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On Maximal Relatively Divisible Submodules

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ON MAXIMAL RELATIVELY DIVISIBLE SUBMODULES

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Abstract. A torsion-free module M over an integral domain R has relatively divisible (RD-) submodules which are maximal with respect to inclusion. There are situations in which the number of non-isomorphic maximal RD-submodules is small; Göbel and Goldsmith have investigated this and related questions in the context of Abelian groups. We address corresponding problems for modules over arbitrary domains. We obtain results relating to the level of coherency of a ring R, and establish connections between the level of coherency and the minimum number of generators of RD-submodules of a given R-module. Under some natural restrictions, we prove that an R-module G, all of whose maximal RD-submodules are isomorphic to a fixed free module X of infinite rank, is itself free. We investigate R-modules G all of whose maximal RD-submodules are isomorphic to R^λ, where $|R|$ and λ are infinite cardinals which are not too large. We first show that, for any slender integral domain R, the module R^λ has infinitely many non-isomorphic maximal RD-submodules. Moreover, when R is a slender valuation domain with p.d.$Q = 1$, and G is an R-module with all maximal RD-submodules isomorphic to R^λ, we prove that G itself is isomorphic to R^λ. Consequently, in a wide range of situations no such module can exist, for instance if R is either a maximal Prüfer domain or a DVR.

Introduction

If M is a torsion-free module over an integral domain R with quotient field Q, then it is easy to establish that M has relatively divisible (RD-) submodules which are maximal with respect to inclusion. In general, one would expect that the number of such modules would be large and, in fact, we can show that if M is of infinite rank λ, then there are $|R|^\lambda$ distinct maximal RD-submodules – see Proposition 4.5. However, there are situations in which the number of *non-isomorphic* maximal RD-submodules is small: for example, if $R = \mathbb{Z}$, the ring of integers and M is a free $\mathbb{Z}$-module of infinite rank, then the number of isomorphism classes of maximal RD-submodules of M is precisely 1. In a recent paper, [GG], Göbel and Goldsmith have investigated this and related questions in the context of (not necessarily torsion-free) Abelian groups. Motivated by this work, we now address the corresponding problem for modules over arbitrary domains which are not fields. The situation is, inevitably, vastly more complicated than for Abelian groups. An immediate complication is that the minimum number of generators of an ideal J of R may be large. Consequently, we devote the first part of the paper to consideration of this issue and obtain results relating to the level of coherency of the ring R; this notion arises in a natural way from that of a coherent ring - an in-depth treatment of such rings may be found in the book [G]. We show, *inter alia*, that for every pair of cardinal numbers $\tau \leq \lambda$, there is a local one-dimensional domain R of cardinality λ having τ as its level of coherency. We also establish connections between the level of coherency of an integral domain R and the minimal number of generators of RD-submodules of a given R-module. The final part of the paper considers the question of the existence of a module G all of whose maximal RD-submodules are

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isomorphic to a fixed module, say X. When X is free of infinite rank, then, under some restrictions, we show that G itself is free. When X is an infinite product of copies of R, the problem is considerably more difficult and a significant amount of the paper is devoted to the consideration of the existence, or otherwise, of modules G with the property that every maximal RD-submodule of G is isomorphic to R^λ, where λ is an arbitrary infinite cardinal. Questions of this type seem to have their origin in a problem posed by Nunke in [Nu] in the context of $R = \mathbb{Z}$ and $\lambda = \aleph_0$; a comprehensive answer for Abelian groups is outlined in Proposition 4.2 of [GG]. Given that we are dealing with products, it is, perhaps, not too surprising that the concept of slenderness plays an important role in our discussions - see Chapter XVI.6 of [FS] for details of this concept in the context of module theory. It is well known that slenderness has strong interactions with set theory and one cannot avoid consideration of ω-measurable cardinals – see Chapter III of [EM] for a clear discussion of these cardinals and their role in slenderness. However, our interests here are not in the set-theoretic aspects and the reader who wishes to avoid these large cardinals may safely assume that no such cardinals exist; formally this may be achieved, for example, by working in the model ZFC + (V = L).

Once we have made the right assumptions on the cardinals, we first show that, for any slender integral domain R, the module R^λ has infinitely many non-isomorphic maximal RD-submodules. Moreover, when R is a slender valuation domain such that Q has projective dimension one, we prove that an R-module G is isomorphic to R^λ, whenever all its maximal RD-submodules are isomorphic to R^λ. As a consequence of these results, in a wide range of situations no such module can exist. In particular, if R is either a maximal Prüfer domain or a discrete valuation ring of rank one, then every R-module has a maximal RD-submodule which is not isomorphic to R^λ.

1. Preliminaries

Unless specified otherwise, in what follows R will be an integral domain, with field of quotients Q. Without loss of generality, we will assume that R is not a field, since otherwise our results are either trivial or void of significance; in particular, $|R| \geq \aleph_0$. Unless stated explicitly to the contrary, all R-modules considered will be torsion-free. Our notation is standard and closely follows that of [FS]; in particular, we denote the completion of a ring or module M in an appropriate natural topology by $\tilde{M}$.

We recall some standard definitions for R-modules, mainly originating from Abelian Group Theory. If M is an R-module, the *rank* of M, denoted by $\mathrm{rank}_R M$, is the dimension of the Q-vector space $M \otimes Q$. Let $A \subseteq B$ be R-modules. Recall that A is said to be *relatively divisible* in B if $rB \cap A = rA$ for every $r \in R$. For short, we will say that A is a RD-submodule of B (see [FS], page 38). When A and B are torsion-free R-modules, the case we are interested in, relative divisibility is equivalent to B/A torsion-free. Moreover, if H is any submodule of the torsion-free module M, we can define the *RD-hull* H_* of H in M as the intersection of the RD-submodules of M that contain H. We have

$$H_* = \{x \in M : rx \in H,\ \exists\ 0 \neq r \in R\},$$

hence, in particular, the ranks of H and H_* coincide. Of course, for torsion-free Abelian groups the notion of RD-hull coincides with the usual purification. The submodule A is said to be *pure* in B if every finite system of equations

$$\sum_j x_j r_{ij} = a_i \in A \quad (r_{ij} \in R)$$

has a solution in A whenever it is solvable in B (see [FS], page 43). Recall that Warfield [W] proved that, when R is a Prüfer domain, A is a RD-submodule of B if and only if A is pure in B. When the integral domain is not Prüfer, there are easy examples of RD-submodules that are not pure, even in case of free modules of finite rank.

