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New estimates for the number of integer polynomials with given discriminants

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Abstract. In this paper, we propose a new method of upper bounds for the number of integer polynomials of the fourth degree with a given discriminant. By direct calculation similar results were established by H. Davenport and D. Kaliada for polynomials of second and third degrees.

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1 Introduction

Denote by $\mathcal{P}_n$ the class of integer polynomials P of degree n. In what follows, we use the Vinogradov symbols $\ll$ (and $\gg$) where $a \ll b$ means that there exists a constant C such that $a \leqslant Cb$. If $a \ll b \ll a$, then we write $a \asymp b$. We denote the cardinality of a set B by $\#B$. Positive constants that depend only on n will be denoted by $c(n)$; where necessary, these constants will be numbered $c_j(n)$, $j = 1, 2, \dots$.

The discriminant of a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}_n$ is defined by

$$D(P) = a_n^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ are the roots of P. Let $H(P) = \max_{0 \leqslant j \leqslant n} |a_j|$ denote the standard (naive) height of $P = \sum_{i=0}^{n} a_i x^i$. Given a parameter $Q \in \mathbb{N}_{>1}$, let

$$\mathcal{P}_n(Q) = \left\{ P(x) \in \mathcal{P}_n \colon H(P) \leqslant Q \right\}$$

denote the set of integer polynomials P of degree n and height $H(P) \leqslant Q$. If P has no repeated roots, then $D(P) \neq 0$. It is well known [16] that $D(P)$ can be represented as a determinant of order $2n-1$, which consists of the coefficients of P. Hence, whenever $D(P) \neq 0$, we have that $|D(P)| \geqslant 1$ and $|D(P)|$ is bounded from above in terms of the height and degree of the polynomial P. We easily verify that for every $n \geqslant 2$, there exists a constant $c_1 > 0$ that depends on n only such that for any $P \in \mathcal{P}_n(Q)$, we have that

$$1 \leqslant \bigl|D(P)\bigr| < c_1 Q^{2n-2}. \tag{1.1}$$

The properties and estimates for $D(P)$ imply the estimates for $|x - \alpha_1|$, where $x \in \mathbb{R}$, and α_1 is the root of P closest to x (see [9, 10, 15]). These estimates were crucial to prove Mahler's conjecture in the case $n = 2, 3$. In a more systematic way, the relation between $|x - \alpha_1|$ and $D(P)$ was investigated by Sprindzuk [15] and others [2,3,4,5,6,11,12,13,14]. In recent years, the problem of counting polynomials with a small discriminant $D(P)$ has become a new branch of the theory of Diophantine approximations.

Given $v \in \mathbb{R}_{\geqslant 0}$, define the subset of $\mathcal{P}_n(Q)$ as follows:

$$\mathcal{P}_n(Q, v) = \bigl\{P(x) \in \mathcal{P}_n(Q)\colon\ 1 \leqslant \bigl|D(P)\bigr| < Q^{2n-2-2v}\bigr\}.$$

Establishing the correct lower and upper bounds for $\#\mathcal{P}_n(Q, v)$ is the goal of this branch of Diophantine approximations. We now briefly recall the results obtained to date. In the case of quadratic polynomials, it was shown in [13] that

$$\#\mathcal{P}_2(Q, v) \asymp Q^{3-2v}, \quad 0 < v < \frac{3}{4}.$$

In the case of cubic polynomials, it was established in [14] that

$$\#\mathcal{P}_3(Q, v) \asymp Q^{4-5v/3}, \quad 0 \leqslant v < \frac{3}{5}.$$

Establishing the correct lower bounds for arbitrary n has been the subject of numerous papers including [2, 3, 6, 13, 14]. The most general and best estimate was found in [3], where it was shown that

$$\#\mathcal{P}_n(Q, v) > c_2 Q^{n+1-(n+2)v/n}, \quad 0 \leqslant v \leqslant n - 1. \tag{1.2}$$

The lower bound (1.2) for the full range of v, $0 \leqslant v \leqslant n - 1$, was obtained for the polynomials that have all $\alpha_2, \ldots, \alpha_n$ roots close to α_1 and x. The method for constructing a large number of polynomials $P \in \mathcal{P}_n(Q, v)$ is based on the results from [1]. Moreover, the following two propositions are key elements of the method for obtaining the lower bound (1.2).

