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Brendan Goldsmith

*Technological University Dublin*, [brendan.goldsmith@tudublin.ie](mailto:brendan.goldsmith@tudublin.ie)

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# On projection-invariant subgroups of Abelian $p$ -groups

P. V. Danchev

Department of Mathematics, Plovdiv University  
“P. Hilendarski”, Plovdiv 4000, Bulgaria

e-mail: pvdanchev@yahoo.com

and

B. Goldsmith

School of Mathematical Sciences, Dublin Institute of Technology,  
Aungier Street, Dublin 8, Ireland

e-mail: brendan.goldsmith@dit.ie

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*To Rüdiger Göbel on his 70<sup>th</sup> birthday*

## Abstract

A subgroup  $P$  of an Abelian  $p$ -group  $G$  is said to be *projection-invariant in  $G$* , if  $P\pi \leq P$  for all idempotent endomorphisms  $\pi$  in  $\text{End}(G)$ . Clearly fully invariant subgroups are projection invariant, but the converse is not true in general. Hausen and Megibben have, however, shown that in many familiar situations these two concepts coincide. In a different direction, the authors have previously introduced the notions of socle-regular and strongly socle-regular groups by focussing on the socles of fully invariant and characteristic subgroups of  $p$ -groups. In the present work the authors examine the socles of projection-invariant subgroups of Abelian  $p$ -groups.

## 1 Introduction

Recall that a subgroup  $P$  of an Abelian  $p$ -group  $G$  is said to be *projection-invariant in  $G$* , if  $P\pi \leq P$  for all idempotent endomorphisms  $\pi$  in  $\text{End}(G)$ . Clearly fully invariant subgroups are projection invariant, but the converse is not true in general. It is an easy exercise to show that  $P$  is projection-invariant in  $G$  if, and only if,  $P\pi = P \cap G\pi$  for every projection  $\pi \in \text{End}(G)$ . Like fully invariant subgroups, projection-invariant subgroups distribute across direct sums i.e., if  $G = A \oplus B$  and  $P$  is projection-invariant, then  $P = (A \cap P) \oplus (B \cap P)$ . In fact in some situations the notions coincide: Hausen [9] and Megibben [14] have established that for separable  $p$ -groups, and for transitive, fully transitive groups

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$G$  satisfying a certain technical condition (\*) on Ulm invariants, all projection-invariant subgroups are in fact fully invariant. For the readers' convenience we list this condition (\*): If  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_m$  are two disjoint finite sequences of ordinals such that the Ulm-Kaplansky invariants  $f_G(\alpha_i) \neq 0$  for each positive integer  $i$ , then there is a direct decomposition  $G = A \oplus B$  where  $f_A(\alpha_i) = 1$  for  $i = 1, \dots, n$  and  $f_A(\beta_j) = 0$  for  $j = 1, \dots, m$ .

In a different direction, the authors have recently investigated the socles of fully invariant and characteristic subgroups of Abelian  $p$ -groups. This led to the notions of socle-regular and strongly socle-regular groups, see [3, 4]. Recall the definitions: a group  $G$  is said to be *socle-regular* (*strongly socle-regular*) if for all fully invariant (characteristic) subgroups  $F$  of  $G$ , there exists an ordinal  $\alpha$  (depending on  $F$ ) such that  $F[p] = (p^\alpha G)[p]$ . It is self-evident that strongly socle-regular groups are themselves socle-regular, whereas the converse is not valid (see [4]). Motivated by these concepts we make the following definition: a  $p$ -group  $G$  is said to be *projectively socle-regular* if, for each projection-invariant subgroup  $P$  of  $G$ , there is an ordinal  $\alpha$  (depending on  $P$ ) such that  $P[p] = (p^\alpha G)[p]$ .

It is immediate that projectively socle-regular groups are socle-regular.

Throughout our discussion all groups will be additively written Abelian groups; our notation and terminology are standard and may be found in the texts [7, 12].

In [3] the following *ad hoc* terminology was introduced; it has been useful in [3, 4, 5] and we find it convenient to use it again here:

Suppose that  $H$  is an arbitrary subgroup of the group  $G$ . Set  $\alpha = \min\{h_G(y) : y \in H[p]\}$  and write  $\alpha = \min_G(H[p])$ ; clearly  $H[p] \leq (p^\alpha G)[p]$ . If there is no possibility of confusion we omit the subscript and just write  $\min(H[p])$ .

In the light of the results of Hausen [9] and Megibben [14], our first result is not too surprising.

