2013

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Recommended Citation
Blackledge, J., Babajanov, B. (2013) : The Fractional Schrodinger-Klein-Gordon Equation and Intermediate Relativism. *Mathematic Aeterna*, Vol.3, 2013, no. 8, p.601 - 615. doi:10.21427/D7F03C

MATHEMATICA AETERNA, Vol. X, 2013, no. XX, XXX - XXX

The Fractional Schrödinger-Klein-Gordon Equation and Intermediate Relativism

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Abstract

By considering a random walk model compounded in Einstein's evolution equation, we show that both the classical Schrödinger and Klein-Gordon equations can be viewed as a consequence of introducing a memory function given by $-i\delta$ and $\delta^{(1)}$, respectively. For a memory function of the type $-i^{1+\alpha}\delta^{(\alpha)}$ where $0 < \alpha < 1$ we derive a fractional Schrödinger-Klein-Gordon equation whose corresponding propagator (free space Green's function) is then evaluated. The purpose of this is to derive a wave equation that, on a phenomenological basis at least, describes the transitional characteristics of wave functions for spin-less particles that may exist in the intermediate or 'semi-relativistic' regime. On the basis of the phenomenology considered, it is shown that such wave functions are self-affine functions of time t with a probability density that scales as $1/t^{1-\alpha}$ for mass-less particles.

Mathematics Subject Classification:
35Q60, 35Q40, 35P25, 74J20, 81U05

Keywords:
Schrödinger equation, Klein-Gordon equation, fractional time evolution, intermediate relativism.

1 Introduction

It is well known that the fundamental difference between non-relativistic and relativistic quantum mechanics is compounded in the application of the Energy E and Momentum $\mathbf{p}$ operators (where $\hbar$ is the Dirac constant) [1]

$$E \to i\hbar\partial_t, \quad \mathbf{p} \to -i\hbar\nabla; \quad \nabla \equiv \hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_x + \hat{\mathbf{z}}\partial_z$$

applied to the energy-momentum equations (for rest mass m and speed of light c)

$$E = \frac{\mathbf{p}^2}{2m} \quad \text{and} \quad E^2 = \mathbf{p}^2 c^2 + m^2 c^4$$

respectively. Using natural units with $c = 1$ and $\hbar = 1$, for a free space in which there is no influencing potential energy, Schrödinger's equation for the non-relativistic wave function $\psi(\mathbf{r}, t)$ is

$$(\nabla^2 + 2im\partial_t)\psi(\mathbf{r}, t) = 0 \tag{1}$$

and the Klein-Gordon equation for the relativistic wave function $\Psi(\mathbf{r}, t)$ is

$$(\nabla^2 - \partial_t^2 - m^2)\Psi(\mathbf{r}, t) = 0 \tag{2}$$

1.1 Spin-less Particles

Both equations (1) and (2) describe spin-less particles, the wave functions being scalar functions of space $\mathbf{r}$ and time t. In terms of the differential operators associated with equations (1) and (2), equation (1) is second order in space and first order in time whereas equation (2) is second order in both space and time. Equation (1) describes non-relativistic quantum systems such as atoms, molecules (subject to interaction with an atomic potential). Equation (2) describes Scalar Bosons such as Mesons which are hadronic subatomic particles composed of one quark and one antiquark, bound together by the strong interaction (subject to interaction with a nuclear potential). One may think of the difference between equation (1) and (2) as being the difference between atomic/molecular physics and nuclear physics, respectively.

In terms of the eigen-functions or standing wave patterns that equations (1) and (2) describe (subject to interaction with a potential), they can loosely be taken to represent the difference between the behaviour of an atom (a non-relativistic system) and the nucleus of an atom (a relativistic system) which provides the potential energy for the standing wave patterns associated with an electron cloud. In this sense, the distinct nature and associated characteristics of equations (1) and (2) are analogous to the distinct components associated with the Bohr model of an atom. There are no intermediate component states and, in this sense, the ideas developed in this paper are an attempt to model the intermediate case. This is not the same the Dirac equation which, for completeness, is briefly discussed in the following section.

