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DISCRIMINANTS OF POLYNOMIALS IN THE ARCHIMEDEAN AND NON-ARCHIMEDEAN METRICS

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Abstract. An upper bound for the number of cubic polynomials which have small discriminant in terms of the Euclidean and p-adic metrics simultaneously is obtained.

1. Introduction

Throughout this paper,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is an integer polynomial with degree $\deg P = n$ and height $H = H(P) = \max_{0 \le j \le n} |a_j|$. Let $Q \in \mathbb{N}_{>1}$ and define the set of polynomials

$$\mathcal{P}_n(Q) = \left\{ P(t) \in \mathbb{Z}[t],\ \deg P = n,\ H(P) \le Q \right\}.$$

Let $\mu_1 A_1$ be the Lebesgue measure of a measurable set $A_1 \subset \mathbb{R}$, and $\mu_2 A_2$ the Haar measure of a measurable set $A_2 \subset \mathbb{Q}_p$. The cardinality of a set B will be denoted by $\#B$. Positive constants which depend only on n will be denoted by $c(n)$; where necessary these constants will be numbered $c_j(n)$, $j = 1, 2, \ldots$. We will use the Vinogradov symbols $\ll$ (and $\gg$) where $a \ll b$

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implies that there exists a constant C such that $a \le Cb$. If $a \ll b \ll a$ then we write $a \asymp b$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the complex roots of the polynomial $P \in \mathbb{Z}[x]$ of degree n. The discriminant of P, denoted by $D(P)$ is defined as

$$D(P) = a_n^{2n-2} \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2.$$

Alternatively, $D(P)$ can be defined as the determinant of a matrix containing only the coefficients of P (see [16]). Hence $D(P) \in \mathbb{Z}$ and if P does not have multiple roots then

$$1 \le |D(P)| < c_1 Q^{2n-2}, \quad c_2 Q^{-2n+2} < |D(P)|_p \le 1,$$

where $|\cdot|_p$ is the standard p-adic valuation. For any $a \in \mathbb{Q}_{\neq 0}$, and therefore for $D(P) \neq 0$, from the product formula we have

$$1 \le |D(P)||D(P)|_p. \tag{1}$$

Finally, let $v_1, v_2 \in \mathbb{R}_+ \cup \{0\}$ and define the set of polynomials

$$\mathcal{P}_n(Q, v_1, v_2) = \left\{ P(x) \in \mathcal{P}_n(Q) : 1 \le |D(P)| < Q^{2n-2-2v_1}, \, |D(P)|_p < Q^{-2v_2} \right\}.$$

For simplicity we will write $\mathcal{P}_n(Q, \bar{v}) = \mathcal{P}_n(Q, v_1, v_2)$.

In this paper we investigate a counting problem regarding the discriminants of polynomials in $\mathcal{P}_n(Q, \bar{v})$. To date the case when $\bar{v} = (v_1, 0)$ has been thoroughly investigated and the following is known. In the case of quadratic polynomials it was shown in [14] that

$$\#\mathcal{P}_2(Q, v_1, 0) = 20(1 + \ln 2) Q^{3-2v_1} + O(Q^{3-3v_1} + Q^2), \quad 0 < v_1 < 1/2.$$

In the case of cubic polynomials it was established in [13] that

$$\#\mathcal{P}_3(Q, v_1, 0) \asymp Q^{4-5v_1/3}, \quad 0 \le v_1 < 3/5. \tag{2}$$

Establishing correct lower bounds for arbitrary n has been the subject of numerous papers including [1,2,6,13,14]. The most general and best estimate was found in [2] where it was shown that

$$\#\mathcal{P}_n(Q, v_1, 0) > c_3 Q^{n+1-(n+2)v_1/n}, \quad 0 \le v_1 \le n-1.$$

In the p-adic case it was shown in [7] that $\#\mathcal{P}_n(Q, 0, v_2) > c_4 Q^{n+1-2v_2}$ for $0 < v_2 < 1/2$. In [11] it was proved that $\#\mathcal{P}_n(Q, v_1, v_1) > c_5 Q^{n+1-4v_1}$ for $0 \le v_1 < 1/3$ and $n \ge 3$. There are also some applications of these estimates, see for example [9,10,12].

The main result of this paper extends the upper bound in (2) to a particular range for v_1+v_2 and to the two metrics simultaneaously. We will be estimating the size of the set $\mathcal{P}_3(Q,\bar{v})$.