**Proposition 1.1** *If  $P$  is a projection-invariant subgroup of the  $p$ -group  $G$  and  $\min_G(P[p]) = n$ , a positive integer, then  $P[p] = (p^n G)[p]$ . Consequently, if  $G$  is separable, then  $G$  is projectively socle-regular.*

*Proof.* Suppose that  $P$  is a projection-invariant subgroup of  $G$  and  $\min(P[p]) = n$ , a finite integer. Then there is an element  $x \in P[p]$  such that  $h_G(x) = n$  and so  $x = p^n y$  where  $y$  is the generator of a direct summand of  $G$ ;  $G = \langle y \rangle \oplus G_1$  say. Suppose that  $z \in (p^n G)[p] \setminus (p^{n+1} G)[p]$ , so that  $z = p^n w$  for some  $w$  of height zero; thus  $G = \langle w \rangle \oplus G_2$ . Note that  $\langle w \rangle \cong \mathbb{Z}(p^{n+1}) \cong \langle y \rangle$ . Then we have that  $w = ry + g_1$  for some integer  $r$  and some  $g_1 \in G_1$ . Now define  $\phi : \langle y \rangle \oplus G_1 \rightarrow \langle y \rangle \oplus G_1$  by  $y\phi = g_1$ ,  $G_1\phi = 0$ . It follows easily, or see the proof of Lemma 5 in [9], that  $\phi$  is the difference of two idempotent endomorphisms of  $G$ . Now define  $\psi : G \rightarrow G$  by  $g\psi = r(g\pi) + g\phi$ , where  $\pi$  is the projection map given by  $y\pi = y$ ,  $G_1\pi = 0$ . Note that  $\psi$  is a sum of integer multiples of idempotents and  $y\psi = ry + g_1 = w$ . Since  $x\psi = p^n(y\psi) = p^n w = z$  and  $x \in P$ , which is a projection-invariant subgroup of  $G$ , we conclude that  $z \in P[p]$ . Hence  $(p^n G)[p] \setminus (p^{n+1} G)[p] \subseteq P[p]$ . However, if  $u \in (p^{n+1} G)[p]$ , then  $z + u \in (p^n G)[p]$ , and so, by the argument above,  $z + u \in P[p]$ . Thus, we have that  $(p^n G)[p] \subseteq P[p]$  and since the reverse inclusion holds by virtue of  $\min(P[p]) = n$ , we have equality. The final deduction is immediate. ■

**Corollary 1.2** *If  $G$  is a  $p$ -group such that  $p^\omega G \cong \mathbb{Z}(p)$ , then  $G$  is projectively socle-regular.*

*Proof.* Suppose  $P$  is a projection-invariant subgroup of  $G$ . If  $P[p] \not\leq p^\omega G$ , then  $\min_G(P[p])$  is finite and we therefore apply Proposition 1.1 above to obtain that  $P[p] = (p^n G)[p]$  for some integer  $n$ . So, we may assume that  $P[p] \leq p^\omega G$ . But since the latter group is cyclic of prime order  $p$ , it follows at once that either  $P = P[p] = 0$ , whence  $P[p] = (p^{\omega+1} G)[p]$ , or  $P[p] = p^\omega G = (p^\omega G)[p]$  as required. ■

The property of a  $p$ -group being projectively socle-regular is inherited by subgroups of the form  $p^\alpha G$ .

**Proposition 1.3** *If  $G$  is a projectively socle-regular  $p$ -group, then so also is  $p^\alpha G$  for all ordinals  $\alpha$ .*

*Proof.* Let  $H = p^\alpha G$  and suppose that  $P$  is a projection-invariant subgroup of  $H$ . Let  $\pi$  be an arbitrary idempotent in  $\text{End}(G)$ . Then  $\pi^* = \pi \upharpoonright H$  is an idempotent endomorphism of  $H$ . Thus  $P\pi = P\pi^* \leq P$  since  $P$  is a projection-invariant subgroup of  $H$ . Consequently  $P$  is a projection-invariant subgroup of  $G$  and so there is an ordinal  $\beta$  such that  $P[p] = (p^\beta G)[p]$ . Since  $P$  is contained in  $p^\alpha G$ , we conclude that  $\beta \geq \alpha$ ; say  $\beta = \alpha + \gamma$ . But then  $P[p] = (p^{\alpha+\gamma} G)[p] = (p^\gamma H)[p]$ , showing that  $H$  is also a projectively socle-regular group. ■

Projectively socle-regular groups of length  $< \omega + \omega$  are easy to construct:

**Proposition 1.4** *If  $H$  is an arbitrary finite  $p$ -group, then there exists a projectively socle-regular group  $G$  with  $p^\omega G = H$ .*

*Proof.* It is convenient to make use of Corner's realization theorem [2, Theorem 6.1]. Let  $A = \text{End}(H)$ , so that  $A$  is certainly countable. Then applying Corner's result we find a group  $G$  such that  $p^\omega G = H$ ,  $\text{End}(G) \upharpoonright H = A$  and  $\text{Aut}(G) \upharpoonright H$  acts as the units of  $A$ . Moreover, there is a semigroup homomorphism, \* say, from the multiplicative semigroup  $A^\times \rightarrow \text{End}(G)^\times$  such that  $\phi^* \upharpoonright H = \phi$  for every  $\phi \in A$ . Note that the group  $G$  is transitive and fully transitive since finite groups have this property; it follows from [3, Theorem 0.3] and [4, Theorem 2.4] that  $G$  is strongly socle-regular and hence socle-regular.

