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Conformal and geometric properties of the Camassa-Holm hierarchy

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Abstract

Integrable equations with second order Lax pair like KdV and Camassa-Holm (CH) exhibit interesting conformal properties and can be written in terms of the so-called conformal invariants (Schwarz form). These properties for the CH hierarchy are discussed in this contribution.

The squared eigenfunctions of the spectral problem, associated to the Camassa-Holm equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform (IST) for the Camassa-Holm hierarchy as a Generalised Fourier Transform (GFT). Using GFT we describe explicitly some members of the CH hierarchy, including integrable deformations for the CH equation. Also we show that solutions of some 2+1-dimensional generalizations of CH can be constructed via the IST for the CH hierarchy.

MSC: 37K10, 37K15, 37K30

Key Words: Schwarz derivative, conformal invariants, Lax pair, Virasoro algebra, inverse scattering, solitons.

1 Introduction

Integrable equations exhibit many extraordinary features, like infinitely many conservation laws, multi- Hamiltonian structures, soliton solutions etc. Many integrable equations in 1+1 dimensions like KdV, MKdV, Harry-Dym, Boussinesq equations possess interesting conformal properties as well [18, 19, 45, 42, 17, 39]. These properties originate from the associated spectral problem, which in most of the cases is related to a second order differential operator.

The Camassa-Holm (CH) [4] equation, which became famous as a model in water-wave theory [33, 34, 20, 21], together with its complete integrability

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[4, 23, 6, 16, 11, 9, 10] exhibits the same type of conformal properties as well [31].

The origin of these properties can be understood noticing that for a second order Lax operator $L = \partial^2 + f(x)$, the Poisson structure is generated by the operator [18]

$$L_+^{3/2} = \partial^3 + \frac{3}{4}(f\partial + \partial \circ f). \tag{1}$$

The CH equation

$$m_t + \frac{c}{12}u_{xxx} + 2mu_x + m_x u = 0, \qquad m = u - u_{xx} \tag{2}$$

can be written in Hamiltonian form

$$m_t = \{m, H_1\}, \tag{3}$$

where

$$\{F, G\} \equiv -\int \frac{\delta F}{\delta m}\Big(\frac{c}{12}\partial^3 + m\partial + \partial \circ m\Big)\frac{\delta G}{\delta m}dx \tag{4}$$

$$H_1 = \frac{1}{2}\int mu dx. \tag{5}$$

The relation to the Virasoro algebra can be seen as follows [18]. Suppose for simplicity that m is 2π periodic in x, i.e.

$$m(x) = \sum_{-\infty}^{\infty} L_n e^{inx} + \frac{c}{24}, \tag{6}$$

(the reality of m can be achieved by $L_{-n} = \bar{L}_n$) and let us modify slightly (4) by a constant multiplier,

$$\{F, G\} \equiv -2\pi i \int_0^{2\pi} \frac{\delta F}{\delta m}\Big(\frac{c}{12}\partial^3 + m\partial + \partial \circ m\Big)\frac{\delta G}{\delta m}dx. \tag{7}$$

Then the Fourier coefficients L_n close a classical Virasoro algebra of central charge c with respect to the Poisson bracket (7):

$$\{L_n, L_m\} = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{8}$$

Further, we use the following form of the CH equation,

$$m_t + 2\omega u_x + 2mu_x + m_x u = 0, \qquad m = u - u_{xx}, \tag{9}$$

which can be obtained from (2) via $u \to u + \frac{c}{12}$, and where apparently $\omega = c/8$.

Let us introduce the so-called independent conformal invariants of the function $\phi = \phi(x, t)$:

$$\begin{aligned} p_1 &= \frac{\phi_t}{\phi_x} \\ p_2 &= \{\phi; x\} \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2} \end{aligned} \tag{10}$$

Here $\{\phi; x\}$ denotes the Schwarz derivative. A quantity is called conformally invariant if it is invariant under the Möbius transformation

$$\phi \quad \to \quad \frac{\alpha\phi + \beta}{\gamma\phi + \delta}, \qquad \alpha\delta \neq \beta\gamma. \tag{11}$$

For example, the KdV equation

$$u_t + au_{xxx} + 3uu_x = 0 \tag{12}$$

(a is a constant) can be written in a Schwarzian form, i.e. in terms of the conformal invariants (10) as $p_1 + ap_2 = 0$ or

$$\frac{\phi_t}{\phi_x} + a\{\phi; x\} = 0 \tag{13}$$

where

$$u = a\{\phi; x\}. \tag{14}$$

The KdV and CH equations are also the geodesic flow equations for the L^2 and H^1 metrics correspondingly on the Bott-Virasoro group [40, 14, 37, 15, 13, 7].