THEOREM 1. *For any $\varepsilon>0$ and for any sufficiently large Q the estimate*

$$\#\mathcal{P}_3(Q,\bar{v}) < Q^{4-5(v_1+v_2)/3+\varepsilon} \tag{3}$$

holds if

$$3\varepsilon/20 \le v_1+v_2 \le 6/5. \tag{4}$$

2. Auxiliary statements

Let $P\in\mathcal{P}_3(Q,\bar{v})$ have complex roots α_1, α_2, α_3 and roots γ_1, γ_2, γ_3 in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. The complex roots of a cubic polynomial P can be always ordered with respect to one of them, say α_1, as follows:

$$|\alpha_1-\alpha_2|\le|\alpha_1-\alpha_3|, \quad |\alpha_1-\alpha_3|\asymp|\alpha_2-\alpha_3|. \tag{5}$$

Let

$$|\alpha_1-\alpha_2|=Q^{-\rho_2}, \quad |\alpha_1-\alpha_3|=Q^{-\rho_3}, \quad |\alpha_2-\alpha_3|\asymp Q^{-\rho_3}. \tag{6}$$

Similarly, order the p-adic roots of P as follows:

$$\begin{gathered}|\gamma_1-\gamma_2|_p\le|\gamma_1-\gamma_3|_p, \quad |\gamma_1-\gamma_3|_p\asymp|\gamma_2-\gamma_3|_p, \\ |\gamma_1-\gamma_2|_p=Q^{-\lambda_2}, \quad |\gamma_1-\gamma_3|_p=Q^{-\lambda_3}, \quad |\gamma_2-\gamma_3|_p\asymp Q^{-\lambda_3}.\end{gathered} \tag{7}$$

Further, define

$$f(\bar{\rho},\bar{\lambda})=\rho_3+\lambda_3.$$

For a given number $\varepsilon_1>0$ let $T=[\varepsilon_1^{-1}]+1$, where $[a]$ is the integer part of $a\in\mathbb{R}$. For a polynomial $P\in\mathcal{P}_3(Q,\bar{v})$ the real numbers ρ_2 and ρ_3 were defined in (6). Also define the integers l_2 and l_3 by

$$(l_2-1)/T<\rho_2\le l_2/T, \quad (l_3-1)/T<\rho_3\le l_3/T,$$

and let

$$m_2=l_2/T, \quad m_3=l_3/T. \tag{8}$$

The number of vectors $\bar{m}=(m_2,m_3)$ is finite and depends only on ε_1 and does not depend on Q. This is easy to prove and proofs can be found in [3,4,15].

The same procedure is carried out with the p-adic roots γ_1, γ_2, γ_3 of $P \in \mathcal{P}_3(Q, \bar{v})$, see (7). Similarly define the integers r_2 and r_3 by

$$(r_2 - 1)/T < \lambda_2 \le r_2/T, \quad (r_3 - 1)/T < \lambda_3 \le r_3/T,$$

and let

$$s_2 = r_2/T, \quad s_3 = r_3/T.$$

Again, it is not difficult to show that the number of vectors $\bar{s} = (s_2, s_3)$ is finite and depends only on ε_1; i.e. it does not depend on Q.

In order for the polynomial $P(x)$ to belong to the class $\mathcal{P}_3(Q, \bar{v})$ it is necessary and sufficient that the inequalities

$$(9) \qquad \rho_2 + 2\rho_3 \ge v_1, \quad \lambda_2 + 2\lambda_3 \ge v_2,$$

hold, as in this case

$$|D(P)| \ll Q^{4-2v_1}, \quad |D(P)|_p \ll Q^{-2v_2}.$$

The latter system follows from the facts that

$$1 \le |D(P)| = |a_3|^4 |\alpha_1 - \alpha_2|^2 |\alpha_1 - \alpha_3|^2 |\alpha_2 - \alpha_3|^2 \ll Q^{4-2v_1},$$

and

$$1 \le |D(P)||D(P)|_p \ll Q^{4-2v_1} |a_3|_p^4 \prod_{1 \le i < j \le 3} |\gamma_i - \gamma_j|_p^2 \ll Q^{4-2(v_1+v_2)}.$$

By (1), (6) and (7), we have

$$(10) \qquad \rho_2 + 2\rho_3 + \lambda_2 + 2\lambda_3 \le 2.$$

Using (9), (10) and the inequalities $\rho_2 \ge \rho_3 \ge 0$, $\lambda_2 \ge \lambda_3 \ge 0$, we obtain that

$$0 \le f(\bar{\rho}, \bar{\lambda}) = \rho_3 + \lambda_3 \le 2/3.$$

In order to satisfy (9) it is sufficient that the inequalities

$$(11) \qquad m_2 + 2m_3 \ge v_1 + 3\varepsilon_1, \quad s_2 + 2s_3 \ge v_2 + 3\varepsilon_1$$

hold.

