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MATHEMATICS

Schrödinger Operators with δ'-Interactions and the Krein–Stieltjes String[1]

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Abstract—We investigate one dimensional symmetric Schrödinger operator $H_{X,\beta}$ with δ'-interactions of strength $\beta = \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ on a discrete set $X = \{x_n\}_{n=1}^{\infty} \subset [0, b)$, $b \le +\infty$ $(x_n \uparrow b)$. We consider $H_{X,\beta}$ as an extension of the minimal operator $H_{\min} := -d^2/dx^2 \lceil W_0^{2,2}(\mathbb{R}\backslash X)$ and study its spectral properties in the framework of the extension theory by using the technique of boundary triplets and the corresponding Weyl functions. The construction of a boundary triplet for $H_{\min}^*$ is given in the case $d_* := \inf_{n\in\mathbb{N}}|x_n - x_{n-1}| = 0$. We show that spectral properties like self-adjointness, lower semiboundedness, nonnegativity, and discreteness of the spectrum of the operator $H_{X,\beta}$ correlate with the corresponding properties of a certain Jacobi matrix. In the case $\beta_n > 0$, $n \in \mathbb{N}$, these matrices form a subclass of Jacobi matrices generated by the Krein–Stieltjes strings. The connection discovered enables us to obtain simple conditions for the operator $H_{X,\beta}$ to be self-adjoint, lower semibounded and discrete. These conditions depend significantly not only on β but also on X. Moreover, as distinct from the case $d_* > 0$, the spectral properties of Hamiltonians with δ- and δ'-interactions in the case $d_* = 0$ substantially differ.

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Let $\mathcal{I} := [0, b)$, $0 < b \le +\infty$, and let $X = \{x_n\}_{n=0}^{\infty} \subset \mathcal{I}$ be a strictly increasing sequence such that $\lim_{n\to\infty} x_n = b$ and $x_0 := 0$. Let also $\beta := \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$.

Consider a symmetric operator $H_{X,\beta}^0$ in $L^2(\mathcal{I})$ defined by the differential expression $-\dfrac{d^2}{dx^2}$ on functions $f \in W_{\text{comp}}^{2,2}(\mathcal{I}\backslash X)$ satisfying the following boundary conditions at the points $x_n \in X$:

$$\begin{aligned} f'(0) &= 0, \quad f'(x_n+) = f'(x_n-), \\ f(x_n+) - f(x_n-) &= \beta_n f'(x_n), \quad n \in \mathbb{N}. \end{aligned} \tag{1}$$

Let us denote its closure by $H_{X,\beta}$. The operator $H_{X,\beta}$ is interpreted as the Hamiltonian with δ'-interactions of strength β_n at the centers x_n [2] and, as a rule, is associated with the formal differential expression

$$l_{X,\beta} := -\frac{d^2}{dx^2} + \sum_{x_n \in X} \beta_n(\cdot, \delta_n')\delta_n', \quad x \in \mathcal{I}, \tag{2}$$

where $\delta_n := \delta(x - x_n)$ and $\delta(\cdot)$ is the Dirac delta-function.

Spectral properties of the operator $H_{X,\beta}$ are widely studied in the case $d_* := \inf_{n\in\mathbb{N}} d_n > 0$, where $d_n := x_n - x_{n-1}$. Thus, it is known that the operator $H_{X,\beta}$ is self-adjoint [2] and its spectrum is not discrete in this case. Further results (investigation of spectrum, resolvent comparability etc.) as well as a comprehensive list of references can be found in [2, 14].

The cases $d_* = 0$ and $d_* > 0$ are significantly different. Recently, it has been found ([5, Theorem 4.7]) that $H_{X,\beta}$ is self-adjoint for any β if $\mathcal{I} = \mathbb{R}_+$ and $d_* = 0$. To the best of our knowledge, other spectral properties of the operator $H_{X,\beta}$ in the case $d_* = 0$ have not been studied yet. Let us also mention the recent papers [3, 15] dealing with spectral properties of Hamiltonians with δ'-interactions on compact subsets of $\mathbb{R}$ with Lebesgue measure zero.

The main aim of this note is the spectral analysis of the Hamiltonian $H_{X,\beta}$ in the case $d_* = 0$. We investigate the operator $H_{X,\beta}$ in the framework of extension theory of symmetric operators using the concept of boundary triplets and the corresponding Weyl functions (see [7, 9]). Let us note that this approach was first applied by Kochubei [12] in the study of operators

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with point interactions. He investigated Hamiltonians with δ-interactions in the case $d_* > 0$. Let us stress that the main difficulty of this approach is the construction of an adequate boundary triplet.