2 The Camassa-Holm equation in Schwarzian form

It is known that the Camassa-Holm equation can be written as a compatibility condition of the following two linear problems (Lax pair) [4]:

$$\begin{aligned}\Psi_{xx} &= \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi, && (15)\\ \Psi_t &= \Big(\frac{1}{2\lambda} - u\Big)\Psi_x + \frac{u_x}{2}\Psi. && (16)\end{aligned}$$

In order to find the Schwarzian form for the CH equation we proceed as follows. Let Ψ_1 and Ψ_2 be two linearly independent solutions of the system (15), (16) and let us define

$$\phi = \frac{\Psi_2}{\Psi_1} \tag{17}$$

Then, from (16) it follows that

$$\frac{\phi_t}{\phi_x} = -u + \frac{1}{2\lambda} \tag{18}$$

According to the Theorem 10.1.1 from [29] due to (15) we also have

$$\{\phi; x\} = -2\lambda(m+\omega) - \frac{1}{2} \tag{19}$$

From (18), (19) and the link between m and u we obtain the Schwarz-Camassa-Holm (S-CH) equation:

$$(1-\partial^2)\frac{\phi_t}{\phi_x} - \frac{1}{2\lambda}\{\phi; x\} = -\frac{3}{4\lambda} + \omega \tag{20}$$

With a Galilean transformation, such that $\partial_t \to \partial_t + b\partial_x$ with a suitable constant b, one can absorb the constant on the right hand side and then the S-CH equation (20) acquires the form $(1-\partial^2)p_1 + ap_2 = 0$ or

$$(1-\partial^2)\frac{\phi_t}{\phi_x} + a\{\phi; x\} = 0, \tag{21}$$

for some constant a. Applying the hodograph transform $x \to \phi$, $t \to t$, $\phi \to x$ to the S-CH (21) and using the transformation properties of the Schwarzian derivative [29]

$$\{\phi; x\} = -\phi_x^2\{x; \phi\}$$

we obtain the following integrable deformation of the Harry Dym equation for the variable $v = 1/x_\phi$:

$$v_t + v^2[v(v\partial_\phi^{-1}(v^{-1})_t)_\phi]_\phi = av^3 v_{\phi\phi\phi}$$

The conformal properties are preserved in some $2+1$ dimensional generalizations. Indeed, consider the equation [5]

$$m_t + 2\omega U_{xy} + 2U_{xy}m + (U_y + \gamma)m_x = 0, \qquad m = U_x - U_{xxx}, \tag{22}$$

where γ is an arbitrary constant. This equation reduces to CH equation in the case $x = y$ and $u = U_x + \gamma$. The associated Lax pair is

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi \tag{23}$$

$$\Psi_t = \frac{1}{2\lambda}\Psi_y - (U_y+\gamma)\Psi_x + \frac{U_{xy}}{2}\Psi. \tag{24}$$

In a similar manner this equation can be expressed in terms of conformal invariants as

$$(\partial - \partial^3)\Big(\frac{\phi_y}{\phi_x} - 2\lambda\frac{\phi_t}{\phi_x}\Big) + \partial_y\{\phi; x\} = 0. \tag{25}$$

The equations (9) and (21) with $u = -\frac{\phi_t}{\phi_x}$ are not equivalent: as a matter of fact (21) implies (9), cf. [45]. It is often convenient to think that the Lax operator belongs to some Lie algebra, and the corresponding eigenfunction - to the corresponding group. Thus the relation between u and ϕ (see (17)) resembles the relation between the Lie group and the corresponding Lie algebra, as pointed out in [45]. More precisely, the following proposition holds:

Let ϕ be a solution of (20). Then one can check easily that $\Psi_1 = \phi_x^{-1/2}$ and $\Psi_2 = \phi\phi_x^{-1/2}$ are two linearly independent solutions of (15). This is consistent with (17). Therefore, the general solution of (15) is

$$\Psi = \frac{A\phi + B}{\sqrt{\phi_x}} \tag{26}$$

where A and B are two arbitrary constants, not simultaneously zero.