For the roots of P we define the sets

$$S_1(\alpha_j) = \Big\{ x \in \mathbb{R} : |x - \alpha_j| = \min_{1 \le i \le 3} |x - \alpha_i| \Big\}, \quad 1 \le j \le 3,$$

$$S_2(\gamma_k) = \Big\{ w \in \mathbb{Q}_p : |w - \gamma_k|_p = \min_{1 \le i \le 3} |w - \gamma_i|_p \Big\}, \quad 1 \le k \le 3.$$

Lemma 1. *Let α_1 be a complex root of an integer cubic polynomial P and $x \in S_1(\alpha_1)$. Then*

$$|x-\alpha_1| \ll \min\Big(\Big|\frac{P(x)}{P'(\alpha_1)}\Big|, \Big|\frac{P(x)(\alpha_1-\alpha_2)}{P'(\alpha_1)}\Big|^{1/2}, \Big|\frac{P(x)}{a_3}\Big|^{1/3}\Big)$$

for $P'(\alpha_1) \neq 0$.

Lemma 2. *Let γ_1 be a p-adic root of an integer cubic polynomial P and $w \in S_2(\gamma_1)$. Then*

$$|w-\gamma_1|_p \ll \min\Big(\Big|\frac{P(w)|_p}{|P'(\gamma_1)}\Big|_p, \Big(\frac{|P(w)|_p|\gamma_1-\gamma_2|_p}{|P'(\gamma_1)|_p}\Big)^{1/2}\Big(\frac{|P(w)|_p}{|a_3|_p}\Big)^{1/3}\Big)$$

for $P'(\gamma_1) \neq 0$.

Lemmas 1 and 2 are proved in [4] and [5] respectively.

Lemma 3. *Suppose that the polynomials $P(u), T(u) \in \mathcal{P}_3(Q)$ have the same vectors $\bar{m}$ and $\bar{s}$ and have no common roots in the parallelepiped $I \times K$, where $\mu_1 I = Q^{-\eta_1}$, $\mu_2 K = Q^{-\eta_2}$, $\eta_1 \geq 0, \eta_2 \geq 0$. Furthermore, let $P(u)$ and $T(u)$ satisfy the systems of inequalities*

$$\max_{x\in I}\big(|P(x)|, |T(x)|\big) < Q^{-\tau_1}, \tag{12}$$

and

$$\max_{w\in K}(|P(w)|_p, |T(w)|_p) < Q^{-\tau_2},$$

where

$$\tau_1 + 1 \geq 2m_2 + m_3, \quad \tau_2 \geq 2s_2 + s_3. \tag{13}$$

Then, for any $\delta > 0$ and $Q > Q_0(\delta)$ the inequality

$$\tau_1+\tau_2+1+2\sum_{j=1}^{2}\max(\tau_1+1-j\eta_1, 0)+2\sum_{r=1}^{2}\max(\tau_2-r\eta_2, 0) < 6+\delta \tag{14}$$

holds.

Note that Lemma 3 is proved for the Archimedean case in [3] and the non–Archimedean case in [7] only when $j = 1$ and $r = 1$. The proof was based on the use of small values of the first derivatives only which is not enough to prove Theorem 1.

Proof of Lemma 3. Let $P \in \mathcal{P}_3(Q)$ have complex roots α_1, α_2, α_3 and roots γ_1, γ_2, γ_3 in $\mathbb{Q}_p^*$. Let $T \in \mathcal{P}_3(Q)$ have complex roots β_1, β_2, β_3 and roots ξ_1, ξ_2, ξ_3 in $\mathbb{Q}_p^*$. We use the fact that the polynomials $P(u)$ and $T(u)$ have the same vectors $\bar{m}$ and $\bar{s}$. We may also assume that $|a_3(V)| \gg Q$ and $|a_3(V)|_p \gg 1$ for $V \in \{P, T\}$. Otherwise, using the same reasoning as below, we obtain a stronger result in this lemma.

First, we rewrite the interval I as $I = \bigcup_{i=1}^{s} (I \cap S_1(\alpha_i(P)))$. Clearly, there exists i, $1 \le i \le 3$, such that $\mu_1(I \cap S_1(\alpha_i(P))) \ge \mu_1 I/3$. We will assume without loss of generality that $i = 1$. The roots of P are ordered with respect to α_1 as in (5.1), and we use the same notation as in (6.1), (6.2) and (8). Precisely the same argument will also be used for T. By Lemma 1 and (12), the inequalities for $|x - \alpha_1|$ and $|x - \beta_1|$ become

$$(15) \qquad |x - \theta_1| \ll \min\left(Q^{-\tau_1 - 1 + m_2 + m_3}, Q^{-(\tau_1 + 1 - m_3)/2}, Q^{-(\tau_1+1)/3}\right),$$

