

(5.2)

one can get a set of recurrence relations:

$$U_0 + Jp_1 = \mathcal{L}_0 + p_1 J \tag{5.3}$$
$$\mathrm{i}p_{1,x} + U_0 p_1 + Jp_2 = \mathcal{L}_1 + p_1 \mathcal{L}_0 + p_2 J \tag{5.4}$$

$$\vdots$$

$$\mathrm{i}p_{k,x} + U_0 p_k + Jp_{k+1} = \mathcal{L}_k + p_{k+1} J + \sum_{m=0}^{k-1} p_{k-m} \mathcal{L}_m \tag{5.5}$$

$$\vdots$$

Here we assume that all coefficients p_l $(l = 1, 2, \ldots)$ are off-diagonal matrices.

In order to solve the recursion relations above, we will split each relation into a diagonal and off-diagonal part. For example, treating this way the first relation above one gets

$$\mathcal{L}_0 = U_0^{\mathrm{d}}, \qquad U_0^{\mathrm{f}} = -[J, p_1], \tag{5.6}$$

where the superscripts d and f above denote projection onto diagonal and off-diagonal part of a matrix respectively. Taking into account the explicit form of U_0 for $\mathcal{L}_0$ we have

$$\mathcal{L}_0 = \frac{\mathrm{i}}{2}(uu_x^* + vv_x^*) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.7}$$

Thus as a density of our first integral we can choose: $\mathcal{I}_0 = u^* u_x + v^* v_x$. It represents momentum density of our system. For the stationary solutions (3.33) and (3.39 the momentum density is depicted on Figure 7. It is evidential, that the integrals of motion are well localized function of x.

Similarly, for the second integral density one gets

$$\mathcal{I}_1 = |uu_x^* + vv_x^*|^2 + 4|uv_x - vu_x|^2.$$

In general, the-k integral of motion can be calculated through the formula

$$\mathcal{L}_k = \left(U_0^f p_k\right)^d. \tag{5.8}$$

The matrix p_k in its turn is obtained from the following recursive formula

$$p_k = -\mathrm{ad}_J^{-1}\left(\mathrm{i}p_{k-1,x} + (U_0 p_{k-1})^f - \sum_{m=0}^{k-1} p_{k-1-m}\mathcal{L}_m\right). \tag{5.9}$$
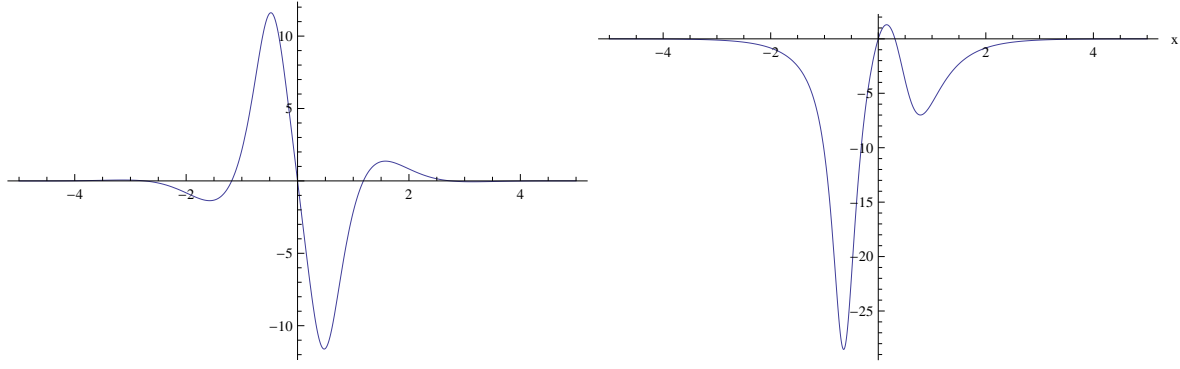
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Figure 7: Plots of the density of the first integral of motion as a function of x evaluated on the stationary quadruplet soliton (3.33) for $\alpha = \beta = \gamma = \omega = 1$, $\delta = 0$ (left panel) and for the stationary doublet soliton (3.39) for $\kappa = \sigma_0 = 1$, $\phi = 0$ (right panel).

Note, that the zero curvature representation is gauge invariant, i. e. $[\mathcal{L}, \mathcal{A}] = 0$ is fulfilled. Since $[\mathcal{L}_k, \mathcal{A}_l] = 0$ the commutativity of $\mathcal{L}$ and $\mathcal{A}$ is equivalent to the following requirements

$$\partial_x \mathcal{A}_{-1} = 0, \qquad \partial_t \mathcal{L}_k - \partial_x \mathcal{A}_k = 0, \qquad k = 0, 1, \dots \qquad (5.10)$$

Hence $\mathcal{L}_k$ represent densities of the integrals of motion we are interested in.

6 Conclusions

The soliton solutions for a hierarchy of NLEEs related to the symmetric space $SU(3)/S(U(1) \times U(2))$ are constructed. In order to obtain the soliton solutions we have applied the dressing procedure with a 2-poles dressing factor. It has been shown that there exist two types of 1-soliton solutions: quadruplet solitons which are associated with 4 symmetrically located eigenvalues of L and doublet solitons which are associated with a pair of purely imaginary eigenvalues. This remarkable fact is a consequence of the simultaneous action of two $\mathbb{Z}_2$ reductions on the Lax pair. The properties of the elementary solitons depend crucially upon the symmetry properties the dispersion law. For example, if the dispersion law is an even polynomial then the elementary soliton of the first type will be stationary (see formula (3.33)) otherwise it is time-dependent (formula (3.34)). In the case of the doublet type solitons the situation changes significantly — the components of the polarization vector $|m_0\rangle$ are no longer independent, see (3.26). This is why we have only two cases possible: generic case and a degenerate case. In the latter case the doublet soliton is stationary if $\mathbf{f}(\lambda)$ is an even polynomial, otherwise they are time-depending. In the generic case a new phenomenon arises. When the dispersion law is an even polynomial the soliton is not a traveling wave. Its behavior resembles that of trappons and boomerons — the soliton velocity is not fixed but varies with time.

We have described the quadruplet soliton interactions for NLEE with odd dispersion laws by calculating explicitly their asymptotics along the soliton trajectories in the generic case (different soliton velocities). The important result consisted in the following:

i) the N-soliton interactions are purely elastic and always split into sequences of elementary 2-soliton interactions;

ii) the effect of each 2-soliton interaction consists in shifts of the relative center of mass and relative phases of each of the solitons;

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iii) the corresponding shifts are different from the ones for the NLS and Heisenberg ferromagnetic equations.

Acknowledgements

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