1.2 The Dirac Equation

The Dirac equation expresses the relativistic energy-momentum relationship in terms of a partial differential equation that is first order in both space and time to yield the equation (using natural units with $\hbar = c = 1$) [2]

$$(a\partial_x + b\partial_y + c\partial_z + i\partial_t - dm)\Psi(\mathbf{r}, t) = 0 \tag{3}$$

where

$$a = i\begin{pmatrix} \mathbf{0} & \sigma_1 \\ \sigma_1 & \mathbf{0} \end{pmatrix}, \quad b = i\begin{pmatrix} \mathbf{0} & \sigma_2 \\ \sigma_2 & \mathbf{0} \end{pmatrix}, \ c = i\begin{pmatrix} \mathbf{0} & \sigma_3 \\ \sigma_3 & \mathbf{0} \end{pmatrix},$$

with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } d = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix},$$

I_2 and $\mathbf{0}$ being 2×2 dimensional identity and zero matrices, respectively. However, in this case, Ψ is not a scalar function but a (Dirac) Spinor and describes particles with 'Spin'. For the case when $m \to 0$, the Weyl equation is obtained which describes massless spin-1/2 particles - 'Weyl Spinors'.

Both equations (2) and (3) satisfy the relativistic energy-momentum relationship and are equally valid along with their appropriate solutions and are generalisations of the Schrödinger equation to the relativistic case in terms of a relativistic scalar wave function and a multi-component Spinor, respectively.

The formulation of a relativistic wave equation that is analogous to the Schrödinger equation in terms of having a first order energy operator leads naturally to the concept of a multi-component scalars which is fundamental to quantum mechanics and its relation with geometric algebra (e.g. Clifford algebra and Dirac algebra). However, neither equations (2) or (3) consider the transitory effects associated with a quantum mechanical system tending from a non-relativistic to a relativistic system (and visa versa). This is because the energy-momentum relationships for a non-relativistic and a relativistic system are fundamentally distinct, being characterised by the energy and the square energy, respectively.

1.3 Schrödinger form of the Klein Gordon Equation

In addition to expressing the Klein-Gordon equation in terms of a partial differential equation that is first order in space and time (which by default introduces Spinors) it is also possible to express the Klein-Gordon equation as a set of coupled Schrödinger type equations thus (using natural units)

$$i\partial_t\phi(\mathbf{r}, t) = -\frac{1}{2m}\nabla^2[\phi(\mathbf{r}.t) + \chi(\mathbf{r}, t)] + m\phi(\mathbf{r}, t)$$

$$i\partial_t \chi(\mathbf{r}, t) = \frac{1}{2m}\nabla^2[\phi(\mathbf{r}.t) + \chi(\mathbf{r}, t)] - m\chi(\mathbf{r}, t)$$

By adding and subtracting these equations, equation (2) is easily recovered where $\Psi = \phi + \chi$. By writing

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

and making using of the Pauli matrices, it is possible to express these coupled equations in terms of a Schrödinger-type equation where each component of Ψ individually satisfies the Klein-Gordon equation. Further, in the non-relativistic limit, the Feshbach-Villard transform [3] allows positively and negatively charges particles to be represented in terms of the upper and lower components of the fields ϕ and χ, respectively. Note that these fields which are not Spinors in the same sense as the component fields associated with Ψ are as defined by equation (3).

1.4 Non-Relativistic Limit and the Schrödinger Equation

It is possible to derive equation (1) from equation (2) by defining a relativistic wave function given by $\Psi = \psi \exp(-imt)$[1] under the condition $\mid \partial_t \ln \psi(\mathbf{r}, t) \mid /m << 0 \forall (\mathbf{r}, t)$ which yields equation (1). This is the non-relativistic limit of the Klein-Gordon equation and on a physical basis, this condition implies that the rest mass energy and the total energy of the system (including the rest mass energy) is small, i.e. $i\hbar\partial_t\psi << mc^2\psi$.

