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MULTIPLE SOLUTIONS OF THE QUASIRELATIVISTIC CHOQUARD EQUATION

M. MELGAARD AND F. ZONGO

ABSTRACT. We prove existence of multiple solutions to the quasirelativistic Choquard equations with a scalar potential.

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1. INTRODUCTION

We study the nonlocal and nonlinear problem

$$L\phi + V\phi - |\phi|^2 * W\phi = -\lambda\phi, \quad (1.1)$$

$$\|\phi\|_{L^2(\mathbb{R}^3)} = 1, \quad (1.2)$$

for a large class of potentials V and W , and $L = \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}$ (the quasirelativistic Laplacian) with α being Sommerfeld's fine structure constant. This Hartree-like Choquard equation arises as the Euler-Lagrange equation associated with a energy functional $\mathcal{E}(\cdot)$ introduced in (3.2). We prove the existence of multiple solutions for two separate cases. Theorem 3.2 concerns the unconstrained problem (1.1), and Theorem 3.4 treats the constrained problem (1.1)-(1.2).

By replacing L by the negative Laplacian and by choosing $V = 0$, and $W(x) = 1/|x|$, we obtain the nonrelativistic Choquard equation which models an electron trapped in its own hole and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory of a one-component plasma [6]. In a meson nucleon theory a system similar to this equation, but with $W(x) = \frac{e^{-\mu|x|}}{|x|}$, arises when one includes the nucleon recoil caused by surrounding mesons [9]; this classical model provides solitary waves. A quantum theory of gravitating particles yields another example [2]. Furthermore, the Choquard equation has become a prototype of nonlocal problems, which arise in many situations [17].

For the nonrelativistic Choquard equation (in the special case $W(x) = 1/|x|$) Lieb proved existence and uniqueness (modulo translations) of a minimizer (for some λ) by using symmetric decreasing rearrangement inequalities. His existence proof can be extended to more general W provided W is symmetric decreasing which, in some sense, has to be considered a severe restriction; regularity of the solution was subsequently studied by Menzala [14].

Within the same setting, always for the negative Laplacian, Lions [13] proved existence of infinitely many spherically symmetric solutions by application of abstract critical point theory both without the constraint (here it suffices that W is spherical symmetric) and with the constraint (more severe restrictions on W must be assumed). Zhang [18, 19] has

studied existence of solutions for the nonhomogenous Choquard equation; considering $\lambda = 1$, a negative V which tends to zero at infinity, and adding a positive function g on the right-hand side of (1.1). Küpper, Zhang, and Xia [10] have studied positive solutions and the bifurcation problem arising when one adds a term $\mu f(x)$ to the (1.1); $\mu > 0$ and f being nonnegative. Furthermore, Zhang, Küpper, Hu and Xia have studied existence of solutions, when the right-hand side is multiplied by a positive function which tends to a constant at infinity [20].

For $V = 0$ and $W = 1/|x|$, the first rigorous study of (1.1) was performed by Lieb and Yau [12] in a slightly different context, when the constraint is replaced by $\|\phi\|_{L^2} = N$. They established the existence of a symmetric decreasing minimizer provided $N < N_b$ for some number N_b .

We prove existence of multiple solutions, including a minimizer of the corresponding energy functional \mathcal{E} . Moreover, we prove some additional properties of the solutions. Our proofs are based upon two classic theorems of critical point theory: in the unconstrained case we apply the mountain pass theorem by Ambrosetti and Rabinowitz [3], and for the constrained case, we apply a suitable variant due to Berestycki and Lions [5].

2. PRELIMINARIES

Throughout the paper we denote by C (with or without indices) various constants whose precise value is of no importance. Let \mathbb{R}^N be the N -dimensional Euclidean space. We set

$$B_R = \{x \in \mathbb{R}^N : |x| < R\}, \quad B(x, R) = \{y \in \mathbb{R}^N : |x - y| < R\}.$$

By \mathbb{S}^{N-1} we will denote the unit sphere in \mathbb{R}^N .

Functions. By C_0^∞ , C^∞ , and L^p we refer to the standard function spaces. For a measure space $\langle M, \mu \rangle$, μ being a σ -finite measure, the weak L^p space (or Marcinkiewicz space) is defined as the space of measurable functions ϕ such that

$$\sup_{t>0} t \mu(\{x : |\phi(x)| > t\})^{1/p} < \infty.$$

The space of bounded measures is denoted \mathcal{M}_b .

Sobolev spaces. Denoting the Fourier-Plancherel transform of $u \in L^2(\mathbb{R}^3)$ by \hat{u} , we define

$$\mathbf{H}^{1/2}(\mathbb{R}^3) = \{\phi \in L^2(\mathbb{R}^3) : (1 + |\xi|)^{1/2} \hat{\phi} \in L^2(\mathbb{R}^3)\}, \quad (2.1)$$

which, equipped with the scalar product

$$\langle \phi, \psi \rangle_{\mathbf{H}^{1/2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|) \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \, d\xi,$$

becomes a Hilbert space; evidently, $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$. We have that $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the continuous embedding $\mathbf{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ holds whenever $r \in [2, 3]$ [1, Theorem 7.57]. Moreover, we shall use that any weakly convergent sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ has a pointwise convergent subsequence. The space of radial (i.e., spherically symmetric) functions belonging to $\mathbf{H}^{1/2}(\mathbb{R}^3)$ will be denoted $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$.

Auxiliary results. We need the following ‘‘radial’’ lemma by Lions [4].