where θ_1 represents α_1 or β_1 as required, and $x \in S_1(\theta_1(V)) \cap I$. It is possible to choose a point x in $I \cap S_1(\theta_1(V))$ such that the left hand side in (15) will be at least $c_4 Q^{-\eta_1}$ for $\theta_1 = \alpha_1$ and for $\theta_1 = \beta_1$ respectively. This will lead to the inequalities

$$\eta_1 \ge \tau_1 + 1 - m_2 - m_3, \quad \eta_1 \ge (\tau_1 + 1 - m_3)/2, \quad \eta_1 \ge (\tau_1 + 1)/3,$$

and

$$(16) \qquad m_2 + m_3 \ge \tau_1 + 1 - \eta_1, \quad m_3 \ge \tau_1 + 1 - 2\eta_1.$$

Similar arguments for $\mathbb{Q}_p$ lead to the inequalities

$$|w - \zeta_1|_p \ll \min\left(Q^{-\tau_2 + s_2 + s_3}, Q^{-(\tau_2 - s_3)/2}, Q^{-\tau_2/3}\right),$$

$$\eta_2 \ge \tau_2 - s_2 - s_3, \quad \eta_2 \ge (\tau_2 - s_3)/2, \quad \eta_2 \ge \tau_2/3,$$

$$s_2 + s_3 \ge \tau_2 - \eta_2, \quad s_3 \ge \tau_2 - 2\eta_2,$$

where ζ_1 represents γ_1 or ξ_1 as required, and $w \in S_2(\zeta_1(V)) \cap K$.

Consider the resultant $R(P,T)$ of $P(u)$ and $T(u)$. Using the fact that $P(u)$ and $T(u)$ have no common roots, the inequality (1) and the fact that $|a_3|^6 |a_3|_p^6 \ll Q^6$, we get

$$(17) \qquad 1 \le |R(P,T)|\,|R(P,T)|_p \ll Q^6 \prod_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} |\alpha_i - \beta_j| \prod_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} |\gamma_i - \xi_j|_p.$$

Now we obtain an upper bound for

$$L_1 = \prod_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} |\alpha_i - \beta_j|.$$

By (13) and (15),

$$|\alpha_1-\beta_1| \ll (|x-\alpha_1|+|x-\beta_1|) \ll Q^{-\tau_1-1+m_2+m_3},$$
$$|\alpha_1-\beta_i| \le |\alpha_1-\beta_1|+|\beta_1-\beta_i|$$
$$\ll \max(Q^{-\tau_1-1+m_2+m_3}, Q^{-m_i+\varepsilon_1}) \ll Q^{-m_i+\varepsilon_1},\ i=2,3,$$
$$|\alpha_i-\beta_1| \le |\alpha_1-\beta_1|+|\alpha_1-\alpha_i|$$
$$\ll \max(Q^{-\tau_1-1+m_2+m_3}, Q^{-m_i+\varepsilon_1}) \ll Q^{-m_i+\varepsilon_1},\ i=2,3,$$
$$|\alpha_i-\beta_j| \le |\alpha_1-\beta_1|+|\alpha_1-\alpha_i|+|\beta_1-\beta_j|$$
$$\ll \max(Q^{-\tau_1-1+m_2+m_3}, Q^{-m_{\max(i,j)}+\varepsilon_1}) \ll Q^{-m_{\max(i,j)}+\varepsilon_1},\ i\ge 2,\ j\ge 2.$$

Thus,

$$L_1 \ll Q^{-\tau_1-1-2m_2-4m_3+8\varepsilon_1}. \tag{18}$$

Using [12] together with (16) and (18), we obtain

$$\tau_1+1+2m_2+4m_3-8\varepsilon_1 \ge \tau_1+1+2(\tau_1+1-\eta_1)+2(\tau_1+1-2\eta_1)-8\varepsilon_1. \tag{19}$$

A similar result can be obtained in exactly the same way in the p-adic case to give

$$L_2=\prod_{\substack{1\le i\le 3\\ 1\le j\le 3}} |\gamma_i-\xi_j|_p \ll Q^{-\tau_2-2(\tau_2-\eta_2)-2(\tau_2-2\eta_2)+8\varepsilon_1}. \tag{20}$$

Finally, using inequalities (17), (19), and (20) with the conditions $m_i\ge 0$, $s_i\ge 0$, $i=2,3$, we obtain

$$\tau_1+\tau_2+1+2\sum_{j=1}^{2}\max(\tau_1+1-j\eta_1,0)+2\sum_{r=1}^{2}\max(\tau_2-r\eta_2,0) < 6+\delta. \qquad \square$$