Although this limiting condition provides a connectivity between the relativistic and non-relativistic extremes of a wave function, it does not yield a statement on how a wave function behaves in a transitory sense which is the issue we explore in this paper as predicated on the following question: What are the properties of the wave function associated with a spin-less particle as that particle approaches the speed of light? In other words, what are the characteristics of a wave function in the 'intermediate zone' when a spin-less particle can not be formally classified as being relativistic or non-relativistic.

1.5 Principal Idea

One of the underlying ideas associated with relativistic quantum mechanics is to find a way of expressing the Klein-Gordon equation in terms of a first order partial differential equation in time, thereby expressing the wave function in

[1] For natural units applied to $\Psi = \psi \exp(-imc^2t/\hbar)$, the quotient $mc^2/\hbar$ being a frequency.

terms of a Schrödinger type equation involving the energy of the system instead of the square energy. Examples of this have been provided in the preceding sections, the Dirac equation being the most famous example of a solution to this problem with regard for its 'naturalness' in terms of introducing the concept of particle spin and thereby address the 1922 Stern-Gerlach experiment in which electrons were shown to possess an intrinsic angular momentum (analogous to the angular momentum of a classically spinning object) but only for certain quantised values.

The non-relativistic equations of quantum mechanics can, in principal, be derived by considering the non-relativistic limit of the corresponding relativistic equations as illustrated in the previous section. However, given that this is a limiting condition, the question then arises as to how it may be possible to consider the intermediate scenario and model a quantum field that is in the semi-relativistic regime. The principal purpose of this paper is two-fold: (i) to introduce the question; (ii) to realise a possible approach to solving it. With regard to point (ii), and, given that the essential difference between equations (1) and (2) relates to the time derivatives being first and second order, the principal idea is to consider an approach that is based on the introduction of a fractional time derivative $\partial_t^{1+\alpha}$ where $0 < \alpha < 1$ to generate a Fractional Shrödinger-Klein-Gordon (FSKG) equation. This is a phenomenology and in order to contextualise this idea further, and working in one-dimension, in the following section we show how it is related (on a phenomenological basis) to the specification of the memory function associated with the generalised Kolmogorov-Feller equation, which, in turn, is a statement of Einstein's evolution equation for a density function associated with random walk processes (random elastic scattering).

2 Derivation of the Schrödinger Equation using a Random Walk Model

For elastic scattering processes associated with a one-dimensional random walk model, Einstein's evolution equation is [4]

$$\psi(x, t+\tau) = \psi(x, t) \otimes_x p(x)$$

where $\psi(x, t)$ is the density function, $p(x)$ is the Probability Density Function (PDF) that characterises the (random) process and $\otimes_x$ denotes the convolution integral of a one-dimensional space x. This equation models the evolution of the density function from a time t to a time $t + \tau$. In conventional random walk theory the density function is taken to represent the concentration of a canonical ensemble of particles undergoing elastic collisions.

Consider a Taylor series for the function $\psi(x, t+\tau)$, i.e.

$$\psi(x,t+\tau) = \psi(x,t) + \tau\partial_t\psi(x,t) + \frac{\tau^2}{2!}\partial_t^2\psi(x,t) + ...$$

For $\tau << 1$

$$\psi(x,t+\tau) \simeq \psi(x,t) + \tau\partial_t\psi(x,t)$$

and we obtain the Classical Kolmogorov-Feller Equation (CKFE), [5], [6]

$$\tau\partial_t\psi(x,t) = -\psi(x,t) + \psi(x,t) \otimes_x p(x) \tag{4}$$

Equation (4) is based on a critical assumption which is that the time evolution of the density field $\psi(x,t)$ is influenced only by short term events and that longer term events have no influence on the behaviour of the field at any time t, i.e. the 'system' described by equation (4) has no 'memory'. This statement is the physical basis upon which the condition $\tau << 1$ is imposed thereby allowing the Taylor series expansion of the $\psi(x,t+\tau)$ to be made to first order.

