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New estimates for the number of integer polynomials with given discriminants

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Abstract. In this paper, we propose a new method of upper bounds for the number of integer polynomials of the fourth degree with a given discriminant. By direct calculation similar results were established by H. Davenport and D. Kaliada for polynomials of second and third degrees.

MSC: 11J83, 11J68

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1 Introduction

Denote by $\mathcal{P}_n$ the class of integer polynomials P of degree n . In what follows, we use the Vinogradov symbols $\ll$ (and $\gg$) where $a \ll b$ means that there exists a constant C such that $a \leq Cb$. If $a \ll b \ll a$, then we write $a \asymp b$. We denote the cardinality of a set B by $\#B$. Positive constants that depend only on n will be denoted by $c(n)$; where necessary, these constants will be numbered $c_j(n)$, $j = 1, 2, \dots$.

The discriminant of a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathcal{P}_n$ is defined by

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ are the roots of P . Let $H(P) = \max_{0 \leq j \leq n} |a_j|$ denote the standard (naive) height of $P = \sum_{i=0}^n a_i x^i$. Given a parameter $Q \in \mathbb{N}_{>1}$, let

$$\mathcal{P}_n(Q) = \{P(x) \in \mathcal{P}_n: H(P) \leq Q\}$$

denote the set of integer polynomials P of degree n and height $H(P) \leq Q$. If P has no repeated roots, then $D(P) \neq 0$. It is well known [16] that $D(P)$ can be represented as a determinant of order $2n-1$, which consists

of the coefficients of P . Hence, whenever $D(P) \neq 0$, we have that $|D(P)| \geq 1$ and $|D(P)|$ is bounded from above in terms of the height and degree of the polynomial P . We easily verify that for every $n \geq 2$, there exists a constant $c_1 > 0$ that depends on n only such that for any $P \in \mathcal{P}_n(Q)$, we have that

$$1 \leq |D(P)| < c_1 Q^{2n-2}. \quad (1.1)$$

The properties and estimates for $D(P)$ imply the estimates for $|x - \alpha_1|$, where $x \in \mathbb{R}$, and α_1 is the root of P closest to x (see [9, 10, 15]). These estimates were crucial to prove Mahler's conjecture in the case $n = 2, 3$. In a more systematic way, the relation between $|x - \alpha_1|$ and $D(P)$ was investigated by Sprindzuk [15] and others [2, 3, 4, 5, 6, 11, 12, 13, 14]. In recent years, the problem of counting polynomials with a small discriminant $D(P)$ has become a new branch of the theory of Diophantine approximations.

Given $v \in \mathbb{R}_{\geq 0}$, define the subset of $\mathcal{P}_n(Q)$ as follows:

$$\mathcal{P}_n(Q, v) = \{P(x) \in \mathcal{P}_n(Q) : 1 \leq |D(P)| < Q^{2n-2-2v}\}.$$

Establishing the correct lower and upper bounds for $\#\mathcal{P}_n(Q, v)$ is the goal of this branch of Diophantine approximations. We now briefly recall the results obtained to date. In the case of quadratic polynomials, it was shown in [13] that

$$\#\mathcal{P}_2(Q, v) \asymp Q^{3-2v}, \quad 0 < v < \frac{3}{4}.$$

In the case of cubic polynomials, it was established in [14] that

$$\#\mathcal{P}_3(Q, v) \asymp Q^{4-5v/3}, \quad 0 \leq v < \frac{3}{5}.$$

Establishing the correct lower bounds for arbitrary n has been the subject of numerous papers including [2, 3, 6, 13, 14]. The most general and best estimate was found in [3], where it was shown that

$$\#\mathcal{P}_n(Q, v) > c_2 Q^{n+1-(n+2)v/n}, \quad 0 \leq v \leq n-1. \quad (1.2)$$

The lower bound (1.2) for the full range of v , $0 \leq v \leq n-1$, was obtained for the polynomials that have all $\alpha_2, \dots, \alpha_n$ roots close to α_1 and x . The method for constructing a large number of polynomials $P \in \mathcal{P}_n(Q, v)$ is based on the results from [1]. Moreover, the following two propositions are key elements of the method for obtaining the lower bound (1.2).

