Spectral Theory of Semibounded Sturm-Liouville Operators with Local Interactions on a Discrete Set

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One-Dimensional Schrödinger Operator with δ-Interactions*

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ABSTRACT. The one-dimensional Schrödinger operator $H_{X,\alpha}$ with δ-interactions on a discrete set is studied in the framework of the extension theory. Applying the technique of boundary triplets and the corresponding Weyl functions, we establish a connection of these operators with a certain class of Jacobi matrices. The discovered connection enables us to obtain conditions for the self-adjointness, lower semiboundedness, discreteness of the spectrum, and discreteness of the negative part of the spectrum of the operator $H_{X,\alpha}$.

KEY WORDS: Schrödinger operator, point interactions, self-adjointness, lower semiboundedness, discreteness.

1. Introduction. Let $\mathcal{J} = [0,b)$, $0 < b \leq +\infty$, and let $X = \{x_n\}_{n=1}^{\infty} \subset \mathcal{J}$ be a strictly increasing sequence such that $\lim_{n \to \infty} x_n = b$. Let also $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

In $\mathcal{J} = L^2(\mathcal{J})$, with the differential expression

$$\ell_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x-x_n)$$

one associates the symmetric operator (see [1])

$$H_{X,\alpha}^0 := -\frac{d^2}{dx^2}, \quad D(H_{X,\alpha}^0) = \{f \in W_{\text{comp}}^2(\mathcal{J} \setminus X) : f'(0) = 0, f(x_n+) = f(x_n-), f'(x_n+) - f'(x_n-) = \alpha_n f(x_n)\}.$$  \hspace{1cm} (2)

Its closure, denoted by $H_{X,\alpha}$, is interpreted as the Hamiltonian with δ-interactions of strength $\alpha_n$ at the centers $x_n$.

The spectral properties of the operator $H_{X,\alpha}$ have been widely studied in the case $d_\alpha := \inf_{n \in \mathbb{N}} d_n > 0$. (Here $d_n := x_n - x_{n-1}$ and $x_0 := 0$.) It is known that in this case the operator $H_{X,\alpha}$ is always self-adjoint [6]. Moreover, it is lower semibounded if and only if there is the sequence $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ (see [3]). Also the condition $\lim_{n \to \infty} d_n = 0$ is necessary for the discreteness of the spectrum.

The cases $d_\alpha = 0$ and $d_\alpha > 0$ are different in essence. It was shown in [4] that for $d_\alpha = 0$ the deficiency indices $n_+(H_{X,\alpha})$ and $n_-(H_{X,\alpha})$ of the operator $H_{X,\alpha}$ coincide and are not greater than 1, and the Weyl alternative was established. In particular, the latter implies that for $n_\pm(H_{X,\alpha}) = 1$ the spectra of the self-adjoint extensions of the operator $H_{X,\alpha}$ are discrete. Moreover, Shubin Christ and Stolz [18] showed that $n_\mp(H_{X,\alpha}) = 1$ if $\mathcal{J} = \mathbb{R}_+$, $d_\alpha = 1/n$, and $\alpha_n = -2n - 1$, $n \in \mathbb{N}$.

The main aim of this note is the spectral analysis of the Hamiltonian $H_{X,\alpha}$ in the case of $d_\alpha = 0$. We study the operator $H_{X,\alpha}$ in the framework of extension theory of symmetric operators using the concept of boundary triplets and the corresponding Weyl functions (see [7] and [5]). For $d_\alpha > 0$, this approach was first applied by Kochubei [12]. Let us mention that the main difficulty in this approach is the construction of an adequate boundary triplet.

In this note, we present the corresponding construction for $d_\alpha = 0$ and show that the properties like self-adjointness, lower semiboundedness, and discreteness of the spectrum of the operator $H_{X,\alpha}$ correlate with the corresponding spectral properties of the Jacobi matrix $B_{X,\alpha}$ of the form (6) (see

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below). This enables us to obtain conditions for the operator $H_{X,\alpha}$ to be self-adjoint and lower semibounded and to have discrete spectrum. It turns out that these conditions significantly depend not only on $\alpha$ but also on $X$ and are completely different from the corresponding conditions for $d_*>0$. In the forthcoming paper, we exploit the discovered connection in the opposite direction. Namely, using the quadratic form approach we obtain new results on spectral properties of both lower semibounded Hamiltonians $H_{X,\alpha}$ and the corresponding matrices $B_{X,\alpha}$ of the form (6).