From equation (4), the Schrödinger operator can be derived if we consider imaginary time and a Gaussian system where the Variance σ^2 (σ being the 'Standard Deviation') approaches zero. Thus let $t := it$, $\tau = \hbar$ and

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right), \quad \sigma^2 \to 0 \tag{5}$$

whose Fourier transform yields the Characteristic Function

$$\widetilde{p}(k) = \int\limits_{-\infty}^{\infty} p(x)\exp(-ikx)dx = \exp(-\sigma^2k^2/2) = 1 - \frac{\sigma^2}{2}k^2, \quad \sigma^2 \to 0$$

Using the convolution theorem, equation (4) transforms to

$$-i\hbar\partial_t\widetilde{\psi}(k,t) = -\widetilde{\psi}(k,t) + \left[1 - \frac{\sigma^2}{2}k^2\right]\widetilde{\psi}(k,t) = -\frac{\sigma^2}{2}k^2\widetilde{\psi}(k,t)$$

which in x-space (using the convolution theorem again) yields the equation

$$-i\hbar\partial_t\psi(x,t) = \frac{\sigma^2}{2}\partial_x^2\psi(x,t)$$

Thus with $\sigma^2 := \hbar^2/m$, equation (1) is recovered in one-dimension using natural units with $\hbar = 1$. Note that this equation is consistent with a model for the PDF given by

$$p(x) = \delta(x) + \frac{\hbar^2}{2m}\delta^{(2)}(x)$$

and has the Green's function

$$G(x \mid x_0, t \mid t_0) = \sqrt{\frac{im}{2\pi(t-t_0)}} \exp\left[\frac{im(x-x_0)^2}{2(t-t_0)}\right]$$

which is the solution of the equation

$$(\partial_x^2 + 2mi\partial_t)G(x \mid x_0, t \mid t_0) = -\delta(x-x_0)\delta(t-t_0)$$

where x_0 and $t_0 < t$ are the spatial origin and initial time, respectively, with $x \mid x_0 \equiv\mid x - x_0 \mid$ and $t \mid t_0 \equiv\mid t - t_0 \mid$.

3 Derivation of the FSKG Equation using a Random Walk Model

Given that equation (4) is memory invariant, the question arises as to how longer term temporal influences can be modelled, other than by taking an increasingly larger number of terms in the Taylor expansion of $\psi(x, t+\tau)$ which is not of practical analytical value, i.e. writing Einstein evolution equation in the form

$$\tau\partial_t\psi(x,t) + \frac{\tau^2}{2!}\partial_t^2\psi(x,t) + ... = -\psi(x,t) + \psi(x,t) \otimes_x p(x)$$

The key to solving this problem is to express the infinite series on the left hand side of the equation above in terms of a 'memory function' $\text{mem}(t)$ and write

$$\tau\text{mem}(t) \otimes_t \partial_t\psi(x,t) = -\psi(x,t) + \psi(x,t) \otimes_x p(x) \tag{6}$$

where $\otimes_t$ is taken to denote the convolution integral over t. This is the Generalised Kolmogorov-Feller Equation (GKFE) which reduces to the CKFE when $\text{mem}(t) = \delta(t)$ For any inverse function or class of inverse functions of the type $\text{mem}^{-1}(t)$, say, such that

$$\text{mem}^{-1}(t) \otimes_t \text{mem}(t) = \delta(t)$$

the GKFE can be written in the form

$$\tau\partial_t\psi(x,t) = -\text{mem}^{-1}(t) \otimes_t \psi(x,t) + \text{mem}^{-1}(t) \otimes_t \psi(x,t) \otimes_x p(x)$$

where the CKFE is again recovered when $\text{mem}^{-1}(t) = \delta(t)$ given that $\delta(t) \otimes_t \delta(t) = \delta(t)$.