Proposition 1. (*See* [3].) *Let $n \geqslant 2$, and let $v_0, v_1, \ldots, v_n$ be a collection of real numbers such that*

$$v_0 + v_1 + \cdots + v_n = 0 \quad \text{and} \quad v_0 \geqslant v_1 \geqslant \cdots \geqslant v_n \geqslant -1.$$

Then there are positive constants c_3 and c_4 depending on n only with the following property. For any interval $J \subset [1/2, 1/2]$, there is a sufficiently large Q_0 such that for all $Q > Q_0$, there is a measurable set $G_J \subset J$ satisfying $|G_J| \geqslant |J|/2$ such that for every $x \in G_J$, there are $n+1$ linearly independent primitive irreducible polynomials $P \in \mathbb{Z}[x]$ of degree exactly n such that

$$c_3 Q^{-v_0} \leqslant \bigl|P(x)\bigr| \leqslant c_4 Q^{-v_0}, \quad c_3 Q^{-v_j} \leqslant \bigl|P^{(j)}(x)\bigr| \leqslant c_4 Q^{-v_j}, \quad 1 \leqslant j \leqslant n. \tag{1.3}$$

Proposition 2. (*See* [3].) *Let* n *and* v_j *be as in Proposition* 1. *Let*

$$d_j = v_{j-1} - v_j, \quad 1 \leqslant j \leqslant n.$$

Suppose that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 0$ *and that for some* $x \in \mathbb{C}$ *and* $Q > 1$, *inequalities* (1.3) *are satisfied by some polynomial* P *over* $\mathbb{C}$ *of degree* $\deg P = n$. *Then there are roots* $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ *of* P *such that*

$$|x - \alpha_j| \leqslant c_{5,j} Q^{-d_j}, \quad 1 \leqslant j \leqslant n,$$

where

$$c_{5,1} = n c_4 c_3^{-1},$$
$$c_{5,\,j+1} = \max\left(\frac{2c_4 n!}{c_3(j+1)!(n-j-1)!}, \frac{2c_{5,j} n!}{j!(n-j!)}\right), \quad 1 \leqslant j \leqslant n-1.$$

It is much harder to get upper bounds for $\#\mathcal{P}_n(Q, v)$ with arbitrary n. Note that the range of v depends on the number of roots of the polynomial close to α_1. For example, if only one root α_2 is close to α_1, then the range for v is $0 \leqslant v \leqslant n/2$.

For results in the p-adic case, see [7]. The upper and lower bounds for the number of polynomials having small discriminants in terms of the Euclidean and p-adic metrics simultaneously are obtained in [5, 11].

Let $\alpha_1, \dots, \alpha_n$ be the roots of the polynomial $P \in \mathcal{P}_n$. An upper bound for the number of integer cubic polynomials with a given discriminant is obtained in [4], where it is established that

$$\#\mathcal{P}_3'(Q, v) \ll Q^{4-5v/3+\epsilon}, \quad 0 \leqslant v \leqslant 2,\ \forall \epsilon > 0,$$

where $\mathcal{P}_3'(Q, v)$ is a subclass of $\mathcal{P}_3(Q, v)$ with a special distribution of roots. The first step of the proof is the ordering the roots $\alpha_1, \alpha_2, \alpha_3$ with respect to one of them α_j, which will denote by α_1, in such way that

$$|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3|, \qquad |\alpha_1 - \alpha_3| \asymp |\alpha_2 - \alpha_3|. \tag{1.4}$$

In the case of the polynomials of fourth degree, we will have another principal case for the ordering of the roots:

$$\begin{aligned} &|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant |\alpha_1 - \alpha_4|, \\ &|\alpha_1 - \alpha_2| \ll |\alpha_3 - \alpha_4| \ll |\alpha_2 - \alpha_3| \ll |\alpha_1 - \alpha_3|. \end{aligned} \tag{1.5}$$

Other cases are similar to (1.4).

