Articles   School of Mathematics and Statistics

2005-01-01

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Abstract. Manev's classical potential is known to successfully modify Newtonian celestial mechanics, in accordance with the general-relativistic description. The idea of replacing the exact solution of Einstein's equations, describing orbiting bodies, with a classical potential is widely used today. These models are not restricted to planetary motion only, but also include numerous interesting astrophysical phenomena such as accretion of matter around black holes or massive stars. There are different known potentials, which originate from metrics, describing a rotating/non-rotating black hole (or a massive body in general). A brief review of these results is given in this contribution, together with an example for the Manev potential, related to the Parametrized Post-Newtonian (PPN) metric, commonly used in general relativity for the description of the solar system.

1 Introduction

During the 20s of the last century, the Bulgarian scientist Prof. Georgi Manev worked on fundamental problems in Astrophysics and Cosmology. He was also the founder of the Sofia University's Department of Theoretical Physics, which he chaired up to 1944, dean of the Faculty of Physics and Mathematics and Rector of the University. The political regime established in Bulgaria after 9 September 1944 expelled Prof. Manev from the University and the scientific community — see [1, 2] for more details.

In the early ninties the works of Manev have been rediscovered by the Romanian scientist Prof. Florin Diacu [3–5].

Manev has introduced a classical potential in order to modify the celestial mechanics in accordance with the general-relativistic description: to describe the motion of a particle of mass m in the static field of universal gravitation due to mass M, Manev [6–10] replaced the mass m with $m = m_0 \exp(GM/r)$, where m_0 is an invariant and G is the Newton's constant. This led to the following modification of the Newton's gravitational law:

$$F = \frac{GMm_0}{r^2} \left(1 + \frac{3GM}{c^2 r} \right) . \tag{1}$$

Potential of the type:

$$V(r) = A\frac{GM}{r} + B\frac{G^2 M^2}{c^2 r^2} , \tag{2}$$

where A and B are some real dimensionless constants, G is the Newton's constant and M is the mass of the central body, was originally introduced by Newton himself in his Principia (Book II, Propositions XLIV and XLV) in order to describe deviations in the Moon's orbit from the Keplerian laws. However, modification of Newtonian physics for mimicing general-relativitic results [7–10] is attributed to Manev. We will refer to potential of the type (2) as to Manev potential. An extensive list of papers, refering to Manev potential is provided in [3].

In this contribution we present the contemporary scheme for modelling general-relativistic results in the Newtonian framework with modified classical potentials of Manev type. In particular, we analyse masses sweeping circular orbits around a rotating and non-rotating black hole (or a massive body in general) as well as parametrized post-Newtonian (PPN) formalism for the former case.

2 The Set-Up

We consider a four-dimensional spacetime parametrized by coordinates x^i $(i = 0, 1, 2, 3)$ and the geodesic motion of a test particle of unit rest mass in gravitational field given by metric $g_{ij}(x^k)$ with signature $(+, -, -, -)$. We will also use system of units for which $G = 1 = c$. The Lagrangian of the particle can conveniently be taken in the usual form [11]:

$$L = g_{ij}\dot{x}^i \dot{x}^j , \tag{3}$$

where the dots represent derivatives with respect to s — the parameter continuously parametrizing the geodesic. A usual choice for this parameter is the arc length ds along the trajectory:

$$ds^2 = g_{ij}dx^i dx^j . \tag{4}$$

An integral of motion is then:

$$L = 1 . \tag{5}$$

We will assume that the gravitational field is generated by a point mass. In spherical coordinates the Lagrangian is:

$$L = g_{tt}\dot{t}^2 - g_{rr}\dot{r}^2 - g_{\theta\theta}\dot{\theta}^2 - g_{\varphi\varphi}\dot{\varphi}^2 . \tag{6}$$

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We will assume that the gravitational field is time-independent. We will also assume independence of the metric on the azimuthal coordinate φ in view of the isotropy of space. Then the Euler–Lagrange equations