Lemma 2.1. *If $u \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, is a radial nonincreasing function (i.e., $0 \leq u(x) \leq u(y)$ whenever $|x| \geq |y|$), then*

$$|u(x)| \leq |x|^{-N/p} \left(\frac{N}{|\mathbb{S}^{N-1}|} \right)^{1/p} \|u\|_{L^p(\mathbb{R}^N)}, \quad x \neq 0.$$

Moreover, we will apply the following compactness lemma due to Strauss [15].

Lemma 2.2. *Let P and $Q : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions satisfying $P(s)/Q(s) \rightarrow 0$ as $s \rightarrow +\infty$. Let (u_n) be a sequence of measurable functions from \mathbb{R}^N into \mathbb{R} such that*

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| dx < \infty$$

and

$$P(u_n(x)) \rightarrow v(x) \text{ a.e. in } \mathbb{R}^N, \quad \text{as } n \rightarrow +\infty.$$

Then for any bounded Borel set Ω one has

$$\int_{\Omega} |P(u_n(x)) - v(x)| dx \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

If, moreover, one assumes that $P(s)/Q(s) \rightarrow 0$ as $s \rightarrow 0$ and $u_n(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ uniformly with respect to n , then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \rightarrow \infty$.

Genus. The genus of any compact symmetric subset A of $\mathbf{H}_r^{1/2}(\mathbb{R}^3) \setminus \{0\}$ will be denoted by $\gamma(A)$. Bear in mind that the boundary ∂A of a symmetric bounded neighborhood of 0 in a d -dimensional space has a genus equal to d . For the definition and properties of the genus, we refer to Struwe [16].

3. ASSUMPTIONS AND MAIN THEOREMS

Functionals. The kinetic energy is defined by

$$\tilde{\mathfrak{l}}_0[\phi] := \alpha^{-1} \|\hat{\phi}(k)\|_{L^2(\mathbb{R}^3, (\sqrt{(2\pi|k|)^2 + \alpha^{-2}} - \alpha^{-1}) dx)}^2$$

on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. It is convenient to introduce

$$\mathfrak{l}_0[\phi] := \alpha^{-1} \|\hat{\phi}(k)\|_{L^2(\mathbb{R}^3, \sqrt{(2\pi|k|)^2 + \alpha^{-2}} dx)}^2.$$

Moreover, we introduce

$$\mathfrak{s}_V : \mathbf{H}^{1/2}(\mathbb{R}^3) \rightarrow \mathbb{R} \text{ by } \phi \mapsto \int_{\mathbb{R}^3} V(x) |\phi(x)|^2 dx \quad (3.1)$$

along with (arising from the direct Coulomb energy)

$$\mathcal{J}_W(\psi, \phi) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(x) \phi(y) W(x-y) dx dy,$$

whenever it makes sense. We consider the following functional $\mathcal{E} : \mathbf{H}^{1/2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$\phi \mapsto \frac{1}{2} \mathfrak{l}_0[\phi] + \frac{1}{2} \mathfrak{s}_V[\phi] + \frac{1}{2} (\lambda - \alpha^{-2}) \|\phi\|_{L^2}^2 - \frac{1}{4} \mathcal{J}_W(|\phi|^2, |\phi|^2), \quad (3.2)$$

At this place we do not focus on whether the functionals are well-defined or not, this will be discussed in detail in the sequel.

Assumptions. We impose the following conditions.

Assumption 3.1. Let V be a real-valued measurable function on \mathbb{R}^3 such that V is nonnegative, the associated form \mathfrak{s}_V is \mathfrak{l}_0 -bounded with bound less than one, and $\mathfrak{l}_0 + \mathfrak{s}_V$ is weakly lower semicontinuous on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Let W be a nonnegative, nonzero, spherically symmetric measure such that there exist $K \geq 1$, $p_k \in (1, \infty)$, with $k \in [1, K]$, and functions W_k satisfying

$$\begin{cases} W = \nu + \sum_{k=1}^K W_k, \\ \nu \in \mathcal{M}_b(\mathbb{R}^3), \quad W_k \in L_{\mathfrak{w}}^{p_k}(\mathbb{R}^3). \end{cases}$$

We have:

Theorem 3.2. *Let Assumption 3.1 be satisfied. Then, for $\lambda > \alpha^{-2}$, there exists a sequence of (nontrivial) solutions $(u_j)_{j \geq 1}$ of (1.1) satisfying:*

1. *The functions u_j are radial and non-increasing.*
2. *The function u_1 is positive and decreasing provided W is non increasing and V is nonnegative and bounded from above.*
3. *One has*

$$0 < \mathcal{E}(u_{j-1}) \leq \mathcal{E}(u_j) \xrightarrow{j \rightarrow \infty} \infty.$$

The general case. We introduce, for $N > 0$, the set

$$\mathcal{C} = \{ u \in \mathbf{H}_r^{1/2}(\mathbb{R}^3) : \|u\|_{L^2} = N \}.$$

We seek critical points of \mathcal{E} restricted to \mathcal{C} .