Let us also mention that there is a series of papers (see [17]) developing a different approach to the study of Sturm–Liouville operators with distributional potentials $q \in W^{-1}_{2,\text{loc}}(\mathcal{S})$.

A detailed treatment of the results discussed in this note is given in [10].

2. A boundary triplet. In the remaining part of the paper, we assume without loss of generality that $d^* := \sup_{n \in \mathbb{N}} d_n < \infty$. Consider the operator

$$H_{\text{min}} := -\frac{d^2}{dx^2}, \quad D(H_{\text{min}}) = W^2_0(\mathcal{S} \setminus X).$$

(3)

The operator $H_{\text{min}}$ is symmetric, $n_\pm(H_{\text{min}}) = \infty$, and $H^*_{\text{min}}$ is defined by the same differential expression $-d^2/dx^2$ on the domain $D(H^*_{\text{min}}) = W^2_0(\mathcal{S} \setminus X)$. It is clear that $H_{\text{min}} = \bigoplus_{n \in \mathbb{N}} H_n$, where $H_n := -d^2/dx^2$, $D(H_n) = W^2_0([x_{n-1},x_n])$. Define mappings $\Gamma^{(n)}_0, \Gamma^{(n)}_1: D(H_n^*) = W^2_0([x_{n-1},x_n]) \to \mathbb{C}^2$ by

$$\Gamma^{(n)}_0 f := \begin{pmatrix} d^{1/2}_n f(x_{n-1}+) \\ -d^{1/2}_n f(x_{n-1}) \end{pmatrix}, \quad \Gamma^{(n)}_1 f := \begin{pmatrix} f'(x_{n-1}+) + f(x_{n-1}) - f(x_{n-1}) \\ f'(x_{n-1}+) + f(x_{n-1}) - f(x_{n-1}) \end{pmatrix}. \quad (4)$$

The collection $\Pi_n = \{\mathbb{C}^2, \Gamma^{(n)}_0, \Gamma^{(n)}_1\}$ forms a boundary triplet for $H_n^*$ in the sense of [7]. Moreover, we have the following

**Theorem 1.** Let $\tilde{\Gamma}^{(n)}_j$ be defined by (4), let $\Gamma_j := \bigoplus_{n \in \mathbb{N}} \Gamma^{(n)}_j$, $j \in \{0,1\}$, and let $\mathcal{H} = l_2(\mathbb{N}, \mathbb{C}^2)$. Then the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \bigoplus_{n \in \mathbb{N}} \Pi_n$ is a boundary triplet for the operator $H^*_{\text{min}}$; that is, the mapping $\Gamma := \text{col}(\Gamma_0, \Gamma_1): D(H^*_{\text{min}}) = W^2_0(\mathcal{S} \setminus X) \to \mathcal{H} \oplus \mathcal{H}$ is surjective, and the Green formula

$$\left(H^*_{\text{min}} f, g\right)_{\mathcal{H}} - \left(f, H^*_{\text{min}} g\right)_{\mathcal{H}} = \left(\Gamma_1 f, \Gamma_0 g\right)_{\mathcal{H}} - \left(\Gamma_0 f, \Gamma_1 g\right)_{\mathcal{H}} \quad (5)$$

holds for all $f, g \in D(H^*_{\text{min}})$. A boundary triplet $\Pi$ for $H^*_{\text{min}}$ may have a simpler form if $d_* > 0$. Namely (see [12]), $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$, where $\Pi_n = \{\mathbb{C}^2, \tilde{\Gamma}^{(n)}_0, \tilde{\Gamma}^{(n)}_1\}$, $\tilde{\Gamma}^{(n)}_0 f = \text{col}(f(x_{n-1}+), -f(x_{n-1})), \tilde{\Gamma}^{(n)}_1 f = \text{col}(f'(x_{n-1}+), f'(x_{n-1}))$, and $\tilde{\Gamma}_j := \bigoplus_{n \in \mathbb{N}} \tilde{\Gamma}^{(n)}_j$, $j \in \{0,1\}$. However, this triplet is not a boundary triplet for $H^*_{\text{min}}$ if $d_* = 0$. Namely, $D(\tilde{\Gamma}_j) \subsetneq D(H^*_{\text{min}}), j \in \{0,1\}$, and the mapping $\tilde{\Gamma} := \text{col}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is not surjective.