Let $\alpha_{1j}, \dots, \alpha_{nj}$ be the roots of the polynomial $P_j \in \mathcal{P}_n$ ordered according to (1.4) or (1.5) depending on the degree of P_j. For $n = 3$, the polynomials P_j are expanded into Taylor series in a neighbourhood of α_{1j}, and the absolute values of P_j are estimated from above. Then we form the new polynomials $R_{j+1} = P_{j+1} - P_j$ of degree $\deg R_{j+1} < n$ from the polynomials P_j with the same oldest coefficients.

For the polynomials of fourth degree, in case (1.4), from the estimates $|P_j|$ in a neighbourhood of α_{1j} we cannot get strong estimates for $|P_j|$ in a neighbourhood of α_{3j}. Therefore the expansion into Taylor series must be carried out in a neighbourhood of α_{1j} and in a neighbourhood of α_{3j}.

The partition of the roots α_j into the clusters is possible for $n = 5, 6$, but for the arbitrary n, we did not find a convenient method to classify the roots. Therefore, from now on, $n = 4$ and the roots α_j satisfy (1.5). Let $\mathcal{P}_4'(Q, v)$ denote the set of polynomials $P \in \mathcal{P}_4(Q, v)$ with distinct roots satisfying (1.5). In this paper, we obtain an upper bound for the number of polynomials $P \in \mathcal{P}_4'(Q, v)$.

Theorem 1. *For any* $\epsilon > 0$ *and any sufficiently large* Q, *we have the estimate*

$$\#\mathcal{P}_4'\bigl(Q, v\bigr) < Q^{5-3v/2+\epsilon}, \quad 0 \leqslant v \leqslant 1. \tag{1.6}$$

2 Auxiliary statements

Let $P \in \mathcal{P}_4'(Q,v)$ have complex distinct roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let

$$|\alpha_1 - \alpha_i| = Q^{-\rho_i}, \quad 2 \leqslant i \leqslant 4,\ \rho_2 \geqslant \rho_3 \geqslant \rho_4, \tag{2.1}$$

and

$$|\alpha_3 - \alpha_4| = Q^{-\rho_5}. \tag{2.2}$$

Similar to other problems of the metric theory regarding polynomials, we assume that $|a_4(P)| \gg H(P)$. If the polynomial P does not satisfy the last condition, then the transformation $S(x) = P(x+m)$ for some $0 \leqslant m \leqslant 4$ can be performed followed by an inversion to obtain $U(x) = x^4 S(1/x)$. Therefore this new polynomial $U(x) = \sum_{j=0}^4 b_j x^j$ satisfies $|b_4| \gg H(S) \asymp H(P)$. For more details, see [15]. If P satisfies $|a_4(P)| \gg H(P)$, then $|\alpha_i| \leqslant c_6$, $1 \leqslant i \leqslant 4$, and $|\alpha_i - \alpha_j| \leqslant 2c_6$ for $1 \leqslant i < j \leqslant 4$. Therefore $\rho_i \geqslant \epsilon_1$, $2 \leqslant i \leqslant 5$, for any $\epsilon_1 > 0$ and any sufficiently large Q.

For a given number $\epsilon_1 > 0$, let $T = [\epsilon_1^{-1}] + 1$, where $[a]$ is the integer part of $a \in \mathbb{R}$. For a polynomial $P \in \mathcal{P}_4(Q,v)$, the real numbers ρ_i, $i = 2,3,4,5$, were defined in (2.1). Also define the integers l_i by

$$\frac{l_i - 1}{T} < \rho_i \leqslant \frac{l_i}{T}, \quad i = 2,3,4,5.$$

It is not difficult to show that the number of vectors $\bar{l} = (l_2, l_3, l_4, l_5)$ is finite, depends only on ϵ_1, and does not depend on Q and $H(P)$.