We claim that  $G$  is also projectively socle-regular. Let  $P$  be a projection-invariant subgroup of  $G$ ; if  $\min(P[p])$  is finite, we are finished by Proposition 1.1. So suppose that  $\min(P[p])$  is infinite, so that  $P[p] \leq p^\omega G$ . Note that  $P[p]$  is also a projection-invariant subgroup of  $G$ . Observe firstly that  $P[p]$  is actually a projection-invariant subgroup of  $p^\omega G = H$ : for if  $\pi \in A$  is an idempotent, there is a mapping  $\pi^* \in \text{End}(G)$  such that  $\pi^* \upharpoonright H = \pi$ ; since \* is a semigroup homomorphism, the mapping  $\pi^*$  is also an idempotent and we have  $(P[p])\pi = (P[p])\pi^* \leq P[p]$ . Since  $H$  is a finite group, it is certainly projectively socle-regular so that  $P[p] = (p^n H)[p]$  for some integer  $n$ . But then  $P[p] = (p^{\omega+n} G)[p]$  and so  $G$  is projectively socle-regular, as required. ■

As observed above, projectively socle-regular groups are socle-regular. However, when  $p \neq 2$ , we can say a little more.

**Proposition 1.5** *If  $p \neq 2$ , then a projectively socle-regular  $p$ -group is strongly socle-regular.*

*Proof.* Let  $C$  be an arbitrary characteristic subgroup of the  $p$ -group  $G$ , where  $p \neq 2$ . If  $\pi$  is an idempotent in  $\text{End}(G)$ , then  $(2\pi - 1)^2 = 1$ , so that  $2\pi - 1$  is an involutory automorphism of  $G$ . Thus  $C(2\pi - 1) \leq C$  implying that  $2(C\pi) \leq C$ . Since, by assumption,  $p \neq 2$ , this implies that  $C\pi \leq C$ . Hence  $C$  is a projection-invariant subgroup of  $G$ . If  $G$  is a projectively socle-regular group, then  $C[p] = (p^\alpha G)[p]$  for some ordinal  $\alpha$ . Since  $C$  was an arbitrary characteristic subgroup of  $G$ , we have that  $G$  is strongly socle-regular, as required. ■

Our next example shows that the class of socle-regular groups is strictly larger than the class of projectively socle-regular groups. First we derive a technical lemma.

**Lemma 1.6** *If  $G$  is a  $p$ -group in which  $p^\omega G$  is homogeneous, then if  $G$  is fully transitive but not transitive, every subgroup of  $p^\omega G$  is a projection-invariant subgroup of  $G$ .*

*Proof.* Let  $P$  be an arbitrary subgroup of  $p^\omega G$  and let  $\pi \in \text{End}(G)$  be an arbitrary idempotent endomorphism. Then  $G = G_1 \oplus G_2$ , where  $G_1 = G\pi$ ,  $G_2 = G(1 - \pi)$ . Now if  $p^\omega G_i > 0$  for  $i = 1, 2$ , it follows from Proposition 2.2 in [2] that  $G$  is transitive - contrary to hypothesis. So either  $p^\omega G_1 = 0$  or  $p^\omega G_2 = 0$ .

It is well known, and easy to show, that if  $\nu$  is an idempotent endomorphism of  $G$ , then  $(p^\omega G)\nu = p^\omega(G\nu)$ . Thus, either  $(p^\omega G)\pi = 0$  or  $(p^\omega G)(1 - \pi) = 0$ . Hence, either  $P\pi = 0$  or  $P(1 - \pi) = 0$ ; in either case  $P\pi \leq P$  and so  $P$  is a projection-invariant subgroup of  $G$ . ■

**Proposition 1.7** *For each prime  $p$ , there exists a socle-regular  $p$ -group which is not projectively socle-regular.*

*Proof.* Let  $G$  be a non-transitive fully transitive  $p$ -group with  $p^\omega G \cong \bigoplus_{\aleph_0} \mathbb{Z}(p)$  - for example the groups of length  $\omega + 1$  constructed by Corner in [2]. Since  $G$  is fully transitive, it is socle-regular by [3, Theorem 0.3]. However,  $G$  is certainly not projectively socle-regular, since it follows from Lemma 1.6 above that any non-zero proper subgroup of  $p^\omega G$  cannot have a socle of the form  $(p^\alpha G)[p]$  for any ordinal  $\alpha$ . ■

**Remark 1.8** Corner's example of a non-transitive fully transitive  $p$ -group  $G$  used in Proposition 1.7 above, provides an easy example of a group which does not satisfy the condition (\*) introduced by Megibben. If it did satisfy (\*), then, since  $f_G(\omega) = \aleph_0$ , there would be a direct decomposition  $G = G_1 \oplus G_2$  in which  $f_{G_1}(\omega) = 1$  and so  $f_{G_2}(\omega) = \aleph_0$ . Thus,  $G$  would decompose as  $G_1 \oplus G_2$  with both  $p^\omega G_1, p^\omega G_2 > 0$ ; as pointed out in Lemma 1.6, no such decomposition exists for  $G$ .

We have seen in Proposition 1.3 that the projective socle-regularity of a group  $G$  is inherited by subgroups of the form  $p^\alpha G$ . We now wish to investigate the converse situation. Before doing so, we establish an elementary result of

independent interest regarding the lifting of idempotents; we remark that we shall need much deeper results than this to handle the situation for subgroups of the form  $p^\alpha G$ , when  $\alpha \geq \omega$ .