Assumption 3.3. Let V satisfy the hypotheses in Assumption 3.1. Let W be a nonnegative, nonzero, spherically symmetric measure such that there exist $K \geq 1$, $p_k \in (3/2, \infty)$, with $k \in [1, K]$, and functions W_k satisfying

$$W = \sum_{k=1}^K W_k, \quad W_k \in L_{\mathfrak{w}}^{p_k}(\mathbb{R}^3).$$

The main result is:

Theorem 3.4. *Let Assumption 3.3 be satisfied and let $d \geq 1$. Suppose there exists a compact symmetric set Ω such that*

$$\Omega \subset \mathcal{C}; \quad \gamma(\Omega) \geq d, \quad : \quad \mathcal{E}(u) < 0 \text{ for } u \in \Omega. \quad (3.3)$$

Then there exists a sequence of pairs $(\lambda_j, u_j)_{1 \leq j \leq d}$ satisfying

$$\begin{cases} \alpha^{-2} < \lambda_j < \infty \\ u_j \text{ is a solution of (1.1) with } \lambda = \lambda_j \end{cases}$$

and, furthermore, one has:

1. The function u_1 is positive and

$$\mathcal{E}(u_1) = \min_{\phi \in \mathcal{C}} \mathcal{E}(\phi) < 0.$$

2. The functions u_j belong to \mathcal{C} .

3. One has $\mathcal{E}(u_1) \leq \mathcal{E}(u_2) \leq \dots \leq \mathcal{E}(u_j) < 0$.

4. All u_j are distinct.

If (3.3) holds for all d , then assertions 1-3 are valid for $j \geq 1$ and $\mathcal{E}(u_j) \nearrow 0$ as $j \rightarrow \infty$.

4. UNCONSTRAINED PROBLEM. PROOF OF THEOREM 3.2

We begin with the following auxiliary result.

Lemma 4.1. *For every $u \in \mathbf{H}^{1/2}(\mathbb{R}^3)$ we have*

$$\frac{1}{2} \|u\|_{\mathbf{H}^{1/2}}^2 \leq \langle u, (\sqrt{-\Delta + \alpha^{-2}})u \rangle \leq \alpha^{-1} \|u\|_{\mathbf{H}^{1/2}}^2. \quad (4.1)$$

Proof. For every real $a \geq 0$ and $b \geq 1$ we have the following inequality

$$\frac{a+1}{2} \leq \sqrt{a^2 + b^2} \leq b(a+1). \quad (4.2)$$

Letting $a = 2\pi|k|$ and $b = \alpha^{-1}$ in (4.2) we get

$$\frac{2\pi|k| + 1}{2} \leq \sqrt{(2\pi|k|)^2 + \alpha^{-1}} \leq \alpha^{-1}(2\pi|k| + 1),$$

and, consequently,

$$\frac{1}{2} \langle (2\pi|k| + \alpha^{-1})\hat{u}, \hat{u} \rangle_{L^2} \leq \langle \sqrt{(2\pi|k|)^2 + \alpha^{-2}}\hat{u}, \hat{u} \rangle_{L^2} \leq \alpha^{-1} \langle (2\pi|k| + \alpha^{-1})\hat{u}, \hat{u} \rangle_{L^2}.$$

Since $\langle u, (\sqrt{-\Delta + \alpha^{-2}})u \rangle_{L^2} = \langle \sqrt{(2\pi|k|)^2 + \alpha^{-2}}\hat{u}, \hat{u} \rangle_{L^2}$ we obtain (4.1). \square

Proof of Theorem 3.2. We apply Theorems 2.1 and 2.8 of Ambrosetti and Rabinowitz [3]. For this purpose we need to verify several conditions. We divide the proof into three steps but first we fix some notation. Let $\mathcal{K} = \mathbf{H}^{1/2}(\mathbb{R}^3)$ and make the decomposition $\mathcal{K} = \mathcal{X} \oplus \mathcal{V}$, where \mathcal{V} is a finite dimensional subspace of \mathcal{K} . Moreover, we let $B_\rho = \{u \in \mathcal{X} : \|u\|_{\mathbf{H}^{1/2}} = \rho\}$.

1. First we show that there exist $\rho, \sigma > 0$ such that $\mathcal{E}_{|\partial B_\rho \cap \mathcal{X}} > \sigma$. For any $u \in \mathcal{X}$, the weak Young inequality implies that

$$\mathcal{J}_W(u^2, u^2) \leq \|W\|_{L_w^p} \|u^2\|_{L^1} \|u^2\|_{L^r} = \|W\|_{L_w^p} \|u\|_{L^2}^2 \|u\|_{L^{2r}}^2,$$

with $1/p + 1/r + 1 = 2$ and $r \in [1, 3/2]$; the latter is a consequence of the Sobolev embedding $\mathbf{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ valid for $s \in [2, 3]$. In particular, $\|u\|_{L^2} \leq C_1 \|u\|_{\mathbf{H}^{1/2}}$ and $\|u\|_{L^r} \leq C_2 \|u\|_{\mathbf{H}^{1/2}}$ and, therefore,

$$\mathcal{J}_W(u^2, u^2) \leq C \|u\|_{\mathbf{H}^{1/2}}^4 \quad (4.3)$$

From the latter inequality, Lemma 4.1, and $\lambda > \alpha^{-2}$, we get that

$$\begin{aligned} \mathcal{E}(u) &\geq \frac{\alpha^{-1}}{4} \|u\|_{\mathbf{H}^{1/2}}^2 + \frac{1}{2}(\lambda - \alpha^{-2}) \|u\|_{L^2}^2 - C \|u\|_{\mathbf{H}^{1/2}}^4 \\ &\geq \frac{\alpha^{-1}}{4} \|u\|_{\mathbf{H}^{1/2}}^2 - C \|u\|_{\mathbf{H}^{1/2}}^4 \\ &\geq \|u\|_{\mathbf{H}^{1/2}}^2 \left(\frac{\alpha^{-1}}{4} - C \|u\|_{\mathbf{H}^{1/2}}^2 \right). \end{aligned}$$

Next we choose $\text{codim } \mathcal{X}$ such that, for every $u \in \mathcal{X}$, $\|u\|_{\mathbf{H}^{1/2}}^2 < \frac{\alpha^{-1}}{4C}$. Then, for every $u \in \partial B_\rho \cap \mathcal{X}$, we conclude that $\mathcal{E}(u) > \sigma > 0$ with $\sigma = \rho^2 \left(\frac{\alpha^{-1}}{4} - C\rho^2 \right)$.