Example 1.1. Let R be a UFD that contains two prime elements α, β such that the ideal $\langle \alpha, \beta \rangle$ is not principal. We consider the R-module $F = Ry_1 \oplus Ry_2$, freely generated by y_1, y_2, and its submodule $F_0 = Rg_0$, where $g_0 = \alpha y_1 + \beta y_2$. Let us verify that F_0 is a RD-submodule of F. In fact, assume that

$$r(f_1 y_1 + f_2 y_2) = s g_0 \in F_0 \cap rF \quad (r, s, f_1, f_2 \in R).$$

It is enough to show that r divides s in R. Equating the coefficients of y_1, y_2 in the above relation, we get

$$r f_1 = s\alpha \ ; \ r f_2 = s\beta,$$

hence any prime factor q of r divides both $s\alpha$ and $s\beta$. It follows that q divides s, since α, β are distinct prime elements of R, and therefore r divides s, as desired. Let us now show that F_0 is not pure in F. Consider the following linear equation

$$\alpha x_1 + \beta x_2 = g_0.$$

It is obviously solvable in F, for $x_1 = y_1, x_2 = y_2$. However, the equation has no solutions in F_0, since

$$\alpha(r g_0) + \beta(s g_0) = g_0 \quad (r, s \in R)$$

implies $\alpha r + \beta s = 1$, contrary to the choice of α, β.

Let M be an R-module; we will denote by $\operatorname{gen} M$ the smallest cardinal number among the cardinalities of generating systems for M. It is useful to note that, if R is an integral domain and G is a torsion-free R-module, then $\operatorname{gen} G \geq \operatorname{rank}_R G$. In fact, if $\operatorname{gen} G = \kappa$, say, then G is an epimorphic image of a free module of rank κ, hence $\operatorname{rank}_R G \leq \kappa$ follows.

Proposition 1.2. *Let R be an integral domain, with field of fractions Q, G a torsion-free R-module that contains a copy of $P = R^{\aleph_0} = \prod_{n<\omega} Re_n$. Then* $\operatorname{rank}_R G = \operatorname{gen} G = |G|$.

Proof. Since $\operatorname{rank}_R G \leq \operatorname{gen} G$, as observed above, and, obviously, $\operatorname{gen} G \leq |G|$, it suffices to show that $\operatorname{rank}_R G \geq |G|$. We claim that $\operatorname{rank}_R G \geq |R|$. If not, we have $\operatorname{rank}_R P < |R|$, a contradiction, since, via Vandermonde matrices, it is readily seen that the elements $u_r = \prod_{n<\omega} r^n e_n$, $0 \neq r \in R$, are linearly independent. Suppose now that $\mu = \operatorname{rank}_R G \geq |R|$. Then $|G| = |G \otimes Q| = \mu|Q| = \mu|R| = \mu$. □

For any commutative ring R, we define the invariant $\gamma_R = \sup\{\operatorname{gen} J\}$, where J ranges over the ideals of R.

In the sequel we will need the following result on valuation domains, which is of independent interest. Recall that, for J a nonzero ideal of R, the prime ideal $J^{\#}$ associated to J is defined by

$$J^{\#} = \{a \in R : aJ \subset J\}.$$

We refer to [FS], Ch.II.4, pages 68 -70, for the main properties of $J^{\#}$.

Proposition 1.3. *Let R be a valuation domain and let $|\operatorname{Spec}(R)| = \beta$. Then* $\operatorname{gen} J \leq \beta\aleph_0$, *for every R-submodule J of Q. In particular, any such J is countably generated when $|\operatorname{Spec}(R)| \leq \aleph_0$.*

Proof. To simplify the notation, let $\gamma = \beta\aleph_0$. We first show that $\operatorname{gen} Q \leq \gamma$. We actually show that there exists a strictly descending sequence of principal ideals $z_0R \supset z_1R \supset \cdots \supset z_\alpha R \supset \ldots$, $\alpha < \gamma_1 \leq \gamma$, such that $\bigcap_{\alpha<\gamma_1} z_\alpha R = 0$. Then Q is generated, as an R-module, by $\{z_\alpha^{-1}\}_{\alpha<\gamma_1}$. First assume that R has a minimal nonzero prime ideal, say H, and let $0 \neq y \in H$. Then $\bigcap_{n>0} y^nR = 0$, and here we take $z_n = y^n$ and $\gamma_1 = \omega \leq \gamma$. Assume now that zero is the intersection of the nonzero prime ideals. Since $|\operatorname{Spec}(R)| \leq \gamma$, we have a strictly descending chain of nonzero prime ideals $P_0 \supset P_1 \supset \cdots \supset P_\alpha \supset \ldots$, $\alpha < \gamma_1 \leq \gamma$, such that $\bigcap_{\alpha<\gamma_1} P_\alpha = 0$. We choose $z_\alpha \in P_\alpha \setminus P_{\alpha+1}$ for all $\alpha < \gamma_1$, and we are done.

Now we consider J to be a proper R-submodule of Q. Then J is a fractional ideal, hence we can assume that J is a proper ideal of R; let $J^\#$ be the prime ideal associated to J. If $\operatorname{gen} J^\# = \gamma_2$, say, we can write $J^\# = \bigcup_{\alpha<\gamma_2} x_\alpha R$, where $x_{\alpha+1}$ divides x_α, for all $\alpha < \gamma_2$. We may assume that J is not principal, otherwise our claim is trivial. Then property (d) on page 69 of [FS] shows that $J = JJ^\#$, so that $J = \bigcup_{\alpha<\gamma_2} x_\alpha J$. Observe that, by the definition of $J^\#$, $x_\alpha J \subset J$ for all $\alpha < \gamma_2$. Then J is the union of an ascending chain of γ_2 ideals properly contained in it, whence it easily follows that $\operatorname{gen} J \leq \gamma_2$.