In this note, we present a corresponding construction in the case $d_* = 0$ and show that the spectral properties like self-adjointness, lower semiboundedness, and discreteness of the spectrum of the operator $H_{X,\beta}$ correlate with the corresponding spectral properties of the Jacobi matrix $B_{X,\beta}$ defined by (15) (see below). It turns out that this class of matrices is closely connected with the class of Krein-Stieltjes string operators. Namely, if $\beta_n > 0$, $n \in \mathbb{N}$, then this class of matrices is a subclass of Krein—Stieltjes operators (see Section 5). Discovered connection enables us to obtain simple criteria for the operator $H_{X,\beta}$ to be self-adjoint, lower semi-bounded, and discrete. These conditions depend substantially not only on β but also on X. Moreover, as distinct from the case $d_* > 0$, the spectral properties of Hamiltonians with δ- and δ'-interactions in the case $d_* = 0$ differ completely.

Detailed treatment of the results discussed in this note are given in [11].

1. BOUNDARY TRIPLETS FOR DIRECT SUMS AND WEYL FUNCTIONS

Let A be a closed symmetric operator with dense domain $\mathfrak{D}(A)$ in a Hilbert space $\mathfrak{H}$ and equal deficiency indices $n_+(A) = n_-(A)$ ($n_\pm(A) := \dim(\mathfrak{N}_{\pm i})$, $\mathfrak{N}_z := \ker(A^* - zI)$).

Definition 1 ([7]). A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space, Γ_0 and Γ_1 are linear mappings from $\mathfrak{D}(A^*)$ to $\mathcal{H}$, is called a boundary triplet for the operator A^* if the abstract Green identity holds

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}} \quad \text{for all } f, g \in \mathfrak{D}(A^*), \tag{3}$$

and the mapping $\Gamma: f \to \{\Gamma_0 f, \Gamma_1 f\}$ from $\mathfrak{D}(A^*)$ to $\mathcal{H} \oplus \mathcal{H}$ is surjective.

A boundary triplet is not unique and, moreover, it enables one to parameterize the set Ext_A of all proper extensions $\tilde{A}$ ($A \subset \tilde{A} \subset A^*$) of the operator A in terms of abstract boundary conditions. Let $A_j := A^*|\ker(\Gamma_j)$, $j \in \{0, 1\}$. It is known that $A_j = A_j^*$, $j \in \{0, 1\}$.

Proposition 1 ([7]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$(\mathrm{Ext}_A \ni)\tilde{A} \to \Gamma\mathfrak{D}(\tilde{A}) = \{\{\Gamma_0 f, \Gamma_1 f\}: f \in \mathfrak{D}(\tilde{A})\} =: \Theta \in \tilde{\mathcal{C}}(\mathcal{H}) \tag{4}$$

establishes a bijective correspondence between Ext_A *and the set* $\tilde{\mathcal{C}}(\mathcal{H})$ *of closed linear relations in* $\mathcal{H}$. *Denote* $A_\Theta := \tilde{A}$, *where* Θ *is determined by Eq.* (4). *Then the following statements hold*

(i) $A_\Theta = A_\Theta^* \Leftrightarrow \Theta = \Theta^*$;

(ii) A_Θ *is symmetric* $\Leftrightarrow$ Θ *is symmetric. Moreover,* $n_\pm(A_\Theta) = n_\pm(\Theta)$;

(iii) $\mathfrak{D}(A_\Theta) \cap \mathfrak{D}(A_0) = \mathfrak{D}(A) \Leftrightarrow \Theta \in \mathcal{C}(\mathcal{H})$. *In this case* Θ *is called a boundary operator and relation* (4) *becomes*

$$A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \tag{5}$$

Definition 2 ([9]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Operator valued function $M(\cdot): \rho(A_0) \to [\mathcal{H}]$ defined by

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad f_z \in \ker(A^* - z), \quad z \in \rho(A_0), \tag{6}$$

is called the Weyl function corresponding to the boundary triplet Π.