3. Proof of Theorem 1

Assume that the estimate (3) does not hold, so

$$\#\mathcal{P}_3(Q,\bar v) \ge Q^{4-5(v_1+v_2)/3+\varepsilon}. \tag{21}$$

Consider the interval $I\subset\mathbb{R}$ with $\mu_1 I = Q^{-m_2}$ and the cylinder $K\subset\mathbb{Q}_p$ with $\mu_2 K = Q^{-s_2}$, where s_2 and m_2 are non-negative numbers. It will be said that the polynomial P belongs to the parallelepiped $M = I \times K$ if $(\alpha_1, \gamma_1) \in M$. From (21) it follows that there exist parallelepipeds M which contain at least

$$\Delta_1 = Q^{4-5(v_1+v_2)/3-m_2-s_2+\varepsilon}$$

polynomials $P \in \mathcal{P}_3(Q, \bar{v})$. If each parallelepiped contains at most Δ_1 polynomials then summing the estimate Δ_1 over all parallelepipeds M we get a contradiction in (21). Fix one of these parallelepipeds M_1 say. Since $\#\bar{m} \ll 1$ and $\#\bar{s} \ll 1$ there exist vectors $\bar{m}$ and $\bar{s}$ satisfying (9) such that

$$\#\mathcal{P}_{3,\bar{m},\bar{s}}(Q, \bar{v}, M_1) \gg Q^{4-5(v_1+v_2)/3-m_2-s_2+\varepsilon}$$

where $\mathcal{P}_{3,\bar{m},\bar{s}}(Q, \bar{v}, M_1)$ denotes the subset of $\mathcal{P}_{3,\bar{m},\bar{s}}(Q, \bar{v})$ consisting of polynomials P belonging to M_1. Fix the vectors $\bar{m}$ and $\bar{s}$ and set

$$h = 4 - 5(v_1 + v_2)/3 - m_2 - s_2. \tag{22}$$

By (10), we have

$$m_2 + 2m_3 + s_2 + 2s_3 \le 2. \tag{23}$$

From (4) and (23) we obtain that $h \ge 0$.

Expand the polynomial $P \in \mathcal{P}_{3,\bar{m},\bar{s}}(Q, \bar{v}, M_1)$ into its Taylor series in the neighbourhood of α_1 to obtain

$$P(x) = P(\alpha_1) + P'(\alpha_1)(x-\alpha_1) + P''(\alpha_1)(x-\alpha_1)^2/2 + P'''(\alpha_1)(x-\alpha_1)^3/6.$$

Estimating each term gives

$$\begin{aligned} |P'(\alpha_1)(x-\alpha_1)| &\le |a_3| \cdot |\alpha_1-\alpha_2| \cdot |\alpha_1-\alpha_3| \cdot |x-\alpha_1| \\ &\ll Q^{1-\rho_2-\rho_3-m_2} \ll Q^{1-2m_2-m_3+2\varepsilon_1}, \\ |P''(\alpha_1)(x-\alpha_1)^2| &\le 2|a_3| \max(|\alpha_1-\alpha_2|, |\alpha_1-\alpha_3|)|x-\alpha_1|^2 \\ &\ll Q^{1-2m_2-m_3+\varepsilon_1}, \\ |P'''(\alpha_1)(x-\alpha_1)^3| &\le 6|a_3| \cdot |x-\alpha_1|^3 \ll Q^{1-3m_2} \ll Q^{1-2m_2-m_3+\varepsilon_1} \end{aligned}$$

for $x \in I$. Thus

$$|P(x)| \ll Q^{1-2m_2-m_3+2\varepsilon_1}. \tag{24}$$

Similarly we obtain

$$|P'(x)| \ll Q^{1-m_2-m_3+2\varepsilon_1}, \quad |P''(x)| \ll Q^{1-m_3+\varepsilon_1}, \quad |P'''(x)| \ll Q \tag{25}$$

for $x \in I$. Also develop the polynomial $P(w)$ and the derivatives $|P^{(k)}(w)|_p$, $1 \le k \le 3$, as Taylor series on the cylinder K at the point γ_1 and obtain the upper bounds for all terms in the series:

$$(26)\qquad \begin{gathered} |P(w)|_p \ll Q^{-2s_2-s_3+2\varepsilon_1}, \quad |P'(w)|_p \ll Q^{-s_2-s_3+2\varepsilon_1}, \\ |P''(w)|_p \ll Q^{-s_3+\varepsilon_1}, \quad |P'''(w)|_p \ll 1. \end{gathered}$$