Proposition 1. (See [3].) *Let $n \geq 2$, and let $v_0, v_1, \dots, v_n$ be a collection of real numbers such that*

$$v_0 + v_1 + \dots + v_n = 0 \quad \text{and} \quad v_0 \geq v_1 \geq \dots \geq v_n \geq -1.$$

Then there are positive constants c_3 and c_4 depending on n only with the following property. For any interval $J \subset [1/2, 1/2]$, there is a sufficiently large Q_0 such that for all $Q > Q_0$, there is a measurable set $G_J \subset J$ satisfying $|G_J| \geq |J|/2$ such that for every $x \in G_J$, there are $n+1$ linearly independent primitive irreducible polynomials $P \in \mathbb{Z}[x]$ of degree exactly n such that

$$c_3 Q^{-v_0} \leq |P(x)| \leq c_4 Q^{-v_0}, \quad c_3 Q^{-v_j} \leq |P^{(j)}(x)| \leq c_4 Q^{-v_j}, \quad 1 \leq j \leq n. \quad (1.3)$$

Proposition 2. (See [3].) Let n and v_j be as in Proposition 1. Let

$$d_j = v_{j-1} - v_j, \quad 1 \leq j \leq n.$$

Suppose that $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ and that for some $x \in \mathbb{C}$ and $Q > 1$, inequalities (1.3) are satisfied by some polynomial P over $\mathbb{C}$ of degree $\deg P = n$. Then there are roots $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ of P such that

$$|x - \alpha_j| \leq c_{5,j} Q^{-d_j}, \quad 1 \leq j \leq n,$$

where

$$c_{5,1} = nc_4 c_3^{-1},$$

$$c_{5,j+1} = \max \left(\frac{2c_4 n!}{c_3(j+1)!(n-j-1)!}, \frac{2c_{5,j} n!}{j!(n-j)!} \right), \quad 1 \leq j \leq n-1.$$

It is much harder to get upper bounds for $\#\mathcal{P}_n(Q, v)$ with arbitrary n . Note that the range of v depends on the number of roots of the polynomial close to α_1 . For example, if only one root α_2 is close to α_1 , then the range for v is $0 \leq v \leq n/2$.

For results in the p -adic case, see [7]. The upper and lower bounds for the number of polynomials having small discriminants in terms of the Euclidean and p -adic metrics simultaneously are obtained in [5, 11].

Let $\alpha_1, \dots, \alpha_n$ be the roots of the polynomial $P \in \mathcal{P}_n$. An upper bound for the number of integer cubic polynomials with a given discriminant is obtained in [4], where it is established that

$$\#\mathcal{P}'_3(Q, v) \ll Q^{4-5v/3+\epsilon}, \quad 0 \leq v \leq 2, \quad \forall \epsilon > 0,$$

where $\mathcal{P}'_3(Q, v)$ is a subclass of $\mathcal{P}_3(Q, v)$ with a special distribution of roots. The first step of the proof is the ordering the roots $\alpha_1, \alpha_2, \alpha_3$ with respect to one of them α_j , which will denote by α_1 , in such way that

$$|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3|, \quad |\alpha_1 - \alpha_3| \asymp |\alpha_2 - \alpha_3|. \quad (1.4)$$

In the case of the polynomials of fourth degree, we will have another principal case for the ordering of the roots:

$$|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3| \leq |\alpha_1 - \alpha_4|,$$

$$|\alpha_1 - \alpha_2| \ll |\alpha_3 - \alpha_4| \ll |\alpha_2 - \alpha_3| \ll |\alpha_1 - \alpha_3|. \quad (1.5)$$

Other cases are similar to (1.4).

Let $\alpha_{1j}, \dots, \alpha_{nj}$ be the roots of the polynomial $P_j \in \mathcal{P}_n$ ordered according to (1.4) or (1.5) depending on the degree of P_j . For $n = 3$, the polynomials P_j are expanded into Taylor series in a neighbourhood of α_{1j} , and the absolute values of P_j are estimated from above. Then we form the new polynomials $R_{j+1} = P_{j+1} - P_j$ of degree $\deg R_{j+1} < n$ from the polynomials P_j with the same oldest coefficients.

For the polynomials of fourth degree, in case (1.4), from the estimates $|P_j|$ in a neighbourhood of α_{1j} we cannot get strong estimates for $|P_j|$ in a neighbourhood of α_{3j} . Therefore the expansion into Taylor series must be carried out in a neighbourhood of α_{1j} and in a neighbourhood of α_{3j} .

The partition of the roots α_j into the clusters is possible for $n = 5, 6$, but for the arbitrary n , we did not find a convenient method to classify the roots. Therefore, from now on, $n = 4$ and the roots α_j satisfy (1.5). Let $\mathcal{P}'_4(Q, v)$ denote the set of polynomials $P \in \mathcal{P}_4(Q, v)$ with distinct roots satisfying (1.5). In this paper, we obtain an upper bound for the number of polynomials $P \in \mathcal{P}'_4(Q, v)$.

