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On the Integrability of a Class of Nonlinear Dispersive Wave Equations

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Abstract

We investigate the integrability of a class of 1+1 dimensional models describing nonlinear dispersive waves in continuous media, e.g. cylindrical compressible hyperelastic rods, shallow water waves, etc. The only completely integrable cases coincide with the Camassa-Holm and Degasperis-Procesi equations.

1 Introduction

In this letter we investigate the integrability of the nonlinear equation

$$u_t - u_{xxt} + \partial_x g[u] = \nu u_x u_{xx} + \gamma u u_{xxx}, \tag{1}$$

where

$$g[u] = \kappa u + \alpha u^2 + \beta u^3 \tag{2}$$

and α, β, γ, κ, ν are constant parameters. The symmetries of (1) for specific choices of the parameters are studied in [4].

The case $\kappa = 0$, $\alpha = 3/2$, $\beta = 0$, $\nu = 2\gamma$ and γ an arbitrary real parameter has been recently studied as a model, describing nonlinear dispersive waves in cylindrical compressible hyperelastic rods [11, 10] – see also [9, 25]. The physical parameters of various compressible materials put γ in the range from -29.4760 to 3.4174 [11, 10].

Other important cases of (1) are:

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Camassa-Holm (CH) equation [3, 14]

$$u_t - u_{xxt} + \kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad (3)$$

$\kappa-$arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom [3, 17]. CH is a completely integrable equation [1, 8, 6, 18], describing permanent and breaking waves [7, 5]. The solitary waves of CH are smooth if $\kappa > 0$ and peaked if $\kappa = 0$ [3, 19]. Integrable generalizations of CH with higher order terms are derived in [15].

Degasperis-Procesi (DP) equation [12]:

$$u_t - u_{xxt} + \kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \qquad (4)$$

κ − arbitrary (real), is another completely integrable equation of this class. It is also known to possess (multi)peakon solutions if $\kappa = 0$ [13, 16].

CH and DP equations are particular cases of the b-family

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \qquad (5)$$

which possesses multipeakon solutions for any real b [13].

Fornberg-Whitham (FW) equation [24]

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx} \qquad (6)$$

appeared in the study of the qualitative behaviors of wave-breaking.

The regularized long-wave (RLW) or BBM equation [2]

$$u_t - u_{xxt} + u_x + uu_x = 0 \qquad (7)$$

and the modified BBM equation

$$u_t - u_{xxt} + u_x + (u^3)_x = 0 \qquad (8)$$

are not completely integrable, although they have three nontrivial independent integrals [22].

In what follows we will demonstrate that the only completely integrable representatives of the class (1) are CH and DP equations (3), (4).

In our analysis we will use the integrability check developed in [20, 23, 21]. This perturbative method can be briefly outlined as follows. Consider the evolution partial differential equation

$$u_t = F_1[u] + F_2[u] + F_3[u] + \dots \qquad (9)$$

where $F_k[u]$ is a homogeneous differential polynomial, i.e. a polynomial of variables u, u_x, u_{xx}, ..., $\partial_x^n u$ with complex constant coefficients, satisfying the condition

$$F_k[\lambda u] = \lambda^k F_k[u], \qquad \lambda \in \mathbb{C}.$$

The linear part is $F_1[u] = L(u)$, where L is a linear differential operator of order two or higher. The representation (9) can be put into correspondence to a symbolic expression of the form

$$u_t = u\omega(\xi_1) + \frac{u^2}{2}a_1(\xi_1,\xi_2) + \frac{u^3}{3}a_2(\xi_1,\xi_2,\xi_3) + \ldots = F \tag{10}$$

where $\omega(\xi_1)$ is a polynomial of degree 2 or higher and $a_k(\xi_1,\xi_2,\ldots\xi_{k+1})$ are symmetric polynomials. Each of these polynomials is related to the Fourier image of the corresponding $F_k[u]$ and can be obtained through a simple procedure, described e.g. in [20]. Each differential monomial $u^{n_0}u_x^{n_1}\ldots(\partial_x^q u)^{n_q}$ is represented by a symbol

$$u^m\langle\xi_1^0\ldots\xi_{n_0}^0\xi_{n_0+1}^1\ldots\xi_{n_0+n_1}^1\xi_{n_0+n_1+1}^2\ldots\xi_{n_0+n_1+n_2}^2\ldots\xi_m^q\rangle$$

where $m = n_0+n_1+\ldots n_q$ and the brackets $\langle\rangle$ denote symmetrization over all arguments ξ_k (i.e. symmetrization with respect to the group of permutations of m elements S_m):

$$\langle f(\xi_1,\xi_2,\ldots,\xi_n)\rangle = \frac{1}{m!}\sum_{\sigma\in S_m} f(\xi_{\sigma(1)},\xi_{\sigma(2)},\ldots,\xi_{\sigma(n)})$$

