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Higher Derivatives of Spectral Functions associated with One-Dimensional Schrödinger Operators

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Abstract We investigate the existence and asymptotic behaviour of higher derivatives of the spectral function in the context of one-dimensional Schrödinger operators on the half-line with integrable potentials. In particular, we identify sufficient conditions on the potential for the existence and continuity of the n-th derivative, and outline a systematic procedure for estimating numerical upper bounds for the turning points of such derivatives. Explicit worked examples illustrate the development and application of the theory.

Introduction

In the context of this paper, we use the term spectral concentration to refer to a significant localised intensification of the spectrum occuring within an interval of purely absolutely continuous spectrum. This terminology and usage in connection with the one-dimensional Schrödinger operator goes back to Titchmarsh, who provided a detailed mathematical analysis in a number of well known cases [22]. Titchmarsh's investigations were themselves developments of earlier work by Schrödinger, Oppenheimer and others on such problems as ionisation of the hydrogen atom in a weak electric field (the Stark effect), and the behaviour of the hydrogen atom in a uniform magnetic field (the Zeeman effect) [16], [19]. A common feature of these problems is that a discrete spectrum is replaced by a continuous spectrum after the introduction of a 'perturbing' field while, in the vicinity of the eigenvalues of the original problem, there are poles of the meromorphic continuation of the perturbed Green's function located just below the real axis, on the so-called unphysical sheet. These poles, also known as resonances or pseudo-eigenvalues, are of significant interest to physicists in connection with impedance theory, resonance scattering and spectral stability, and are correlated with the existence of scattering states which remain localised for a long time [5], [12].

In Titchmarsh's examples, the influence of the eigenvalues of the unperturbed problem is also reflected in the occurence nearby of sharp peaks in the perturbed spectral density function, and away from these peaks the perturbed spectral density is small. Thus close to the eigenvalues of the original operator, there are so-called points of spectral concentration of the perturbed operator, so that in some sense the spectrum of the perturbed operator is 'close' to that of the original [11]. Moreover, the fact that both resonance poles and points of spectral concentration are located in the vicinity of the eigenvalues of the unperturbed problems suggests that a correlation between spectral concentration and resonances may hold more widely, and indeed there is considerable evidence to support this idea, whether or no the phenomena arise in connection with the perturbation of a discrete spectrum. For example, in those cases for which the Kodaira formula for the spectral density holds ([13], p.940), it is evident that if a resonance pole in the meromorphic continuation of the Green's function into the unphysical sheet lies sufficiently close to the real axis, then it is liable to increase the spectral density on part of the real axis closest to the pole. These related phenomena have given rise to a considerable literature over the years and useful references may be found in [12], [17].

Independently of the connection with resonance poles, the study of spectral concentration in its own right introduces a valuable additional dimension into analysis of the absolutely continuous part of the spectrum. In recent years, there has been a significant focus on the development of analytical and numerical methods for investgating the existence and location of this phenomenon (see e.g. [3], [6]). The present paper provides an overview of some recent contributions of the authors in this direction.

Mathematical Background

We consider the time independent one-dimensional Schrödinger operator H associated with the system

$$Ly := -y'' + q(x)y = zy, \qquad x \in [0, \infty), \ \ z \in \mathbf{C},$$

$$y(0) = 0,$$

where the potential q is real valued and integrable on $[0, \infty)$. Here L is in Weyl's limit point case at infinity, so that for each $z \in \mathbf{C} \backslash \mathbf{R}$ there exists precisely one linearly independent solution of $Lu = zu$ in $L_2[0, \infty)$. The corresponding selfadjoint operator H acting on $\mathcal{H} = L_2[0, \infty)$ is defined by

$$Hf = Lf, f \in D(H),$$

where

$$D(H) = \{f \in \mathcal{H} : Lf \in \mathcal{H}; f, f' \text{ locally a.c.}; f(0) = 0\}.$$

It is well known that the essential spectrum of H fills the semi-axis $[0, \infty)$ and is purely absolutely continuous for $\lambda > 0$. The negative spectrum, if any, consists of isolated eigenvalues, possibly accumulating at $\lambda = 0$, which may itself be an eigenvalue when $xq(x) \notin L_2([0,\infty))$. However, in this paper we are only concerned with the absolutely continuous part of the spectrum.