Consider the memory function for which

$$\tau\text{mem}(t) = -i^{1+\alpha}\delta^{(\alpha)}(t), \quad 0 < \alpha < 1$$

and the PDF

$$p(x) = (1 - \alpha m^2)\delta(x) + \left(\frac{1}{2m}\right)^{1-\alpha} \delta^{(2)}(x)$$

so that equation (6) becomes

$$-i^{1+\alpha}\partial_t^{(1+\alpha)}\psi(x,t) = -\alpha m^2 \psi(x,t) + \left(\frac{1}{2m}\right)^{1-\alpha} \partial_x^2 \psi(x,t) \qquad (7)$$

It is then clear that when $\alpha \to 0$ and $\alpha \to 1$, equations (1) and (2) are recovered, respectively. In the following section, we study the free space Green's function for equation (7) - the Fractional Schr̈dingier-Klein-Gordon (FSKG) equation.

4 Evaluation of the FSKG Green's Function

The Green's function G for equation (7) is the solution of

$$(\partial_x^2 + i^{1+\alpha}\beta\partial_t^{1+\alpha} + \gamma)G(x \mid x_0, t \mid t_0) = -\delta(x - x_0)\delta(t - t_0), \quad 0 < \alpha < 1$$

where

$$\beta = (2m)^{1-\alpha} \text{ and } \gamma = -\alpha\beta m^2$$

and describes the propagation of a particle whose wave function is given by the solution of equation (7) from a space-time singularity at (x_0, t_0).

In Fourier space this equation transforms to

$$(-k^2 + \Omega^2 + \gamma)\widetilde{g} = -1 \qquad (8)$$

where

$$g(X, \omega) = \int_{-\infty}^{\infty} G(X, \tau)\exp(-i\omega\tau)d\tau, \quad \tau \equiv\mid t - t_0 \mid$$

$$\widetilde{g}(k, \tau) = \int_{-\infty}^{\infty} G(X, \tau)\exp(-ikX)dX, \quad X \equiv\mid x - x_0 \mid$$

and

$$\Omega^2 = i^{1+\alpha}\beta(i\omega)^{1+\alpha}$$

so derived by considering the following Fourier transform based definition of a fractional derivative:

$$\partial_t^{1+\alpha}\psi(X,t) = \frac{\partial_t^{1+\alpha}}{2\pi}\int_{-\infty}^{\infty} \widetilde{\psi}(X,\omega)\exp(i\omega t)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^{1+\alpha}\widetilde{\psi}(X,\omega)\exp(i\omega t)d\omega$$

Theorem 4.1 The time independent Green's function $g(X,\omega)$ for equation (7) is given by

$$g(X,\omega) = \frac{i}{(2\Omega - i\gamma)}\exp(i\Omega X)$$

Proof From equation (8) we can write

$$\widetilde{g} = \frac{1}{k^2 - \Omega^2 - \gamma} = \frac{1}{k^2 - \Omega^2}\left(1 - \frac{\gamma}{k^2 - \Omega^2}\right)^{-1}$$

$$= \frac{1}{k^2 - \Omega^2}\left(1 + \frac{\gamma}{k^2 - \Omega^2} + \frac{\gamma^2}{(k^2 - \Omega^2)^2} + ...\right)$$

The first term of this series is the Fourier space representation for the conventional Green's function for the classical (one-dimensional) wave equation and is given by $i\exp(i\Omega X)/2\Omega$. Hence, using the convolution theorem, we obtain a multiple convolution series expression for the function $g(X,\omega)$ given by

$$g(X,\omega) = \frac{i}{2\Omega}\exp(i\Omega X) + \gamma\frac{i}{2\Omega}\exp(i\Omega X) \otimes_X \frac{i}{2\Omega}\exp(i\Omega X)$$

$$+\gamma^2\frac{i}{2\Omega}\exp(i\Omega X) \otimes_X \frac{i}{2\Omega}\exp(i\Omega X) \otimes_X \frac{i}{2\Omega}\exp(i\Omega X) + ...$$