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \tag{7}$$

give the following two integrals of motion:

$$E = g_{tt}\dot{t}, \tag{8}$$

$$J = g_{\varphi\varphi}\dot{\varphi}. \tag{9}$$

Here E is the total energy and J is the angular momentum of the test particle (of unit rest mass). We will be interested in orbits in the equatorial plane which corrsponds to $\theta = \pi/2$. Substituting (5), (8) and (9) into (6), we get:

$$\frac{\dot{r}^2}{2} + \frac{1}{2g_{rr}}(1 - \frac{E^2}{g_{tt}} + \frac{J^2}{g_{\varphi\varphi}}) = 0, \tag{10}$$

where the metric depends only on r.

If we consider the energy conservation law of a classical particle of unit mass in a field with potential $U(r)$,

$$\frac{\dot{r}^2}{2} + U(r) = \mathcal{E}, \tag{11}$$

with $\mathcal{E} = $ const being the specific energy of this one-dimensional motion, we can interpret (10) as the energy conservation equation of a one-dimensional motion in potential

$$V(r) = \frac{1}{2g_{rr}}(1 - \frac{E^2}{g_{tt}} + \frac{J^2}{g_{\varphi\varphi}}) = U(r) - \mathcal{E} = U(r) + \text{const}. \tag{12}$$

We now focus on circular orbits for which $\dot{r} = 0$ both instantaneously and at all subsequent times (orbit at a perpetual turning point) [12, 13]. This implies:

$$V(r) = 0, \tag{13}$$

$$\frac{dV(r)}{dr} = 0. \tag{14}$$

Identifying the Keplerian angular momentum distribution J/E, the centripetal force acting on the particle can be expressed as [13]:

$$F = \frac{J^2}{E^2 r^3}, \tag{15}$$

where E and J are expressed as functions of r from (13) and (14) for any specific metric $g_{ij} = g_{ij}(r, \theta = \pi/2)$.

The pseudo-potential $V_M(r)$ giving rise to this force is the Manev potential:

$$V_M(r) = -\int F dr. \tag{16}$$

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3 Examples

3.1 Schwarzschild metric

The line element in Schwarzschild geometry is given by (see, for example, [14]):

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (17)$$

where M is the mass of the central body.

Equations (13) and (14) then produce:

$$\frac{2M}{r} - \left(1 - \frac{2M}{r}\right)\frac{J^2}{r^2} - 1 + E^2 = 0, \quad (18)$$

$$\frac{J^2}{r} - \frac{3MJ^2}{r^2} - M = 0, \quad (19)$$

respectively. One can solve (18) and (19) to obtain:

$$J^2 = Mr\left(1 - \frac{3M}{r}\right)^{-1}, \quad (20)$$

$$E^2 = \left(1 - \frac{2M}{r}\right)^2\left(1 - \frac{3M}{r}\right)^{-1}. \quad (21)$$

Then the centripetal force (15) is:

$$F = \frac{M}{r^2\left(1 - \frac{2M}{r}\right)^2} \quad (22)$$

and the Manev potential is given by:

$$V_M(r) = \frac{M}{r - 2M}. \quad (23)$$

The pseudo-Newtonian potential (23) has been used in [15] to correctly reproduce at classical level the model of accretion disk around a non-rotating black hole.

Upon expanding (23) far from the center, we obtain

$$V_M(r) = \frac{M}{r} + 2\frac{M^2}{r^2} + \cdots, \quad (24)$$

which corresponds to $A = 1$ and $B = 2$.

3.2 Kerr geometry

We consider Kerr geometry in Boyer–Lindquist coordinates [14], which describes a rotating black hole [14]:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\varphi - a\, dt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \quad (25)$$

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where

$$\Delta = r^2 - 2Mr + a^2, \qquad (26)$$
$$\rho^2 = r^2 + a^2 \cos^2 \theta, \qquad (27)$$

where M is, again, the mass of the centre and a — the angular momentum of the centre (per unit mass).