Associated with H is a non-decreasing spectral function, $\rho(\lambda)$, and we define the spectrum, $\sigma(H)$, to be the complement in $\mathbf{R}$ of the set of points in a neighbourhood of which $\rho(\lambda)$ is constant (which is consistent with the more usual definition in terms of the resolvent operator).

The spectrum of H may also be studied through properties of the related Titchmarsh-Weyl m-function, $m(z)$, which is defined and analytic on $\mathbf{C}\backslash\mathbf{R}$ and is a Herglotz function in the upper half-plane. The spectral density, $\rho'(\lambda)$, is related to the boundary values of the m-function on the real axis through the formula

$$\rho'(\lambda) = \lim_{\epsilon\to 0} \frac{1}{\pi} \Im m(\lambda + i\epsilon), \tag{1}$$

which holds for all $\lambda \in \mathbf{R}$ for which the respective limits exist. The m-function is closely related to the Green's function for H and hence reflects the analyticity properties of the resolvent operator. It is also related to solutions of the differential equation in various ways. For example, for $z \in \mathbf{C}^+$, let $\psi(x,z)$ denote the so-called Jost solution of $Lu = zu$, which is in $L_2[0,\infty)$ and satisfies $\psi(x,z) \sim \exp ikx$ as $x \to \infty$, where $k^2 = z$ and the principal branch is chosen. Then the logarithmic derivative of $\psi(x,z)$ evaluated at $x = 0$ is equal to the value of the Dirichlet m-function at z, i.e.

$$\left.\frac{\psi'(x,z)}{\psi(x,z)}\right|_{x=0} = m(z),$$

where $'$ denotes differentiation with respect to x. We remark that for $z \in \mathbf{C}^+$, $\psi(x,z)$ cannot vanish at $x = 0$, since to do so would imply that H had a non-real eigenvalue at z, and thus contradict the selfadjointness of H. In a similar way, $\psi(x_0,z) \neq 0$ for any $x_0 > 0$, $z \in \mathbf{C}^+$. We may therefore set

$$m(x,z) := \frac{\psi'(x,z)}{\psi(x,z)} \tag{2}$$

for $x \geq 0, z \in \mathbf{C}^+$, and it is straightforward to check that for each $x_0 \geq 0$, $m(x_0,z)$ is the Dirichlet m-function associated with the system: $Lu = zu$, $x \geq x_0$, $u(x_0,z) = 0$. We will refer to $m(x,z)$ as the generalised Dirichlet m-function, and note that in terms of our earlier notation, $m(0,z) \equiv m(z)$. Since $\psi(x,z)$ is a solution of $Lu = zu$, it follows from (2) that $m(x,z)$ satisfies the Riccati equation

$$\frac{\partial}{\partial x} m(x,z) = -z + q(x) - (m(x,z))^2 \tag{3}$$

for $x \in [0,\infty)$, $z \in \mathbf{C}^+$.

In the present context, we formally define spectral concentration as follows (cf. [3]).

Definition The point $\lambda_c \in \mathbf{R}$ is said to be a *point of spectral concentration* of H if

(i) $\rho'(\lambda)$ exists finitely and is continuous in a neighbourhood of λ_c, and

(ii) $\rho'(\lambda)$ has a local maximum at λ_c.

We note that since $\rho(\lambda)$ is non-decreasing, the definition implies that $\rho'(\lambda)$ exists and satisfies $0 \leq \rho'(\lambda) < \rho'(\lambda_c) < \infty$ in a deleted neighbourhood of λ_c, from which it follows by the local continuity of $\rho'(\lambda)$ that $\rho'(\lambda) > 0$ in a neighbourhood of λ_c. Thus the definition effectively restricts attention to points of spectral concentration which occur in subintervals of the essential spectrum in which the spectral function is purely absolutely continuous and strictly increasing. A further consequence of the definition is that if $\rho''(\lambda)$ exists and has one sign for $\lambda > M$, then $\rho'(\lambda)$ exists, is absolutely continuous, but has no local maxima in (M, ∞), so that M is an upper bound for points of spectral concentration of H.