Theorem 1. For any $\epsilon > 0$ and any sufficiently large Q , we have the estimate

$$\#\mathcal{P}'_4(Q, v) < Q^{5-3v/2+\epsilon}, \quad 0 \leq v \leq 1. \quad (1.6)$$

2 Auxiliary statements

Let $P \in \mathcal{P}'_4(Q, v)$ have complex distinct roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let

$$|\alpha_1 - \alpha_i| = Q^{-\rho_i}, \quad 2 \leq i \leq 4, \quad \rho_2 \geq \rho_3 \geq \rho_4, \quad (2.1)$$

and

$$|\alpha_3 - \alpha_4| = Q^{-\rho_5}. \quad (2.2)$$

Similar to other problems of the metric theory regarding polynomials, we assume that $|a_4(P)| \gg H(P)$. If the polynomial P does not satisfy the last condition, then the transformation $S(x) = P(x + m)$ for some $0 \leq m \leq 4$ can be performed followed by an inversion to obtain $U(x) = x^4 S(1/x)$. Therefore this new polynomial $U(x) = \sum_{j=0}^4 b_j x^j$ satisfies $|b_4| \gg H(S) \asymp H(P)$. For more details, see [15]. If P satisfies $|a_4(P)| \gg H(P)$, then $|\alpha_i| \leq c_6$, $1 \leq i \leq 4$, and $|\alpha_i - \alpha_j| \leq 2c_6$ for $1 \leq i < j \leq 4$. Therefore $\rho_i \geq \epsilon_1$, $2 \leq i \leq 5$, for any $\epsilon_1 > 0$ and any sufficiently large Q .

For a given number $\epsilon_1 > 0$, let $T = [\epsilon_1^{-1}] + 1$, where $[a]$ is the integer part of $a \in \mathbb{R}$. For a polynomial $P \in \mathcal{P}_4(Q, v)$, the real numbers ρ_i , $i = 2, 3, 4, 5$, were defined in (2.1). Also define the integers l_i by

$$\frac{l_i - 1}{T} < \rho_i \leq \frac{l_i}{T}, \quad i = 2, 3, 4, 5.$$

It is not difficult to show that the number of vectors $\bar{l} = (l_2, l_3, l_4, l_5)$ is finite, depends only on ϵ_1 , and does not depend on Q and $H(P)$.

In order for the polynomial $P(x)$ to belong to the class $\mathcal{P}'_4(Q, v)$, it is necessary and sufficient that the inequality

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \geq v \quad (2.3)$$

holds. Note that inequality (2.3) follows from (1.1), (1.5), (2.1), (2.2), and the triangle inequalities for the roots of the polynomial P . For (2.3), the inequality

$$\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + \frac{l_5}{T} \geq v + 6\epsilon_1$$

is sufficient. By (1.1), (2.1), and (2.2) we have

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \leq 3. \quad (2.4)$$

For the roots of $P \in \mathcal{P}_4$, we define the sets

$$S(\alpha_j) = \{x \in \mathbb{R}: |x - \alpha_j| = \min_{1 \leq i \leq 4} |x - \alpha_i|\}, \quad 1 \leq j \leq 4.$$

Lemma 1. *Let α_1 be a complex root of an integer polynomial $P \in \mathcal{P}_4$, and let $x \in S(\alpha_1)$. Then*

$$|x - \alpha_1| \leq \min_{2 \leq j \leq 4} \left(2^{4-j} |P(x)| |P'(\alpha_1)|^{-1} \prod_{k=2}^j |\alpha_1 - \alpha_k| \right)^{1/j}$$

for $P'(\alpha_1) \neq 0$.

Lemma 1 is proved in [10].

Lemma 2. Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(x), T(x) \in \mathcal{P}_k(Q)$, $k \leq 4$, have the same vector $\bar{l}$ and have no common roots. Let I denote interval of length $|I| = Q^{-\gamma}$ with $\gamma \in \mathbb{R}_+$. If there exists a real number $\tau > 0$ such that for all $x \in I$,

$$\max_{x \in I} (|P(x)|, |T(x)|) < Q^{-\tau},$$

then

$$\tau + 1 + 2 \sum_{j=1}^k \max(\tau + 1 - j\gamma, 0) < 2k + \delta.$$

Lemma 2 can be proved similarly to Lemma 3 in [5]. In this case, we need to add the summands related to the root α_4 .

