




2008-01-01

Algebraic Discretization of the Camassa-Holm and Hunter-Saxton Equations

Rossen Ivanov

Technological University Dublin, rivanov@dit.ie

Follow this and additional works at: <https://arrow.dit.ie/scschmatart>

 Part of the [Dynamic Systems Commons](#), [Mathematics Commons](#), [Non-linear Dynamics Commons](#), and the [Partial Differential Equations Commons](#)

Recommended Citation

R. Ivanov, Algebraic Discretization of the Camassa-Holm and Hunter-Saxton Equations, *Journal of Nonlinear Mathematical Physics* 15, 2008, supplement 2, 1--12.

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact yvonne.desmond@dit.ie, arrow.admin@dit.ie, brian.widdis@dit.ie.



This work is licensed under a [Creative Commons Attribution-NonCommercial-Share Alike 3.0 License](#)



Algebraic Discretization of the Camassa-Holm and Hunter-Saxton Equations

Rossen I. Ivanov¹

Abstract

The Camassa-Holm (CH) and Hunter-Saxton (HS) equations have an interpretation as geodesic flow equations on the group of diffeomorphisms, preserving the H^1 and \dot{H}^1 right-invariant metrics correspondingly. There is an analogy to the Euler equations in hydrodynamics, which describe geodesic flow for a right-invariant metric on the infinite-dimensional group of diffeomorphisms preserving the volume element of the domain of fluid flow and to the Euler equations of rigid body with a fixed point, describing geodesics for a left-invariant metric on $SO(3)$. The CH and HS equations are integrable bi-hamiltonian equations and one of their Hamiltonian structures is associated to the Virasoro algebra. The parallel with the integrable $SO(3)$ top is made explicit by a discretization of both equations based on Fourier modes expansion. The obtained equations represent integrable tops with infinitely many momentum components.

An emphasis is given on the structure of the phase space of these equations, the momentum map and the space of canonical variables.

1 Introduction

This geometric interpretation of the Camassa-Holm equation [6] as a geodesic flow equation on the group of diffeomorphisms, preserving the H^1 right-invariant metric was noticed firstly by Misiólek [42] and developed further in many recent publications, e.g. [37, 25, 14, 15, 39, 13]. The CH equation has an interpretation in the context of water waves propagation [6, 34, 35, 19, 20, 32, 29]. The spectral problem for the CH equation on the line is developed in [2, 8, 10, 11, 17, 36], the periodic spectral problem – in [7, 16, 48]. The CH solutions are investigated in a variety of recent papers, e.g. in [4, 5, 9, 12, 21, 22, 23, 26, 30, 46]. Hierarchies of CH equations are studied in [11, 31, 33], different modifications are studied in [41, 47].

There are different forms of the CH equation, containing linear term with a first derivative u_x ; with a third derivative u_{xxx} (called sometimes Dullin-Gottwald-Holm equation [19, 20, 43, 44, 49]), or without such terms. These terms can be put in or removed from the equation independently by Galilean transformations.

¹Present address: School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland, Email: rivanov@dit.ie

We will be interested in the CH equation of the form

$$m_t + au_{xxx} + 2mu_x + m_xu = 0, \quad m = u - u_{xx}, \quad (1.1)$$

with a being an arbitrary constant. It can be written in Hamiltonian form

$$m_t = \{m, H_1\}, \quad (1.2)$$

where, assuming that m is 2π periodic in x , i.e. $m(x) = m(x + 2\pi)$, the Poisson bracket and the Hamiltonian are

$$\{F, G\} \equiv - \int_0^{2\pi} \frac{\delta F}{\delta m} \left(a\partial^3 + m\partial + \partial \circ m \right) \frac{\delta G}{\delta m} dx, \quad (1.3)$$

$$H_1 = \frac{1}{2} \int_0^{2\pi} mudx. \quad (1.4)$$

The equation (1.1) is bi-Hamiltonian with a second Hamiltonian representation $m_t = \{m, H_2\}_2$, where

$$\{F, G\}_2 \equiv - \int_0^{2\pi} \frac{\delta F}{\delta m} (\partial - \partial^3) \frac{\delta G}{\delta m} dx, \quad (1.5)$$

$$H_2 = \frac{1}{2} \int_0^{2\pi} (u^3 + uu_x^2 - \frac{a}{2}u_x^2) dx. \quad (1.6)$$

One can notice that the integral

$$H_0 = \int_0^{2\pi} m dx \quad (1.7)$$

is a Casimir for the second Poisson bracket (1.5).