Short Range Potentials

In the case where $q \in L_1[0, \infty)$, it was shown by Titchmarsh [21] that for $\lambda > 0$,

$$m_\alpha(\lambda) := \lim_{\epsilon \downarrow 0} m_\alpha(\lambda + i\epsilon)$$

exists and satisfies

$$m_\alpha(\lambda) = \frac{\mu_1(\lambda) + i\nu_1(\lambda)}{\mu(\lambda) + i\nu(\lambda)},$$

where $\mu_1(\lambda), \nu_1(\lambda), \mu(\lambda), \nu(\lambda)$ are continuous functions of λ, and $\mu(\lambda), \nu(\lambda)$ do not vanish simultaneously. It follows from the properties of these functions and (1) above that

$$\rho'_\alpha(\lambda) = \frac{1}{\pi} \Im m_\alpha(\lambda) = \frac{1}{\pi\sqrt{\lambda}(\mu^2(\lambda) + \nu^2(\lambda))},$$

so that $\rho'_\alpha(\lambda)$ is continuous with $0 < \rho'_\alpha(\lambda) < \infty$ for $\lambda > 0$, and hence $\sigma(H_\alpha)$ is purely absolutely continuous on $(0, \infty)$ [9]. It is not hard to show that the generalised Dirichlet m-function and corresponding Dirichlet spectral functions on $L_2[x, \infty)$ have similar properties, so that $m(x, z)$ may be continuously extended on to the non-negative real axis, $z = \lambda \in \mathbf{R}^+$, for $x \geq 0$. We then have (cf. (2)) that $m(x, \lambda)$ is well defined, continuous and non-real for $x \geq 0, \lambda > 0$, and satisfies

$$m(x, \lambda) = \frac{\psi'(x, \lambda)}{\psi(x, \lambda)},$$

where $\psi(x, \lambda) \notin L_2[0, \infty)$ is the pointwise limit as $z \downarrow \lambda$ of the Weyl solution, and is itself a solution of $Lu = \lambda u$. It follows that the Riccati equation (3) is also satisfied for $z = \lambda > 0$, so that

$$\frac{\partial}{\partial x} m(x,\lambda) = -\lambda + q(x) - (m(x,\lambda))^2 \,, \tag{4}$$

where $m(x,\lambda)$ is the finite non-real limit as $z \downarrow \lambda$ of the generalised Dirichlet m-function, $m(x,z)$.

We now show that we can investigate the behaviour of the spectral density, $\rho_0'(\lambda)$, using the Riccati equation. For $x \geq 0, \lambda > 0$, we have

$$m(x,\lambda) := \Re m(x,\lambda) + i\Im m(x,\lambda),$$

from which by (1),

$$m(0,\lambda) := m_0(\lambda) = \Re m_0(\lambda) + i\pi\rho_0'(\lambda). \tag{5}$$

Hence, in principle, $\rho_0'(\lambda)$ can be obtained for $\lambda > 0$ by finding the appropriate solution of the Riccati equation and evaluating at $x = 0$. If in addition it can be shown that $m(x,\lambda)$ is differentiable with respect to λ for sufficiently large λ, we can also seek conditions under which $\rho_0''(\lambda)$ exists and satisfies

$$\rho_0''(\lambda) = \frac{1}{\pi}\left[\Im \frac{\partial}{\partial\lambda} m(x,\lambda)\right]_{x=0}. \tag{6}$$

Equations (4) and (6) will form the basis for our investigation into the existence of upper bounds for points of spectral concentration of H_0.