2. For each finite dimensional subspace \mathcal{V} of \mathcal{K} there exists $R = R(\mathcal{V})$ such that $\mathcal{E} < 0$ on $\mathcal{V} \setminus B_R$; B_R is defined similarly to B_ρ above. With a slight abuse of notation we let $J(u) = \mathcal{J}_W(u^2, u^2)$. Then we see that $J'(u)u = 4J(u)$ for all $u \in \mathcal{K}$. Let \mathcal{V} be a finite dimensional subspace of \mathcal{K} . For every $u \in \mathcal{K}$ with $\|u\|_{\mathbf{H}^{1/2}} \geq 1$ and, for any $t > 0$, let $g(t) = J(tu/\|u\|_{\mathbf{H}^{1/2}})$. Then $g(t) > 0$ and

$$\begin{aligned} g'(t) &= J' \left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \right) \frac{u}{\|u\|_{\mathbf{H}^{1/2}}} = \frac{1}{t} J' \left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \right) \frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \\ &= \frac{4}{t} J \left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \right) = 4t^{-1} g(t). \end{aligned}$$

Thus

$$\frac{g'(t)}{g(t)} = \frac{4}{t} \Rightarrow \int_1^{\|u\|_{\mathbf{H}^{1/2}}} \frac{g'(t)}{g(t)} dt = \int_1^{\|u\|_{\mathbf{H}^{1/2}}} \frac{4}{t} dt,$$

and, consequently,

$$\begin{aligned} \ln[J(u)] - \ln[J(tu/\|u\|_{\mathbf{H}^{1/2}})] &= \ln[\|u\|_{\mathbf{H}^{1/2}}^4] \\ \Rightarrow J(u) &= \|u\|_{\mathbf{H}^{1/2}}^4 J \left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \right). \end{aligned} \quad (4.4)$$

Let $\delta = \inf \{J(u) : \|u\|_{\mathbf{H}^{1/2}} = 1, u \in \mathcal{V}\}$ and let $\mathcal{S}_\mathcal{V}$ be the unit sphere of \mathcal{V} , and let $(u_j)_{j \geq 1}$ be a sequence in $\mathcal{S}_\mathcal{V}$. Then (u_j) is bounded and therefore there exists a subsequence of (u_j) still denoted by (u_j) that converges weakly to u in \mathcal{K} . Since $\dim \mathcal{V} < \infty$ we can assume that (u_j) is a minimizing sequence of $J(\cdot)$ and also (u_j) converges strongly to u in \mathcal{V} . The weakly lower semicontinuity of $J(\cdot)$ implies that

$$\delta = \inf_{v \in \mathcal{S}_\mathcal{V}} J(v) = \liminf_j J(u_j) \geq J(u) > 0, \quad \text{because } u \neq 0.$$

From (4.4) and above it follows that

$$J(u) \geq \|u\|_{\mathbf{H}^{1/2}}^4 \inf_{\mathcal{S}_\mathcal{V}} J(u) \text{ i.e. } J(u) \geq \delta \|u\|_{\mathbf{H}^{1/2}}^4.$$

This, in conjunction with Lemma 4.1, gives us that

$$\mathcal{E}(u) \leq \alpha^{-1} \|u\|_{\mathbf{H}^{1/2}}^2 + (\lambda - \alpha^{-2}) \|u\|_{L^2}^2 - \delta \|u\|_{\mathbf{H}^{1/2}}^4$$

It is not hard to see that $\mathcal{E}(u) \rightarrow -\infty$ as $\|u\|_{\mathbf{H}^{1/2}} \rightarrow +\infty$. This ends step 2.

3. Within the framework of Ambrosetti and Rabinowitz we look for critical points of $\mathcal{E}(\cdot)$ in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$. It is easy to see that $\mathcal{E} \in C^1(\mathbf{H}^{1/2}(\mathbb{R}^3); \mathbb{R})$. It remains to check the

Palais-Smale (PS) condition, i.e., if $(u_j)_{j \geq 1}$ is a sequence of non increasing functions in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$ such that

$$\begin{cases} \mathcal{E}(u_j) \text{ is bounded,} \\ \mathcal{E}'(u_j) = (\alpha^{-1}\sqrt{-\Delta + \alpha^{-2}})u_j + (\lambda - \alpha^{-2})u_j + Vu_j - (W * |u_j|^2)u_j \xrightarrow{\mathbf{H}^{-1/2}} 0. \end{cases}$$

then there exists a subsequence of (u_j) which converges in $\mathbf{H}^{1/2}(\mathbb{R}^3)$.

