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Abstract

The integrability of a complex generalisation of the 'elegant' system, proposed by D. Fairlie [1] and its relation to the Nahm equation and the Manakov top is discussed.

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1 A complex nonlinear system with symmetry

The Euler top is probably the best known example of a nonlinear integrable system. Generalizations of this system are under intense investigation for many years. In general, the (generalized) top equations contain quadratic nonlinearity. A problem of the form $dY/dt = Y^2$, where Y is an $n \times n$ symmetric matrix with vanishing diagonal elements is analyzed in the work of Fairlie [1] and leads to the following n-dimensional nonlinear system:

$$\frac{du_j}{dt} = \prod_{k \neq j}^{n} u_k, \qquad j = 1, \ldots, n. \tag{1}$$

In [1] this system is explicitly integrated. The system (1) for $n = 4$ appears also in the classification of a class of double-elliptic systems [2]. Other higher dimensional integrable tops have been recently found in [3, 4, 5].

The problem of generalizations of the standard Hamiltonian dynamics to complex dynamical variables often contains many interesting aspects, e.g. [6, 7]. In particular, an integrable complexification of (1) for $n = 3$ and the related Halphen system are considered in [8]. A natural generalization of these systems is the following system in n complex dimensions:

$$\frac{d\bar{u}_j}{dt} = \prod_{k \neq j}^{n} u_k, \qquad j = 1, \ldots, n, \tag{2}$$

where the bar stands for complex conjugation. The system (2) is also integrable. Indeed, it possesses the following independent algebraic constants of the motion:

$$H \equiv iI_1 = \prod_{k=1}^{n} \bar{u}_k - \prod_{k=1}^{n} u_k, \tag{3}$$

$$I_k = |u_1|^2 - |u_k|^2, \ k = 2, \ldots n. \tag{4}$$

There are n real integrals, I_1, ..., I_n, but they are sufficient for the integration of the system.

The system (2) can be realised as a reduction of the following Hamiltonian system with $\mathbf{Z_2}$ symmetry:

$$H(p, q) \equiv \prod_{k=1}^{n} p_k - \prod_{k=1}^{n} q_k,$$

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} = \prod_{k \neq j}^{n} q_k \tag{5}$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} = \prod_{k \neq j}^{n} p_k \tag{6}$$

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Considering $q_k \equiv u_k$ as coordinate variables and $p_k \equiv \bar{u}_k$ as momentum variables, the system (2) admits a Hamiltonian formulation with a Hamiltonian H given by (3). The equations of motion (5) and (6) then coincide with (2). Note that in the real case (1) $H \equiv 0$.

Now we are able to integrate explicitly the system (2). Let $u_k = e^{i\phi_k} r_k$ where r_k, ϕ_k are real functions and $\phi = \phi_1 + \phi_2 + ... + \phi_n$. Then

$$u_k d\bar{u}_k + \bar{u}_k du_k = rdt$$

where $r = \prod_{k=1}^{n} u_k + \prod_{k=1}^{n} \bar{u}_k$ is a real function. Let

$$rdt = d\tau. \tag{7}$$

Then

$$dr_k^2 = rdt \tag{8}$$

and

$$r_k^2 = \tau + c_k \tag{9}$$

for some real constants c_k. It is obvious that

$$I_1 = -2 \left(\prod_{k=1}^{n} r_k \right) \sin\phi \tag{10}$$

$$r = 2 \left(\prod_{k=1}^{n} r_k \right) \cos\phi \tag{11}$$

and therefore from (7), (9), (10) and (11) we obtain

$$\left(\frac{d\tau}{dt} \right)^2 = 4 \left(\prod_{k=1}^{n} r_k \right)^2 - I_1^2 = 4 \left(\prod_{k=1}^{n} (\tau + c_k) \right) - I_1^2. \tag{12}$$

i.e. formally

$$t = \int \frac{d\tau}{\sqrt{4 \left(\prod_{k=1}^{n} (\tau + c_k) \right) - I_1^2}} \tag{13}$$

For a specific choice of the constants c_k and I_1, τ can be expressed via t through a hyperelliptic function. Furthermore

$$\frac{d}{dt} \left(\frac{u_k}{\bar{u}_k} \right) = i \frac{I_1}{\bar{u}_k^2},$$

i.e.

$$\frac{d\phi_k}{dt} = \frac{I_1}{2r_k^2}, \tag{14}$$

and from (9) and (12) we have formally

$$\phi_k(\tau) = \frac{I_1}{2} \int \frac{d\tau}{(\tau + c_k) \sqrt{4 \left(\prod_{k=1}^{n} (\tau + c_k) \right) - I_1^2}}. \tag{15}$$

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The integrals (13) and (15) can be expressed in terms of hyperellyptic functions.