Let $d = d(I)$ denote the centre of I and let w_0 be any point in K. The Vinogradov symbol $\ll$ in inequalities (24), (25) and (26) is replaced by the maximum value of c_j; call it c_3 say. We obtain the following systems of inequalities:

$$(27)\qquad \begin{gathered} |P(d)| < c_3 Q^{1-2m_2-m_3+2\varepsilon_1}, \quad |P'(d)| < c_3 Q^{1-m_2-m_3+2\varepsilon_1}, \\ |P''(d)| < c_3 Q^{1-m_3+\varepsilon_1}, \quad |P'''(d)| < c_3 Q, \end{gathered}$$

and

$$(28)\qquad \begin{gathered} |P(w_0)|_p < c_3 Q^{-2s_2-s_3+2\varepsilon_1}, \quad |P'(w_0)|_p < c_3 Q^{-s_2-s_3+2\varepsilon_1}, \\ |P''(w_0)|_p < c_3 Q^{-s_3+\varepsilon_1}, \quad |P'''(w_0)|_p \ll c_3. \end{gathered}$$

Denote the quantities on the right hand sides of (27) and (28) by U_0, U_1, U_2, U_3 and G_0, G_1, G_2, G_3, so that the index number corresponds to the order of the derivative. From (22) it is easy to obtain that there exists $g \ge 0$, for which

$$k_1' = g - 5v_1/3 - m_2 \ge 0, \quad k_2' = 4 - g - 5v_2/3 - s_2 \ge 0$$

hold. Define

$$k_1 = k_1' + \varepsilon/8, \quad k_2 = k_2' + \varepsilon/8.$$

For example, take G_2 which is the right hand side of the inequality $|P''(w_0)|_p < c_3 Q^{-s_3+\varepsilon_1}$. Divide the interval $[0, c_3 Q^{-s_3+\varepsilon_1}]$ into $Q^{k_2/4}$ smaller intervals, and also the range of the ith derivative ($i = 0, 1, 3$) namely $[0, c_3 Q^{-n_i}]$ into $Q^{k_2/4}$ smaller intervals. Therefore, the parallelepiped $\Pi_1 = \prod_{0 \le i \le 3} [0, c_3 Q^{-n_i}]$ has been divided into Q^{k_2} parallelepipeds where

$$n_0 = 2s_2 + s_3 + 2\varepsilon_1, \quad n_1 = s_2 + s_3 + 2\varepsilon_1, \quad n_2 = s_3 + \varepsilon_1, \quad n_3 = 0.$$

By assumption there are at least Δ_1 polynomials $P(w)$ in the parallelepiped M_1:

$$P_1(w), P_2(w), \dots, P_l(w), \quad 1 \le l \le \Delta_1.$$

Since $\Delta_1 > Q^{k_2}$ then using Dirichlet's box principle there exist at least $\Delta_2 = \Delta_1 Q^{-k_2}$ polynomials $P_j(x)$, $1 \le j \le \Delta_2$, whose values belong to the parallelepiped

$$\Pi_2 = \prod_{j=0}^{3} [jc_3 Q^{-n_j - k_2/4}, (j+1)c_3 Q^{-n_j - k_2/4}] \subset \Pi_1$$

for some set (j_0, j_1, j_2, j_3), where $j_i \in [0, [Q^{k_2/4}] - 1]$, $i = 0, 1, 2, 3$. We form new polynomials

$$R_1(w) = P_2(w) - P_1(w), \quad \dots, \quad R_{\Delta_2 - 1}(w) = P_{\Delta_2}(w) - P_1(w)$$

with P_{j+1} and P_1 from the same Π_2. These new polynomials satisfy the inequalities

$$(29) \quad |R(w_0)|_p < c_3 Q^{-2s_2 - s_3 - k_2/4 + 2\varepsilon_1}, \quad |R'(w_0)|_p < c_3 Q^{-s_2 - s_3 - k_2/4 + 2\varepsilon_1},$$
$$|R''(w_0)|_p < c_3 Q^{-s_3 - k_2/4 + \varepsilon_1}, \quad |R'''(w_0)|_p < c_3 Q^{-k_2/4}.$$

Note that the polynomials $R_j(d)$ satisfy the system of inequalities (27) with $c_4 = 2c_3$. Let us now analyze this system for the polynomial

$$R(w) = a_3 w^3 + a_2 w^2 + a_1 w + a_0.$$

From the last inequality in (29) we conclude that $|a_3|_p < c_5 Q^{-k_2/4}$. This means that

$$(30) \quad a_3 = p^{b_3} a_3', \quad (a_3', p) = 1, \quad p^{b_3} > c_5^{-1} Q^{k_2/4}.$$