The relation of the first Poisson bracket (1.3) to the Virasoro algebra can be seen as follows [18]. The 2π -periodic function allows a Fourier decomposition

$$m(x, t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} L_n(t) e^{inx} + \frac{a}{2}, \quad (1.8)$$

(the reality of m can be achieved by $L_{-n} = \bar{L}_n$). Then the Fourier coefficients L_n close a classical Virasoro algebra of central charge $c = -24\pi a$ with respect to the Poisson bracket (1.3):

$$i\{L_n, L_m\} = (n - m)L_{n+m} - 2\pi a(n^3 - n)\delta_{n+m,0}. \quad (1.9)$$

The CH equation in the form

$$m_t + 2\omega u_x + 2mu_x + m_xu = 0, \quad m = u - u_{xx}, \quad (1.10)$$

can be obtained from (1.1) via $u \rightarrow u + a$, and apparently $\omega = 3a/2$.

Since

$$H_0 = \frac{L_0}{2\pi} + \pi a \quad (1.11)$$

is an integral of motion (Casimir), so is L_0 .

The first Hamiltonian is

$$H_1 = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{L_n L_{-n}}{1+n^2} + \frac{a}{2} L_0 + \frac{2\pi a^2}{8}. \quad (1.12)$$

From (1.9) and (1.12) we obtain the 'Camassa-Holm top' equations on the Virasoro group, which are a discretization of the Camassa-Holm equation (1.1)

$$i\dot{L}_k = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{k+n}{1+n^2} L_n L_{k-n} + \frac{a}{2} \frac{3k-k^3}{1+k^2} L_k, \quad (1.13)$$

(the dot is a t -derivative). This equation is analogous to the Euler top (rigid body) equation on the Lie group $\text{SO}(3)$

$$\dot{M}_k = \sum_{p,l=1}^3 \varepsilon_{kpl} \Omega_p M_l, \quad M_k \equiv I_k \Omega_k$$

for the quadratic Hamiltonian

$$H_E = \frac{1}{2} \sum_{p=1}^3 M_p \Omega_p,$$

where I_k ($k = 1, 2, 3$) are three constants – the principle inertia momenta. The phase space is embedded in the Lie coalgebra $\text{so}(3)^*$ as a coadjoint orbit. The Lie-Poisson bracket, related to the $\text{so}(3)^*$ coalgebra is

$$\{M_n, M_m\} = \varepsilon_{nmk} M_k. \quad (1.14)$$

The inertia operator $\mathbf{I}: \text{so}(3) \rightarrow \text{so}(3)^*$ (see e.g. [1]) relates the parametrization on the $\text{so}(3)$ algebra given by the functions Ω_k and the parametrization on the co-algebra $\text{so}(3)^*$ given by the functions $M_k = I_k \Omega_k$. Note that the Poisson bracket (1.14) has a Casimir

$$K = \Omega_1^2 + \Omega_2^2 + \Omega_3^2, \quad (1.15)$$

constraining the phase space on a sphere. Since the Lie-Poisson bracket is degenerate on $\text{so}(3)^*$, the coadjoint orbits (which are spheres centered at the origin) are labelled by the value of the Casimir K .

For the CH top (1.13) the coadjoint orbits are embedded in the Virasoro algebra (parameterized by the functions L_k) due to the Lie-Poisson bracket (1.9).

2 Lax representation for the discrete Camassa-Holm equation and integrals of motion

The Lax pair for the discrete CH equation (1.13) can be obtained from the Lax pair for (1.10),

$$\Psi_{xx} = \left(\frac{1}{4} + \lambda \left(m + \frac{a}{2} \right) \right) \Psi \quad (2.1)$$

$$\Psi_t = \left(\frac{1}{2\lambda} - u + a \right) \Psi_x + \frac{u_x}{2} \Psi, \quad (2.2)$$

as follows. We take the expansions

$$\Psi = \sum_{n \in \mathbb{Z}} \Psi_{\frac{n}{2}} e^{i \frac{n}{2} x}, \quad (2.3)$$

$$u = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} u_n e^{inx} + \frac{a}{2}, \quad u_n = \frac{L_n}{1+n^2}. \quad (2.4)$$