Also, for any function F (10) there exists a formal recursion operator

$$\Lambda = \eta + u\phi_1(\xi_1,\eta) + u^2\phi_2(\xi_1,\xi_2,\eta) + \ldots \tag{11}$$

where the coefficients $\phi_m(\xi_1,\xi_2,\ldots\xi_m,\eta)$ can be determined recursively:

$$\phi_1(\xi_1,\eta) = N^\omega(\xi_1,\eta)\xi_1 a_1(\xi_1,\eta) \tag{12a}$$

$$\begin{aligned}\phi_m(\xi_1,\xi_2,\ldots\xi_m,\eta) = N^\omega(\xi_1,\xi_2,\ldots\xi_m,\eta)\Big\{(\sum_{p=1}^{m}\xi_p)a_m(\xi_1,\xi_2,\ldots\xi_m,\eta)+\\ +\sum_{n=1}^{m-1}\Big\langle\frac{n}{m-n+1}\phi_n(\xi_1,\ldots\xi_{n-1},\xi_n+\ldots+\xi_m,\eta)a_{m-n}(\xi_n,\ldots\xi_m)+\\ +\phi_n(\xi_1,\ldots\xi_n,\eta+\xi_{n+1}+\ldots+\xi_m)a_{m-n}(\xi_{n+1},\ldots\xi_m,\eta)-\\ -\phi_n(\xi_1,\ldots\xi_n,\eta)a_{m-n}(\xi_{n+1},\ldots\xi_m,\eta+\xi_1+\ldots+\xi_n)\Big\rangle\Big\}\end{aligned} \tag{12b}$$

with

$$N^\omega(\xi_1,\xi_2,\ldots\xi_m) = \Big(\omega(\sum_{n=1}^{m}\xi_n) - \sum_{n=1}^{m}\omega(\xi_n)\Big)^{-1} \tag{13}$$

and the symbols $\langle\rangle$ denote symmetrization with respect to $\xi_1,\xi_2,\ldots\xi_m$, (the symbol η is not included in the symmetrization). Before formulating the integrability criterion it is necessary to introduce the following

Definition 1. *The function* $b_m(\xi_1, \xi_2, \ldots \xi_m, \eta)$, $m \geq 1$ *is called local if all coefficients* $b_{mn}(\xi_1, \xi_2, \ldots \xi_m)$, $n = n_s, n_{s+1}, \ldots$ *of its expansion as* $\eta \to \infty$

$$b_m(\xi_1, \xi_2, \ldots \xi_m, \eta) = \sum_{n=n_s}^{\infty} b_{mn}(\xi_1, \xi_2, \ldots \xi_m)\eta^{-n} \tag{14}$$

are symmetric polynomials.

Now the integrability criterion can be summarized as follows [20]:

Theorem 1. *The complete integrability of the equation (9), i.e. the existence of an infinite hierarchy of local symmetries or conservation laws, implies that all the coefficients (12) of the formal recursion operator(11) are local.*

2 The integrability test

After shifting $u \to -(u + \delta)$ and $x \to x - \lambda t$ where δ and λ are arbitrary constants, the equation (1) can be written in the form

$$u_t = (1 - \partial_x^2)^{-1}\Big(Ku_x + Bu_{xxx} + Cuu_x + Au^2u_x - \nu u_x u_{xx} - \gamma uu_{xxx}\Big) \tag{15}$$

where the new constants A, B, C and K are related to the old ones as follows:

$$A = -3\beta \tag{16a}$$
$$B = \lambda - \gamma\delta \tag{16b}$$
$$C = 2\alpha - 6\beta\delta \tag{16c}$$
$$K = 2\alpha\delta - \kappa - 3\beta\delta^2 - \lambda \tag{16d}$$

Since the linear part of the equation must contain second derivative or higher, the applicability of the test requires $B \neq 0$, i.e. $\lambda \neq \gamma\delta$ which always can be achieved by a proper choice of the arbitrary constant λ.