To prove Theorem 1, we need to consider a generalization of Lemma 2 for the simultaneous approximations of the polynomials on two intervals (see Lemma 3). We consider a new classification of the roots α_i , $1 \leq i \leq 4$, of $P \in \mathcal{P}'_4(Q)$ with respect to α_1 (as before) and α_3 simultaneously. We obtain

$$\begin{aligned} |\alpha_1 - \alpha_2| &\leq |\alpha_1 - \alpha_3| \leq |\alpha_1 - \alpha_4|, \\ |\alpha_3 - \alpha_4| &\leq |\alpha_3 - \alpha_2| \leq |\alpha_3 - \alpha_1|. \end{aligned} \quad (2.5)$$

Let $|\alpha_3 - \alpha_2| = Q^{-\rho_6}$ and define the integer l_6 by $(l_6 - 1)/T < \rho_6 \leq l_6/T$. It is not difficult to see that by (1.5)

$$\rho_4 \leq \rho_3 \leq \rho_2, \quad \rho_3 \leq \rho_6 \leq \rho_5, \quad (2.6)$$

where ρ_i , $2 \leq i \leq 5$, are defined in (2.1)–(2.2). We also define the vector $\bar{l}' = (\bar{l}, l_6)$. Define the class $\mathcal{P}'_{4, \bar{l}'}(Q, v)$ consisting of the polynomials $P \in \mathcal{P}'_4(Q, v)$ corresponding to a vector $\bar{l}'$.

Lemma 3. Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(t), T(t) \in \mathcal{P}_k(Q)$, $k \leq 4$, have the same vector $\bar{l}'$ and have no common roots in the rectangle $I_1 \times I_2$, where $|I_1| = Q^{-\gamma_1}$ and $|I_2| = Q^{-\gamma_2}$ with $\gamma_j \in \mathbb{R}_+$, $j = 1, 2$. Furthermore, let $P(t)$ and $T(t)$ satisfy the system of inequalities

$$\max_{x \in I_1} (|P(x)|, |T(x)|) < Q^{-\tau_1}, \quad \max_{y \in I_2} (|P(y)|, |T(y)|) < Q^{-\tau_2}. \quad (2.7)$$

Then for any $\delta > 0$ and $Q > Q_0(\delta)$, we have the inequality:

$$\tau_1 + \tau_2 + 2 + l_2 + 2l_3 + 3l_4 + l_5 < 2k + \delta. \quad (2.8)$$

The proof of Lemma 3 follows from the new classification (2.5) of the roots of polynomials, using inequalities (2.6) and (2.7), and can be proved similarly to Lemma 2 in [8].

3 Proof of Theorem 1

Assume that estimate (1.6) does not hold, so that

$$\#\mathcal{P}'_4(Q, v) \geq Q^{5-3v/2+\epsilon}. \quad (3.1)$$

Consider two intervals $I_1, I_2 \subset \mathbb{R}$ with $|I_1| = Q^{-l_2/T}$ and $|I_2| = Q^{-l_5/T}$. We will say that the polynomial P belongs to $M = I_1 \times I_2$ if $(\alpha_1, \alpha_3) \in M$, where α_1 and α_3 are the roots of P in the ordering (1.5). From (3.1) it follows that there exist rectangles $I_1 \times I_2$ that contain at least

$$\Delta = Q^{5-3v/2-l_2/T-l_5/T+\epsilon}$$

polynomials $P \in \mathcal{P}'_4(Q, v)$ satisfying (2.7). Fix one of these rectangles, say M . Since $\#\bar{l}' \ll 1$, there exists a vector $\bar{l}'$ satisfying (2.3) such that

$$\#\mathcal{P}'_{4, \bar{l}'}(Q, v, M) \gg Q^{5-3v/2+\epsilon-l_2/T-l_5/T+\epsilon},$$

where $\mathcal{P}'_{4, \bar{l}'}(Q, v, M)$ denotes the subset of $\mathcal{P}'_{4, \bar{l}'}(Q, v)$ consisting of polynomials P belonging to M . Fix the vector $\bar{l}'$ and set

$$h = 5 - \frac{3v}{2} - \frac{l_2}{T} - \frac{l_5}{T} + \frac{\epsilon}{2}.$$

By (2.4) we have

$$\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + \frac{l_5}{T} \leq 3. \quad (3.2)$$

From (3.2) we obtain that $h > 0$ for $v \leq 4/3$.