In order to better understand the underlying symmetry of the system let us rewrite it in terms of real variables as follows. The summation over k of (14) gives

$$\frac{d\phi}{dt} = \frac{I_1}{2} \sum_{k=1}^{n} \frac{1}{r_k^2},\tag{16}$$

which, due to (10) is equivalent to

$$\frac{d\phi}{dt} = -\sin\phi \left(\prod_{k=1}^{n} r_k\right) \sum_{k=1}^{n} \frac{1}{r_k^2}.\tag{17}$$

From (8) and (11) we obtain the following set of equations:

$$\frac{dr_k}{dt} = \cos\phi \prod_{i\neq k}^{n} r_i.\tag{18}$$

Now it is clear that (2) is equivalent to the $n+1$ dimensional real system (17), (18). The reduction to the 'real' version (1) can be achieved as follows. Let us choose for ϕ the initial value $\phi(0) = 2K\pi$ for some arbitrary integer K. Then the integral $I_1 = 0$, according to (10), and consequently $\frac{d\phi}{dt} = 0$, according to (16). Thus the solution in this case is $\phi(t) = 2K\pi$ and (18) simply coincides with (1).

2 Example: $su(3)$ Nahm top $(n = 3)$

It is well known that the one-dimensional reductions of the Self-dual Yang Mills (SDYM) equations in four dimensions yield various generalizations of some classical systems as the Euler-Arnold-Manakov and Nahm equations [9, 10]. In [8] the SDYM equations in seven dimensions have been investigated, in the case when their invariance group is the $\mathbf{G}_2$ subgroup of the rotation group $SO(7)$. One of the six-dimensional reductions of these equations leads to (2) with $n = 3$:

$$\frac{d\bar{u}_1}{dt} = u_2 u_3 \quad \text{and cyclic permutations,}\tag{19}$$

In [8] the question about the existence of a Lax pair for the system (19) is posed, but the Lax pair is not found. Although this system is in principle known for some time, since it is an interesting one here we pay some more attention to it.

In physical terms, the system (19) describes membrane instantons in three complex dimensions [11, 12]. It can also be seen as a "stationary

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point" for the 3–wave equations with a "blow-up" instability (i.e. $\frac{\partial u_i}{\partial x} = 0$):

$$\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} = i\epsilon \bar{u}_2 \bar{u}_3,$$

where v_k and ϵ are real constants [13, 14, 15]. The equations (19) possess a complete symmetry and in this respect are similar to the Kaup equations, where the derivatives are over the three space variables [16].

Firstly, one can show that (19) is related to the well known Nahm equation. Using the matrices

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -u_1 \\ 0 & \bar{u}_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & \bar{u}_2 \\ 0 & 0 & 0 \\ -u_2 & 0 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & -u_3 & 0 \\ \bar{u}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{20}$$

the system (19) can be written in the usual form for the Nahm equation

$$\frac{dT_i}{dt} = \frac{1}{2}\varepsilon_{ijk}[T_j, T_k]. \tag{21}$$

Equation (21) was introduced initially in relation to $SU(2)$ monopoles on $\mathbf{R}^3$ [10]. However, the Nahm matrices, T_i in (21) can take values in various simple Lie algebras (not necessarily $su(2)$). This is explained in details in [17]. Since the matrices (20) are anti-Hermitian, i.e. take values in $su(3)$, we will call (19) an $su(3)$ Nahm top. Now it is straightforward to represent (21) in the Lax form $\frac{dL}{dt} = [L, M]$ (e.g. see [18, 19]) with

$$L(\zeta) = T_1 + iT_2 - 2\zeta iT_3 + \zeta^2(T_1 - iT_2) \tag{22}$$
$$M(\zeta) = -iT_3 + \zeta(T_1 - iT_2). \tag{23}$$

The computation of $\det(L(\zeta) - \lambda \mathbf{1})$ immediately gives the integrals of motion (3), (4). For another form of the Lax pair for the Nahm equation see also [12].