Then (2.1) gives

$$\frac{1}{\lambda} \Psi_{\frac{n}{2}} = \sum_{p \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p}, \quad (2.5)$$

where

$$\mathcal{L}_{\frac{n}{2}, \frac{n}{2}-p} = -\frac{4}{n^2+1} \left(\frac{L_p}{2\pi} + a\delta_{p,0} \right),$$

or

$$\mathcal{L}_{\frac{n}{2}-q, \frac{n}{2}-p} = -\frac{4}{(n-2q)^2+1} \left(\frac{L_{p-q}}{2\pi} + a\delta_{p,q} \right) \quad (2.6)$$

Now from (2.2), (2.3), (2.4) and (2.5) it follows

$$\dot{\Psi}_{\frac{n}{2}} = \sum_{p \in \mathbb{Z}} \mathcal{A}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p}, \quad (2.7)$$

where

$$\mathcal{A}_{\frac{n}{2}, \frac{n}{2}-p} = -\frac{i}{4\pi} \left(2n \frac{p^2+1}{n^2+1} + n - 3p \right) u_p + in \left(\frac{1}{4} - \frac{1}{n^2+1} \right) a\delta_{p,0},$$

or

$$\begin{aligned} \mathcal{A}_{\frac{n}{2}-q, \frac{n}{2}-p} &= -\frac{i}{4\pi} \left(2(n-2q) \frac{(p-q)^2+1}{(n-2q)^2+1} + n - 3p + q \right) u_{p-q} \\ &+ i(n-2q) \left(\frac{1}{4} - \frac{1}{(n-2q)^2+1} \right) a\delta_{p,q}. \end{aligned} \quad (2.8)$$

Differentiating (2.5) with respect to t we obtain

$$\frac{1}{\lambda} \dot{\Psi}_{\frac{n}{2}} = \sum_{p \in \mathbb{Z}} \dot{\mathcal{L}}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p} + \sum_{p \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}, \frac{n}{2}-p} \dot{\Psi}_{\frac{n}{2}-p},$$

and with the further substitution from (2.7),

$$\begin{aligned} \frac{1}{\lambda} \sum_{q \in \mathbb{Z}} \mathcal{A}_{\frac{n}{2}, \frac{n}{2}-q} \Psi_{\frac{n}{2}-q} &= \sum_{p \in \mathbb{Z}} \dot{\mathcal{L}}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p} + \sum_{p, q \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}, \frac{n}{2}-q} \mathcal{A}_{\frac{n}{2}-q, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p}, \\ \sum_{q \in \mathbb{Z}} \mathcal{A}_{\frac{n}{2}, \frac{n}{2}-q} \left(\frac{1}{\lambda} \Psi_{\frac{n}{2}-q} \right) &= \sum_{p \in \mathbb{Z}} \dot{\mathcal{L}}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p} + \sum_{p, q \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}, \frac{n}{2}-q} \mathcal{A}_{\frac{n}{2}-q, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p}, \end{aligned}$$

and finally, the substitution of (2.5) gives

$$\begin{aligned} \sum_{p, q \in \mathbb{Z}} \mathcal{A}_{\frac{n}{2}, \frac{n}{2}-q} \mathcal{L}_{\frac{n}{2}-q, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p} &= \\ \sum_{p \in \mathbb{Z}} \dot{\mathcal{L}}_{\frac{n}{2}, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p} + \sum_{p, q \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}, \frac{n}{2}-q} \mathcal{A}_{\frac{n}{2}-q, \frac{n}{2}-p} \Psi_{\frac{n}{2}-p}, \end{aligned} \quad (2.9)$$

or in matrix form,

$$\dot{\mathcal{L}} = [\mathcal{A}, \mathcal{L}]. \quad (2.10)$$

After some lengthy computations one can verify that (2.10) gives (1.13). The integrals of motion are given by $I_k = \text{tr}(\mathcal{L}^k)$. For example,

$$I_1 = \text{tr}(\mathcal{L}) = \sum_{p \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}-p, \frac{n}{2}-p} = -4 \left(\frac{L_0}{2\pi} + a \right) \sum_{p \in \mathbb{Z}} \frac{1}{(n-2p)^2 + 1}$$

produces, up to an overall constant, the Casimir H_0 , (1.11).