Further, let S_n be a closed densely defined symmetric operator in $\mathfrak{H}_n$ such that $n_+(S_n) = n_-(S_n) \le \infty$, $n \in \mathbb{N}$. In the Hilbert space $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$, consider the operator $A := \bigoplus_{n=1}^{\infty} S_n$. Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for the operator S_n^*. Define a direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets Π_n by setting

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad \Gamma_j := \bigoplus_{j=1}^{\infty} \Gamma_j^{(n)}, \quad \Gamma_j: \mathfrak{D}(A^*) \to \mathcal{H}, \quad j \in \{0, 1\}. \tag{7}$$

Green's formulae (3) for operators S_n^*, $n \in \mathbb{N}$, yield validity of Green's formula for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n$ and also vectors $f = \bigoplus_{n=1}^{\infty} f_n$, $g = \bigoplus_{n=1}^{\infty} g_n \in \mathfrak{D}(\Gamma_0) \cap \mathfrak{D}(\Gamma_1) \subset \mathfrak{D}(A^*)$, where

$$\mathfrak{D}(\Gamma_j) := \left\{\varphi = \bigoplus_{n=1}^{\infty} \varphi_n: \sum_{n \in \mathbb{N}} \left\|\Gamma_j^{(n)}\varphi_n\right\|_{\mathcal{H}_n}^2 < \infty\right\}.$$

It is shown in [12] that a boundary triplet Π for the operator $A^* = \bigoplus_1^{\infty} S_n^*$ can be chosen in the form (7) if there exist $\varepsilon > 0$ and a sequence $\{\rho_n\}_{n=1}^{\infty} \subset [\varepsilon, \varepsilon^{-1}]$ such that all pairs $\{\rho_n S_n, \rho_n S_{n0}\}$, $n \in \mathbb{N}$, are unitary equivalent to $\{S_1, S_{10}\}$. However, it is not difficult to give an example of the operator $A = \bigoplus_1^{\infty} S_n$ for which a triplet (7) is not a boundary triplet for A^*. The reason is the following. The mapping $\Gamma = \{\Gamma_0, \Gamma_1\}: \mathfrak{D}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ may not fulfill the following necessary conditions:

$$\mathfrak{D}(\Gamma_0) \cap \mathfrak{D}(\Gamma_1) = \mathfrak{D}(A^*), \quad \mathrm{ran}(\Gamma) = \mathcal{H} \oplus \mathcal{H}.$$

Next theorem provides criteria for the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of the form (7) to be a boundary triplet for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

Theorem 1. *Let* $A = \bigoplus_{n=1}^{\infty} S_n$, $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ *be a boundary triplet for the operator* S_n^*, $n \in \mathbb{N}$, *and let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be defined by* (7). *Then*:

(i) *the triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary relation* (*in the sense of* [8]) *for the operator* A^*;

(ii) *the collection* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for the operator* A^* *precisely when the mappings* Γ_j *are bounded, i.e.*,

$$\sup_{n \in \mathbb{N}} \left\|\Gamma_0^{(n)}\right\| < \infty, \quad \sup_{n \in \mathbb{N}} \left\|\Gamma_1^{(n)}\right\| < \infty; \tag{8}$$

(iii) *the collection* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for the operator* A^* if *and only if the following conditions are satisfied*

$$\sup_n \|M_n(\mathrm{i})\| < \infty, \quad \sup_n \left\|(\mathrm{Im}\, M_n(\mathrm{i}))^{-1}\right\| < \infty. \tag{9}$$

Here $M_n(\cdot)$ *is the Weyl function corresponding to the triplet* Π_n.

(iv) *If* $a_0 = \bar{a}_0$ $(\in \mathbb{R})$ *is the point of a regular type of the operator A, then the collection* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for the operator* A^* *precisely when*

$$\sup_{n \in \mathbb{N}} \|M_n(a_0)\| < \infty, \quad \sup_{n \in \mathbb{N}} \left\|(M_n'(a_0))^{-1}\right\| < \infty. \tag{10}$$

Theorem 1 enables one to regularize an arbitrary collection of boundary triplets $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ in such a way that the direct sum of the regularized triplets is a boundary triplet for the operator A^*. Namely, assume that $Q_n = Q_n^* \in [\mathcal{H}_n]$ and $R_n \in [\mathcal{H}_n]$ with $R_n^{-1} \in [\mathcal{H}_n]$. We set

$$\Gamma_0^{(n)} := R_n \tilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} := (R_n^{-1})^* (\tilde{\Gamma}_1^{(n)} - Q_n \tilde{\Gamma}_0^{(n)}). \tag{11}$$