Defining the convolution integral in normalised form, i.e. for two piecewise continuous functions $f_1(X)$ and $f_2(X)$

$$f_1(X) \otimes_X f_2(X) \equiv \lim_{L\to\infty}\int_{-L}^{L} f_1(X-Y)f_2(Y)dY$$

the primary convolution integral of this series is

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\exp[i\Omega(X-Y)]\exp(i\Omega Y)dY = \exp(i\Omega X)$$

and hence the series reduces to the form

$$g(X,\omega) = \frac{i}{2\Omega}\exp(i\Omega X)\left[1 + \frac{i\gamma}{2\Omega} + \left(\frac{i\gamma}{2\Omega}\right)^2 + ...\right]$$

$$= \frac{i}{2\Omega}\exp(i\Omega X)\left(1 - \frac{i\gamma}{2\Omega}\right)^{-1} = \frac{i}{(2\Omega - i\gamma)}\exp(i\Omega X)$$

5 Temporal Properties of Mass-Less Particles

The time dependent Green's function is given by

$$G(X,\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g(X,\omega)\exp(i\omega\tau)d\omega$$

We consider the time dependent properties of this function for the case when $x \to x_0$ and

$$G(0,\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{i}{2\Omega}\left(1-\frac{i\gamma}{2\Omega}\right)^{-1}\exp(i\omega\tau)d\tau$$

This function expresses the temporal properties of a free spin-less particle in the proximity of its origin.

Theorem 5.1 For a mass-less particle

$$\mid G(0,\tau)\mid^2 \sim \frac{1}{\tau^{1-\alpha}}, \quad \tau > 0$$

Proof

$$\frac{i}{2\Omega}\left(1-\frac{i\gamma}{2\Omega}\right)^{-1} = \frac{i}{2\Omega} - \frac{\gamma}{4\Omega^2} + ... = \frac{i}{2\Omega}, \quad m \to 0 \tag{9}$$

so that

$$G(0,\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\exp(i\omega\tau)}{2i^{(1+\alpha)/2}\sqrt{\beta}(i\omega)^{(1+\alpha)/2}}d\omega = \frac{1}{2i^{(1+\alpha)/2}\sqrt{\beta}\Gamma\left(\frac{1+\alpha}{2}\right)\tau^{(1-\alpha)/2}}$$

given that

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} \frac{\exp(p\tau)}{p^q}dp = \frac{1}{\Gamma(q)\tau^{1-q}}, \quad q > 0$$

and hence it is clear that $\mid G(0,\tau)\mid^2$ - the probability density (of finding a particle in a given place at a given time, if the particle's position is measured) - scales with time according to $1/\tau^{1-\alpha}$.

This result is of course based on the asymptote $m \to 0$ to recover just the first term of the series expansion given in equation (9). However, we can relax the condition by taking the inverse Fourier transform of higher order terms given that the principal component of each term is $\Omega^{-n}, n = 1, 2, 3, ...$ Repeating the same calculation as that applied to the first term, we obtain the series

$$G(0,\tau) = \frac{c_1(\alpha,m)}{\tau^{(1-\alpha)/2}} + c_2(\alpha,m)\tau^{\alpha} + c_3(\alpha,m)\tau^{(1+3\alpha)/2} + ..$$

where c_n are complex coefficients. It is then clear the first term is the only term that has an (fractional) inverse scaling with time.