$$\begin{aligned} I_2 &= \text{tr}(\mathcal{L}^2) = \sum_{p, q \in \mathbb{Z}} \mathcal{L}_{\frac{n}{2}-p, \frac{n}{2}-q} \mathcal{L}_{\frac{n}{2}-q, \frac{n}{2}-p} \\ &= \frac{4}{\pi^2} \sum_{p, q \in \mathbb{Z}} \frac{L_{p-q} L_{q-p}}{[(n-2p)^2 + 1][(n-2q)^2 + 1]} \\ &+ \frac{16a}{\pi} (L_0 + \pi a) \sum_{p \in \mathbb{Z}} \frac{1}{[(n-2p)^2 + 1]^2}. \end{aligned} \quad (2.11)$$

With partial fractions decomposition with respect to n one can derive the identity

$$\begin{aligned} &\frac{1}{[(n-2p)^2 + 1][(n-2q)^2 + 1]} \\ &= \frac{1/4}{(p-q)^2 + 1} \left\{ \frac{(n-2q) + (p-q)}{(p-q)[(n-2q)^2 + 1]} - \frac{(n-2p) + (q-p)}{(p-q)[(n-2p)^2 + 1]} \right\}. \end{aligned}$$

Further, using the fact that all expressions that change sign under $p - q \rightarrow -(p - q)$ are zero, due to the summation over all integer numbers, we have

$$\begin{aligned} \frac{4}{\pi^2} \sum_{p,q \in \mathbb{Z}} \frac{L_{p-q} L_{q-p}}{[(n-2p)^2+1][(n-2q)^2+1]} \\ = \frac{1}{\pi^2} \sum_{p,q \in \mathbb{Z}} \frac{L_{p-q} L_{q-p}}{1+(p-q)^2} \left\{ \frac{1}{(n-2p)^2+1} + \frac{1}{(n-2q)^2+1} \right\} \\ = \frac{2}{\pi^2} \sum_{p \in \mathbb{Z}} \frac{L_p L_{-p}}{1+p^2} \sum_{q \in \mathbb{Z}} \frac{1}{(n-2q)^2+1}. \end{aligned}$$

Thus, the new integral that appears is $\sum_{p \in \mathbb{Z}} \frac{L_p L_{-p}}{1+p^2}$, giving H_1 , the first Hamiltonian (1.12).

3 Oscillator algebra, Miura transformation and momentum map

Let us introduce now the oscillator algebra

$$i\{a_n, a_m\} = \frac{2\pi a}{\kappa^2} n \delta_{n+m,0}, \quad (3.1)$$

where κ is an arbitrary constant. Clearly, a_0 is a Casimir due to (3.1). One can easily verify the following oscillator representation of the Virasoro algebra [38, 24]:

$$L_n = -\kappa(n-1)a_n + \frac{\kappa^2}{4\pi a} \sum_{k \in \mathbb{Z}} a_k a_{n-k}. \quad (3.2)$$

This representation is also known as Sugawara construction. Further, it is evident that

$$i\{a_n, L_m\} = n a_{n+m} + \frac{2\pi a}{\kappa} n(n+1) \delta_{n+m,0}. \quad (3.3)$$

Since a_k satisfy the 'canonical' Poisson brackets they are natural candidates for the coordinates in the phase-space. Thus, L_n has an interpretation of a momentum and (3.2) gives the momentum map. The Sugawara construction relates to the Miura transformation, which in terms of field variables can be obtain as follows. Defining

$$v = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} a_k e^{ikx} + \frac{a}{\kappa} \quad (3.4)$$

from (3.2) and (1.8) we have the analog of the Miura transformation:

$$m = i\kappa v_x + \frac{\kappa^2}{2a} v^2 + \frac{a}{2} \quad (3.5)$$

The reality can be achieved by taking κ purely imaginary, $a_k = \bar{a}_{-k}$ for $k \neq 0$ and $\kappa = 2\pi ia/\mathfrak{S}(a_0)$.

Here we notice that the Casimir (1.7) due to (3.5) leads to the restriction

$$\int_0^{2\pi} v^2(x, t) dx = \text{const}, \quad (3.6)$$

which reduces the evolution of $v(x, t)$ on the L^2 -sphere. In terms of the canonical coordinates this condition is

$$\sum_{k>0} |a_k|^2 = \text{const}, \quad (3.7)$$

since a_0 is a constant. It shows that the time evolution of the canonical variables, given by

$$\dot{a}_n = \{a_n, H_1\}$$

is constrained on the infinite-dimensional l_2 -sphere, a condition, similar to the one that we see in the $so(3)$ example (1.15).