Corollary 1. *Let* $\tilde{\Pi}_n = \{\mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$ *be a boundary triplet for* S_n^*, *and let* $\tilde{M}_n(\cdot)$ *be the corresponding Weyl function,* $n \in \mathbb{N}$. *If*

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \{\left\|(R_n^{-1})^* (\tilde{M}_n(\mathrm{i}) - Q_n) R_n^{-1}\right\| < \infty, \\ &\sup_{n \in \mathbb{N}} \left\|R_n (\mathrm{Im}(\tilde{M}_n(\mathrm{i})))^{-1} R_n^*\right\|\} < \infty, \end{aligned} \tag{12}$$

then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ (*see* (7)) *of triplets* $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ *defined by* (11) *is a boundary triplet for the operator* $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

Remark 1. *The existence of the regularization* (11) *was first established in* [13, *Theorem* 5.3]. *Namely, in relations* (11) *it is taken* $Q_n := \mathrm{Re}\, \tilde{M}_n(\mathrm{i})$ *and* $R_n := \sqrt{\mathrm{Im}\, \tilde{M}_n(\mathrm{i})}$, $n \in \mathbb{N}$ (*cf.* [13]). *However, let us stress that the latter regularization is not suitable for our aims since it does not lead to a parametrization of the Hamiltonians* $H_{X, \beta}$ *by Jacobi matrices.*

2. BOUNDARY TRIPLETS FOR SCHRODINGER OPERATORS

Consider the minimal (closed) symmetric operator

$$H := H_{\min} := -\frac{d^2}{dx^2}, \quad \mathfrak{D}(H_{\min}) = W_0^{2,2}(\mathcal{I} \backslash X). \tag{13}$$

It is clear that $n_{\pm}(H_{\min}) = \infty$ and the operator $H_{\min}^*$ is defined by the same differential expression on the domain $\mathfrak{D}(H_{\min}^*) = W^{2,2}(\mathcal{I} \backslash X)$. It is easy to see that $H_{\min} = \bigoplus_{n \in \mathbb{N}} H_n$, where $H_n := -\frac{d^2}{dx^2}$, $\mathfrak{D}(H_n) = W_0^{2,2}[x_{n-1}, x_n]$. Define the mappings

$$\Gamma_0^{(n)} f := \begin{pmatrix} d_n^{1/2} f(x_{n-1}+) \\ -d_n^{1/2} f(x_n-) \end{pmatrix},$$

$$\Gamma_1^{(n)} f := \begin{pmatrix} \dfrac{d_n f'(x_{n-1}+) + (f(x_{n-1}+) - f(x_n-))}{d_n^{3/2}} \\ \dfrac{d_n f'(x_n-) + (f(x_{n-1}+) - f(x_n-))}{d_n^{3/2}} \end{pmatrix}. \tag{14}$$

A straightforward calculation shows that the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is a boundary triplet for the operator H_n^*. Moreover, we have the following

Theorem 2. *Let* $d^* := \sup_{n \in \mathbb{N}} d_n < \infty$ *and let* $\Gamma_0^{(n)}, \Gamma_1^{(n)}$ *be defined by* (14). *Then the triplet* $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *defined by* (7) *is a boundary triplet for the operator* $H_{\min}^*$.

Proof. The condition $d^* < \infty$ yields that $z = 0$ is a point of a regular type for the operator $H_{\min}$. To complete the proof of Theorem 2 it suffices to check (10) at the point $z = 0$.

3. PARAMETRIZATION OF HAMILTONIANS WITH δ'-INTERACTIONS

For the remainder of this note, we can assume without loss of generality that $\beta_n \neq 0$ for all $n \in \mathbb{N}$. Further, consider the semi-infinite Jacobi matrix

$$B_{X,\beta} := \begin{pmatrix} d_1^{-2} & d_1^{-2} & 0 & 0 & 0 & \dots \\ d_1^{-2} & \frac{d_1^{-1}}{\beta_1} + d_1^{-2} & \frac{d_1^{-1/2}d_2^{-1/2}}{\beta_1} & 0 & 0 & \dots \\ 0 & \frac{d_1^{-1/2}d_2^{-1/2}}{\beta_1} & \frac{d_2^{-1}}{\beta_1} + d_2^{-2} & d_2^{-2} & 0 & \dots \\ 0 & 0 & d_2^{-2} & \frac{d_2^{-1}}{\beta_2} + d_2^{-2} & \frac{d_2^{-1/2}d_3^{-1/2}}{\beta_2} & \dots \\ 0 & 0 & 0 & \frac{d_2^{-1/2}d_3^{-1/2}}{\beta_2} & \frac{d_3^{-1}}{\beta_2} + d_3^{-2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{15}$$

In $l^2(\mathbb{N})$, the minimal symmetric operator is naturally associated with the matrix $B_{X,\beta}$ (see [4]). We denote it also by $B_{X,\beta}$.