From the inequality for $|P''(w_0)|_p$ in (29) and using (30), we have

$$(31) \quad |6a_3 w + 2a_2|_p < c_3 Q^{-s_3 - k_2/4 + \varepsilon_1}, \quad |a_2|_p < c_6 Q^{-k_2/4 + \varepsilon_1},$$
$$a_2 = p^{b_2} a_2', \quad (a_2', p) = 1, \quad p^{b_2} > c_6^{-1} Q^{k_2/4 - \varepsilon_1}.$$

Further, from the inequality for $|P'(w_0)|_p$, (30) and (31), we obtain

$$|a_1|_p < c_7 Q^{-k_2/4 + 2\varepsilon_1}, \quad a_1 = p^{b_1} a_1', \quad (a_1', p) = 1, \quad p^{b_1} > c_8 Q^{k_2/4 - 2\varepsilon_1},$$

and from the first inequality in (29)

$$|a_0|_p < c_9 Q^{-k_2/4 + 2\varepsilon_1}, \quad a_0 = p^{b_0} a_0', \quad p^{b_0} > c_{10} Q^{k_2/4 - 2\varepsilon_1}.$$

This means that all the coefficients of the polynomials $R(w)$ are divisible by a large power of the prime number p. Let $b = \min(b_j)$, $0 \le j \le 3$. Then for the polynomials $N(w) = p^{-b}R(w)$ of height $H(N)$ the system of inequalities (29) can be rewritten as follows

$$(32)\qquad |N(w_0)|_p < c_{11}Q^{-2s_2-s_3+2\varepsilon_1}, \quad |N'(w_0)|_p < c_{11}Q^{-s_2-s_3+2\varepsilon_1},$$
$$|N''(w_0)|_p < c_{11}Q^{-s_3+\varepsilon_1}, \quad |N'''(w_0)|_p < c_{11}, \quad H(N) < c_{12}Q^{1-k_2/4-2\varepsilon_1}.$$

Given that the polynomials $N(x)$ and $R(x)$ have a special form then for $N(x)$ the system of inequalities (27) can be written as

$$(33)\quad |N(d)| < c_{13}Q^{1-2m_2-m_3-k_2/4+4\varepsilon_1}, \quad |N'(d)| < c_{13}Q^{1-m_2-m_3-k_2/4+4\varepsilon_1},$$
$$|N''(d)| < c_{13}Q^{1-m_3-k_2/4+3\varepsilon_1}, \quad |N'''(d)| < c_{13}Q^{1-k_2/4+2\varepsilon_1}.$$

Using $\Delta_2 - 1$ polynomials $N(x)$, we will construct new polynomials with the absolute values of these polynomials and their derivatives taking smaller values than the upper bounds in (33). Again, we use Dirichlet's box principle. Each of the right hand sides in (33) is divided into intervals of length $c_{14}U_jQ^{-k_1/4}$ respectively. As a result, there are at most $c_{15}Q^{k_1}$ different combinations of smaller intervals, and the number of polynomials $N(x)$ is greater than $c_{15}Q^{k_1}$ by $c_{16}Q^{3\varepsilon/4}$ times. Thus, there exist at least $c_{17}Q^{3\varepsilon/4}$ polynomials $N_j(x)$ belonging to the parallelepiped

$$\Pi_3 = \prod_{i=0}^{3}[j_ic_{13}Q^{-t_i}, (j_i+1)c_{13}Q^{-t_i}]$$

for some set (j_0, j_1, j_2, j_3), where $j_i \in [0, [c_{14}Q^{k_1/4}] - 1]$, $i = 0, 1, 2, 3$, and

$$t_0 = 1 - 2m_2 - m_3 - (k_1 + k_2)/4 + 4\varepsilon_1,$$
$$t_1 = 1 - m_2 - m_3 - (k_1 + k_2)/4 + 4\varepsilon_1,$$
$$t_2 = 1 - m_3 - (k_1 + k_2)/4 + 3\varepsilon_1, \quad t_3 = 1 - (k_1 + k_2)/4 + 2\varepsilon_1.$$

Therefore, the new polynomials

$$K_j(x) = N_j(x) - N_1(x),\ j = 2, \dots, c_{17}Q^{3\varepsilon/4}, \quad H(K_j) < c_{18}Q^{1-(k_1+k_2)/4+2\varepsilon_1}$$

with N_j and N_1 from the same parallelepiped Π_3 satisfy the following system

$$(34)\qquad |K_j(d)| < c_{19}Q^{1-2m_2-m_3-(4-5(v_1+v_2)/3-m_2-s_2+\varepsilon/4)/4+4\varepsilon_1},$$
$$|K_j'(d)| < c_{19}Q^{1-m_2-m_3-k_1/4-k_2/4+4\varepsilon_1},$$
$$|K_j''(d)| < c_{19}Q^{1-m_3-k_1/4-k_2/4+3\varepsilon_1}, \quad |K_j'''(d)| < c_{19}Q^{1-k_1/4-k_2/4+2\varepsilon_1}.$$