Note that the expression (26) is covariant with respect to the Möbius transformation (11), i.e. under (11), the expression (26) transforms into an expression of the same form but with constants

$$A \to A' = \frac{\alpha A + \gamma B}{\sqrt{\alpha\delta - \beta\gamma}}, \qquad B \to B' = \frac{\beta A + \delta B}{\sqrt{\alpha\delta - \beta\gamma}}. \tag{27}$$

3 Other equations of the CH hierarchy

Let us write the second equation of the CH Lax pair in the form

$$\Psi_t = -\mathcal{U}(x,\lambda)\Psi_x + \frac{1}{2}\mathcal{U}(x,\lambda)\Psi. \tag{28}$$

Taking $\mathcal{U}(x,\lambda) = \lambda v(x)$, the compatibility condition gives $v = (m+\omega)^{-1/2}$ and the evolution equation

$$m_t + (\partial - \partial^3)(m+\omega)^{-1/2} = 0. \tag{29}$$

Taking $\mathcal{U}(x,\lambda) = -\frac{1}{2\lambda} + u(x) + \lambda v(x)$, we obtain the following integrable deformation of the CH equation:

$$m_t + 2\omega u_x + 2mu_x + m_x u + \alpha(\partial - \partial^3)(m+\omega)^{-1/2} = 0, \tag{30}$$

where $m = u - u_{xx}$ and α is an arbitrary constant. (The compatibility condition gives $v = 2\alpha(m+\omega)^{-1/2}$ for an arbitrary constant α.)

An 'extension' of the CH hierarchy can be obtained if one considers a more general Lax pair:

$$\begin{aligned}\Psi_{xx} &= \mathcal{Q}(x,\lambda)\Psi, &(31)\\ \Psi_t &= -\mathcal{U}(x,\lambda)\Psi_x + \frac{1}{2}\mathcal{U}_x(x,\lambda)\Psi, &(32)\end{aligned}$$

where

$$\begin{aligned}\mathcal{Q}(x,\lambda) &= \lambda^n q_n(x) + \lambda^{n-1}q_{n-1}(x) + \ldots + \lambda q_1(x) + \frac{1}{4}, &(33)\\ \mathcal{U}(x,\lambda) &= u_0(x) + \frac{u_1(x)}{\lambda} + \ldots \frac{u_k(x)}{\lambda^k}. &(34)\end{aligned}$$

The compatibility condition of (31), (32) gives the equation

$$\mathcal{Q}_t = \frac{1}{2}\mathcal{U}_{xxx} - 2\mathcal{U}_x\mathcal{Q} - \mathcal{U}\mathcal{Q}_x, \tag{35}$$

which, due to (33), (34), is equivalent to a chain of n evolution equations with $k+1$ differential constraints for the $n+k+1$ variables q_1, q_2, …, q_n, u_0, u_1, …, u_k (n and k are arbitrary natural numbers, i.e. positive integers):

$$
\begin{aligned}
q_{n-r,t} &= -\sum_{s=\max(0,r-k)}^{r}(2u_{r-s,x}q_{n-s}+u_{r-s}q_{n-s,x}), \qquad r=0,1,\dots,n-1,\\
0 &= \frac{1}{2}(u_{r,xxx}-u_{r,x})-\sum_{s=1}^{\min(n,k-r)}(2u_{r+s,x}q_s+u_{r+s}q_{s,x}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r=0,1,\dots,k-1,\\
0 &= \frac{1}{2}(u_{k,xxx}-u_{k,x}).
\end{aligned}
$$

The two-component Camassa-Holm equation ($k=1$, $n=2$) was derived earlier in [43]. More details and examples on the 'extended' CH hierarchy can be found in [32].

4 Description of the whole CH hierarchy

For the description of the whole CH hierarchy we need to introduce the so-called recursion operator.

CH is a bi-hamiltonian equation, i.e. it admits two compatible hamiltonian structures $J_1=(2\omega\partial+m\partial+\partial m)$, $J_2=\partial-\partial^3$:

$$
\begin{aligned}
m_t &= -J_2\frac{\delta H_2[m]}{\delta m}=-J_1\frac{\delta H_1[m]}{\delta m}, &(36)\\
H_1 &= \frac{1}{2}\int mu\mathrm{d}x, &(37)\\
H_2 &= \frac{1}{2}\int(u^3+uu_x^2+2\omega u^2)\mathrm{d}x. &(38)
\end{aligned}
$$