It is rarely possible to solve the Riccati equation explicitly, so we proceed by postulating a series representation for $m(x,\lambda)$, which is substituted into (4). We then choose the terms of the series, establish sufficient conditions for the validity of the representation, and investigate the existence of $\rho_0''(\lambda)$. Based on the known asymptotic behaviour of $m(x,\lambda)$ as $\lambda \to \infty$ (cf. [10]; [18], Theorem 5.1), we seek a series representation in the form

$$m(x,\lambda) = i\sqrt{\lambda} + g(x,\lambda) \tag{7}$$

where

$$g(x,\lambda) := \sum_{n=0}^{\infty} m_n(x,\lambda)$$

is in $L_1([0,\infty);dx)$, and satisfies $g(x,\lambda) \to 0$ as $x \to \infty$. It may be shown that the Riccati equation has at most one solution of this form (see [7]), from which it follows that if the series representation in (7) is a valid solution of (4), then it does indeed represent the extension of the generalised Dirichlet m-function onto the real axis, as sought.

Substituting for $m(x,\lambda)$ from (7) into (4) and rearranging yields

$$m_1' + 2i\sqrt{\lambda}m_1 + m_2' + 2i\sqrt{\lambda}m_2 + \sum_{n=3}^{\infty}(m_n' + 2i\sqrt{\lambda}m_n)$$

$$= q - m_1^2 - \sum_{n=3}^{\infty} \left(m_{n-1}^2 + 2m_{n-1} \sum_{k=1}^{n-2} m_k \right)$$

where $'$ denotes differentiation with respect to x. Choosing the $\{m_n\}$ to satisfy

$$\begin{aligned} m_1' + 2i\sqrt{\lambda} m_1 &= q \\ m_2' + 2i\sqrt{\lambda} m_2 &= -m_1^2 \\ m_n' + 2i\sqrt{\lambda} m_n &= -\left(m_{n-1}^2 + 2m_{n-1} \sum_{k=1}^{n-2} m_k \right), \qquad n \geq 3, \end{aligned}$$

yields

$$\begin{aligned} m_1(x,\lambda) &= -e^{-2i\sqrt{\lambda}x} \int_x^{\infty} e^{2i\sqrt{\lambda}t} q(t) dt, \\ m_2(x,\lambda) &= e^{-2i\sqrt{\lambda}x} \int_x^{\infty} e^{2i\sqrt{\lambda}t} m_1^2(t,\lambda) dt, \\ m_n(x,\lambda) &= e^{-2i\sqrt{\lambda}x} \int_x^{\infty} e^{2i\sqrt{\lambda}t} \left(m_{n-1}^2 + 2m_{n-1} \sum_{k=1}^{n-2} m_k \right) dt, \qquad n \geq 3. \end{aligned} \tag{8}$$

In order to determine under what circumstances $\rho_0''(\lambda)$ exists for sufficiently large λ, we now define

$$w_n(x,\lambda) := \frac{\partial}{\partial \lambda} m_n(x,\lambda), \qquad n = 1, 2, 3, ...$$

for those λ for which the derivatives exist. The following lemma establishes some key properties of $\{m_n(x,\lambda)\}$ and $\{w_n(x,\lambda)\}$, and is proved in [7].

Lemma 1 *Let $q(x) \in L_1[0,\infty)$ and suppose that there exists $\Lambda_1 > 0$ such that for $x \geq 0$ and $\lambda > \Lambda_1$*

$$\left| \int_x^{\infty} e^{2i\sqrt{\lambda}t} q(t) dt \right| \leq a(x)\eta(\lambda),$$

where $a(x) \in L_1[0,\infty)$ is decreasing, $\eta(\lambda) \to 0$ as $\lambda \to \infty$, and $32\eta(\lambda)\int_0^{\infty} a(t)dt \leq 1$. Then for $x \geq 0$, $\lambda > \Lambda_1$ and $n = 1, 2, 3, ...$

$$\begin{aligned} | m_n(x,\lambda) | &\leq \frac{a(x)\eta(\lambda)}{2^{n-1}}, \\ | w_n(x,\lambda) | &\leq \frac{\eta(\lambda)}{2^{n-1}\sqrt{\lambda}} \int_x^{\infty} a(t) dt, \end{aligned}$$

so that the series $\sum_{n=1}^{\infty} m_n(x,\lambda)$ and $\sum_{n=1}^{\infty} w_n(x,\lambda)$ are uniformly absolutely convergent in x and λ.