**Proposition 1.9** *If  $G$  is a  $p$ -group which has no non-trivial  $p^n$ -bounded pure subgroups and  $\phi \in \text{End}(G)$  is such that  $\phi \upharpoonright p^n G$  is an idempotent endomorphism of  $p^n G$ , then there is an idempotent endomorphism  $\theta$  of  $G$  such that  $\theta \upharpoonright p^n G = \phi \upharpoonright p^n G$ .*

*Proof.* Assume, for the moment that we have shown that if  $G$  has no non-trivial  $p^n$ -bounded pure subgroups, then an endomorphism  $\psi$  of  $G$  such that  $p^n \psi = 0$  satisfies the relation  $\psi^{n+1} = 0$ .

Let  $I = \{\varphi \in \text{End}(G) \mid \varphi \upharpoonright p^n G = 0\}$ ; it is easily checked that  $I$  is a 2-sided ideal of  $\text{End}(G)$ . Moreover, any  $\varphi \in I$  satisfies the relationship  $p^n \varphi = 0$ , and so, by the observation above,  $\varphi^{n+1} = 0$ ; in particular  $I$  is a nil ideal of  $\text{End}(G)$ . Since  $\phi \upharpoonright p^n G$  is an idempotent, it is immediate that  $\phi + I$  is an idempotent of  $\text{End}(G)/I$ . It now follows from standard ring theory - see e.g. [1, Proposition 27.1] - that idempotents lift modulo  $I$ . Thus there is an idempotent endomorphism  $\theta$  of  $G$  such that  $\theta + I = \phi + I$  and so  $\theta \upharpoonright p^n G = \phi \upharpoonright p^n G$ . It remains only to verify the claim in the first paragraph. This follows from the next lemma. ■

The next result is true in a wider context than  $p$ -groups; the second author learned it from an unpublished manuscript of Tony Corner and the proof below is taken from that source.

**Lemma 1.10** *If  $G$  is a  $p$ -group which has no non-trivial  $p^k$ -bounded pure subgroups and  $\psi$  is an endomorphism of  $G$  such that  $p^k \psi = 0$ , then  $\psi^{k+1} = 0$ .*

*Proof.* If  $k = 0$  there is nothing to prove; so we suppose that  $k \geq 1$ . Note first that if  $x \in G$  and the exponent of  $x$ ,  $E(x) = 1$ , then  $h_G(x) \geq k$ . For if  $h_G(x) = l < k$ , then  $x = p^l y$  for some  $y \in G$ , and it is clear that  $\langle y \rangle$  is a pure subgroup of order  $p^{l+1}$ , a factor of  $p^k$ , contrary to our hypothesis.

Let  $\mathcal{P}(n)$  denote the proposition:  $x \in G, E(x) = n \leq k \Rightarrow x\psi^n = 0$ . We prove  $\mathcal{P}(n)$  by induction on  $n$ . Since  $\mathcal{P}(0)$  is trivial, we may suppose that  $1 \leq n \leq k$  and that  $\mathcal{P}(r)$  is true for  $r < n$ . If  $x \in G$  and  $E(x) = n$ , then  $E(p^{n-1}x) = 1$ , so  $h_G(p^{n-1}x) \geq k$  and therefore  $p^{n-1}x = p^k z$  for some  $z \in G$ . So  $p^{n-1}(x\psi) = z(p^k \psi) = 0$ , whence  $E(x\psi) \leq n - 1$  and so  $(x\psi)\psi^{n-1} = 0$  i.e.  $x\psi^n = 0$ .

Since for each  $x \in G$  we have  $p^k(x\psi) = 0$ , so that  $E(x\psi) \leq k$ , therefore it follows that  $x\psi^{k+1} = (x\psi)\psi^k = 0$ . Thus  $\psi^{k+1} = 0$ . ■

We now have all the ingredients to prove:

**Theorem 1.11** *If  $G$  is a  $p$ -group and  $\pi \in \text{End}(p^n G)$  is an idempotent, then there is an idempotent  $\theta \in \text{End}(G)$  such that  $\theta \upharpoonright p^n G = \pi$ .*

*Proof.* Let  $G = (B_1 \oplus \cdots \oplus B_n) \oplus X$  be the Baer decomposition of  $G$ ; thus  $(B_1 \oplus \cdots \oplus B_n)$  is a  $p^n$ -bounded summand of a basic subgroup  $B$  of  $G$ . It follows by a well-known theorem of Szele [7, Theorem 33.2], that  $X$  has no non-trivial  $p^n$ -bounded pure subgroup (or equivalently no non-trivial  $p^n$ -bounded summand). Note that  $p^n G = p^n X$ , so that  $\pi \in \text{End}(p^n X)$  and  $\pi^2 = \pi$ .