In order for the polynomial $P(x)$ to belong to the class $\mathcal{P}_4'(Q,v)$, it is necessary and sufficient that the inequality

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \geqslant v \tag{2.3}$$

holds. Note that inequality (2.3) follows from (1.1), (1.5), (2.1), (2.2), and the triangle inequalities for the roots of the polynomial P. For (2.3), the inequality

$$\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + \frac{l_5}{T} \geqslant v + 6\epsilon_1$$

is sufficient. By (1.1), (2.1), and (2.2) we have

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \leqslant 3. \tag{2.4}$$

For the roots of $P \in \mathcal{P}_4$, we define the sets

$$S(\alpha_j) = \big\{x \in \mathbb{R}\colon\ |x - \alpha_j| = \min_{1 \leqslant i \leqslant 4} |x - \alpha_i|\big\}, \quad 1 \leqslant j \leqslant 4.$$

Lemma 1. *Let α_1 be a complex root of an integer polynomial $P \in \mathcal{P}_4$, and let $x \in S(\alpha_1)$. Then*

$$|x - \alpha_1| \leqslant \min_{2 \leqslant j \leqslant 4} \left(2^{4-j} \big|P(x)\big| \big|P'(\alpha_1)\big|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k|\right)^{1/j}$$

for $P'(\alpha_1) \neq 0$.

Lemma 1 is proved in [10].

Lemma 2. *Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(x), T(x) \in \mathcal{P}_k(Q)$, $k \leqslant 4$, have the same vector $\bar{l}$ and have no common roots . Let I denote interval of length $|I| = Q^{-\gamma}$ with $\gamma \in \mathbb{R}_+$. If there exists a real number $\tau > 0$ such that for all $x \in I$,*

$$\max_{x\in I}\bigl(|P(x)|, |T(x)|\bigr) < Q^{-\tau},$$

then

$$\tau + 1 + 2\sum_{j=1}^{k}\max(\tau + 1 - j\gamma,\, 0) < 2k + \delta.$$

Lemma 2 can be proved similarly to Lemma 3 in [5]. In this case, we need to add the summands related to the root α_4.

To prove Theorem 1, we need to consider a generalization of Lemma 2 for the simultaneous approximations of the polynomials on two intervals (see Lemma 3). We consider a new classification of the roots α_i, $1 \leqslant i \leqslant 4$, of $P \in \mathcal{P}'_4(Q)$ with respect to α_1 (as before) and α_3 simultaneously. We obtain

$$\begin{aligned} &|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant |\alpha_1 - \alpha_4|,\\ &|\alpha_3 - \alpha_4| \leqslant |\alpha_3 - \alpha_2| \leqslant |\alpha_3 - \alpha_1|. \end{aligned} \tag{2.5}$$

Let $|\alpha_3 - \alpha_2| = Q^{-\rho_6}$ and define the integer l_6 by $(l_6 - 1)/T < \rho_6 \leqslant l_6/T$. It is not difficult to see that by (1.5)

$$\rho_4 \leqslant \rho_3 \leqslant \rho_2, \qquad \rho_3 \leqslant \rho_6 \leqslant \rho_5, \tag{2.6}$$

where ρ_i, $2 \leqslant i \leqslant 5$, are defined in (2.1)–(2.2). We also define the vector $\bar{l}' = (\bar{l}, l_6)$. Define the class $\mathcal{P}'_{4,\bar{l}'}(Q, v)$ consisting of the polynomials $P \in \mathcal{P}'_4(Q, v)$ corresponding to a vector $\bar{l}'$.