It is shown in the recent paper [13] that for an arbitrary sequence of symmetric operators $\{S_n\}_{n \in \mathbb{N}}$ satisfying $n_+(S_n) = n_-(S_n)$ there exists a sequence of boundary triplets $\Pi_n$ for $S^*_n$ such that $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n$ is a boundary triplet for the operator $S^* = \bigoplus_{n \in \mathbb{N}} S^*_n$. Furthermore, $\{\Pi_n\}_{n \in \mathbb{N}}$ can be constructed as a special regularization of an arbitrary sequence $\{\tilde{\Pi}_n\}_{n \in \mathbb{N}}$ of boundary triplets for $S^*_n$. Developing the construction in [13, §5], we propose a general regularization procedure for $\tilde{\Pi}_n$, which significantly exploits the Weyl functions $\tilde{M}_n(\cdot)$ corresponding to the triplets $\tilde{\Pi}_n$, $n \in \mathbb{N}$. In the case under consideration, this construction gives Theorem 1.

3. Connection with Jacobi matrices. Consider the semi-infinite Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} \frac{1}{r_1^2} (\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}) & \frac{1}{r_1 r_2 d_2} & \frac{1}{r_1 r_2 d_3} & 0 & \cdots \\ \frac{1}{r_1 r_2 d_2} & \frac{1}{r_1 r_2 d_3} & \frac{1}{r_2 r_3 d_3} & \cdots & \ddots \\ \frac{1}{r_1 r_2 d_3} & \frac{1}{r_2 r_3 d_3} & \frac{1}{r_3 r_4 d_4} & \cdots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}, \quad (6)$$

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where \( r_n := \sqrt{d_n + d_{n+1}} \) and \( d_n = x_n - x_{n-1}, \) \( n \in \mathbb{N}. \) The matrix \( B_{X,\alpha} \) induces the minimal symmetric operator in \( l^2(\mathbb{N}) \), also denoted by \( B_{X,\alpha} \) (see [2] for details).

**Lemma 1.** Let \( \Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be the boundary triplet constructed in Theorem 1 for \( H_{\min}^* \), and let \( \Theta_{X,\alpha} \) be the following linear relation in \( \mathcal{H} \):

\[
\Theta_{X,\alpha} := \Gamma D(H_{X,\alpha}) := \{(\Gamma_0 f, \Gamma_1 f) : f \in D(H_{X,\alpha})\}.
\]

Then \( \Theta_{X,\alpha} = \Theta^{\text{op}} \oplus \Theta^{\infty} \), where \( \Theta^{\text{op}} \) and \( \Theta^{\infty} = \{0, f\} : \{0, f\} \in \Theta_{X,\alpha}\) are the operator and pure parts of \( \Theta_{X,\alpha}. \) Moreover, \( \Theta^{\text{op}} \) is unitarily equivalent to the minimal operator \( B_{X,\alpha}. \)

Theorem 1, Lemma 1, and some results in [5] and [7] provide the following connection between the spectral properties of the operators \( H_{X,\alpha} \) and \( B_{X,\alpha}. \)

**Theorem 2.** Let \( \sup_{n \in \mathbb{N}} d_n < \infty. \) Then the following assertions hold.

(i) \( n_{\pm}(H_{X,\alpha}) = n_{\pm}(B_{X,\alpha}). \)

(ii) The operator \( H_{X,\alpha} \) is lower semibounded (nonnegative) if and only if so is \( B_{X,\alpha}. \)

(iii) Suppose that \( H_{X,\alpha} = B_{X,\alpha}. \) Then the spectrum \( \sigma(H_{X,\alpha}) \) is discrete if and only if \( \lim_{n \to \infty} d_n = 0 \) and the spectrum \( \sigma(B_{X,\alpha}) \) is discrete.