Thus it suffices to show that $\operatorname{gen} J^\# \leq \gamma$. If $J^\#$ is a union of a strictly ascending chain of prime ideals, say $J^\# = \bigcup_{\alpha<\gamma_3} P_\alpha$, then $\operatorname{gen} J^\# \leq \gamma_3 \leq |\operatorname{Spec}(R)| \leq \gamma$. Otherwise, $J^\#$ contains a prime ideal H such that $R_{J^\#}/H$ is a one-dimensional valuation domain. Then $J^\#/H$ is a countably generated ideal of $R_{J^\#}/H$, since the value group of this latter valuation domain is an ordered subgroup of the real numbers. It easily follows that $J^\#$, as an $R_{J^\#}$-module, is countably generated, as well, say by the set $\{x_i : i < \omega\}$. This concludes our proof in the case when $R = R_{J^\#}$, i.e., when $J^\#$ is maximal.

If $J^\#$ is not maximal, from $\operatorname{gen} Q \leq \gamma$ it follows that $J^\# = \bigcap_{\alpha<\gamma_4} s_\alpha R$, where the $s_\alpha R$ form a strictly descending sequence of principal ideals, and $\gamma_4 \leq \gamma$. Then $J^\#$ is generated, as an R-module, by the set $\{x_i/s_\alpha : i < \omega, \alpha < \gamma_4\}$. In fact, for all $s \in R \setminus J^\#$ there exists $\delta < \gamma_4$ such that $s_\delta \in sR$, so that $x_i/s \in (x_i/s_\delta)R$, for all $i < \omega$. It follows that $J^\# = J^\# R_{J^\#} \subseteq \sum_{i,\alpha}(x_i/s_\alpha)R \subseteq J^\#$. In conclusion, $\operatorname{gen} J^\# \leq \omega\gamma_4 \leq \gamma$, as desired. □

It is worth remarking that, within the class of the so-called strongly discrete valuation domains – see [FS], Ch.II.8, one may find a valuation domain R such that $\gamma_R = \aleph_0$ and $|\operatorname{Spec}(R)| = \operatorname{gen} Q > \aleph_0$.

2. The level of coherency

In the present section, R denotes an arbitrary commutative ring. We say that the cardinal number τ is the *level of coherency* of R if τ is the smallest cardinal such that, for every short exact sequence of R-modules

$$0 \to N \to X \to J \to 0,$$

where J is a finitely generated ideal of R and X is finitely generated, we have $\operatorname{gen} N \leq \tau$. It is clear that the level of coherency is always an infinite cardinal. In particular, if R is a coherent ring, then its level of coherency is $\aleph_0$. For a treatment of coherent rings see Glaz's book [G]; clearly, a valuation domain is coherent. Note that such a τ always exists, since, in any case, if N is a submodule of a finitely generated R-module, we have $\operatorname{gen} N \leq |R|\aleph_0$. The next result may be regarded as a generalization of [G], Lemma 2.1.1.

Proposition 2.1. *Let R be a commutative ring. If for every short exact sequence of R-modules $0 \to K \to Y \to J \to 0$, where J is an ideal of R and Y is finitely* *generated free module, we have* $\operatorname{gen} K \leq \gamma$*, then the level of coherency of* R *is less or equal to* γ*.*

Proof. Take any epimorphism $g : X \to J$, with X finitely generated, and build an epimorphism $f : Y \to X$, with Y finitely generated and free. Then $g \circ f : Y \to J$ is onto. If $K = \ker(g \circ f)$, then our hypothesis yields $\operatorname{gen} K \leq \gamma$. Since $N = \ker(g)$ is an epic image of K, we have $\operatorname{gen} N \leq \gamma$, as well. The desired conclusion follows. □

The next proposition gives an upper bound for the level of coherency, sharper than the cardinality of the ring times $\aleph_0$.

Proposition 2.2. *Let* R *be a commutative ring,* Y *a finitely generated free* R*-module,* N *a submodule of* Y*. Then* $\operatorname{gen} N \leq \gamma_0 = \gamma_R \aleph_0$*. As a consequence, the level of coherency of an integral domain* R *is less or equal to* γ_0*.*

Proof. We make induction on $n = \operatorname{gen} Y$. If $n = 1$, then N is isomorphic to an ideal of R, hence $\operatorname{gen} N \leq \gamma_R \leq \gamma_0$. Let $n > 1$, and write $Y = C \oplus Y_1$, where $C \cong R$. Then $N/N \cap C$ embeds into Y_1, hence, by inductive hypothesis, $\operatorname{gen}(N/N \cap C) \leq \gamma_0$. Let $\{y_\alpha : \alpha < \gamma_0\}$ be a subset of N such that $N/N \cap C = \langle y_\alpha + N \cap C : \alpha < \gamma_0 \rangle$, and consider the submodule $A = \langle y_\alpha : \alpha < \gamma_0 \rangle$. Then $N = A + N \cap C$. Since $N \cap C$ is isomorphic to an ideal of R, we get $\operatorname{gen} N \leq \gamma_0$.

Finally, an application of Proposition 2.1 readily shows that the level of coherency of R is $\leq \gamma_0$. □

In our next result we give a general construction of integral domains with preassigned level of coherency.

Theorem 2.3. *For every pair of infinite cardinal numbers* $\tau \leq \lambda$ *there exists a local one-dimensional domain* R *with* $|R| = \lambda$ *and* τ *as its level of coherency.*