Since every endomorphism of  $p^n X$  lifts to an endomorphism of  $X$  - this follows by a simplification of the proof of [7, Proposition 113.3] - there is an endomorphism  $\phi$  of  $X$  such that  $\phi \upharpoonright p^n X = \pi$ . Now apply Proposition 1.9 to obtain an idempotent endomorphism  $\theta_1$  of  $X$  such that  $\theta_1 \upharpoonright p^n X = \phi \upharpoonright p^n X = \pi$ . Define  $\theta : G \rightarrow G$  by  $\theta = 0 \oplus \theta_1$ ; clearly  $\theta$  is an idempotent endomorphism of  $G$  and  $\theta \upharpoonright p^n G = \pi$ . ■

We can now prove the desired partial converse to Proposition 1.3.

**Theorem 1.12** *If  $n$  is a non-negative integer and  $p^n G$  is a projectively socle-regular  $p$ -group, then  $G$  is a projectively socle-regular  $p$ -group.*

*Proof.* Let  $P$  be a projection-invariant subgroup of  $G$ . If  $\min(P[p])$  is finite,  $k$  say, then  $P[p] = (p^k G)[p]$  by Proposition 1.1. If  $\min(P[p]) \geq \omega$ , then  $P[p] \leq p^\omega G \leq p^n G$ . We claim that  $P[p]$  is a projection-invariant subgroup of  $p^n G$ : if  $\pi$  is any idempotent endomorphism of  $p^n G$ , then by Theorem 1.11, there is an idempotent endomorphism  $\theta$  of  $G$  such that  $\theta \upharpoonright p^n G = \pi$ . Hence,  $(P[p])\pi = (P[p])\theta \leq P[p]$  since  $P\theta \leq P$ . As  $p^n G$  is projectively socle-regular, we have that  $P[p] = (p^\alpha(p^n G)[p])$  for some ordinal  $\alpha$ . Thus  $P[p] = (p^\gamma G)[p]$  where  $\gamma = n + \alpha$  and  $G$  is projectively socle-regular. ■

We have already seen in Proposition 1.7 that there is a socle-regular  $p$ -group which is not projectively socle-regular. We now exhibit a strongly socle-regular 2-group which is not projectively socle-regular.

**Proposition 1.13** *There is a strongly socle-regular 2-group which is not projectively socle-regular.*

*Proof.* Consider any of the groups  $C$  constructed by Corner in [2] which were transitive but not fully transitive; these groups had the property that  $2^\omega C = \langle a \rangle \oplus \langle b \rangle$  where  $o(a) = 2, o(b) = 8$  and  $\text{End}(C) \upharpoonright 2^\omega C = \Phi$ , where  $\Phi$  is the subring generated by the automorphisms of  $\langle a \rangle \oplus \langle b \rangle$ .

It is shown in [8, Example 3.16] that the elements of  $\Phi$  can be described by two families  $\{\theta_{i\lambda}\}$  and  $\{\phi_{j\mu}\}$  with the parameters  $1 \leq i, j \leq 4$  and  $\lambda \in \{\pm 1, \pm 3\}$ ,  $\mu \in \{0, \pm 1, 2\}$ . A straightforward check using the definitions given in Example 3.16 of [8], reveals that the only idempotent endomorphisms in  $\Phi$  are 0 and 1.

Let  $P = \langle a \rangle$ ; we claim that  $P$  is a projection-invariant subgroup of  $C$ . For if  $\pi \in \text{End}(C)$  is an idempotent, then  $\pi \upharpoonright P$  is an idempotent in  $\Phi$  and so is either 0 or 1. In either case  $P\pi \leq P$  and so  $P$  is a projection-invariant subgroup of  $C$ . However direct calculation shows that  $P[2] = P$  is not any of the subgroups  $(2^\omega C)[2], (2^{\omega+1} C)[2], (2^{\omega+2} C)[2], (2^{\omega+3} C)[2] = 0$ . Since  $a \in P[2]$  has height  $\omega$  in  $C$ ,  $P[2] \neq (p^n C)[2]$  for any finite  $n$ . Thus  $C$  is not projectively socle-regular but it is strongly socle-regular since it is transitive - see [4, Theorem 2.4]. ■

The class of projectively socle-regular groups is, however, large. Recall that Megibben [14] has shown that a transitive fully transitive group  $G$ , which satisfies the technical condition (\*) on its Ulm invariants, has the property that every projection-invariant subgroup  $P$  of  $G$  is fully invariant; in particular he noted that the class of totally projective groups satisfies the property (\*). Consequently we have:

**Proposition 1.14** *Transitive, fully transitive  $p$ -groups satisfying the condition (\*) are projectively socle-regular. In particular, totally projective  $p$ -groups and  $C_\lambda$ -groups of length  $\lambda$ , where  $\lambda$  has cofinality  $\omega$ , are projectively socle-regular.*

*Proof.* If  $P$  is a projection-invariant subgroup of  $G$ , then it follows from Megibben's theorem [14] noted above, that  $P$  is fully invariant in  $G$ . In view of [3, Theorem 0.3],  $G$  is socle-regular. Thus  $P[p] = (p^\alpha G)[p]$  for some ordinal  $\alpha$ , whence  $G$  is projectively socle-regular as required. Since a totally projective group is both transitive and fully transitive by Theorems 1.2 of [10] and satisfies the condition (\*), it is projectively socle-regular.