The Lagrangian for a particle in the equatorial plane is:

$$L = (1 - \frac{2M}{r})\dot{t}^2 + \frac{4Ma}{r}\dot{t}\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 - (r^2 + a^2 + \frac{2a^2M}{r})\dot{\phi}^2. \qquad (28)$$

The appearance of the cross-term leads to the following integrals of motion (instead of (8) and (9)) [13]:

$$J = -\frac{2Ma}{r}\dot{t} + (r^2 + a^2 + \frac{2a^2M}{r})\dot{\phi}, \qquad (29)$$
$$E = (1 - \frac{2M}{r})\dot{t} + \frac{2Ma}{r}\dot{\phi}. \qquad (30)$$

Solving this system of simultaneous equations for $\dot{t}$ and $\dot{\phi}$ yields:

$$\dot{t} = \frac{-2aMJ + (2a^2M + a^2r + r^3)E}{r(a^2 - 2Mr + r^2)}, \qquad (31)$$

$$\dot{\phi} = \frac{(r - 2M)J + 2aME}{r(a^2 - 2Mr + r^2)}. \qquad (32)$$

The effective potential $V(r)$ is [13]:

$$V(r) = \frac{1}{2}(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3})E^2 - \frac{1}{2r^2}(1 - \frac{2M}{R})J^2 - \frac{2MaEJ}{r^3} - \frac{\Delta}{2r^2}. \qquad (33)$$

Solving the simultaneous equations (13) and (14) for E and J and substituting into (15) gives the following centripetal force:

$$F = \frac{\left[a^3M + aMr(3r - 4M) \pm r(a^2 - 2Mr + r^2)\sqrt{Mr}\right]^2}{r^3\left[a^2M - r(r - 2M)^2\right]^2}. \qquad (34)$$

The positve sign corresponds to the case when the angular momentum of the particle is in direction opposite to the direction of the angular momentum of the rotating central mass (counter-rotation) and the negative sign corresponds to co-rotation [12].

Upon expansion over the powers of $1/r$ and integration, the Manev potential is determined as:

$$V_M(r) = \frac{M}{r} + 2\frac{M^2}{r^2} \pm \frac{12a}{5}\frac{\sqrt{M^3}}{r^{5/2}} + \frac{2}{3}\frac{M(a^2 + 6M^2)}{r^3} + O(r^{-7/2}). \qquad (35)$$

Obviously, to detect the effects of the rotation of the central mass, the Manev potential will have to include higher-order terms (otherwise, Schwarzschild geometry is recovered).

3.3 Parametrised Post-Newtonian Metric (PPN Formalism)

To describe the Solar System, it is useful to make spacetime look as Euclidean as possible by introducing isotropic spatial coordinates. The PPN metric used for this description is [14]:

$$ds^2 \;=\; (1 - \frac{2M}{r} + \frac{2\beta M^2}{r^2})dt^2 - (1 + \frac{2\gamma M}{r})\big[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\big]\,,$$

$$(36)$$

where β and γ are the PPN parameters.

The integrals of motion are:

$$E \;=\; (1 - \frac{2M}{r} + \frac{2\beta M^2}{r^2})\dot{t}\,, \tag{37}$$

$$J \;=\; (1 + \frac{2\gamma M}{r})r^2\dot{\phi}\,. \tag{38}$$

Following the same procedure, we obtain:

$$V_M(r) \;=\; \frac{M}{r} + \frac{4 - 2\beta + 3\gamma}{2}\frac{M^2}{r^2} + \frac{12 + 12\gamma + \gamma^2 - 6\beta(2+\gamma)}{3}\frac{M^3}{r^3} + \cdots.$$

$$(39)$$

General relativity predicts $\beta = 1 = \gamma$, which corresponds to $A = 1$ and $B = 5/2$.

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