We remark that the conditions of Lemma 1 are satisfied, for example, if $(1 + x)q(x) \in L_1[0, \infty)$ or if $q(x) \in L_1[0, \infty)$ is monotonic [10]. The convergence and continuity properties of the series in Lemma 1 imply that

(i) $m(x, \lambda)$, as defined in (7) and (8), is a valid series representation of the generalised Dirichlet m-function for $x \geq 0$, $\lambda > \Lambda_1$, and

(ii) $\rho_0''(\lambda)$ exists for $\lambda > \Lambda_1$ and is given by

$$\rho_0''(\lambda) = \frac{1}{\pi}\left(\frac{1}{2\sqrt{\lambda}} + \sum_{n=1}^{\infty} \Im w_n(0, \lambda)\right).$$

This leads to the following result.

Theorem 1 *Let $q(x) \in L_1[0, \infty)$ and suppose that the hypothesis of Lemma 1 is satisfied. Then for all $\lambda > \Lambda_1$, $\rho_0''(\lambda)$ exists and satisfies*

$$\left|\rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}}\right| \leq \frac{4}{\pi\sqrt{\lambda}}\eta(\lambda)\int_0^\infty a(t)dt,$$

so that $\rho_0''(\lambda) > 0$ for all $\lambda > \Lambda_1$. In particular, there are no points of spectral concentration of H_0 for $\lambda > \Lambda_1$.

Theorem 1 enables explicit upper bounds for points of spectral concentration to be calculated, as illustrated in Example 1 below. Details of the proof of this theorem are given in [7].

Long Range Potentials

In the case where $q(x) \to 0$ as $x \to \infty$, but $q(x) \notin L_1[0, \infty)$, the situation is more delicate. It is no longer true in general that $\rho_0'(\lambda)$ exists, is continuous and satisfies $\rho_0'(\lambda) > 0$ for all $\lambda > 0$. If $q(x)$ decays more slowly than the Coulomb potential, examples can be constructed where $\rho_0(\lambda)$ is discontinuous on a dense set of eigenvalues in $[0, \infty)$ [15]. If $q(x)$ fails to be in $L_2[0, \infty)$, then the absolutely continuous spectrum may be empty, in which case there is a dense set of points in $[0, \infty)$ on which $\rho_0'(\lambda)$ does not exist as a finite limit [4], [9], [20]. The situation is more fully understood in the case of von Neumann-Wigner type potentials, where the spectrum is purely absolutely continuous on $(0, \infty)$ apart from an at most countable set of isolated eigenvalues, known as resonances [1], [2]. Moreover, under fairly minimal smoothness conditions, classes of decaying, but non-integrable potentials do exist for which the the spectrum is purely absolutely continuous on $(0, \infty)$ or on (M, ∞) for sufficiently large M [2], [6]. In such circumstances, we can investigate whether it is possible to extend $m(x, z)$ continuously onto part of the non-negative real axis in such a way that $\rho_0''(\lambda)$ exists and (4), (5) and (6) are satisfied for sufficiently large λ. If this can be achieved, then estimates of upper bounds for both embedded singular spectrum and points of spectral concentration can normally be obtained.

We proceed as before by postulating the existence of a series representation for the generalised Dirichlet m-function, $m(x, z)$, in this case for $\Re z > 0$,