There exists an infinite sequence of conservation laws (multi-Hamiltonian structure) $H_n[m]$, $n=0,\pm1,\pm2,\ldots$, [4, 22, 10] such that

$$
J_2\frac{\delta H_n[m]}{\delta m} = J_1\frac{\delta H_{n-1}[m]}{\delta m}. \qquad (39)
$$

The recursion operator is $L\sim J_2^{-1}J_1=(1-\partial^2)^{-1}[2(m+\omega)-\partial^{-1}m_x]\cdot$. The eigenfunctions of the recursion operator are the squared eigenfunctions of the CH spectral problem. More specifically, let us for simplicity consider the case where m is a Schwartz class function, $\omega>0$ and $m(x,0)+\omega>0$. Then $m(x,t)+\omega>0$ for all t, e.g. see [6]. It is convenient to introduce the notation: $q\equiv m+\omega$. Let $k^2=-\frac{1}{4}-\lambda\omega$, i.e.

$$
\lambda(k)=-\frac{1}{\omega}\Big(k^2+\frac{1}{4}\Big). \qquad (40)
$$

A basis in the space of solutions of (15) can be introduced: $f^+(x,k)$ and $\bar{f}^+(x,\bar{k})$. For all real $k \neq 0$ it is fixed by its asymptotic when $x \to \infty$ [6], see also [41, 11, 9]:

$$\lim_{x\to\infty} e^{-ikx} f^+(x,k) = 1, \tag{41}$$

Another basis can be introduced, $f^-(x,k)$ and$\bar{f}^-(x,\bar{k})$ fixed by its asymptotic when $x \to -\infty$ for all real $k \neq 0$:

$$\lim_{x\to-\infty} e^{ikx} f^-(x,k) = 1, \tag{42}$$

Since $m(x)$ and ω are real one gets that if $f^+(x,k)$ and $f^-(x,k)$ are solutions of (15) then

$$\bar{f}^+(x,\bar{k}) = f^+(x,-k), \qquad \text{and} \qquad \bar{f}^-(x,\bar{k}) = f^-(x,-k), \tag{43}$$

are also solutions of (15). The squared solutions are

$$F^\pm(x,k) \equiv (f^\pm(x,k))^2, \qquad F_n^\pm(x) \equiv F(x,i\kappa_n), \tag{44}$$

where $F_n^\pm(x)$ are apparently related to the discrete spectrum $k = i\kappa_n$,

$$0 < \kappa_1 < \ldots < \kappa_n < 1/2.$$

Using the asymptotics (41), (42) and the Lax equation (15) one can show that

$$L_\pm F^\pm(x,k) = \frac{1}{\lambda} F^\pm(x,k). \tag{45}$$

where

$$L_\pm = (\partial^2 - 1)^{-1}\Big[4q(x) - 2\int_{\pm\infty}^{x} \mathrm{d}y\, m'(y)\Big] \tag{46}$$

is the Recursion operator. The inverse of this operator is also well defined.

If $\Omega(z) = \frac{P_1(z)}{P_2(z)}$ is a ratio of two polynomials one can define $\Omega(L_\pm) \equiv P_1(L_\pm)P_2^{-1}(L_\pm)$ (provided $P_2(L_\pm)$ is an invertible operator). Then we can write the following nonlinear evolution integro-differential (in general) equation

$$q_t + 2q\tilde{u}_x + q_x\tilde{u} = 0, \qquad \tilde{u} = \frac{1}{2}\Omega(L_\pm)\Big(\sqrt{\frac{\omega}{q}} - 1\Big). \tag{47}$$

Example 1: With $\Omega(z) = z$ one can easily check that

$$\tilde{u} = \frac{1}{2}L_\pm\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = u \tag{48}$$

and thus the equation (47) becomes the Camassa-Holm equation (9) with Hamiltonian $H = H_1^{CH} = \frac{1}{2}\int mu\mathrm{d}x$.

Example 2: $\Omega(z) = 1/z$. The equation (47) has the form

$$q_t + \frac{1}{4}\partial_x(\partial_x^2 - 1)\sqrt{\frac{\omega}{q}} = 0, \tag{49}$$

i.e. the extended Dym equation [4, 24, 10] with Hamiltonian

$$H = \frac{1}{8}\int_{-\infty}^{\infty}\Big[\Big(\sqrt[4]{\frac{\omega}{q}} - \sqrt[4]{\frac{q}{\omega}}\Big)^2 + \frac{\sqrt{\omega}q_x^2}{4q^{5/2}}\Big]\mathrm{d}x, \tag{50}$$

which is, up to a constant, the (-1)-st Hamiltonian for the CH equation, H_{-1}^{CH}.