(iv) If \( H_{X,\alpha} = B_{X,\alpha}^* \), then the negative part of the spectrum \( \sigma(H_{X,\alpha}) \) is discrete (finite) if and only if so is the negative part of \( \sigma(B_{X,\alpha}). \) In particular, \( \sigma_{c}(H_{X,\alpha}) \subseteq \mathbb{R}_+ \) if and only if \( \sigma_{c}(B_{X,\alpha}) \subseteq \mathbb{R}_+. \)

**Remark 1.** An analog of Theorem 2(i) for \( d_s > 0 \) was given in [1, Sec. III.2.1, Theorem 2.1.5]. However, the Jacobi matrices in [1] were completely different from the matrices \( B_{X,\alpha} \) defined by (6), and for \( d_s = 0 \) the connection between the Hamiltonians \( H_{X,\alpha} \) and the Jacobi matrices remained unclear.

Another approach to establishing the connection between the operators \( H_{X,\alpha} \) and \( B_{X,\alpha} \) has been brought to our attention by the referee. Namely, let us identify a function \( f \) in the space \( \mathcal{L} \) of functions having compact support and linear on intervals \([x_{k-1}, x_k]\) with the sequence \( y := \{y_k\}, y_k := f(x_k). \) Then the formulas \( (H_{X,\alpha} f, f) = \sum (d_{k-1}^1 y_k - y_{k-1}^1)^2 + \alpha_k |y_k|^2 \) and \( (f, f) = \sum d_k (y_{k-1}^2 + y_k^2 + y_{k-1} y_k)/3 \) give rise to a new Hilbert space \( \mathcal{H}'' \) and a form \( \mathcal{K}' \) on it. Clearly, \( (f, f)' = \sum d_k (y_{k-1}^2 + y_k^2) = \sum (d_k + d_{k+1}) |y_k|^2 \) defines an equivalent norm on \( \mathcal{H}' \). Therefore, \( U : f \to \bar{y}, \) where \( \bar{y}_k = y_k \sqrt{d_k + d_{k+1}}, \) is an isometry from \( \mathcal{L} \) into \( l_2(\mathbb{N}). \)

This approach allows one to prove Theorem 2(ii).

**Remark 2.** The mapping \( T^2 \) in [14, Theorem 1] coincides with the mapping \( \bar{\Gamma}_0 = \bigoplus_{n \in \mathbb{N}} \bar{\Gamma}_0(n). \) Therefore, the triplet constructed in [14] is not a boundary triplet for \( H_{\min}^* \) if \( d_s = 0, \) since \( D(\bar{\Gamma}_0) \not\subseteq D(H_{\min}^* \) in this case. Because of this, the parametrization of extensions of the operator \( H_{X,\alpha} \) in [14] and [15], as well as the conclusions on their spectral properties, is not correct.

4. **Self-adjointness.** By applying Carleman’s criterion [2, Theorem VII.1.5] to the matrix \( B_{X,\alpha} \) and by using Theorem 2(ii), we obtain

**Proposition 1.** If \( \sum_{n \in \mathbb{N}} d_n^2 = \infty, \) then the operator \( H_{X,\alpha} \) is self-adjoint for any real sequence \( \alpha = \{\alpha_n\}_{n \in \mathbb{N}}. \)

Note that a similar condition occurs in a paper by R. S. Ismagilov (see [8, Theorem 1]), where operators with continuous potentials are studied. Also, Proposition 1 can be deduced from [16, Theorem 2].

Proposition 1 is sharp. Namely, the following holds.

**Proposition 2.** Let \( \sum_{n \in \mathbb{N}} d_n^2 < \infty \) and \( d_{n-1} d_{n+1} \geq d_n^2, \) \( n \in \mathbb{N}. \) If

\[
\sum_{n=1}^{\infty} d_{n+1} |\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}}| < \infty,
\]

then the operator \( H_{X,\alpha} \) is symmetric with \( n_{\pm}(H_{X,\alpha}) = 1. \)
Proposition 2 is inspired by the result of Shubin and Stolz [18, p. 496], and its proof is based on the recent improvement by Kostyuchenko and Mirzoev [11, Theorem 1] of the well-known Berezanskii condition [2, Theorem VII.1.5].