To show the projective socle-regularity of  $C_\lambda$ -groups of length  $\lambda$ , where  $\lambda$  has cofinality  $\omega$  - see e.g., [16, Chapter 5], [13] or [17] for further details of  $C_\lambda$ -groups - note that, as observed by Megibben [14, p. 179], such a  $C_\lambda$ -group satisfies the technical condition (\*) - this is essentially because a  $C_\lambda$ -group of length  $\lambda$ , where  $\lambda$  has cofinality  $\omega$ , has a  $\lambda$ -basic subgroup whose decompositions lift to the whole group (see [16, Lemma 30.2] and the  $\lambda$ -basic subgroup satisfies (\*) since it is totally projective. Hence, if we can show that  $G$  is both transitive and fully transitive, the desired result will follow from the argument above.

Now such a  $C_\lambda$ -group  $G$  of length  $\lambda$ , is  $\lambda$ -separable when  $\lambda$  has cofinality  $\omega$  - see [16, Corollary 31.3]. Hence if  $x, y \in G$  with  $U_G(x) = U_G(y)$  (resp.  $U_G(x) \leq U_G(y)$ ), there is a direct summand  $H$  of  $G$ , say  $G = H \oplus K$ , such that  $H$  is totally projective and  $\langle x, y \rangle \leq H$ . Since  $H$  is a direct summand of  $G$ , we have  $U_H(x) = U_G(x)$  and  $U_H(y) = U_G(y)$ . However, a totally projective group is both transitive and fully transitive, so there exists an automorphism  $\theta$  (resp. an endomorphism  $\phi$ ) of  $H$  with  $x\theta = y$  (resp.  $x\phi = y$ ). Since  $\theta, \phi$  extend to maps  $\theta \oplus 1_K, \phi \oplus 1_K$  which are, respectively, an automorphism and an endomorphism of  $G$ , we have that  $G$  is both transitive and fully transitive, as required. ■

We return now to consideration of the converse of Proposition 1.3; recall that in Theorem 1.12 we showed that the property of being projectively socle-regular lifts from the subgroup  $p^n G$  to the whole group  $G$ . To extend this result to the ordinal  $\omega$  and beyond, it seems inevitable that we need some condition relating to total projectivity.

**Theorem 1.15** *(i) If  $G/p^\omega G$  is a direct sum of cyclic groups and  $p^\omega G$  is projectively socle-regular, then  $G$  is projectively socle-regular;*

*(ii) If  $\alpha$  is an ordinal strictly less than  $\omega^2$  such that  $p^\alpha G$  is projectively socle-regular and  $G/p^\alpha G$  is totally projective, then  $G$  is projectively socle-regular;*

*(iii) If  $G/p^\beta G$  is totally projective and  $p^\beta G$  is separable, then  $G$  is projectively socle-regular.*



*Proof.* (i) Arguing as in the proof of Theorem 1.12, it suffices to consider an arbitrary projection-invariant subgroup  $P$  of  $G$ , where  $P \leq p^\omega G$ . We claim that  $P[p]$  is a projection-invariant subgroup of  $p^\omega G$ . Assuming this, it follows immediately that  $P[p] = (p^\alpha(p^\omega G))[p] = (p^{\omega+\alpha}G)[p]$  for some ordinal  $\alpha$  and so  $G$  is projectively socle-regular. To complete the proof of (i) it remains only to substantiate the claim.

Suppose  $\pi$  is an arbitrary idempotent endomorphism of  $p^\omega G$ , then it follows from [11, Theorem 11] that there is an idempotent endomorphism  $\eta$  of  $G$  such that  $\eta \upharpoonright p^\omega G = \pi$ . But then  $P[p]\pi = P[p]\eta \leq P[p]$  since  $P$  is a projection-invariant subgroup of  $G$ .

The proof of (ii) is by transfinite induction; note that when  $\alpha$  is finite or equal to  $\omega$ , the result follows from Theorem 1.12 and (i) above. Since the argument follows exactly as in the proof of Proposition 1.6 (v) of [4], we simply refer the reader there.

For the final part (iii) we note firstly that if  $l(p^\beta G) = n < \omega$ , then  $p^\beta G$  is a direct sum of cyclic groups and so  $G$  itself is totally projective by a well-known theorem of Nunke [15]. The result then follows from Proposition 1.14 above. Suppose then that  $l(p^\beta G) = \omega$ . Observe that  $G$  is then a  $C_\lambda$ -group of length  $\lambda = \beta + \omega$  and  $\lambda$  has cofinality  $\omega$ . To see this note that if  $\sigma \leq \beta$ , then  $p^\sigma(G/p^\beta G) = p^\sigma G/p^\beta G$  and so  $G/p^\sigma G \cong (G/p^\beta G)/p^\sigma(G/p^\beta G)$  is totally projective by the previously quoted result of Nunke. If  $\beta < \sigma < \lambda$ , then  $\sigma$  has the form  $\beta + n$  and so if  $X = G/p^{\beta+n}G$ ,  $p^\beta X = p^\beta G/p^{\beta+n}G$  implying that  $X/p^\beta X \cong G/p^\beta G$  is totally projective. Since  $p^\beta X = p^\beta G/p^{\beta+n}G$ , we have that  $p^\beta X$  is a direct sum of cyclic groups and so  $X$  is again totally projective. It follows then  $G$  is a  $C_\lambda$ -group and clearly it has length  $\lambda$ . The result then follows from Proposition 1.14 above.