The symbolic representation of the operator $(1 - \partial_x^2)^{-1}$ is $\frac{1}{1-\eta^2}$ and the symbol, corresponding to $(1 - \partial_x^2)^{-1}F_k[u]$ is $\frac{u^k}{k}\frac{a_{k-1}(\xi_1,\xi_2,\ldots\xi_k)}{1-(\xi_1+\xi_2+\ldots+\xi_k)^2}$, where $\frac{u^k}{k}a_{k-1}(\xi_1, \xi_2, \ldots \xi_k)$ is the symbol corresponding to $F_k[u]$; see [20] for details. Moreover, Theorem 1 can be applied in this case as well. Therefore, the equation (15) can be represented in the form (10) with

$$\omega(\xi_1) = \frac{K\xi_1 + B\xi_1^3}{1 - \xi_1^2} \tag{17a}$$

$$a_1(\xi_1, \xi_2) = \frac{C(\xi_1 + \xi_2) - \nu\xi_1\xi_2(\xi_1 + \xi_2) - \gamma(\xi_1^3 + \xi_2^3)}{1 - (\xi_1 + \xi_2)^2} \tag{17b}$$

$$a_2(\xi_1, \xi_2, \xi_3) = \frac{A(\xi_1 + \xi_2 + \xi_3)}{1 - (\xi_1 + \xi_2 + \xi_3)^2} \tag{17c}$$

Then from (12):

$$\phi_1(\xi_1,\eta) = \frac{(1-\xi_1^2)(1-\eta^2)\Big(-C+\gamma\xi_1^2+(\nu-\gamma)\xi_1\eta+\gamma\eta^2\Big)}{(B+K)\eta(-3+\xi_1^2+\xi_1\eta+\eta^2)} \tag{18a}$$

$$\phi_2(\xi_1,\xi_2,\eta) = \Phi_{21}(\xi_1,\xi_2)\eta + \Phi_{20}(\xi_1,\xi_2) + \Phi_{2,-1}(\xi_1,\xi_2)\eta^{-1} + \Phi_{2,-2}(\xi_1,\xi_2)\eta^{-2} + \dots \tag{18b}$$

All coefficients in the expansion of $\phi_1(\xi_1,\eta)$ (18a) with respect to η are polynomials on ξ_1 and therefore there are no obstacles to the integrability of (15). However, the expansion of $\phi_2(\xi_1,\xi_2,\eta)$ (18b) may contain, in general, non-polynomial contributions.

Let us start with the case $\gamma \neq 0$. Then

$$\Phi_{21}(\xi_1,\xi_2) = \gamma\frac{(1-\xi_1^2)(1-\xi_2^2)\Big(-C+(\gamma+\nu)\xi_1\xi_2\Big)}{(B+K)^2(1-\xi_1\xi_2)} \tag{19}$$

is a polynomial iff

$$C = \gamma+\nu. \tag{20}$$

Then

$$\Phi_{2,-1}(\xi_1,\xi_2) = \frac{(1-\xi_1^2)(1-\xi_2^2)P_1(\xi_1,\xi_2)}{(B+K)^2(1-\xi_1\xi_2)} \tag{21}$$

where

$$P_1(\xi_1,\xi_2) = -A(B+K) + (\xi_1\xi_2-1)\times$$
$$\times\Big(\gamma^2(2\xi_1+\xi_2)(\xi_1+2\xi_2)+\gamma\nu(1+\xi_1^2+\xi_1\xi_2+\xi_2^2)-\nu^2(1+(\xi_1+\xi_2)^2)\Big).$$

$\Phi_{2,-1}(\xi_1,\xi_2)$ is a polynomial iff $A=0$. From (16), (20) we get

$$\beta = 0, \qquad \alpha = \frac{\gamma+\nu}{2}. \tag{23}$$

With α and β as in (23), the term $\Phi_{2,-3}(\xi_1,\xi_2)$ has the form

$$\Phi_{2,-3}(\xi_1,\xi_2) = \frac{P_2(\xi_1,\xi_2)}{(B+K)^2(1-\xi_1\xi_2)} \tag{24}$$

where $P_2(\xi_1,\xi_2)$ is a symmetric polynomial. The polynomial remainder of the division of $P_2(\xi_1,\xi_2)$ with $1-\xi_1\xi_2$ (e.g. if ξ_2 is treated as a constant, and ξ_1 as a polynomial variable) is proportional to the factor $6\gamma^2-5\gamma\nu+\nu^2$. Thus, for complete integrability it is necessary

$$6\gamma^2-5\gamma\nu+\nu^2 = 0. \tag{25}$$

There are two nonzero solutions of (25): $\nu = 2\gamma$ and $\nu = 3\gamma$. From (23), $\alpha = \frac{3\gamma}{2}$ and $\alpha = 2\gamma$ in these two cases correspondingly. The requirement