For \( \{d_n\} \in I_2 \), the question concerning the self-adjointness of the operator \( H_{X,\alpha} \) is quite subtle. For instance, the following holds.

\[ \text{Proposition 3. Let } \mathcal{F} = \mathbb{R}_+ \text{ and } d_n = n^{-1}, \; n \in \mathbb{N}. \text{ Then } \]
\[ (i) \; n_+(H_{X,\alpha}) = 0 \text{ if } \sum_{n=1}^{\infty} |\alpha_n|n^{-3} = \infty. \]
\[ (ii) \; n_+(H_{X,\alpha}) = 0 \text{ if } \alpha_n \leq -2(2n + 1) + O(n^{-1}). \]
\[ (iii) \; n_-(H_{X,\alpha}) = 0 \text{ if } \alpha_n \geq -Cn^{-1}, \; n \in \mathbb{N}, \; C \equiv \text{const} > 0. \]
\[ (iv) \; n_-(H_{X,\alpha}) = 1 \text{ if } \alpha_n = -2n - 1 + O(n^{-\varepsilon}) \text{ with some } \varepsilon > 0. \]
\[ (v) \; n_+(H_{X,\alpha}) = 1 \text{ if } \alpha_n = -A(2n + 1) + O(n^{-1}), \; A \in (0, 2). \]

5. Lower semiboundedness. By (6) and Theorem 2(ii), we obtain

\[ \text{Proposition 4. The operator } H_{X,\alpha} \text{ is lower semibounded if } \]
\[ \inf_{n \in \mathbb{N}} \alpha_n(d_{n-1} + d_n)^{-1} > -\infty. \] (8)

A stronger assertion is given in [3, Theorem 3]. If \( d_* > 0 \), then condition (8) is equivalent to \( \inf_{n \in \mathbb{N}} \alpha_n > -\infty \) and is also necessary. It also follows from [3] that \( H_{X,\alpha} \) is not lower semibounded if \( d_n = n^{-1/2} \) and \( \alpha_n = -n^{-\varepsilon}, \; \varepsilon \in (0, 1/2) \); however, \( \lim_{n \to \infty} \alpha_n = 0 \).

6. Discreteness of spectrum. Applying Chihara’s condition for the discreteness of spectra of Jacobi matrices (see [19]), we obtain

\[ \text{Proposition 5. Let } X = \{x_n\}_{n=1}^{\infty} \text{ and } \alpha = \{\alpha_n\}_{n=1}^{\infty} \text{ be such that } H_{X,\alpha} = H_{X,\alpha}^*, \lim_{n \to \infty} d_n = 0, \]
\[ \text{and } \lim_{n \to \infty} d_n/d_{n+1} = 1. \text{ If } \]
\[ \lim_{n \to \infty} |\alpha_n|d_n^{-1} = \infty \text{ and } \lim_{n \to \infty} (\alpha_n d_n)^{-1} > -1/4, \] (9)

then the spectrum of the operator \( H_{X,\alpha} \) is discrete.

The second condition in (9) is sharp. (Under additional mild assumptions on \( d_n \) and \( \alpha_n \), the spectrum of the matrix \( B_{X,\alpha} \) is absolutely continuous whenever \( \lim_{n \to \infty} (\alpha_n d_n)^{-1} < -1/4 \) [19, Theorem 2.2].) Proposition 5 also enables us to construct lower unbounded operators \( H_{X,\alpha} \) with discrete spectrum. (Another approach can be derived from [9, §2].) Let us illustrate this by considering the following example.

\[ \text{Example. Let } d_n = n^{-1/2} \text{ and } \alpha_n = -cn^{-1/2}, \; c \in \mathbb{R}_+ \setminus \{4\}. \text{ Then } H_{X,\alpha} \text{ is self-adjoint and lower unbounded. The spectrum } \sigma(H_{X,\alpha}) \text{ is discrete precisely for } c > 4. \]
\[ \text{If } c \in (0, 4), \text{ then the spectrum } \sigma(B_{X,\alpha}) \text{ is absolutely continuous, } \sigma_{ac}(B_{X,\alpha}) = \mathbb{R} \text{ (see [19])}, \text{ and hence, by Theorem 2(iv), the negative spectrum of } H_{X,\alpha} \text{ is not discrete.} \]

By setting \( d_n = n^{-1/2} \) and \( \alpha_n = n^{-\varepsilon}, \; \varepsilon \in (0, 1/2) \), we obtain the operator \( H_{X,\alpha} \) with discrete spectrum for the case in which \( \lim_{n \to \infty} \alpha_n = 0 \).

References


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