The next assertion demonstrates that certain subgroups inherit projective socle-regularity.

**Proposition 1.16** *If  $G$  is a projectively socle-regular  $p$ -group and  $P$  is a projection-invariant subgroup of  $G$  with the same first Ulm subgroup, then  $P$  is projectively socle-regular.*

*Proof.* Suppose  $K$  is an arbitrary projection-invariant subgroup of  $P$ . Since the projection-invariant property is obviously transitive, it follows that  $K$  is a projection-invariant subgroup of  $G$ . Therefore there is an ordinal  $\alpha$  such that  $K[p] = (p^\alpha G)[p]$ . If  $\alpha \geq \omega$ , it follows at once that  $K[p] = (p^\alpha P)[p]$  and we are done. If now  $\alpha$  is a finite ordinal number, say  $t$ , then  $K[p] = (p^t G)[p] \geq (p^t P)[p]$  and so it is easy to check that  $\min_P(K[p])$  is finite. Furthermore, Proposition 1.1 applies to infer that  $K[p] = (p^s P)[p]$  for some natural  $s$ , as required. ■

Let  $L$  denote a large subgroup of a  $p$ -group  $G$ . Using standard group-theoretic facts about  $L$  (see, e.g., [16]), a direct consequence is the following:

**Corollary 1.17** *If  $G$  is projectively socle-regular  $p$ -group and  $L$  is a large subgroup of  $G$ , then  $L$  is projectively socle-regular.*

**Remark 1.18** It is worthwhile noticing that the direct sum of two projectively socle-regular groups need not be projectively socle-regular. In fact, consider the example based on an idea of Megibben that has been used in [3] and [4]: let  $A$

and  $B$  be  $p$ -groups with  $p^\omega A \cong p^\omega B \cong \mathbb{Z}(p)$  such that  $A/p^\omega A$  is a direct sum of cyclic groups and  $B/p^\omega B$  is torsion-complete. We have shown in [3, Theorem 1.6] that their direct sum  $G = A \oplus B$  is not socle-regular and hence it is clearly not projectively socle-regular. However, utilizing Corollary 1.2, we observe that both  $A$  and  $B$  are projectively socle-regular but their direct sum is not.

However, as observed in Proposition 1.1, separable  $p$ -groups are always projectively socle-regular and so one would expect that the addition of a separable summand to a projectively socle-regular group would result in a projectively socle-regular group; this is, indeed, the case: if  $G$  is projectively socle-regular and  $H$  is separable, then  $A = G \oplus H$  is projectively socle-regular. The proof follows exactly as the proof of the corresponding statement for strongly socle-regular groups - see [4, Proposition 3.2]. The converse situation i.e., if  $G$  is a summand, with separable complement, of a projectively socle-regular group  $A$ , whether  $G$  is necessarily projectively socle-regular is not clear and one encounters similar difficulties to those experienced for strongly socle-regular groups - see the discussion and results following the proof of [4, Proposition 3.2].

## 2 Direct Powers

There is another source of projectively socle-regular groups which can be easily exhibited. In fact groups  $G$  of the form  $G = H^{(\kappa)}$ , where  $H$  is any  $p$ -group and  $\kappa$  is a cardinal, have the property that every projection-invariant subgroup of  $G$  is fully invariant in  $G$ . We begin with a result which is presumably well known but we could not find an explicit reference to it.

**Proposition 2.1** *Let  $R$  be an arbitrary ring, then every  $\Delta \in M_n(R)$ , the ring of (finite)  $n \times n$  matrices over  $R$  ( $n > 1$ ), can be expressed as a finite sum  $\Delta = \Delta_1 + \dots + \Delta_k$ , where each  $\Delta_i$  is either idempotent or a product of two idempotents.*

*Proof.* We consider first the case where  $n = 2$ . Let  $\Delta = \begin{pmatrix} r & s \\ u & v \end{pmatrix}$  and set  $\Delta_1 = \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}$ ,  $\Delta_2 = \begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix}$ ,  $\Delta_3 = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$  and  $\Delta_4 = \begin{pmatrix} 0 & 0 \\ 0 & v-2 \end{pmatrix}$ ; clearly  $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ .

A straightforward check shows that  $\Delta_1, \Delta_2$  are both idempotent.

However,  $\Delta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 1 \\ r-1 & 0 \end{pmatrix} = XY$  say, and it is easy to check that both  $X, Y$  are idempotent.

Finally  $\Delta_4 = \begin{pmatrix} 0 & 0 \\ v-3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = ZW$  say, and again it is straightforward to check that both  $Z, W$  are idempotents.