The system (19) can also be seen as a Manakov top [20] with a complex inertia tensor. Indeed, consider the following Lax pair:

$$L(\zeta) = [J^2, Q] - \zeta J^2 \tag{24}$$
$$M(\zeta) = [J, Q] - \zeta J, \tag{25}$$

where the cubic root of unity, $\omega = e^{2\pi i/3}$ is used to define the inertia tensor

$$J = \frac{1}{\omega(\omega - 1)}J_0, \qquad J_0 = \text{diag}(\omega, \omega^2, 1); \tag{26}$$

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the matrix Q is defined as $[J_0, Q] = -(T_1 + T_2 + T_3)$ where where T_k are the matrices (20), i.e. $Q = -\mathrm{ad}_{J_0}^{-1}(T_1 + T_2 + T_3)$. Explicitly, the operators L (24), and M (25) are (the irrelevant overall factor $[\omega(\omega - 1)]^{-2}$ in L is omitted)

$$L(\zeta) = \sum_{k=1}^{3} \omega^k T_k - \zeta J_0^2, \tag{27}$$

$$M(\zeta) = -\frac{1}{\omega(\omega - 1)} \Big(\sum_{k=1}^{3} T_k + \zeta J_0 \Big). \tag{28}$$

The characteristic polynomial of $L(\zeta)$, (27) is

$$\det L(\zeta) = -\zeta^3 - \zeta \Big(\omega |u_1|^2 + \omega^2 |u_2|^2 + |u_3|^2 \Big) + (\bar{u}_1 \bar{u}_2 \bar{u}_3 - u_1 u_2 u_3) \tag{29}$$

Its coefficients can be split into real and imaginary parts to give the integrals of motion (3), (4).

The representation of the system (19) as a Manakov top allows immediately the classical r-matrix to be written [21, 22]. Defining a Poisson bracket

$$\{A \overset{\otimes}{,} B\} \equiv \sum_{k=1}^{3} \Big(\frac{\partial A}{\partial u_k} \otimes \frac{\partial B}{\partial \bar{u}_k} - \frac{\partial A}{\partial \bar{u}_k} \otimes \frac{\partial B}{\partial u_k} \Big), \tag{30}$$

it is straightforward to check, that

$$\{L(\zeta) \overset{\otimes}{,} L(\mu)\} = [r(\zeta - \mu), L(\zeta) \otimes \mathbf{1} + \mathbf{1} \otimes L(\mu)], \tag{31}$$

where L is given by (27), and the classical r matrix is

$$r(\zeta) = -\frac{P}{\omega(\omega - 1)\zeta}. \tag{32}$$

Here P is the permutation matrix in $\mathrm{C}^3 \otimes \mathrm{C}^3$, i.e.

$$P(x \otimes y) = y \otimes x, \qquad x, y \in \mathrm{C}^3. \tag{33}$$

3   Example: $so(4)$ Nahm top ($n = 4$)

As a second example we consider the system (1) with $n = 4$. It is known that with a change of variables [2] it can be represented as two $so(3)$ Nahm tops. Since $so(4) = so(3) \oplus so(3)$ it is natural to expect that (1) with $n = 4$ is an $so(4)$ Nahm top. Indeed, using the $so(4)$ parametrization given in [23] we can define the following $so(4)$ Nahm matrices:

$$T_1 \;=\; \begin{pmatrix} 0 & 0 & 0 & u_1 u_4 \\ 0 & 0 & -u_2 u_3 & 0 \\ 0 & u_2 u_3 & 0 & 0 \\ -u_1 u_4 & 0 & 0 & 0 \end{pmatrix},$$

$$T_2 \;=\; \begin{pmatrix} 0 & 0 & u_1 u_3 & 0 \\ 0 & 0 & 0 & u_2 u_4 \\ -u_1 u_3 & 0 & 0 & 0 \\ 0 & -u_2 u_4 & 0 & 0 \end{pmatrix},$$

$$T_3 \;=\; \begin{pmatrix} 0 & -u_3 u_4 & 0 & 0 \\ u_3 u_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 u_2 \\ 0 & 0 & -u_1 u_2 & 0 \end{pmatrix}. \tag{34}$$

With the help of (34), one can write (1) explicitly in the canonical form (21). Now the Lax pair (22), (23) follows automatically. We also note that the construction

$$L \;=\; \sum_{k=1}^{3} \omega^k T_k, \tag{35}$$

$$M \;=\; -\frac{1}{\omega(\omega-1)} \sum_{k=1}^{3} T_k \tag{36}$$

also provides a Lax pair for this system, indeed, without a spectral parameter.

The Lie group interpretation, the Lax pairs etc. for the system (2) with $n \geq 4$ is a very interesting problem, which deserves a further investigation.

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