Proof. Let F be a field with $|F| = \lambda$, K a field extension such that $[K : F] = \tau$. Let $\{z_\alpha\}_{\alpha<\tau}$ be a basis of K over F. Consider the field $Q = K(X)$ of the rational functions over K, and define

$$R = \{f \in Q : f(0) \in F\}$$

(of course, when we write $f(0) \in F$, we automatically mean that the rational function f has no poles in zero). Clearly we have $|R| = \lambda$. It is easily seen that R is a local one-dimensional domain, with maximal ideal $\mathfrak{M} = \{f \in Q : f(0) = 0\}$. In fact, one verifies that $f \in R$ is a unit if and only if $f(0) \neq 0$, and any $g \in \mathfrak{M}$ may be written in the form $g = X^n z u$, where $n > 0$, $z \in K$, and u is a unit of R. Then, if $g_1 = X^m z_1 u_1 \in \mathfrak{M}$, with $m > n$, we have $g_1/g = X^{m-n}(z_1/z)(u_1/u) \in \mathfrak{M}$. It is then straightforward to check that every ideal I of R is generated by elements of the form $X^k z$, where $z \in K$ and $k > 0$ is the minimum exponent that appears in the elements of I. Let $\mathcal{A} = \{z \in K : X^k z \in I\}$, and consider the F-vector space V generated by $\mathcal{A}$. Let $\{v_\beta : \beta < \gamma\}$ be a basis of V. Since $F \subset R$ we easily see that

$$I = \langle X^k v_\beta : \beta < \gamma \rangle.$$

However, V is a subspace of K, hence $\gamma \leq \tau$. Therefore we conclude that $\operatorname{gen} I \leq \tau$ for every ideal I of R. Let now σ (say) be the level of coherency of R. We want to prove that $\sigma = \tau$. Since $\operatorname{gen} I \leq \tau$ for every ideal I of R, using Proposition 2.2 we readily get $\sigma \leq \tau$. To prove the reverse inequality, pick any $t \in K \setminus F$, and consider the ideal $J = \langle X, tX \rangle$ of R, a free R-module $Y = Ry_1 \oplus Ry_2$, and the onto homomorphism $\phi : Y \to J$, defined by $y_1 \mapsto X$, $y_2 \mapsto tX$. Let $N = \operatorname{Ker} \phi$; we will show that $\operatorname{gen} N = \tau$, so that $\tau \leq \sigma$, and the desired conclusion will follow. Pick $ry_1 - sy_2 \in N$; then $rX - stX = 0$, whence $r = st$. Since $t \notin R$, necessarily $s \in \mathfrak{M}$;

we may write $s = X^n zu$, where $n > 0$, $z \in K$, and u is a unit of R. It follows that $ry_1 - sy_2 \in R(Xtzy_1 - Xzy_2)$. Now, since $z \in \sum_{\alpha<\tau} Fz_\alpha$, and $F \subset R$, we get

$$Xtzy_1 - Xzy_2 \in \langle Xtz_\alpha y_1 - Xz_\alpha y_2 : \alpha < \tau \rangle = M.$$

We have thus proved that $N \subseteq M$. However, since $\phi(Xtz_\alpha y_1 - Xz_\alpha y_2) = Xtz_\alpha X - Xz_\alpha tX = 0$, we get $N = M$. We have $\operatorname{gen} N \geq \dim_{R/\mathfrak{M}}(N/\mathfrak{M}N)$, then, since $\operatorname{gen} N \leq \tau$, we only need to prove that $\dim_{R/\mathfrak{M}}(N/\mathfrak{M}N) \geq \tau$. It suffices to show that any relation

$$\sum_{\alpha<\tau} r_\alpha(Xtz_\alpha y_1 - Xz_\alpha y_2) = \sum_{\alpha<\tau} q_\alpha(Xtz_\alpha y_1 - Xz_\alpha y_2)$$

with $r_\alpha \in R$ and $q_\alpha \in \mathfrak{M}$ almost all zero, yields $r_\alpha \in \mathfrak{M}$, for all $\alpha < \tau$. Looking at the coefficients of y_2 in the above relation, we get $\sum_{\alpha<\tau}(r_\alpha - q_\alpha)z_\alpha = 0$. We specialize at $X = 0$; then $r_\alpha(0) \in F$ and $q_\alpha(0) = 0$, since the q_α lie in $\mathfrak{M}$. We get $\sum_{\alpha<\tau} r_\alpha(0)z_\alpha = 0$, hence $r_\alpha(0) = 0$ for all α, which means that $r_\alpha \in \mathfrak{M}$ for all $\alpha < \tau$, as desired. □

3. maximal RD-submodules

For the remainder of the paper, R will be an integral domain, not a field, and all the R-modules considered will be assumed torsion-free. We will investigate the RD-submodules of the (torsion-free) R-module M which are maximal with respect to inclusion.

Lemma 3.1. *Let M be an R-module. Then H is a maximal RD-submodule of M if and only if $\operatorname{rank}_R M/H = 1$.*

Proof. By the definition of a maximal RD-submodule, M/H has no RD-submodules other than the trivial ones. This forces M/H to be of rank 1, for otherwise the RD-hull of a cyclic submodule of M/H would be such a rank 1 RD-submodule. Conversely, assume that $M/H \cong J \subseteq Q$. We show that no proper nonzero submodule K of J can be relatively divisible in J. Pick any $0 \neq z \in K$ and $y \in J \setminus K$. We have $z = a/b$, $y = c/d$, for suitable $a, b, c, d \in R$. Then $ady = bcz \in K$ and $ad \neq 0$ implies that J/K has nonzero torsion elements, hence K is not relatively divisible in J. □

Proposition 3.2. *Every R-module M contains maximal RD-submodules.*

Proof. Let E be the injective envelope of M. We write $E = Q_1 \oplus E_1$, where $Q_1 \cong Q$, and consider the projection $\pi : E \to Q_1$. If $H = M \cap \operatorname{Ker}(\pi)$, we get $M/H \cong \pi(M) \subseteq Q_1$, hence $\operatorname{rank}_R M/H = 1$ and the result follows from Lemma 3.1. □

In view of the preceding results, it is clear that the level of coherency of the integral domain R may be also defined as the sup of the cardinals $\operatorname{gen} N$, where N ranges over the maximal RD-submodules of finitely generated modules. (Recall that a rank-one finitely generated module is isomorphic to an ideal.)