$\Im z \geq 0$, and $\mid z \mid$ sufficiently large. It is necessary to consider the m-function in the upper half-plane as well as on the real axis, so as to ensure that if $m(x, \lambda)$ exists and is continuous in x and λ for $\mid \lambda \mid > M$, then it is the unique continuous extension of $m(x, z)$, $z \in \mathbf{C}^+$, as $z \downarrow \lambda \in \mathbf{R}$. Also, in order to construct a series with the desired convergence properties, it is helpful to introduce an additional term, $R(x, z)$, into the expression for $m(x, z)$, so we now seek a representation in the form

$$m(x,z) = i\sqrt{z} + R(x,z) + g(x,z) \tag{9}$$

where

$$g(x,z) := \sum_{n=0}^{\infty} m_n(x,z)$$

is in $L_1([0,\infty); dx)$, and satisfies $g(x,z) \to 0$ as $x \to \infty$. The introduction of $R(x,z)$ enables the terms of the series to be generated iteratively with $Q(x,z) := q(x) - R' - R^2 - 2i\sqrt{\lambda}R \in L_1([0,\infty); dx)$ having an analogous role to that of $q(x)$ in the short range case. We observe that in general the choice of $R(x,z)$ is not unique.

Proceeding as before, we substitute for $m(x,z)$ from (9) into (3), and after rearrangement the $\{m_n\}$ are chosen to satisfy:

$$\begin{aligned} m_1' + (2i\sqrt{z} + 2R)m_1 &= Q \\ m_2' + (2i\sqrt{z} + 2R)m_2 &= -m_1^2 \\ m_n' + (2i\sqrt{z} + 2R)m_n &= -\left(m_{n-1}^2 + 2m_{n-1}\sum_{k=1}^{n-2} m_k\right), \qquad n \geq 3, \end{aligned}$$

The solutions $\{m_n\}$ of these equations, and their derivatives, form the basis of our analysis of the long range case. This leads to the following theorem which is proved in [8].

Theorem 2 *Let $q(x)$ be continuously differentiable and satisfy $q(x) \to 0$ as $x \to \infty$, $q(x) \notin L_1[0,\infty)$. Define*

$$Q(x,z) := q(x) - R' - R^2 - 2i\sqrt{z}R$$

for $\Re z > 0$, $\Im z \geq 0$, where $R = R(x,z)$ is chosen so that $Q(\cdot, z) \in L_1[0,\infty)$, R' denotes differentiation with respect to x, and Q, R, $\frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial z}$ are continuous in x and z. Suppose that there exists $M > 0$ so that

(a) for $\Re z > 0$, $\Im z \geq 0$, $\mid z \mid > M$,

(i) there exists $K \in \mathbf{R}$ so that for $0 \leq x < t$,

$$\Re\left(2i\sqrt{z}(t-x) + 2\int_x^t R(s,z)ds\right) \leq K,$$

(ii) for $0 \le x < t$,

$$\left|\int_x^\infty e^{2i\sqrt{z}(t-x)+2\int_x^t R(s,z)ds} Q(t,z)dt\right| \le a(x)\eta(z),$$

where $a(x)$, $\eta(z)$, *are real valued functions with* $a(x) \in L_1[0,\infty)$ *and decreasing,* $\eta(z) \to 0$ *as* $|z| \to \infty$ *and* $32\eta e^K \int_0^\infty a(t)dt \le 1$,

(iii)

$$\left|\frac{\partial}{\partial z}\int_x^t R(s,z)ds\right| \le \text{const}(t-x)$$

for $0 \le x < t < \infty$,

(b) for $\lambda = \Re z > M$, $\Im z = 0$, *there exists a decreasing function* $b(x)$ *such that for* $x \ge 0$,

$$e^K \int_x^\infty \left|\frac{\partial Q}{\partial \lambda}\right| + \left|\frac{i}{\sqrt{\lambda}} + 2\frac{\partial R}{\partial \lambda}\right| a(t)\eta(\lambda)dt \le \frac{\eta(\lambda)}{\sqrt{\lambda}} b(x).$$

Then $\rho_0''(\lambda)$ *exists for* $\lambda > M$, *and satisfies*

$$\left|\rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}} - \frac{1}{\pi}\Im R(0,\lambda)\right| \le \frac{3}{\pi\sqrt{\lambda}}\eta(\lambda)b(0).$$

The following corollary may be inferred from the proof of Theorem 2 (see [8]).