So the result is true for  $n = 2$ . Proceeding by induction, assume the result holds for  $n = k$  and consider  $\Delta = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ , a  $(k+1) \times (k+1)$  matrix, where  $A$  is  $k \times k$ ,  $d \in R$ ,  $b$  is a  $k \times 1$  column vector and  $c$  is a  $1 \times k$  row vector. Let  $O$  denote the  $k \times k$  zero matrix.

Then  $\Delta = \begin{pmatrix} O & b \\ c & 1 \end{pmatrix} + \begin{pmatrix} O & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O & 0 \\ 0 & d-2 \end{pmatrix}$ .

The first two matrices in this sum are easily seen to be idempotent and an identical argument to that used in the  $2 \times 2$  case, shows that the final matrix is a product of two idempotents.

By induction the matrix  $A$  may be expressed as  $A = A_1 + \dots + A_t$  where each  $A_i$  is either idempotent or a product of two idempotents. However, if  $X$  is an idempotent  $k \times k$  matrix, then  $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$  is an idempotent  $(k+1) \times (k+1)$  matrix, while if  $X, Y$  are idempotent then  $\begin{pmatrix} XY & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$  is a product

of two  $(k+1) \times (k+1)$  idempotent matrices. Thus the remaining matrix  $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$  can be expressed as a sum of idempotents and products of two idempotents. Therefore the matrix  $\Delta$  has the same property and the proof is completed by induction. ■

**Proposition 2.2** *If  $G$  has the form  $G = H^{(\kappa)}$ , for some group  $H$  and some cardinal  $\kappa$ , then every projection-invariant subgroup of  $G$  is fully invariant in  $G$ .*

*Proof.* If  $\kappa$  is finite, then every endomorphism of  $G$  can be expressed as a  $\kappa \times \kappa$  matrix  $\Delta$  over the ring  $R = \text{End}(H)$ . Thus if  $P$  is a projection-invariant subgroup of  $G$ , then  $P\Delta = P(\Delta_1 + \cdots + \Delta_t)$  where each  $\Delta_i$  is either idempotent or a product of idempotents. It is immediate that  $P\Delta \leq P$  and hence  $P$  is fully invariant in  $G$ . If  $\kappa$  is infinite, then  $G = A \oplus A$ , where  $A \cong H^{(\kappa)}$  and so every endomorphism of  $G$  is a  $2 \times 2$  matrix over the ring  $S = \text{End}(A)$ . Again every endomorphism of  $G$  is a sum of products of idempotent matrices over  $S$  by Proposition 2.1 and so is a sum of products of idempotent endomorphisms of  $G$ . The result follows as before. ■

It is now easy to exhibit many projectively socle-regular groups.

**Theorem 2.3** *If  $G$  is socle-regular, then  $G^{(\kappa)}$  is projectively socle-regular for all  $\kappa > 1$ . In particular, if  $G$  is projectively socle-regular, then so also is  $G^{(\kappa)}$  for all  $\kappa$ .*

*Proof.* Suppose that  $P$  is a projection-invariant subgroup of  $G^{(\kappa)}$  for some  $\kappa > 1$ , then, by Proposition 2.2,  $P$  is a fully invariant subgroup of  $G^{(\kappa)}$ . However, it follows from [3, Theorem 1.4] that  $G^{(\kappa)}$  is socle-regular and so  $P[p] = (p^\alpha G^{(\kappa)})[p]$  for some  $\alpha$ . Hence  $G^{(\kappa)}$  is projectively socle-regular. The final comment is immediate. ■

The next surprising statement illustrates that in some cases socle-regularity, strong socle-regularity and projective socle-regularity do coincide.

**Theorem 2.4** *Suppose  $\kappa > 1$ . Then the following four points are equivalent:*

- (i)  $G$  is socle-regular;
- (ii)  $G^{(\kappa)}$  is socle-regular;
- (iii)  $G^{(\kappa)}$  is strongly socle-regular;
- (iv)  $G^{(\kappa)}$  is projectively socle-regular.

*Proof.* The equivalences (i)  $\iff$  (ii)  $\iff$  (iii) follow from [4, Corollary 3.7] and [5, Corollary 3.2]. Moreover, the implication (i)  $\implies$  (iv) follows from Theorem 2.3, whereas the implication (iv)  $\implies$  (ii) is trivial. ■

However, the class of projectively socle-regular groups is not closed under taking summands.

**Proposition 2.5** *There exists a projectively socle-regular group  $H$  having a direct summand  $G$  which is not projectively socle-regular.*

*Proof.* Let  $G$  be a socle-regular group which is not projectively socle-regular—for example, the group in Proposition 1.7. Then if  $H = G \oplus G$ , it follows from Theorem 2.3 that  $H$  is projectively socle-regular, but clearly its direct summand  $G$  is not projectively socle-regular. ■

We close the paper with some open questions:

**Problem 1.** Are Krylov transitive  $p$ -groups satisfying condition (\*) projectively socle-regular groups?

**Problem 2.** If  $G$  is a socle-regular  $p$ -group with finite  $p^\omega G$ , does it follow that  $G$  is projectively socle-regular?

**Problem 3.** Is it true that projectively socle-regular 2-groups are strongly socle-regular?

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