Proposition 3.3. *Let τ be the level of coherency of R. Let N be an RD-submodule of a finitely generated R-module X. Then $\operatorname{gen} N \leq \tau$. Moreover, N is finitely generated whenever R is coherent.*

Proof. Obviously we may assume $N \neq X$. We use induction on $\operatorname{rank}_R X$. If $\operatorname{rank}_R X = 1$ and N is a proper RD-submodule of X, then $N = 0$. Now assume that $\operatorname{rank}_R X > 1$; choose a maximal RD-submodule H of X, containing N. Then $\operatorname{rank}_R X/H = 1$ implies that X/H is a finitely generated fractional ideal of R. Since τ is the level of coherency, and X is finitely generated, we get $\operatorname{gen} H = \kappa \leq \tau$. Assume firstly that κ is finite. Then H is finitely generated, N is a RD-submodule of

H, and $\operatorname{rk} H < \operatorname{rank}_R X$, hence $\operatorname{gen} N \leq \tau$ follows by induction. Now assume that κ is an infinite cardinal. Choose a set of generators $\{x_\alpha\}_{\alpha<\kappa}$ of H, indexed by κ. For every finite subset F of κ, let $N_F = N \cap \langle x_\alpha : \alpha \in F\rangle$. Since N_F is relatively divisible in $\langle x_\alpha : \alpha \in F\rangle$, and $\operatorname{rank}_R\langle x_\alpha : \alpha \in F\rangle \leq \operatorname{rank}_R H < \operatorname{rank}_R X$, by induction we get $\operatorname{gen} N_F \leq \tau$. Then from $N = \sum_F N_F$, we get $\operatorname{gen} N \leq \sum_F \operatorname{gen} N_F \leq \kappa\tau = \tau$, as desired.

In the case when R is a coherent domain, we argue as above, and use the same notation (recall that here $\tau = \aleph_0$). In these circumstances we get that H is finitely generated, by [G] Lemma 2.1.1. Hence, by induction, we get that $\operatorname{gen} N$ is finite, since N is relatively divisible in H and $\operatorname{rank}_R H < \operatorname{rank}_R X$. □

The following result and its corollary are related to Lemma 3.5 page 208 of [FS].

Theorem 3.4. *Let τ be the level of coherency of R. Let N be an RD-submodule of the R-module M. Then* $\operatorname{gen} N \leq \tau \operatorname{gen} M$.

Proof. Let $\operatorname{gen} M = \kappa$; choose a set of generators $\{x_\alpha\}_{\alpha<\kappa}$ of M, indexed by κ. For every finite subset F of κ, let $N_F = N \cap \langle x_\alpha : \alpha \in F\rangle$. Since N_F is relatively divisible in $\langle x_\alpha : \alpha \in F\rangle$, by Proposition 3.3 we get $\operatorname{gen} N_F \leq \tau$. Then, since $N = \sum_F N_F$, we have that $\operatorname{gen} N \leq \sum_F \operatorname{gen} N_F \leq \tau\kappa$, as required. □

Corollary 3.5. *If R is a coherent domain, N an RD-submodule of the non-finitely generated R-module M, then* $\operatorname{gen} N \leq \operatorname{gen} M$.

For R a valuation domain, the above result is extendable to the case when M is finitely generated, as proved in [FS], Lemma 3.5, page 208. In fact, all finitely generated torsion-free modules over valuation domains are free, and hence their pure submodules are direct summands. Of course, for general coherent domains the corollary is not extendable to finitely generated modules. It is appropriate to provide a concrete example, where R is even Noetherian, local and one-dimensional.

Example 3.6. The construction is similar to that made in the proof of Theorem 2.3. Let $K = F[t]$ be an extension of the field F, where t is an algebraic element of degree $m > 2$, and let X be an indeterminate. Consider the field $Q = K(X)$ of the rational functions over K, and define

$$R = \{f \in Q : f(0) \in F\}.$$

It is easily seen that R is a local one-dimensional domain,with maximal ideal $\mathfrak{M} = \{f \in Q : f(0) = 0\}$; R is Noetherian, since $\mathfrak{M} = \langle X, tX, \dots, t^{m-1}X\rangle$.

Consider the ideal $J = \langle X, tX\rangle$ of R, a free R-module $Y = Ry_1 \oplus Ry_2$, and the onto map $\phi : Y \to J$, defined by $y_1 \mapsto X$, $y_2 \mapsto tX$. Then, obviously, $N = \operatorname{Ker}\phi$ is relatively divisible in Y. One can show that

$$N = \langle Xt^{i+1}y_1 - Xt^i y_2 : 0 \leq i \leq m-1\rangle$$

and $\operatorname{gen} N = m > 2 = \operatorname{gen} Y$, by using an argument similar to that used in the proof of Theorem 2.3.

4. Maximal RD-submodules which are direct sums or products

In this section we focus on R-modules having the property that all maximal RD-submodules are either a direct sum or a direct product of copies of R. In the latter case, investigations of this type can be traced back a problem proposed by Nunke [Nu] and to unpublished work of A.L.S. Corner [C] on Abelian groups.

Given the integral domain R, with field of fractions Q, we define the cardinal invariant $\delta_R = \sup\{\operatorname{gen} J\}$, where J ranges over the R-submodules of Q. By the definition, we get $\delta_R \geq \gamma_R$. Note that $\delta_R \geq \aleph_0$, since Q is never a finitely generated R-module.

The first result of this section extends Proposition 2.6 of [GG].

Theorem 4.1. *Let R be an integral domain, M an R-module satisfying the following conditions*
(i) every maximal RD-submodule of M is free;
(ii) $\operatorname{gen} M > \delta_R$.
Then M is free.

Proof. Let H be a maximal RD-submodule of M. Then $\operatorname{rank}_R M/H = 1$, and hence $\operatorname{gen} M/H \leq \delta_R$. Select a submodule A of M such that $\operatorname{gen} A \leq \delta_R$ and $M = A + H$. Then we see at once that $\operatorname{gen} H = \operatorname{gen} M$. Now $A \cap H$ has the property that $A/A \cap H \cong (A+H)/H = M/H$, so that $A \cap H$ is relatively divisible in H. Since $\operatorname{gen} A \leq \delta_R$, and $\delta_R \geq \gamma_R \aleph_0$ is greater or equal to the level of coherency of R, by Proposition 2.2, it follows from Theorem 3.4 that $\operatorname{gen}(A \cap H) \leq \delta_R$. By hypothesis, H is free, hence there is a direct decomposition $H = H_0 \oplus H_1$ such that H_0 and H_1 are free, $A \cap H \subseteq H_0$ and $\operatorname{gen} H_1 > \delta_R$. Set $N = A + H_0$, so that $M = N + H_1$ and it is easy to see that $N \cap H_1 = 0$, whence $M = N \oplus H_1$. Now decompose the free module H_1 as $H_1 = H_2 \oplus R$. Then $N \oplus H_2$ is a maximal RD-submodule of M, and hence is free. However $M = (N \oplus H_2) \oplus R$, so that M is also free. □

The above theorem and Proposition 1.3 readily give the following corollary, valid for modules over valuation domains.