Lemma 3. *Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(t), T(t) \in \mathcal{P}_k(Q)$, $k \leqslant 4$, have the same vector $\bar{l}'$ and have no common roots in the rectangle $I_1 \times I_2$, where $|I_1| = Q^{-\gamma_1}$ and $|I_2| = Q^{-\gamma_2}$ with $\gamma_j \in \mathbb{R}_+$, $j = 1, 2$. Furthermore, let $P(t)$ and $T(t)$ satisfy the system of inequalities*

$$\max_{x\in I_1}\bigl(|P(x)|, |T(x)|\bigr) < Q^{-\tau_1}, \qquad \max_{y\in I_2}\bigl(|P(y)|, |T(y)|\bigr) < Q^{-\tau_2}. \tag{2.7}$$

Then for any $\delta > 0$ and $Q > Q_0(\delta)$, we have the inequality:

$$\tau_1 + \tau_2 + 2 + l_2 + 2l_3 + 3l_4 + l_5 < 2k + \delta. \tag{2.8}$$

The proof of Lemma 3 follows from the new classification (2.5) of the roots of polynomials, using inequalities (2.6) and (2.7), and can be proved similarly to Lemma 2 in [8].

3 Proof of Theorem 1

Assume that estimate (1.6) does not hold, so that

$$\#\mathcal{P}'_4(Q, v) \geqslant Q^{5-3v/2+\epsilon}. \tag{3.1}$$

Consider two intervals $I_1, I_2 \subset \mathbb{R}$ with $|I_1| = Q^{-l_2/T}$ and $|I_2| = Q^{-l_5/T}$. We will say that the polynomial P belongs to $M = I_1 \times I_2$ if $(\alpha_1, \alpha_3) \in M$, where α_1 and α_3 are the roots of P in the ordering (1.5). From (3.1) it follows that there exist rectangles $I_1 \times I_2$ that contain at least

$$\varDelta = Q^{5-3v/2-l_2/T-l_5/T+\epsilon}$$

polynomials $P \in \mathcal{P}'_4(Q,v)$ satisfying (2.7). Fix one of these rectangles, say M. Since $\#\bar{l}' \ll 1$, there exists a vector $\bar{l}'$ satisfying (2.3) such that

$$\#\mathcal{P}'_{4,\bar{l}'}\big(Q,v,M\big) \gg Q^{5-3v/2+\epsilon-l_2/T-l_5/T+\epsilon},$$

where $\mathcal{P}'_{4,\bar{l}'}(Q,v,M)$ denotes the subset of $\mathcal{P}'_{4,\bar{l}'}(Q,v)$ consisting of polynomials P belonging to M. Fix the vector $\bar{l}'$ and set

$$h = 5 - \frac{3v}{2} - \frac{l_2}{T} - \frac{l_5}{T} + \frac{\epsilon}{2}.$$

By (2.4) we have

$$\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + \frac{l_5}{T} \leqslant 3. \tag{3.2}$$

From (3.2) we obtain that $h > 0$ for $v \leqslant 4/3$.

Expand the polynomial $P \in \mathcal{P}'_{4,\bar{l}'}(Q,v,M)$ into its Taylor series in a neighbourhood of α_1 to obtain

$$\begin{aligned} P(x) &= P(\alpha_1) + P'(\alpha_1)(x-\alpha_1) \\ &\quad + \frac{1}{2}P''(\alpha_1)(x-\alpha_1)^2 + \frac{1}{6}P'''(\alpha_1)(x-\alpha_1)^3 + \frac{1}{24}P^{(4)}(\alpha_1)(x-\alpha_1)^4. \end{aligned}$$