Let $(u_j)_{j \geq 1}$ be such a sequence and let $\epsilon_j = \mathcal{E}'(u_j)$. We begin by proving that $(u_j)_{j \geq 1}$ is a bounded sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Now,

$$\mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] - \mathcal{J}_W(u_j^2, u_j^2) = \langle \epsilon_j, u_j \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} \quad (4.5)$$

Since, by hypothesis, $\mathcal{E}(u_j)$ is bounded, we have that

$$\begin{aligned} \mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] &= 2\mathcal{E}(u_j) + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) \\ &\leq C + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) \end{aligned} \quad (4.6)$$

On the other hand,

$$\langle \mathcal{E}'(u_j), u_j \rangle = \mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] + \mathcal{J}_W(u_j^2, u_j^2),$$

i.e.,

$$\langle \mathcal{E}'(u_j), u_j \rangle = 2\mathcal{E}(u_j) - \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2),$$

which implies that

$$\langle \epsilon_j, u_j \rangle + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) = 2\mathcal{E}(u_j) \leq C$$

and, consequently,

$$|\langle \mathcal{E}'(u_j), u_j \rangle| \leq C \text{ and } \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) \leq C.$$

This, in conjunction with (4.6) implies that

$$\mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] \leq C,$$

whence

$$\mathfrak{I}_0[u_j] + \alpha(\lambda - \alpha^{-2})\|u_n\|_{L^2}^2 \leq C$$

because V is nonnegative. Then by (4.1) we obtain

$$\frac{1}{2}\|u_j\|_{\mathbf{H}^{1/2}}^2 + \alpha(\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 \leq C$$

Since $\lambda - \alpha^{-2} \geq 0$, then we immediately conclude that $\|u_j\|_{\mathbf{H}^{1/2}} \leq C$.

Now, by the Banach-Alaoglu theorem there exists a subsequence of u_j (still denoted u_j) such that $u_j \rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and a.e. on \mathbb{R}^3 . It is worth to mention that u is radial and non increasing because all u_j are. Since u_j is radial and non increasing, Lemma 2.1 implies that

$$|u_j(x)| \leq c|x|^{-3/2}, \quad x \neq 0.$$

Therefore $\lim_{|x| \rightarrow \infty} u_j(x) = 0$ and, consequently, $\lim_{|x| \rightarrow \infty} u(x) = 0$. Let $v_j = u_j - u$. Then it is not hard to see that $(v_j)_{j \geq 1}$ is bounded in $\mathbf{H}^{1/2}$ and $\lim_{|x| \rightarrow \infty} v_j(x) = 0$. An application of Sobolev's embedding theorem shows that each v_j belongs to $L^p(\mathbb{R}^3)$, $p \in [2, 3]$. Hence

we can apply Lemma 2.2, i.e., Strauss' compactness principle [15], wherein we choose $P(s) = |s|^r$ and $Q(s) = |s|^2 + |s|^3$, and $v = 0$. It follows that

$$\int_{\mathbb{R}^3} |v_n|^r dx \xrightarrow{n \rightarrow \infty} 0, \text{ i.e. } \|u_j - u\|_{L^r} \xrightarrow{n \rightarrow \infty} 0, \quad r \in [2, 3].$$

Next we show that $\mathcal{E}'(u_j) \rightarrow \mathcal{E}'(u)$ in $\mathbf{H}^{-1/2}(\mathbb{R}^3)$. We have $(u_j^2)_{j \geq 1}$ bounded in $L^s(\mathbb{R}^3)$, $s \in [1, \frac{3}{2}]$ since u_j is bounded in $L^r(\mathbb{R}^3)$, $r \in [2, 3]$ and, together with $W \in L^p_w(\mathbb{R}^3)$ and the generalized Young inequality, we deduce that $W * u_j^2$ is bounded in $L^q(\mathbb{R}^3)$ with $3/2 < q < \infty$. Moreover, by the dominated convergence theorem we infer that $W * u_j^2$ converges strongly to $W * |u|^2$ in $L^q(\mathbb{R}^3)$. Let $\psi_j = W * |u_j|^2$, and $w \in \mathbf{H}^{1/2}$. Then

$$\begin{aligned} |\langle \psi_j u_j - \psi u, w \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}}| &= |\langle \psi_j u_j - \psi_j u + \psi_j u - \psi u, w \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}}| \\ &\leq C [\|\psi_j(u_j - u)\|_{L^2} + \|(\psi_j - \psi)u\|_{L^2}] \end{aligned}$$

By Hölder's inequality we have that

$$\|\psi_j(u_j - u)\|_{L^2} \leq \|\psi_j^2\|_{L^l} \|(u_j - u)^2\|_{L^m}$$

with $(1/l) + (1/m) = 1$; valid because $m \in [1, 3/2]$ and $l \in (3/4, \infty)$. Then, by the uniform boundedness of ψ_j in $L^q(\mathbb{R}^3)$, $q \in (3/2, \infty)$, and the strong convergence of u_j to u in L^r , $r \in [2, 3]$, and the strong convergence of ψ_j to ψ in $L^q(\mathbb{R}^3)$, it follows that $\langle \psi_j u_j - \psi u, w \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} \rightarrow 0$ as $j \rightarrow \infty$. Hence

$$\psi_j u_j = (W * u_j^2) u_j \xrightarrow{\mathbf{H}^{-1/2}} \psi u = (W * u^2) u. \quad (4.7)$$

On the other hand, by the boundedness of u_j in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the boundedness of $W * u_j^2$ in L^q , we have that $(W * u_j^2) u_j^2$ is bounded in L^1 . These facts, together with the pointwise convergence of $(W * u_j^2) u_j^2$ to $(W * u^2) u^2$ in \mathbb{R}^3 imply that Lebesgue's dominated convergence theorem yields

$$\mathcal{J}_W(u_j^2, u_j^2) \longrightarrow \mathcal{J}_W(u^2, u^2).$$

By passing to the limit in (4.5) as $j \rightarrow \infty$, we get that

$$\lim_j \{ \mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2}) \|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] \} = \mathcal{J}_W(u^2, u^2).$$