The following Lemma shows that the boundary operator Θ parameterizing $H_{X,\beta}$ via (5) is a Jacobi matrix of the form (15).

Lemma 1. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $H^*_{\min}$ defined in Theorem 2. Then $H_{X,\beta} = H_{B_{X,\beta}}$, that is*

$$H_{X,\beta} = H^*_{\min}\lceil \mathfrak{D}(H_{X,\beta}),$$

$$\mathfrak{D}(H_{X,\beta}) = \{f \in W^{2,2}(\mathbb{R}\backslash X)\colon \Gamma_1 f = B_{X,\beta}\Gamma_0\}.$$

Theorem 2, Lemma 1 and some results from [7, 9] enable us to establish the following connection between spectral properties of the Hamiltonian $H_{X,\beta}$ and the Jacobi matrix (boundary operator) $B_{X,\beta}$.

Theorem 3. *Assume that $d^* < \infty$. Then*:

(i) *the following relation holds $n_\pm(H_{X,\beta}) = n_\pm(B_{X,\beta})$;*

(ii) *the operator $H_{X,\beta}$ is lower semibounded precisely when the operator $B_{X,\beta}$ is lower semibounded;*

(iii) *if $n_\pm(H_{X,\beta}) = 0$, i.e., $H_{X,\beta} = H^*_{X,\beta}$, then the spectrum of the operator $H_{X,\beta}$ is discrete precisely when $\lim_{n\to\infty} d_n = 0$ and the spectrum of the operator $B_{X,\beta}$ is discrete.*

4. A CONNECTION WITH THE KREIN–STIELTJES STRING

It is clear that the parameterizing matrix of the form (15) admits the following factorization

$$B_{X,\beta} = R_X^{-1}(I+U)D_{X,\beta}^{-1}(I+U^*)R_X^{-1}, \tag{16}$$

where U is the unilateral shift operator in $l^2(\mathbb{N})$ and

$$D_{X,\beta} := \bigoplus_{n=1}^{\infty}\begin{pmatrix} d_n & 0 \\ 0 & \beta_n \end{pmatrix}, \quad R_X := \bigoplus_{n=1}^{\infty}\begin{pmatrix} \sqrt{d_n} & 0 \\ 0 & \sqrt{d_n} \end{pmatrix}. \tag{17}$$

We put $l_{2n-1} := d_n$, $l_{2n} := \beta_n$, $m_{2n-1} = m_{2n} := d_n$, $n \in \mathbb{N}$. In the case $\beta_n > 0$, $n \in \mathbb{N}$, the difference equation associated with the matrix $B_{X,\beta}$ describes the motion of an inhomogeneous string (Krein–Stieltjes string) with mass distribution $\mathcal{M}(y) = \sum_{y_n<y} m_n$, where $y_n - y_{n-1} := l_n$ and $y_0 := 0$. This class of matrices is studied sufficiently well. In particular, criterion of self-adjointness (Hamburger's Theorem [1, Theorem 0.5]) and criterion of discreteness of spectrum (the Kac-Krein Theorem [10]) are known and are formulated in terms of sequences $l = \{l_n\}_{n=1}^\infty$ and $m = \{m_n\}_{n=1}^\infty$. We hardly exploit these criteria as well as Theorem 3 in the following sections.

5. SELF-ADJOINTNESS

Combining Theorem 3 (i) with the representation (16), (17), we obtain

Theorem 4. *Deficiency indices of the operator $H_{X,\beta}$ are equal and are not greater than one, $n_+(H_{X,\beta}) = n_-(H_{X,\beta}) \le 1$. Furthermore, $H_{X,\beta}$ is self-adjoint if and only if at least one of the following conditions hold:*

(1) $\sum_{n=1}^{\infty} d_n = \infty$, *i.e.* $\mathcal{I} = \mathbb{R}_+$;

(2) $\sum_{n=1}^{\infty}\left[d_{n+1}\left|\sum_{i=1}^{n}(\beta_i + d_i)\right|^2\right] = \infty.$

Remark 3. In the case of positive strengths β_n, Theorem 4 follows from Hamburger's Theorem [1]. In the case of arbitrary strengths, we use the formulae for $P_n(0)$ and $Q_n(0)$ [1, p. 291] and the self-adjointness criterion [4, Lemma VII.1.5].