Proposition 4.2. *Let R be a valuation domain, M an uncountably generated R-module such that* $\operatorname{gen} M > |\operatorname{Spec}(R)|$ *and every maximal RD-submodule of M is free. Then M is free.*

It is worth noting that easy examples show that the preceding results are no longer valid if we drop the condition that $\operatorname{gen} M$ is enough large. For instance, for any domain R which is not a PID, take a non-principal ideal J. Then J is not free, but, trivially, all its maximal RD-submodules are free, since they all coincide with zero.

However, it is worth giving counterexamples where the maximal RD-submodules are nonzero.

Example 4.3. Let $R = \mathbb{Z}_p$ be the ring of integers localized at the prime p. Using the methods described in [FS] Ch.XV.6, for all $n > 0$ we may construct an indecomposable torsion-free R-module M of rank $n+1$ whose basic submodules are free of rank n. Under the present circumstances, the maximal RD-submodules of M coincide with the basic submodules, and so they are all free, while M is not free. Note that the construction of M by generators (see [FS], page 510) shows that $\operatorname{gen} M = \operatorname{gen} \mathbb{Q} = \aleph_0$.

In the remainder of this section, for R an integral domain and λ an infinite cardinal number, we will denote by P_λ the direct product $\prod_\lambda R$ of λ many copies of the integral domain R, and by S_λ the corresponding direct sum $S_\lambda = \bigoplus_\lambda R$. For any R-module X we denote by X^* its R-dual, namely $X^* = \operatorname{Hom}_R(X, R)$ (cf. [FS], Ch.IV.5).

Clearly, P_λ has infinitely many maximal RD-submodules isomorphic to itself. An obvious question is whether all the maximal RD-submodules are isomorphic to P_λ, at least for a suitable choice of the cardinal λ. Questions of this type seem to have originated from a problem posed by Nunke in [Nu] in the context of $R = \mathbb{Z}$ and $\lambda = \aleph_0$; in the latter case the question was answered in the negative by Meijer [Me]. A comprehensive answer for Abelian groups is outlined in Proposition 4.2 of [GG]. We want to investigate the same problem for general integral domains R, with specific focus on the case of Prüfer and valuation domains. Recall that a Prüfer domain is called maximal if it is pure-injective, and that for Prüfer domains the notions of purity and relative divisibility are equivalent, hence a maximal Prüfer domain is also RD-injective. In the next result we make use of the existence of the so-called pure hull of a submodule M of a pure-injective R-module A (see [FS] Theorem 3.6, page 438, and related discussion).

Theorem 4.4. *Let R be a maximal Prüfer domain and λ an infinite cardinal number. Then P_λ has non-isomorphic maximal RD-submodules.*

Proof. Since λ is fixed, to simplify the notation we write $P = P_\lambda$. Note that as a result of the present hypotheses P is pure-injective. It suffices to show that there exist maximal RD-submodules of P not isomorphic to P. Let $S = S_\lambda = \bigoplus_\lambda R \subseteq P$, and let $H(S)$ be its pure hull. Recall that $H(S)$ is pure in P and isomorphic to the pure-injective envelope of S. In particular, we have $P = H(S) \oplus N$ for some N. Since λ is infinite, S is not pure-injective, hence $S \neq H(S)$. Moreover $H(S)/S$ is torsion-free, since it is a submodule of P/S. Now we pick $S \subseteq T \subset H(S)$ such that T/S is maximal pure in $H(S)/S$; then T is maximal pure in $H(S)$, and it cannot be pure-injective, by the definition of pure hull. We consider the submodule $T \oplus N$ of P. Clearly it is a maximal pure, and hence relatively divisible, submodule of P, but it cannot be isomorphic to P, since it is not pure-injective. The desired conclusion follows. □

Proposition 4.5. *Let R be an integral domain with field of fractions Q, G a torsion-free R-module of rank μ. Then (1) $\mathrm{Hom}_R(G,Q)$ is a Q-vector space of dimension $|Q|^\mu$; (2) $\phi_1, \phi_2 \in \mathrm{Hom}_R(G,Q)$ are linearly dependent if and only if $\ker\phi_1 = \ker\phi_2$.*

Proof. (1) Let $\mathcal{B}$ be a maximal linearly independent set of elements of G, and set $F = \langle\mathcal{B}\rangle$. Then from the exact sequence $0 \to F \to G \to G/F = T \to 0$, where T is a torsion R-module, we get

$$0 = \mathrm{Hom}_R(T,Q) \to \mathrm{Hom}_R(G,Q) \to \mathrm{Hom}_R(F,Q) \to \mathrm{Ext}^1_R(T,Q) = 0.$$

So we have that $\mathrm{Hom}_R(G,Q) \cong \mathrm{Hom}_R(F,Q)$, and this latter, since F is a free module of rank μ, is a Q-space of dimension $|Q|^\mu$, by a well-known result of Jacobson. (2) If $\phi_1 = q\phi_2$ for some nonzero $q \in Q$, then torsion-freeness and RD-divisibility of the kernels give $\ker\phi_1 = \ker\phi_2$. Conversely, suppose that $\ker\phi_1 = \ker\phi_2 = K$, say. Then ϕ_1, ϕ_2 can be regarded as homomorphisms from G/K into Q. Since G/K is a submodule of Q, we get $\mathrm{Hom}_R(G/K,Q) \cong Q$, and so $\phi_1 = q\phi_2$ for some $0 \neq q \in Q$. □

We recall that an integral domain R is said to be slender if for every homomorphism $\phi : \prod_{n<\omega} Re_n \to R$ the set $\{n < \omega : \phi(e_n) \neq 0\}$ is finite. For the general notion of slenderness in the context of module theory see [FS], Ch.XVI.6; for a detailed discussion of the underlying set-theoretic concepts see [EM], Chapter III. As we stated earlier, our interests are not primarily set-theoretic in nature, so we shall avoid the deeper complications of the theory of products by usually assuming that the indexing cardinal of our products, λ, is not ω-measurable. Notice that with this assumption if R is slender, then $\mathrm{Hom}_R(R^\lambda, R) \cong R^{(\lambda)}$ – this follows from Corollary 3.3 in Chapter III, [EM].