An application of Fatou's lemma yields

$$\begin{aligned} \mathfrak{I}_0[u] + (\lambda - \alpha^{-2}) \|u\|_{L^2}^2 + \mathfrak{s}[u] &\leq \liminf_j \{ \mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2}) \|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] \} \\ &= \lim_j \{ \mathfrak{I}_0[u_j] + (\lambda - \alpha^{-2}) \|u_j\|_{L^2}^2 + \mathfrak{s}[u_j] \} \\ &= \mathcal{J}_W(u^2, u^2). \end{aligned}$$

Moreover, since u_j converges strongly to u in $L^r(\mathbb{R}^3)$, $r \in [2, 3]$, we have that

$$\alpha^{-1} \left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1} \right) u_j + \lambda u_j + V u_j \xrightarrow{\mathbf{H}^{-1/2}} \alpha^{-1} \left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1} \right) u + \lambda u + V u$$

in the sense of distributions. The latter, in conjunction with (4.7), implies that

$$\mathcal{E}'(u_j) \xrightarrow{\mathbf{H}^{-1/2}} \mathcal{E}'(u) = \left(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2} \right) u + \lambda u + V u + (W * u^2) u.$$

Then, by hypothesis, we deduce that $\mathcal{E}'(u) = 0$. In particular, $\langle \mathcal{E}'(u), u \rangle = 0$ and we infer that

$$\mathfrak{I}_0[u] + (\lambda - \alpha^{-2})\|u\|_{L^2}^2 + \mathfrak{I}[u] = \mathcal{J}_W(u^2, u^2).$$

Furthermore,

$$\begin{aligned} & \langle u_j - u, \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u) \rangle \\ &= \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u, u - u_j \rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j \rangle \\ &= \left\langle \left(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2} \right) u + \lambda u + Vu - (W * u^2)u, u - u_j \right\rangle + \int (W * |u|^2)u(u - u_j) dx \\ & \quad + (\alpha^{-2} - \lambda)\langle u, u - u_j \rangle - \langle Vu, u - u_j \rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j \rangle. \end{aligned}$$

The first term on the right-hand side is equal to $\langle \mathcal{E}'(u), u - u_j \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} = 0$, the third term from the right-hand side, viz. $\langle u, u - u_j \rangle$ tends to zero (because u_j converges weakly to u in $\mathbf{H}^{1/2}$), the same argument applies to fourth term. As for the second term we apply Hölder's inequality twice. Since both $W * u^2$ and u are bounded in L^q , $3/2 < q < \infty$ and u_j converges strongly to u in L^r , $r \in [2, 3]$, this implies that the second term tends to zero. For the last term we need the uniform boundedness of $\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j$ in $L^2(\mathbb{R}^3)$, together with the strong convergence of u_j to u in $L^2(\mathbb{R}^3)$ to conclude. In view of the above, we obtain

$$\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u \rangle_{L^2} \longrightarrow 0$$

Since $\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u \rangle \geq \langle |\nabla|(u_j - u), u_j - u \rangle$, we have $\langle |\nabla|(u_j - u), u_j - u \rangle \rightarrow 0$. We conclude that $\|u_j - u\|_{\mathbf{H}^{1/2}} \rightarrow 0$. \square

It is worth to mention that Assumption 3.1 is optimal for a nonnegative, radial W because there exists $W \in L^\infty(\mathbb{R}^3)$ such that (1.1) has no $\mathbf{H}^{1/2}(\mathbb{R}^3)$ solutions. For instance, we may choose $W \equiv 1$. Then (1.1), with $V \equiv 0$, takes the form $Lu + (1 - \|u\|_{L^2}^2)u = 0$ and this implies that $u \equiv 0$.

5. CONSTRAINED PROBLEM. PROOF OF THEOREM 3.4

We prove Theorem 3.4 and we establish two corollaries.

Proof of Theorem 3.4. Without loss of generality we consider $W \in L_w^{p_i}(\mathbb{R}^3)$. The idea is to apply the critical point theory by Berestycki and Lions [5] in the following framework: $\mathcal{H} = L^2(\mathbb{R}^3)$ and $\mathcal{K} = \mathbf{H}_r^{1/2}(\mathbb{R}^3)$. In order to apply the abstract theorem, we need to establish the following requirements:

1. $\mathcal{E}|_{\mathcal{C}}$ is bounded below;
2. \mathcal{E} is weakly lower semicontinuous on $\mathcal{T} = \{u \in \mathcal{C} : \mathcal{E}(u) \leq 0\}$;
3. $\mathcal{E}|_{\mathcal{C}}$ satisfies the (PS)₋ condition.

Verification of item 1. From Lemma 4.1 we find that

$$\mathcal{E}(u) \geq \frac{\alpha^{-1}}{4} (\|u\|_{\mathbf{H}^{1/2}}^2 - \|u\|_{L^2}^2) - 1/4 \mathcal{J}_W(u^2, u^2). \quad (5.1)$$

An application of the weak Young inequality and Sobolev's inequality yield

$$\mathcal{J}_W(u^2, u^2) \leq \|W\|_{L_w^p} \|u^2\|_{L^1} \|u^2\|_{L^{r/2}} \leq CN^2 \|W\|_{L_w^p} \|u\|_{\mathbf{H}^{1/2}}^2 \quad (5.2)$$

where $1/p+2/r+1 = 2$, i.e., $1/p+2/r = 1$ which is possible to satisfy because $r \in [2, 3]$ and $p \geq 3$. Since u belongs to \mathcal{C} , it is not hard to see that $\|u^2\|_{L^1} = \|u\|_{L^2}^2 = N^2$. Moreover, $\|u^2\|_{L^{r/2}} = \|u\|_{L^r}^2 \leq C\|u\|_{\mathbf{H}^{1/2}}^2$. Without loss of generality, we choose $\|W\|_{L_w^p} = 1/2\alpha CN^2$. Then inequality (5.2) becomes

$$\mathcal{J}_W(u^2, u^2) \leq \frac{\alpha^{-1}}{2} \|u\|_{\mathbf{H}^{1/2}}^2$$

while (5.1) becomes simply $\mathcal{E}(u) \geq -N^2$.