$B+K\neq 0$ (18a) or $\kappa\neq\nu\delta$ can be achieved for suitable δ, if $\nu\neq 0$, or even for $\nu=0$ if $\kappa\neq 0$. The test is inconclusive if $\kappa=\nu=0$, which corresponds to the equation (5) with $b=0$. This case is not integrable, although it admits a Hamiltonian formulation [13].

Without loss of generality one can choose now $\gamma=1$ (e.g. after rescaling of t), which gives precisely the integrable Camassa-Holm and Degasperis-Procesi equations (3), (4).

Now suppose $\gamma=0$, $\nu\neq 0$. In this case

$$\Phi_{20}(\xi_1,\xi_2)=\nu\frac{(1-\xi_1^2)(1-\xi_2^2)(\xi_1+\xi_2)(C-\nu\xi_1\xi_2)}{(B+K)^2(1-\xi_1\xi_2)} \tag{26}$$

is a polynomial iff $C=\nu$. Then

$$\Phi_{2,-1}(\xi_1,\xi_2)=\nu\frac{(1-\xi_1^2)(1-\xi_2^2)\Big(-A(B+K)+\nu^2(1-\xi_1\xi_2)(1+(\xi_1+\xi_2)^2)\Big)}{(B+K)^2(1-\xi_1\xi_2)} \tag{27}$$

is a polynomial iff $A=0$, i.e. $\beta=0$. If $C=\nu$ and $\beta=0$ (i.e. $\alpha=\nu/2$) a further computation gives

$$\Phi_{2,-3}(\xi_1,\xi_2)=-\nu^2\frac{(1-\xi_1^2)(1-\xi_2^2)P_3(\xi_1,\xi_2)}{(B+K)^2(1-\xi_1\xi_2)} \tag{28}$$

where

$$\begin{aligned}P_3(\xi_1,\xi_2)=3+7\xi_1^2+7\xi_2^2-\xi_1^4-\xi_2^4+12\xi_1\xi_2-29\xi_1^2\xi_2^2+8\xi_1^3\xi_2^3+\xi_1^5\xi_2+\xi_1\xi_2^5+\\+8\xi_1^2\xi_2^4+8\xi_1^4\xi_2^2-12\xi_1^3\xi_2-12\xi_1\xi_2^3.\end{aligned}$$

Therefore $\Phi_{2,-3}$ (28) is not a polynomial for $\nu\neq 0$. Note that the restriction $B+K\neq 0$ (18a) is again secured by the choice $\delta\neq\kappa/\nu$. Thus, if $\gamma=0$ and $\nu\neq 0$ no completely integrable equations emerge.

Finally, let us take $\gamma=\nu=0$. In this case

$$\Phi_{2,-1}(\xi_1,\xi_2)=\frac{\Big(A(B+K)-C^2\Big)(1-\xi_1^2)(1-\xi_2^2)}{(B+K)^2(1-\xi_1\xi_2)} \tag{30}$$

is a polynomial iff $C^2=A(B+K)$. But then

$$\Phi_{2,-3}(\xi_1,\xi_2)=\frac{A(1-\xi_1^2)(1-\xi_2^2)(1-4\xi_1\xi_2)}{(B+K)(1-\xi_1\xi_2)} \tag{31}$$

is apparently not a polynomial if $A\neq 0$, i.e. $\beta\neq 0$ (if $\beta\neq 0$, $B+K=2\alpha\delta-\kappa-3\beta\delta^2$ can be arranged to be nonzero by a proper choice of δ). Therefore, the only possibility, leading to an integrable equation could be $A=0$. Then it is obvious that for $C\neq 0$ (30) is not a polynomial (in this case $B + K = C\delta - \kappa$, $\delta \neq \kappa/C$). Thus for integrability it is necessary $A = C = 0$ but then the equation (15) becomes linear.

Therefore, the only nonlinear completely integrable representatives of the class (1) are the Camassa-Holm and Degasperis-Procesi equations (3), (4).

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