Example 3: $\Omega(z) = z + \varepsilon/z$, where ε is an arbitrary constant.

$$q_t + 2qu_x + q_x u + \frac{\varepsilon}{4}(\partial - \partial^3)q^{-1/2} = 0, \tag{51}$$

The Hamiltonian of this equation is the first CH Hamiltonian with an integrable perturbation, given by the (-1)-st CH Hamiltonian (50):

$$\begin{aligned} H &= \frac{1}{2}\int_{-\infty}^{\infty} mu\mathrm{d}x + \frac{\varepsilon}{8}\int_{-\infty}^{\infty}\Big[\Big(\sqrt[4]{\frac{\omega}{q}} - \sqrt[4]{\frac{q}{\omega}}\Big)^2 + \frac{\sqrt{\omega}q_x^2}{4q^{5/2}}\Big]\mathrm{d}x \\ &= H_1^{CH} + \varepsilon H_{-1}^{CH}. \end{aligned}$$

Let us introduce the notation $\partial_{\pm}^{-1} \equiv \int_{\pm\infty}^{x}\mathrm{d}x$. The equations from the CH Hierarchy can be written in the form

$$\frac{\partial_{\pm}^{-1}(\sqrt{q})_t}{\sqrt{q}} + \Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = 0. \tag{52}$$

The squared solutions (44) form a complete basis in the space of the Schwartz class functions $m(x)$, and y, t, can be treated as some additional parameters. Also, the Generalised Fourier Transform (GFT) for q and its variation over this basis is [10]

$$\sqrt{\frac{\omega}{q(x)}} - 1 = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\omega\lambda(k)}F^{\pm}(x,k)\mathrm{d}k + \sum_{n=1}^{N}\frac{2\kappa_n}{\omega\lambda_n}R_n^{\pm}F_n^{\pm}(x), \tag{53}$$

$$\begin{aligned} \frac{\partial_{\pm}^{-1}\delta(\sqrt{q})}{\sqrt{q}} &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\mathcal{R}^{\pm}(k)}{\omega\lambda(k)}\delta F^{\pm}(x,k)\mathrm{d}k \\ &\pm \sum_{n=1}^{N}\Big[\frac{\delta R_n^{\pm} - R_n^{\pm}\delta\lambda_n}{\omega\lambda_n}F_n^{\pm}(x) + \frac{R_n^{\pm}}{i\omega\lambda_n}\delta\kappa_n\dot{F}_n^{\pm}(x)\Big]. \end{aligned} \tag{54}$$

Here $\dot{F}_n^{\pm}(x) \equiv \frac{\partial}{\partial k} F^{\pm}(x,k)|_{k=i\kappa_n}$. The generalized Fourier coefficients $\mathcal{R}^{\pm}(k)$, $R_n^{\pm}$, together with the set of discrete eigenvalues, are called scattering data. The variation is with respect to any additional parameter, e.g. y, t. Due to the completeness of squared eigenfunctions basis, from (52), (53) and (54) we have linear differential equations for the scattering data:

$$\mathcal{R}_t^{\pm} \mp ik\Omega(\lambda^{-1})\mathcal{R}^{\pm}(k) = 0, \tag{55}$$

$$R_{n,t}^{\pm} \pm \kappa_n \Omega(\lambda_n^{-1}) R_n^{\pm} = 0, \tag{56}$$

$$\lambda_{n,t} = 0. \tag{57}$$

The GFT for other integrable systems is derived e.g. in [35, 27, 25, 26, 28, 30].

Example 4: Consider again the two dimensional CH generalisation

$$q_t + 2U_{xy}q + (U_y + \gamma)q_x = 0, \qquad q = U_x - U_{xxx} + \omega, \tag{58}$$

with arbitrary constants ω and γ. This equation can be written as

$$(\sqrt{q})_t + [(U_y + \gamma)\sqrt{q}]_x = 0. \tag{59}$$

Then

$$\partial_{\pm}^{-1}(\sqrt{q})_t + (U_y + \gamma)\sqrt{q} + \beta = 0, \tag{60}$$

where β is an integration constant. Further, with the choice $\beta = -\gamma\sqrt{\omega}$ and due to the identity

$$U_y = -\frac{1}{2} L_{\pm}\Big(\frac{\partial_{\pm}^{-1}(\sqrt{q})_y}{\sqrt{q}}\Big), \tag{61}$$

the equation can be written in the form

$$\frac{\partial_{\pm}^{-1}(\sqrt{q})_t}{\sqrt{q}} - \frac{1}{2} L_{\pm}\Big(\frac{\partial_{\pm}^{-1}(\sqrt{q})_y}{\sqrt{q}}\Big) - \gamma\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = 0. \tag{62}$$