Theorem 4.6. *Let R be a slender domain and let λ be a non-ω-measurable cardinal number. Then P_λ has non-isomorphic maximal RD-submodules.*

Proof. To simplify the notation, we write $P = P_\lambda$. Since $\mathrm{rank}_R P = |P| = |R|^\lambda$ (see Proposition 1.2), then, by Proposition 4.5(1), $\mathrm{Hom}_R(P,Q)$ is a Q-vector space of dimension $|Q|^\mu = |R|^\mu$, where $\mu = |R|^\lambda$. Therefore, by Proposition 4.5(2), there are $|R|^\mu$ distinct kernels of homomorphisms from P into Q. It follows that there are at least $|R|^\mu$ distinct maximal RD-submodules of P. Now we observe that $\mathrm{End}_R(P)$ has cardinality $|R|^\lambda$. In fact, since R is slender and λ is a non-ω-measurable cardinal, we have $\mathrm{Hom}_R(P,P) \cong \prod_\lambda(\bigoplus_\lambda R)$ and $|\,\mathrm{End}_R(P)| = |R|^\lambda$ follows. Then, if all the maximal RD-submodules were isomorphic, necessarily to P, this would give $|R|^{|R|^\lambda} > |R|^\lambda$ distinct endomorphisms of P, impossible. □

We conjecture that *it never happens* that all the maximal RD-submodules of an R-module are isomorphic to P_λ (except for $R = Q$, of course). We will give a partial solution to this conjecture in Corollary 4.8 and Theorem 4.11.

Theorem 4.7. *Let R be a maximal Prüfer domain, λ an infinite cardinal number, G an R-module such that every maximal RD-submodule of G is isomorphic to P_λ. Then G itself is isomorphic to P_λ.*

Proof. Take any maximal RD-submodule of G, say H. Then $H \cong P_\lambda$ is pure-injective, and so H is a direct summand of G, hence, in particular, G has a direct summand isomorphic to R, say $G = K \oplus R$. Since also $K \cong P_\lambda$, we immediately get $G \cong P_\lambda$. □

Corollary 4.8. *Let R be a maximal Prüfer domain and λ an infinite cardinal number. Then every R-module admits maximal RD-submodules not isomorphic to P_λ.*

Proof. Assume, for a contradiction, that all the maximal RD-submodules of the R-module G are isomorphic to P_λ. Then $G \ncong P_\lambda$ by Theorem 4.4, and $G \cong P_\lambda$, by Theorem 4.7, impossible. □

In the following results we assume that R is a slender valuation domain and $\mathrm{gen}\, Q = \aleph_0$. This last condition is very natural, since $\mathrm{gen}\, Q = \aleph_0$ is equivalent to p.d.$Q = 1$ (see e.g. [FS], Theorem 3.4, page 208). This assumption has the useful consequence that R is slender if and only if it is not complete – see [D], Corollary 21.

Following [FS], for R a valuation domain we denote by $\tilde{X}$ the completion of the R-module X in its R-topology.

We need a technical lemma.

Lemma 4.9. *Let R be a valuation domain with p.d.$Q = 1$. If $\mathrm{rank}_R\tilde{R} > m$ (m a positive integer), then every reduced torsion-free R-module M of rank $\leq m$ is slender.*

Proof. The module M cannot contain $\tilde{R}$, since $\mathrm{rank}_R\tilde{R} > \mathrm{rank}_R M$. Moreover M cannot contain a copy of $P_\omega = \prod_{n<\omega} Re_n$, since P_ω has infinite rank. From [FS], Theorem 6.9, page 557, it follows that M is slender. □

Theorem 4.10. *Let R be a slender valuation domain with p.d.$Q = 1$ and G an R-module such that $\mathrm{gen}\, G > \gamma_R$. If λ is a non-ω-measurable cardinal number and every maximal RD-submodule of G is isomorphic to P_λ, then G itself is isomorphic to P_λ.*

Proof. Since λ is fixed, to simplify the notation we write $P_\lambda = P$. Observe that G is necessarily reduced, otherwise some maximal RD-submodule of G has a summand isomorphic to Q and cannot be isomorphic to P. Moreover, Proposition 1.2 yields $\lambda < |R|^\lambda = \mathrm{gen}\, P \leq |G| = \mathrm{gen}\, G$. We consider three different cases.

Case 1. $G^* \neq 0$.

Let J be a nonzero ideal of R such that there is an onto map $f : G \to J$. Then $H = \ker f$ is a maximal RD-submodule of G, and J is slender, being a submodule of the slender module R. Pick $z_i \in G$, $i \in C$ such that the $f(z_i)$ generate J. We can assume that $|C| = \operatorname{gen} J$. Let N_0 be the submodule of G generated by the z_i. We note that $\operatorname{gen} N_0 \leq \operatorname{gen} J \leq \gamma_R$. Moreover, since G is reduced, the RD-hull Z of any cyclic submodule Rz of G satisfies $\operatorname{gen} Z \leq \gamma_R$. It follows that the RD-hull N of N_0 satisfies $\operatorname{gen} N \leq \gamma_R < \operatorname{gen} G$. Hence $N \neq G$ and, in particular, there exists a maximal RD-submodule M of G containing N. The composition $M \to G \to G/H = J$ is then an epimorphism and, since $M \cong P$ and λ is a non-ω-measurable cardinal, this epimorphism must vanish on a summand of finite corank in M – see Corollary 3.3 in Chapter III, [EM]. Hence J is finitely generated, and so is a principal ideal, i.e. $G/H \cong R$, which implies that $G \cong H \oplus R \cong P$, as required. This completes Case 1.

In the remaining two cases we essentially prove that $G^* = 0$ is impossible, hence Case 1 is the only admissible one, and our statement follows.