Estimating each term gives

$$\begin{aligned} \big|P'(\alpha_1)(x-\alpha_1)\big| &\leqslant |a_4|\cdot|\alpha_1-\alpha_2|\cdot|\alpha_1-\alpha_3|\cdot|\alpha_1-\alpha_4|\cdot|x-\alpha_1| \\ &\leqslant Q^{1-\rho_2-\rho_3-\rho_4-l_2/T} < Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, \\ \big|P''(\alpha_1)(x-\alpha_1)^2\big| &\leqslant 6|a_4|\max\big(|\alpha_1-\alpha_2||\alpha_1-\alpha_3|,\, |\alpha_1-\alpha_2||\alpha_1-\alpha_4|,\, |\alpha_1-\alpha_3||\alpha_1-\alpha_4|\big) \\ &\quad\times|x-\alpha_1|^2 \\ &< 6Q^{1-2l_2/T-l_3/T-l_4/T+2\epsilon_1}, \\ \big|P'''(\alpha_1)(x-\alpha_1)^3\big| &\leqslant 18|a_4|\max\big(|\alpha_1-\alpha_2|,\, |\alpha_1-\alpha_3|,\, |\alpha_1-\alpha_4|\big)\cdot|x-\alpha_1|^3 \\ &< 18Q^{1-3l_2/T-l_4/T+\epsilon_1} \\ \big|P^{(4)}(\alpha_1)(x-\alpha_1)^4\big| &\leqslant 24|a_4||x-\alpha_1|^4 \leqslant 24Q^{1-4l_2/T} \end{aligned}$$

for $x \in I_1$. Thus

$$\big|P(x)\big| \ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, \quad x \in I_1.$$

Also develop the polynomial P as Taylor series on the interval I_2 at the point α_3 and obtain the upper bounds for all terms in the series. Thus we obtain

$$\big|P(y)\big| \ll Q^{1-2l_5/T-l_3/T-l_6/T+3\epsilon_1}, \quad y \in I_2.$$

Further, for Q^h polynomials P, we use the Dirichlet box principle. We will assume that the fractional part of h does not exceed ϵ_1. If the last condition is not satisfied, then we rewrite h as $h = [h] + \{h\}$. As a result, using the number $Q^{[h]}$, we reduce the degree of polynomials, and using the number $Q^{\{h\}}$, we reduce the height of polynomials $R_{j+1}(t) = P_{j+1}(t) - P_1(t)$, $j = 1, 2, \dots$, as in [5]. Therefore the new polynomials R_j satisfy

$$\begin{aligned} \left|R_j(x)\right| &\ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon_1}, \quad x \in I_1, \\ \left|R_j(y)\right| &\ll Q^{1-2l_5/T-l_3/T-l_6/T+3\epsilon_1}, \quad y \in I_2, \end{aligned} \tag{3.3}$$

$$H(R_j) \leqslant Q^{1-\epsilon_1}, \qquad \deg R_j \leqslant 4 - \left(5 - \frac{3v}{2} - \frac{l_2}{T} - \frac{l_5}{T} + \frac{\epsilon}{2} - \epsilon_1\right). \tag{3.4}$$

If there exist two polynomials R_1 and R_2 with no common roots, then Lemma 3 can be applied. The values of τ_1 and τ_2 are found from estimates (3.3) and (3.4). Thus

$$\tau_1 = \frac{-1 + 2l_2/T + l_3/T + l_4/T - 3\epsilon_1}{1 - \epsilon_1} \quad \text{and} \quad \tau_2 = \frac{-1 + 2l_5/T + l_3/T + l_6/T - 3\epsilon_1}{1 - \epsilon_1}.$$

The left-hand side of (2.8) is equal to

$$\frac{3l_2/T + 4l_3/T + 4l_4/T + 3l_5/T + l_6/T - 6\epsilon_1}{1 - \epsilon_1}.$$

This leads to a contradiction in (2.8) for $v \leqslant 1$ and $\delta \leqslant \epsilon - 2\epsilon_1$.

If, among polynomials $R_j(t)$, there exist no two polynomials without common roots, then the polynomials $R_j(t)$ are reducible. It is not difficult to see that $\deg R_j \leqslant 2$ for $v \leqslant 1$. Thus the polynomials $R_j(t)$ are decomposed into the product of two linear polynomials. Again, as, for example, in [4], we will use Lemma 2 to get a contradiction.

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