Finally, we note that since

$$g(X,\omega) = \frac{i}{(2\Omega - i\gamma)}(1 + i\Omega X + ...) = \frac{i}{2\Omega} - \frac{X}{2}, \quad m \to 0$$

we can write

$$G(X,\tau) = \frac{c_1(\alpha,m)}{\tau^{(1-\alpha)/2}} - \frac{X}{2}\delta(\tau), \quad m \to 0$$

and the inverse scaling law with time is preserved over all space. Thus, for any source of semi-relativistic mass-less particles $\Upsilon(x,t)$ of compact support $x \in [-X, X]$, say, the wave function is given by

$$\psi(x,t) \sim \frac{1}{t^{(1-\alpha)/2}} \otimes_t \int\limits_{-X}^{X} \Upsilon(x,t)dx - \frac{x}{2} \otimes_x \Upsilon(x,t)$$

6 Conclusion

By introducing a fractional time derivative we have derived an equation, i.e. equation (7), that models the intermediate relativistic case for spin-less free particles using an approach that is based on the GKFE. We have then shown that for mass-less particles, the time dependent behaviour of the density function scales as $1/t^{1-\alpha}$. This scaling function is the kernel of the fractional (Riemann-Liouville) integral which can be used to define a fractional differential. This is discussed in Appendix A which provides a brief introduction for the benefit of readers who are not familiar with the fractional calculus.

6.1 Open Questions

The methodology considered in this paper is entirely phenomenological and there are a number of open questions that should be considered:

1. Given that there are, in principal, a number of routes to constructing an equation that full-fills the requirements of equation (7) on a phenomenological basis, is there a uniqueness criteria that can be applied (or otherwise)?

2. Given 1. above, what are the properties associated with a two- and three-dimensional version of equation (7)?

3. Given 1. above, what modifications to equation (7) should be considered in order to model an interacting system determined by a time independent potential $V(\mathbf{r})$, for example, given that equation (1) and (2) extend to the forms
$$[\nabla^2 + 2im\partial_t - 2mV(\mathbf{r})]\psi(\mathbf{r},t) = 0$$
$$[\nabla^2 - \partial_t^2 - m^2 - 2iV(\mathbf{r})\partial_t + V^2(\mathbf{r})]\Psi(\mathbf{r},t) = 0$$
respectively?

6.2 Discussion

Unlike the fractional Schrdinger equation [8] which has a fractional spatial derivative and is based on using a Lévy distribution (with Characteristic Function $\exp(-a \mid k \mid^\gamma) \simeq 1 - a \mid k \mid^\gamma, \quad a \to 0$ where γ is the Lévy index) in place of the Gaussian distribution as given in equation (5), the use of a memory function to fractionalise the time derivative associated with the GKFE may be of significance in terms of modelling intermediate-relativism. If so, it is not coincidence that the time evolution of such semi-relativistic fields should be characterised by a $1/t^{1-\alpha}$ scaling law as this kernel is fundamental to fractional calculus (as discussed briefly in Appendix A) and through fractional calculus, the behaviour of self-affine or fractal fields (also discussed briefly in Appendix A). Thus, the possibility exists that there is a correlation between the physics of semi-relativistic states and the 'world of Fractal Geometry'.

The phenomenology associated with the approach considered in this paper is consistent with the derivation of the Schrödinger and Klein-Gordon equations using a Feynman path integral approach (e.g. [8], [9] and [10]). It is also compatible with an approach based on fundamental properties such as homogeneity, isotropy and randomness to justify the emergence of the continuity equation through fluctuations described in terms of the probability density [11], quantum Brownian motion [12] and maximising the Gibbs-Boltzmann entropy (or equivalently minimising the Szilard-Shannon-Kotelnikov information entropy measure) [13].

In addition to the Klein-Gordon equation considered in this paper, the fundamental field equations of physics are relativistic forms of non-relativistic counterparts. Just as the Klein-Gordon equation is a relativistic spin-0 version of the spin-0 Schrödinger equation, so the Proca equations [7], for example, are a relativistic version of Maxwell's equations which describe electromagnetic fields with mass and a spin of 1 characterising a Vector Boson. Similarly, the Proca equations decouple to give the Klein-Gordon equation whereas Maxwell's equation decouple to yield the classical wave equation, i.e. Maxwell's equations are Proca's equations in the non-relativistic limit. Thus, the issue of considering the nature of physical fields associated with the 'relativistic to non-relativistic continuum' applies to other systems of equations in addition to those considered in this paper, a path of enquiry that is left for future investigation.