Again, from (62), (53) and (54), considering variations with respect to y and t we obtain linear equations for the scattering data:

$$\mathcal{R}_t^{\pm} - \frac{1}{2\lambda}\mathcal{R}_y^{\pm} \pm 2ik\gamma\mathcal{R}^{\pm} = 0, \tag{63}$$

$$R_{n,t}^{\pm} - \frac{1}{2\lambda_n} R_{n,y}^{\pm} \mp 2\gamma\kappa_n R_n^{\pm} = 0. \tag{64}$$

E.g. when $\gamma = 0$ the solution is any function (with appropriate decaying properties) of $t - 2\lambda y$:

$$\mathcal{R}^{\pm}(y,t) = \mathcal{R}^{\pm}(t + 2\lambda y), \qquad R_n^{\pm}(y,t) = R_n^{\pm}(t + 2\lambda_n y). \tag{65}$$

Other possible choices for $\Omega(z)$ (47) produce the other members of the Camassa-Holm hierarchy.

5 Inverse scattering transform

Inverse scattering method for the hierarchy (47) is the same as the one for the CH equation [9]. The only difference is the time-dependence of the scattering data (and/or the y-dependence, etc). For example, the inverse scattering is simplified in the important case of the so-called reflectionless potentials, when the scattering data is confined to the discrete spectrum. This class of potentials corresponds to the N-soliton solutions of the CH hierarchy. In this case the time evolution of the scattering data is R_n^+ is

$$R_n^+(t) = R_n^+(0)\exp\Big(-\kappa_n \Omega(\lambda_n^{-1})t\Big). \tag{66}$$

The N-soliton solution is [9]

$$q(x,t) = \int_0^\infty \delta(x - g(\xi,t))p(x,t)\mathrm{d}\xi, \tag{67}$$

where $g(\xi,t)$ can be expressed through the scattering data as

$$g(\xi,t) \quad \equiv \quad \ln \int_0^\xi \Big(1 - \sum_{n,p} \frac{R_n^+(t)\underline{\xi}^{-2\kappa_n}}{\kappa_n + 1/2} A_{np}^{-1}[\underline{\xi},t]\Big)^{-2} \mathrm{d}\underline{\xi}, \tag{68}$$

with

$$A_{pn}[\xi,t] \quad \equiv \quad \delta_{pn} + \frac{R_n^+(t)\xi^{-2\kappa_n}}{\kappa_p + \kappa_n}$$

and

$$p(x,t) = \omega\xi^{-2}g_\xi^{-1}(\xi,t). \tag{69}$$

In particular, for the CH equation $q_t + uq_x = -2qu_x$, from (67) it follows

$$\dot{g}(\xi,t) = \frac{1}{2}\int_0^\infty e^{-|g(\xi,t)-g(\underline{\xi},t)|}p(\underline{\xi},t)d\underline{\xi} - \omega, \qquad \dot{g}(\xi,t) = u(g(\xi,t),t),$$

therefore $g(x,t)$ in (68) is the diffeomorphism (Virasoro group element) in the purely solitonic case [12]. The situation when the condition $q(x,0) \equiv m(x,0) + \omega > 0$ on the initial data does not hold is more complicated and requires separate analysis [36] (if $m(x,0)+\omega$ changes sign there are infinitely many positive eigenvalues accumulating at infinity and singularities might appear in finite time [8, 7, 6]).

The explicit construction of the peakon solutions ($\omega = 0$) is also known [4, 1, 2], e.g. a single peakon travelling with speed c is $u_c(x,t) = ce^{-|x-ct|}$. The peakons are the only solitary waves if $\omega = 0$, cf. [38]. They have to be interpretted as weak solutions due to the fact that they are not continuously differentiable - e.g. see [3]. The peakons however interact like solitons [4, 2]. Some nonintegrable generalizations of the CH equation also have been studied recently, e.g. [44].

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