Case 2. $G^* = 0$, and $\operatorname{rank}_R \tilde{R} > 2$.

Note that the assumption $G^* = 0$ implies that $G/X \cong Q$ for every maximal RD-submodule X of G. Take a maximal RD-submodule H of G, and fix $g_0 \in G \setminus H$; then the RD-hull of $Rg_0 + H$ is all of G. By hypothesis, we can write $H = \prod_{i<\lambda} Re_i$. Since $\operatorname{gen} G > \lambda$, arguing as in Case 1 we may choose a maximal RD-submodule M that contains g_0 and all the e_i, $i < \lambda$. By hypothesis, we also have $M \cong P$, and so there is a homomorphism from M onto R, say $\phi : M \to R$. As noted above, we have $G/M \cong Q$, hence, in particular, $\tilde{M} = \tilde{G}$. Then ϕ extends to a homomorphism $\tilde{\phi} : \tilde{G} \to \tilde{R}$. Now $\ker \phi$ is of corank 1 in M, and hence of corank ≤ 2 in G. Hence, $\tilde{\phi}(G)$ is a nonzero R-submodule of rank ≤ 2 of $\tilde{R}$, and so is slender, by Lemma 4.9 above. Now $\tilde{\phi}|_H : H \to \tilde{\phi}(G)$ is a map from P into a slender module, hence almost all the $\tilde{\phi}(e_i)$ vanish, again by the above quoted result in [EM]. We claim that $\tilde{\phi}(G) \subseteq R$.

To see this consider an arbitrary $g \in G$. Since the RD-hull $(Rg_0 + H)_* = G$, there is a $0 \neq s \in R$ such that $sg = h + tg_0$ where $t \in R$ and $h = (t_i e_i)_{i<\lambda} \in H$, $t_i \in R$. Then $s\tilde{\phi}(g) = \tilde{\phi}(h) + t\tilde{\phi}(g_0)$. Now, since almost all the $\tilde{\phi}(e_i)$ vanish and $\tilde{\phi}(H)$ is slender, being a submodule of the slender module $\tilde{\phi}(G)$, it follows that $\tilde{\phi}(h) = \sum_{n \in F} t_n \tilde{\phi}(e_n)$, for some finite subset F of λ. Moreover, since M contains g_0 and all the e_i, we have $\tilde{\phi}(g_0) = \phi(g_0) \in R$ and $\tilde{\phi}(e_n) = \phi(e_n) \in R$, for all $n \in F$. We conclude that $s\tilde{\phi}(g) \in R \cap s\tilde{R}$, and it follows by torsion-freeness that $\tilde{\phi}(g) \in R$. Thus $0 \neq \tilde{\phi}(G) \subseteq R$ and the claim is established. However, this cannot happen if $G^* = 0$, and so the present case is impossible.

Case 3. $G^* = 0$ and $\operatorname{rank}_R \tilde{R} = 2$.

It is clear that $\operatorname{rank}_R \tilde{R} = 2$ if and only if $\tilde{R}/R \cong Q$. Let H be a maximal RD-submodule of G. Then $G/H \cong Q$, and from the short exact sequence

$$0 \to H \to G \to Q \to 0$$

we get the exact sequence

$$0 = G^* \to H^* \to \operatorname{Ext}^1_R(Q, R).$$

Since in [M] Theorem 10 it is shown that $\operatorname{Ext}^1_R(Q, R) \cong \tilde{R}/R$, it follows that H^* embeds into $\tilde{R}/R \cong Q$. But $H \cong P$ and R slender yield $H^* \supseteq P^*_\omega \cong S_\omega$, and S_ω is free of countable rank, hence we get a contradiction. So this case is also impossible. □

Remark 1. In Example 6.10, page 557 of [FS] it is stated that a valuation domain R is always slender, unless it is complete with p.d.$Q = 1$. This is an oversight, since, when R is maximal, i.e. pure-injective, it cannot be slender, by [FS], property D), page 554. When p.d.$Q > 1$, a characterization of slender valuation domains by excluding submodules seems to be a challenging problem. This difficulty justifies our current assumption that p.d.$Q = 1$.

Theorem 4.11. *Let R be a slender valuation domain with p.d.$Q = 1$, λ an infinite non-ω-measurable cardinal number and G an R-module such that every maximal RD-submodule of G is isomorphic to P_λ. Then G itself is isomorphic to P_λ, provided that one of the following conditions holds (1) $|R|^\lambda > |R|$; (2) $\operatorname{gen} G > |\operatorname{Spec}(R)|$; (3) R is a DVR. Consequently, no such module G can exist.*

Proof. When R is a DVR the hypotheses of Theorem 4.10 are satisfied since $\gamma_R = 1$. If condition (2) holds, our statement is immediate from Proposition 1.3 of the first section and Theorem 4.10. Assume now that condition (1) holds. We clearly have $|R| \geq \gamma_R$, so $\operatorname{gen} G = |G| \geq |R|^\lambda > |R|$, allows us to apply Theorem 4.10.

Since λ is a non-ω-measurable cardinal, we are in a position to apply Theorem 4.6, and so P_λ has non-isomorphic maximal RD-submodules. It follows that no such module can exist. □

The general question whether the above result extends to any valuation domain remains open.

Remark 2. It is not true that a finite rank R-module M is slender if R is slender, not even assuming that M is a submodule of $\tilde{R}$. Indeed, there do exist discrete valuation domains R such that $\operatorname{rank}_R \tilde{R} < \infty$. Nagata, in [N] Example E33, page 207, was the first to construct such rings, called Nagata valuation domains in [Z]. The importance of these rings have been illustrated in several other papers; see, for instance, [AD], [GZ1], [GZ2]. A Nagata valuation domain R has positive characteristic, and the rank of $\tilde{R}$ is a power of the characteristic. In particular, the case $\operatorname{rank}_R \tilde{R} = 2$, examined in Case 3 of Theorem 4.10, is meaningful. Also Vámos [V] constructed a valuation domain R, one-dimensional but not a DVR, such that $\operatorname{rank}_R \tilde{R} = 2$. Constructions inspired by Nagata's examples, but with Krull dimension arbitrarily large, have been made in [FZ].

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