Appendix A: A Short Overview on the Fractional Differential

For $0 < q < 1$, if we define the (Riemann-Liouville) derivative of order q as

$$\hat{D}^q u(t) \equiv \frac{d}{dt}[\hat{I}^{1-q}u](t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{-\infty}^{t}(t-\tau)^{-q}u(\tau)d\tau,$$

then,

$$\hat{D}^q u(t) = \frac{1}{\Gamma(1-q)}\int_{-\infty}^{t}(t-\tau)^{-q}u'(\tau)d\tau \equiv \hat{I}^{1-q}u'(t).$$

Hence,

$$\hat{I}^q[\hat{D}^q u] = \hat{I}^q[\hat{I}^{1-q}u'] = \hat{I}^1 u' = u$$

and $\hat{D}^q$ is the formal inverse of the operator $\hat{I}^q$. Given any $q > 0$, we can always write $\lambda = n - 1 + q$ and then define

$$\hat{D}^\lambda u(t) = \frac{1}{\Gamma(1-q)}\frac{d^n}{dt^n}\int_{-\infty}^{t}u(\tau)(t-\tau)^{-q}d\tau.$$

D^q is an operator representing a time invariant linear system consisting of a cascade combination of an ideal differentiator and a fractional integrator of order $1-q$. For D^λ we replace the single ideal differentiator by n such that

$$\hat{D}^0 u(t) = \frac{1}{\Gamma(1)}\frac{d}{dt}\int_{-\infty}^{t}u(\tau)d\tau = u(t) \equiv \int_{-\infty}^{\infty}u(\tau)\delta(t-\tau)d\tau$$

and

$$\hat{D}^n u(t) = \frac{1}{\Gamma(1)}\frac{d^{n+1}}{dt^{n+1}}\int_{-\infty}^{t}u(\tau)d\tau = u^{(n)}(t) \equiv \int_{-\infty}^{\infty}u(\tau)\delta^{(n)}(t-\tau)d\tau.$$

In addition to the conventional and classical definitions of fractional derivatives and integrals, more general definitions have recently been developed including the Erdélyi-Kober operators [14], hypergeometric operators and operators involving other special functions such as the Maijer G-function and the

Fox H-function [15]. Moreover, all such operators leading to a fractional integral of the Riemann-Liouville type and the Weyl type would appear (through induction) to have the general forms

$$\hat{I}^q f(t) = t^{q-1} \int_{-\infty}^{t} \Phi\left(\frac{\tau}{t}\right) \tau^{-q} f(\tau) d\tau \quad \text{and} \quad \hat{I}^q f(t) = t^{-q} \int_{t}^{\infty} \Phi\left(\frac{t}{\tau}\right) \tau^{q-1} f(\tau) d\tau$$

respectively, where the kernel Φ is an arbitrary continuous function so that the integrals above make sense in sufficiently large functional spaces.

Although it is possible to compute fractional integral and differential operators using the results discussed above, to the best of the authors knowledge, no direct relationship between fractional calculus and fractal geometry has yet been established and we arrive at an open question: Is there a geometrical representation of a fractional derivative? If not, can one prove that a graphical representation of a fractional derivative does not exist? The general consensus of opinion is that there is no simple geometrical interpretation of a derivative of fractional order and that if there is, then as Virginia Kiryakova concludes in her book on 'Generalised Fractional Calculus and Applications' [16], '... it is likely to be found in our fractal world'.

Acknowledgement

Jonathan Blackledge is the Science Foundation Ireland Stokes Professor. Bazar Babajanov is currently funded by the Erasmus Mundus Action II co-operation and the mobility programme EASTANA (EuroAsian Starter for Technical Academic Network Applications). Both authors are grateful to Dr Marek Rebow (Dublin Institute of Technology) of his support.

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