Theorem 4 immediately yields the following result of Buschmann–Stolz–Weidmann [5, Theorem 4.7].

Corollary 2 ([5]). *If $\mathcal{I} = \mathbb{R}_+$, then the operator $H_{X,\beta}$ is self-adjoint.*

6. LOWER SEMIBOUNDED HAMILTONIANS

Using Theorem 3(ii) and the form of the matrix $B_{X,\beta}$, we arrive at the following result.

Proposition 2. *For the operator $H_{X,\beta}$ to be lower semibounded it is necessary that*

$$\beta_n^{-1} \ge -C_1 d_n - \min\{d_n^{-1}, d_{n+1}^{-1}\}, \quad n \in \mathbb{N} \tag{18}$$

and sufficient that

$$\beta_n^{-1} \ge -C_2 \min\{d_n, d_{n+1}\}, \quad n \in \mathbb{N}. \tag{19}$$

Here C_1 are C_2 are positive constants independent of $n \in \mathbb{N}$.

Corollary 3. *If $0 < d_* \le d^* < \infty$, then the operator $H_{X,\beta}$ is lower semi-bounded precisely when $\inf_{n\in\mathbb{N}} \beta_n^{-1} > -\infty$.*

7. DISCRETENESS OF SPECTRUM

We set $\beta_n^- := \beta_n$ if $\beta_n < 0$ and $\beta_n^- := -\infty$ if $\beta_n > 0$. Applying the discreteness criterion [10] to the matrix $B_{X,\beta}$ and combining this with Theorem 3 (iii), we arrive at the following necessary conditions for discreteness of the spectrum of the operator $H_{X,\beta}$.

Proposition 3. *Let $\mathcal{I} = \mathbb{R}_+$ and $d_n \to 0$. The spectrum of the operator $H_{X,\beta}$ is not discrete if at least one of the following conditions hold:*

(i) $\lim_{n\to\infty} x_n \sum_{j=n}^{\infty} d_j^3 > 0$;

(ii) $\beta_n \ge -C d_n^3$, $n \in \mathbb{N}$, $C > 0$;

(iii) $\beta_n^- \le -C(d_n^{-1} + d_{n+1}^{-1})$, $n \in \mathbb{N}$, $C > 0$.

Corollary 4. *If either $\beta_n > 0$ for all $n \in \mathbb{N}$ or $\{d_n\}_{n=1}^{\infty} \notin l^3(\mathbb{N})$, then the spectrum of the operator $H_{X,\beta}$ is not discrete.*

Proposition 3 shows that the spectrum of the operator $H_{X,\beta}$ is discrete in the very exceptional cases. For instance, the spectrum is not discrete if either $\beta \subset \mathbb{R}_+$ or if there exists a subsequence $\{\beta_{n_k}\} \subset \mathbb{R}_-$, which tends to $-\infty$ sufficiently fast. Furthermore, d_n must tend to zero sufficiently fast for the spectrum to be discrete. Sufficient conditions for discreteness of the spectrum of the Hamiltonian $H_{X,\beta}$ are given in the following

Theorem 5. *Assume that $\beta_n + d_n \ge 0$ for all $n \in \mathbb{N}$.*

(i) *If $b < +\infty$, and X and β are such that the operator $H_{X,\beta}$ is self-adjoint, then the spectrum of $H_{X,\beta}$ is discrete if and only if*

$$\lim_{n\to\infty} (b - x_n) \sum_{j=1} (\beta_j + d_j) = 0. \tag{20}$$

(ii) *$\mathcal{I} = \mathbb{R}_+$, then the spectrum of $H_{X,\beta}$ is discrete precisely when*

$$\lim_{n\to\infty} x_n \sum_{j=n}^{\infty} d_j^3 = 0, \quad \lim_{n\to\infty} x_n \sum_{j=n}^{\infty} (\beta_j + d_j) = 0. \tag{21}$$

Remark 4. Let us note that we use another parametrization of the Hamiltonian $H_{X,\beta}$ for proving Propositions 3 (i), (ii) and Theorem 5 (for the details, see [11, Sect. 6].