Expand the polynomial $P \in \mathcal{P}'_{4, \bar{l}'}(Q, v, M)$ into its Taylor series in a neighbourhood of α_1 to obtain

$$\begin{aligned} P(x) &= P(\alpha_1) + P'(\alpha_1)(x - \alpha_1) \\ &\quad + \frac{1}{2}P''(\alpha_1)(x - \alpha_1)^2 + \frac{1}{6}P'''(\alpha_1)(x - \alpha_1)^3 + \frac{1}{24}P^{(4)}(\alpha_1)(x - \alpha_1)^4. \end{aligned}$$

Estimating each term gives

$$\begin{aligned} |P'(\alpha_1)(x - \alpha_1)| &\leq |a_4| \cdot |\alpha_1 - \alpha_2| \cdot |\alpha_1 - \alpha_3| \cdot |\alpha_1 - \alpha_4| \cdot |x - \alpha_1| \\ &\leq Q^{1-\rho_2-\rho_3-\rho_4-l_2/T} < Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, \\ |P''(\alpha_1)(x - \alpha_1)^2| &\leq 6|a_4| \max(|\alpha_1 - \alpha_2||\alpha_1 - \alpha_3|, |\alpha_1 - \alpha_2||\alpha_1 - \alpha_4|, |\alpha_1 - \alpha_3||\alpha_1 - \alpha_4|) \\ &\quad \times |x - \alpha_1|^2 \\ &< 6Q^{1-2l_2/T-l_3/T-l_4/T+2\epsilon_1}, \\ |P'''(\alpha_1)(x - \alpha_1)^3| &\leq 18|a_4| \max(|\alpha_1 - \alpha_2|, |\alpha_1 - \alpha_3|, |\alpha_1 - \alpha_4|) \cdot |x - \alpha_1|^3 \\ &< 18Q^{1-3l_2/T-l_4/T+\epsilon_1} \\ |P^{(4)}(\alpha_1)(x - \alpha_1)^4| &\leq 24|a_4||x - \alpha_1|^4 \leq 24Q^{1-4l_2/T} \end{aligned}$$

for $x \in I_1$. Thus

$$|P(x)| \ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, \quad x \in I_1.$$

Also develop the polynomial P as Taylor series on the interval I_2 at the point α_3 and obtain the upper bounds for all terms in the series. Thus we obtain

$$|P(y)| \ll Q^{1-2l_5/T-l_3/T-l_6/T+3\epsilon_1}, \quad y \in I_2.$$

Further, for Q^h polynomials P , we use the Dirichlet box principle. We will assume that the fractional part of h does not exceed ϵ_1 . If the last condition is not satisfied, then we rewrite h as $h = [h] + \{h\}$. As a result, using the number $Q^{[h]}$, we reduce the degree of polynomials, and using the number $Q^{\{h\}}$, we reduce the height

of polynomials $R_{j+1}(t) = P_{j+1}(t) - P_1(t)$, $j = 1, 2, \dots$, as in [5]. Therefore the new polynomials R_j satisfy

$$\begin{aligned} |R_j(x)| &\ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, & x \in I_1, \\ |R_j(y)| &\ll Q^{1-2l_5/T-l_3/T-l_6/T+3\epsilon_1}, & y \in I_2, \end{aligned} \quad (3.3)$$

$$H(R_j) \leq Q^{1-\epsilon_1}, \quad \deg R_j \leq 4 - \left(5 - \frac{3v}{2} - \frac{l_2}{T} - \frac{l_5}{T} + \frac{\epsilon}{2} - \epsilon_1\right). \quad (3.4)$$

If there exist two polynomials R_1 and R_2 with no common roots, then Lemma 3 can be applied. The values of τ_1 and τ_2 are found from estimates (3.3) and (3.4). Thus

$$\tau_1 = \frac{-1 + 2l_2/T + l_3/T + l_4/T - 3\epsilon_1}{1 - \epsilon_1} \quad \text{and} \quad \tau_2 = \frac{-1 + 2l_5/T + l_3/T + l_6/T - 3\epsilon_1}{1 - \epsilon_1}.$$

The left-hand side of (2.8) is equal to

$$\frac{3l_2/T + 4l_3/T + 4l_4/T + 3l_5/T + l_6/T - 6\epsilon_1}{1 - \epsilon_1}.$$

This leads to a contradiction in (2.8) for $v \leq 1$ and $\delta \leq \epsilon - 2\epsilon_1$.

If, among polynomials $R_j(t)$, there exist no two polynomials without common roots, then the polynomials $R_j(t)$ are reducible. It is not difficult to see that $\deg R_j \leq 2$ for $v \leq 1$. Thus the polynomials $R_j(t)$ are decomposed into the product of two linear polynomials. Again, as, for example, in [4], we will use Lemma 2 to get a contradiction.

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