We now move from the systems of inequalities (32), (34) at the point (d, w_0) to the systems of inequalities at any point of the parallelepiped M_1. Replace the value of Q in the right hand sides of inequalities by the value $Q_1 = Q^{1-(k_1+k_2)/4+2\varepsilon_1}$. Write only the first inequality for $|K_j(x)|$ and $|K_j(w)|_p$ with $r_1 = 4 - k_1 - k_2 + 8\varepsilon_1$:

$$|K_j(x)| < c_{20} Q_1^{4(1-2m_2-m_3-(k_1+k_2)/4+4\varepsilon_1)/r_1}, \quad \mu_1 I = Q_1^{-4m_2/r_1}, \tag{35}$$

$$|K_j(w)|_p < c_{20} Q_1^{4(-2s_2-s_3+2\varepsilon_1)/r_1}, \quad \mu_2 K = Q_1^{-4s_2/r_1}, \tag{36}$$

$1 \le j \le c_{21} Q^{3\varepsilon/4}$. Among $c_{21} Q^{3\varepsilon/4}$ polynomials K_j there are $c_{22} Q^{3\varepsilon/4}$ polynomials with the same vectors $\bar{m_1}$ and $\bar{s_1}$, which do not necessarily coincide with the vectors $\bar{m}$ and $\bar{s}$. If there exist two polynomials K_1 and K_2 which have no common roots Lemma 3 can be applied. The values of τ_1, η_1 we find from the estimates (35). Thus $\eta_1 = 4m_2/r_1$ and

$$\tau_1 = 4(2m_2 + m_3 + (k_1 + k_2)/4 - 1 - 4\varepsilon_1)/r_1. \tag{37}$$

Then we have

$$\begin{cases} \tau_1 + 1 = (8m_2 + 4m_3 - 8\varepsilon_1)/r_1, \\ 2(\tau_1 + 1 - \eta_1) = (8m_2 + 8m_3 - 16\varepsilon_1)/r_1, \\ 2(\tau_1 + 1 - 2\eta_1) = (8m_3 - 16\varepsilon_1)/r_1. \end{cases} \tag{38}$$

Similarly, from estimates (36) we get

$$\begin{cases} \tau_2 = (8s_2 + 4s_3 - 8\varepsilon_1)/r_1, \\ 2(\tau_2 - \eta_2) = 2(\tau_2 - 4s_2/r_1) = (8s_2 + 8s_3 - 16\varepsilon_1)/r_1, \\ 2(\tau_2 - 2\eta_2) = (8s_3 - 16\varepsilon_1)/r_1. \end{cases} \tag{39}$$

Next, from (37)–(39) compute the sum S of the numbers which are included in the left hand side of the inequality (14) in Lemma 3 to obtain

$$S = \frac{16m_2 + 20m_3 + 16s_2 + 20s_3 - 80\varepsilon_1}{5(v_1 + v_2)/3 + m_2 + s_2 - \varepsilon/4 + 8\varepsilon_1}. \tag{40}$$

From (11) it follows that

$$v_1 + v_2 \le m_2 + 2m_3 + s_2 + 2s_3 - 6\varepsilon_1,$$

therefore, the denominator r_1 in (40) satisfies the inequality

$$r_1 \le 8m_2/3 + 10m_3/3 + 8s_2/3 + 10s_3/3 - 2\varepsilon_1 - \varepsilon/4 = r_2$$

and

$$6r_1 \leq 16m_2 + 20m_3 + 16s_2 + 20s_3 - 12\varepsilon_1 - 3\varepsilon/2. \tag{41}$$

Then using (23) and the previous inequalities, we get $r_2 \leq 16/3 - \varepsilon/4 - 2\varepsilon_1$. Using the fact that $v_1 + v_2 \geq 3\varepsilon/20$, we obtain that $r_1 > 0$. Inequalities (41) and (40) lead to the inequality

$$S \geq 6 + (-68\varepsilon_1 + 3\varepsilon/2)/r_2 > 6 + \varepsilon/10,$$

which leads to a contradiction for $\delta < \varepsilon/10$ and $\varepsilon > 2^7\varepsilon_1$.

If, among polynomials $K_j(t)$, there exist no two polynomials without common roots then the polynomials $K_j(t)$ are reducible. At least one of the divisors of $K_j(t)$ is linear and here the argument is simple. We pass from the estimates for $|K_j(x)|$ and $|K_j(w)|_p$ to upper bounds for polynomials of the first and second degrees. Again, as, for example, in [8,11] we get a contradiction.

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