Verification of item 2. Let $(u_j) \subset \mathcal{T} := \{u \in \mathcal{C} : \mathcal{E}(u) \leq 0\}$ such that $u_j \rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Obviously, as for item 1, it follows that

$$\sup_j \mathcal{J}_W(u_j^2, u_j^2) < \infty$$

and, by Fatou's lemma, we get that

$$\mathcal{J}_W(u^2, u^2) \leq \liminf_j \mathcal{J}_W(u_j^2, u_j^2).$$

Since the remaining terms are obviously weakly lower semicontinuous, it follows that \mathcal{E} is weakly lower semicontinuous on \mathcal{T} .

Verification of item 3. Let $(u_j)_{j \geq 1}$ be a sequence in \mathcal{C} satisfying

$$\begin{cases} -\infty < \beta \leq \mathcal{E}(u_j) \leq \sigma < 0 \\ (\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}} - \alpha^{-2})u_j + Vu_j - (W * u_j^2)u_j + \lambda_j u_j = \epsilon_j \xrightarrow{\mathbf{H}^{-1/2}} 0, \end{cases}$$

where

$$-\lambda_j = \mathcal{E}(u_j) = \frac{1}{2} \mathfrak{I}_0[u_j] + \frac{1}{2} \mathfrak{S}[u_j] - \frac{1}{4} \mathcal{J}_W(u_j^2, u_j^2)$$

We have

$$\begin{aligned} & \frac{1}{2} \langle (\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}} - \alpha^{-2})u_j, u_j \rangle + \frac{1}{2} (\lambda - \alpha^{-2}) \|u_j\|_{L^2}^2 \\ & + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u_j(x)|^2 dx - \frac{1}{4} \int \int W(x-y) |u_j(x)|^2 |u_j(y)|^2 dx dy \leq \sigma. \end{aligned}$$

Since we have already proved that, for any $v \in \mathcal{C}$, $\mathcal{J}_W(v^2, v^2) \leq C$, we obtain

$$\frac{1}{2} \langle \sqrt{-\alpha^{-2}\Delta - \alpha^{-4}} u_j, u_j \rangle + \frac{1}{2} (\lambda - \alpha^{-2}) \|u_j\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u_j(x)|^2 dx \leq C,$$

whence

$$C \geq \frac{1}{2} \langle (\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}}) u_j, u_j \rangle \geq \|u_j\|_{\mathbf{H}^{1/2}}^2.$$

Therefore, $C \geq \|u_j\|_{\mathbf{H}^{1/2}}^2$, i.e., (u_j) is bounded in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$. Furthermore,

$$-\lambda_j \leq 2\mathcal{E}(u_j) \leq 2\sigma, \quad -2\sigma \leq \lambda_j \leq \lambda.$$

Indeed,

$$\begin{aligned} -\frac{1}{2} \lambda_j &= \frac{1}{2} \langle (\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}) u_j, u_j \rangle + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u_j(x)|^2 dx \\ & - \frac{1}{4} \int \int W(x-y) |u_j(x)|^2 |u_j(y)|^2 dx dy - \frac{1}{4} \int \int W(x-y) |u_n(x)|^2 |u_n(y)|^2 dx dy \end{aligned}$$

i.e.

$$\frac{-1}{2}\lambda_j = \mathcal{E}(u_j) - \frac{1}{4} \int \int W(x-y)|u_j(x)|^2|u_j(y)|^2 dx dy,$$

This shows that $\frac{-1}{2}\lambda_j \leq \mathcal{E}(u_j)$ and then $-\lambda_j \leq 2\mathcal{E} \leq 2\sigma$.

On the other hand, since $\mathcal{J}_W(u_j^2, u_j^2)$ is uniformly bounded with respect to j and from the facts above we conclude that $\lambda_j \leq \lambda$. Now we can follow the proof of Theorem 3.2 and conclude that u_j converges strongly to u in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$. This verifies item 3. Then the assertions of the theorem follows immediately from Berestycki and Lions [5, Theorems 7 and 9]. □

Corollary 5.1. *Let the hypotheses of Theorem 3.4 be satisfied. Then there there exists a nondecreasing and positive sequence $(N_d)_{d \geq 1}$ such that, if $N \geq N_d$, then the conclusions of Theorem 3.4 hold.*

Proof. Let $(\mathcal{V}_d)_{d \geq 1}$ be a sequence of d -dimensional subspaces of $\mathbf{H}_r^{1/2}$ such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$ and let $\mathcal{C}_1 = \{u \in \mathbf{H}_r^{1/2} : \|u\|_{L^2} = 1\}$. By definition of the genus, $\gamma(\mathcal{C}_1 \cap \mathcal{V}_d) = d$. For any positive real number N and any $u \in \mathcal{C}_1 \cap \mathcal{V}_d$, we have that

$$\begin{aligned} \mathcal{E}(Nu) &\leq \frac{N^2}{2} \mathfrak{I}_0[u] + \frac{N^2}{2} \mathfrak{s}[u] - \frac{N^4}{4} \mathcal{J}_W(u^2, u^2) \\ &\leq \frac{N^2}{2} \left\{ \sup_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} (\mathfrak{I}_0[u] + \mathfrak{s}[u]) - \frac{N^2}{2} \inf_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} \mathcal{J}_W(u^2, u^2) \right\}. \end{aligned}$$