Corollary 1 *Let* $q(x)$, $Q(x,z)$ *and* $R(x,z)$ *be as in Theorem 2 and suppose that* $\Lambda_0 > 0$ *exists such that for* $\Re z > 0$, $\Im z \ge 0$, $|z| > \Lambda_0$, *conditions (a)(i)and (ii) of Theorem 2 are satisfied. Then for* $\lambda = \Re z > \Lambda_0$, $\rho_0'(\lambda)$ *exists as a finite limit, and hence the spectrum of* H_0 *is purely absolutely continuous on* (Λ_0, ∞).

Applications

Example 1 Let $q(x) = (1+x)^{-\gamma}\sin(1+x)$, $\gamma > 1$. By expressing $\sin(1+x)$ in exponential form and integrating by parts we obtain

$$\left|\int_x^\infty e^{2i\sqrt{\lambda}t} q(t)dt\right| \le \frac{2}{(2\sqrt{\lambda}-1)(1+x)^\gamma}$$

for $\lambda > \frac{1}{4}$, from which we may choose $\eta(\lambda) = 2(2\sqrt{\lambda}-1)^{-1}$ and $a(x) = (1+x)^{-\gamma}$. It is then straightforward to show from Theorem 1 that

$$\Lambda_1 = \left(\frac{1}{2} + \frac{32}{\gamma - 1}\right)^2$$

is an upper bound for points of spectral concentration of H_0.

Example 2 Let $q(x) = \sin(1+x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$. In this case $q(x)$, $q'(x)$ and $(q(x))^2$ are not in $L_1[0,\infty)$, so we take

$$R(x,z) = \frac{q}{2i\sqrt{z}} - \frac{q'}{(2i\sqrt{z})^2} - \frac{q^2}{(2i\sqrt{z})^3}$$

to give $Q(x,z) = O(1+x)^{-\frac{3}{2}} \in L_1[0,\infty)$. This leads to the choice

$$a(x) = \frac{1}{(1+x)^{\frac{3}{2}}}, \qquad \eta(z) = \frac{1}{5 \mid z \mid}, \qquad b(x) = \frac{47}{5\sqrt{1+x}},$$

from which it follows by Theorem 2 that $\rho_0''(\lambda) > 0$ if $\lambda > 30$, so that $\Lambda_1 = 30$ is an upper bound for points of spectral concentration of H_0. Note that $\sigma(H_0)$ is known to be purely absolutely continuous on $(0,\infty)$ [2], so that the issue of embedded singular spectrum does not arise.

Example 3 We consider the von Neumann Wigner type potential

$$q(x) = \sum_{k=-M}^{M} h_k(x) e^{2ic_k x},$$

where for each $k = -M, ..., M$, $c_k \in \mathbf{R}\backslash\{0\}$, $h_k(x) \to 0$ as $x \to \infty$, $h_k(x) \in C^L[0,\infty)$ and $h_k^{L+1}(x) \in AC[0,\infty)$. We suppose also that there exists a real valued non-negative function $p(x)$ such that $x(p(x))^{L+2}$ is decreasing with $(p(x))^{L+2}$, $x(p(x))^{L+2} \in L_1[0,\infty)$, and that for $j = 0, ..., L+1$, $k = -M, ..., M$,

$$\mid h_k^{(j)}(x) \mid \leq (p(x))^{j+1}.$$

Then $R(x,z)$ may be chosen so that, after successive integrations by parts,

$$\mid Q(x,z) \mid \leq c \left(\frac{p(x)}{\mid z \mid^{\frac{1}{2}} - 2Lc_*} \right)^{L+2},$$

where c and c_* are real constants which are computable for given $q(x)$. We may then take

$$a(x) = \int_x^\infty (p(t))^{L+2} dt, \quad \eta(z) = \frac{c}{(\mid z \mid^{\frac{1}{2}} - 2Lc_*)^{L+2}}, \quad b(x) = \int_x^\infty a(t) dt,$$

from which the existence of computable upper bounds, Λ_0 and Λ_1, for resonances (embedded eigenvalues) and points of spectral concentration follows from Corollary 1 and Theorem 2 respectively. Further details may be found in [8].

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