8. ON THE NEGATIVE SPECTRUM OF THE OPERATOR $H_{X,\beta}$

Let T be a self-adjoint operator in $\mathfrak{H}$ and let $E_T(\cdot)$ be its spectral function. Dimension of the negative subspace $E_T(-\infty, 0)\mathfrak{H}$ of the operator T is denoted by $\kappa_-(T)$, $\kappa_-(T) := \dim(E_T(-\infty, 0)\mathfrak{H})$.

Proposition 4. *Let $d^* < \infty$ and let $\kappa_-(\beta)$ be the number of negative elements in the sequence $\beta = \{\beta_n\}_{n=1}^{\infty}$. If $H_{X,\beta} = H_{X,\beta}^*$, then*

$$\kappa_-(H_{X,\beta}) = \kappa_-(\beta). \tag{22}$$

In particular, $H_{X,\beta} > 0$ if $\beta_n > 0$, $n \in \mathbb{N}$.

Proof. Let $M(\cdot)$ be the Weyl function of the operator $H_{\min}$, which corresponds to the boundary triplet Π defined in Theorem 2. Using the form of the Weyl function, we get the following equality $M(0) := s\text{-}\lim_{x\to-0} M(x) = 0$. By [9, Theorem 4], $\kappa_-(H_{X,\beta}) = \kappa_-(B_{X,\beta} - M(0)) = \kappa_-(B_{X,\beta})$. On the other hand, by (16), (17), $\kappa_-(B_{X,\beta}) = \kappa_-(D_{X,\beta}) = \kappa_-(\beta)$.

Remark 5. Using rather different approaches, Proposition 4 was obtained in [15] under assumptions that $\kappa_-(\beta) = |X| < \infty$, and also in [6] assuming that $|X| = \infty$ and $d_* > 0$.

9. HAMILTONIANS WITH δ-INTERACTIONS

Let $\alpha = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$. Consider the differential expression

$$l_{X,\alpha} = -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n), \quad x \in \mathcal{I}. \tag{23}$$

In $L^2(\mathcal{I})$, one associates with (23) a closed symmetric operator $H_{X,\alpha}^0$, defined by the differential expression $-\frac{d^2}{dx^2}$ on functions $f \in W_{\text{comp}}^{2,2}(\mathcal{I}\backslash X)$ satisfying the following boundary conditions at the points $x_n \in X$:

$$\begin{aligned} f'(0) &= 0, \quad f(x_n+) = f(x_n-), \\ f'(x_n+) - f'(x_n-) &= \alpha_n f(x_n), \quad n \in \mathbb{N}. \end{aligned} \tag{24}$$

Let $H_{X,\alpha}$ be its closure. The operator $H_{X,\alpha}$ is known as the Hamiltonian with δ-interactions of strengths α_n at the centers x_n [2].

The analogs of Lemma 1 and Theorem 3 hold true for the operator $H_{X,\alpha}$ (see [11, Sect. 5]) and the role of the matrix (15) is played by the Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} \frac{1}{r_1^2}\left(\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}\right) & -\frac{1}{r_1 r_2 d_2} & 0 & \dots \\ -\frac{1}{r_1 r_2 d_2} & \frac{1}{r_2^2}\left(\alpha_2 + \frac{1}{d_2} + \frac{1}{d_3}\right) & -\frac{1}{r_2 r_3 d_3} & \dots \\ 0 & -\frac{1}{r_2 r_3 d_3} & \frac{1}{r_3^2}\left(\alpha_3 + \frac{1}{d_3} + \frac{1}{d_4}\right) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad (25)$$

where $r_n := \sqrt{d_n + d_{n+1}}$ and $d_n = x_n - x_{n-1}$, $n \in \mathbb{N}$. It is easy to see that

$$B_{X,\alpha} = R_X^{-1}((I - U)D_X^{-1}(I - U^*) + A_\alpha)R_X^{-1}, \quad (26)$$

where $A_\alpha = \mathrm{diag}(\alpha_n)$, $D_X = \mathrm{diag}(d_n)$, and $R_X = \mathrm{diag}(r_n)$. By (16) and (26), the structure of matrices (15) and (25) are completely different in the case $d_* = 0$. Therefore, the spectral properties of Hamiltonians with δ' and δ-interactions on X are essentially different in this case (see [11]).

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