Then there exists N_d such that for $N \geq N_d$ the right-hand side is negative and, therefore, \mathcal{E} is negative. Thus, for $N \geq N_d$, $\tilde{A} = \{Nu : u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ satisfies (3.3) and, consequently, the assertions of Theorem 3.4 hold true. □

Corollary 5.2. *Let the hypotheses of Theorem 3.4 be satisfied. If, moreover,*

$$\liminf_{r \rightarrow +\infty} r^2 W(r) \geq L, \tag{5.3}$$

then there exists L_d such that (3.3) holds true provided $L \geq L_d$. If $L = +\infty$, then (3.3) holds true for all $d \geq 1$. In particular, the assertions of Theorem 3.4 are valid.

Proof. Without loss of generality we may suppose $N = 1$. Let $A = \mathcal{C}_1 \cap \mathcal{V}_d$ where $(\mathcal{V}_d)_{d \geq 1}$ is a sequence of d -dimensional subspaces of $\mathbf{H}_r^{1/2}$ (to be specified below) such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$.

Choose $u \in A$ and let $u_\kappa(x) = u(x/\kappa)$. Then $\|\kappa^{-3/2}u_\kappa\|_{L^2} = 1$ and

$$\mathcal{E}(\kappa^{-3/2}u_\kappa) \leq \frac{1}{2} \mathfrak{I}_0[\kappa^{-3/2}u_\kappa] + \frac{1}{2} \int_{\mathbb{R}^3} V(\kappa x)|u(x)|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u^2(x)u^2(y)W(\kappa|x-y|) dx dy.$$

Using that $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}$ and, specifically,

$$\mathfrak{I}_0[\phi] \leq C\|\phi\|_{\mathbf{H}^1}^2, \quad \forall \phi \in \mathbf{H}^1(\mathbb{R}^3),$$

in conjunction with

$$\int \int_{\frac{1}{2} \leq |x-y| \leq 1} u^2(x)u^2(y)W(\kappa|x-y|) dx dy \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u^2(x)u^2(y)W(\kappa|x-y|) dx dy$$

we have that

$$\begin{aligned}
\mathcal{E}(\kappa^{-3/2}u_\kappa) &\leq \frac{C}{2}\lambda^{-2} \int_{\mathbb{R}^3} |\nabla u|^2 + 1 + \frac{1}{2} \int_{\mathbb{R}^3} V(\kappa x)|u(x)|^2 dx \\
&\quad - \frac{1}{4} \int \int_{1/2 \leq |x-y| \leq 1} u^2(x)u^2(y)W(\kappa|x-y|) dx dy \\
&\leq \frac{C_1}{2}\kappa^{-2} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{\kappa^2}{2} \int \int_{1/2 \leq |x-y| \leq 1} u^2(x)u^2(y)W(\kappa|x-y|) dx dy \right\} + C_2 \\
&\leq \frac{C_1}{2}\kappa^{-2} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{L}{2} \int \int_{1/2 \leq |x-y| \leq 1} u^2(x)u^2(y) dx dy \right\} + C_2.
\end{aligned}$$

where, in the last inequality, we used the assumption in (5.3). For $u \in A$ we may suppose that $u^2(x) > 0$ for $\Xi = \{|x| \leq 2\}$. Indeed, we may choose \mathcal{V}_d to be the subspace spanned by the first d eigenfunctions u_n of $-\Delta$ with Dirichlet boundary conditions on $\partial\Xi$. Since each $u_n \in \mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$ is radial, we have that $u_n \in \mathbf{H}_r^1(\mathbb{R}^3) \subset \mathbf{H}_r^{1/2}(\mathbb{R}^3)$ as required. This choice of \mathcal{V}_d will ensure that

$$\inf_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} \int \int_{1/2 \leq |x-y| \leq 1} u^2(x)u^2(y) dx dy > 0$$

and, by taking L large enough, we find that

$$\sup_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} \mathcal{E}(\kappa^{-3/2}u_\kappa) < 0 \quad \text{for } \kappa \geq \kappa_0.$$

Finally, with $\tilde{A} = \{\kappa_0^{-3/2}u_{\kappa_0} : u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ we conclude that $\gamma(\tilde{A}) = \gamma(A) = d$ and, therefore, (3.3) is satisfied for \tilde{A} . \square

If one takes $W(x) = 1/|x|^\alpha$, $2 < \alpha < 4$, then \mathcal{E} is not even bounded below; this observation alone shows that Assumption 3.3 is necessary.

A posteriori it can be shown that solutions u_j of (1.1) satisfy the following properties:

- (i) $u_j \in C^\infty(\mathbb{R}^3 \setminus \{0\})$;
- (ii) For all $R > 0$ and $\beta < \nu := \sqrt{\lambda(2\alpha^{-2} - \lambda)}$, there exists $C = C(\beta, R) > 0$ such that

$$|u_j(x)| \leq Ce^{-\beta|x|}, \quad \text{for } |x| \geq R.$$

Indeed, the proof of properties (i) and (ii) for the quasirelativistic Choquard equation (1.1) is carried over, with minor changes, from the proof of similar properties, valid for the quasirelativistic Hartree-Fock equations, found in Dall-Aqua *et al.* [7].

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