When $a = 0$, the Sugawara construction for the Virasoro modes in the case of zero central charge is

$$L_n = \frac{1}{2\tilde{\kappa}} \sum_{k \in \mathbb{Z}} a_k a_{n-k},$$

where $\tilde{\kappa}$ is an arbitrary constant and

$$i\{a_n, a_m\} = \tilde{\kappa} n \delta_{n+m, 0}. \quad (3.8)$$

The Casimir with respect to the first Poisson bracket (1.3) with $a \equiv 0$ is

$$\int_0^{2\pi} \sqrt{m} dx = \sqrt{\frac{\pi}{\tilde{\kappa}}} \int_0^{2\pi} v dx = 2\pi \sqrt{\frac{\pi}{\tilde{\kappa}}} a_0,$$

i.e. this is the Casimir a_0 of (3.8).

With the expansions

$$m = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} L_n e^{inx}, \quad v = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

the Sugawara construction takes the form $m = \frac{\pi}{\tilde{\kappa}} v^2$. Since the integral $\int_0^{2\pi} m dx = \text{const}$ is a Casimir, we have again

$$\int_0^{2\pi} v^2 dx = \text{const},$$

leading to (3.7).

4 The Hunter-Saxton equation

The Hunter-Saxton (HS) equation

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0$$

describes the propagation of waves in a massive director field of a nematic liquid crystal [27], with the orientation of the molecules described by the field of unit 1 vectors $n(x, t) = (\cos u(x, t), \sin u(x, t))$, where x is the space variable in a reference frame moving with the linearized wave velocity, and t is a 'slow time variable'. A linear term au_{xxx} can be generated by a shift $u \rightarrow u + a$:

$$u_{xxt} + au_{xxx} + 2u_x u_{xx} + uu_{xxx} = 0. \quad (4.1)$$

The HS equation is a short-wave limit of the CH equation, and can be obtained if one takes $m = -u_{xx}$. The Hamiltonian representation (1.2) – (1.4) for this equation is also valid. The HS equation (4.1) is an integrable, bi-Hamiltonian equation with a second Hamiltonian representation $m_t = \{m, H_2\}_2$, where

$$\{F, G\}_2 \equiv \int_0^{2\pi} \frac{\delta F}{\delta m} \partial^3 \frac{\delta G}{\delta m} dx, \quad (4.2)$$

$$H_2 = \frac{1}{2} \int_0^{2\pi} \left(u - \frac{a}{2}\right) u_x^2 dx. \quad (4.3)$$

The HS Lax pair is

$$\Psi_{xx} = \lambda m \Psi, \quad (4.4)$$

$$\Psi_t = \left(\frac{1}{2\lambda} - u - a\right) \Psi_x + \frac{u_x}{2} \Psi. \quad (4.5)$$

The analytic and geometric aspects of the HS equation are discussed in a variety of recent papers, e.g. [28, 3, 40] and the references therein. Again, assuming periodicity and using the expansions

$$\Psi = \sum_{n \in \mathbb{Z}} \Psi_n e^{inx}, \quad u = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} u_n e^{inx}$$

we obtain the discrete form of the HS equation:

$$in\dot{u}_n - an^2 u_n - \frac{1}{2} \sum_{k \in \mathbb{Z}} k(n+k) u_k u_{n-k} = 0.$$

In a similar manner from the Lax pair we obtain the matrix Lax representation for the discrete HS equation

$$\dot{\mathcal{L}}^{HS} = [\mathcal{A}^{HS}, \mathcal{L}^{HS}]. \quad (4.6)$$

where

$$\mathcal{L}_{n,n-p}^{HS} = -\frac{p^2}{n^2}u_p, \quad \mathcal{A}_{n,n-p}^{HS} = \frac{i}{2}\left(-\frac{p^2}{n} - 2n + 3p\right)u_p - ina\delta_{p,0}.$$

The momentum map (the Sugawara construction) for the HS equation remains the same as for the CH equation. However, it becomes degenerated in the case $a = 0$, since $m = -u_{xx}$ and the Casimir $\int_0^{2\pi} m dx = 0$. Then $\int_0^{2\pi} v^2 dx = 0$, which, for real variables is only possible when $v \equiv 0$, i.e. $m \equiv 0$.

5 Conclusions

At the examples of the CH and HS equations we have shown that the integrable systems with quadratic Hamiltonians are equivalent to integrable tops (possibly with infinitely many components), associated to the algebra of their Poisson brackets. An example for the two dimensional Euler equations in fluid mechanics is presented in [50], another example for the KdV superequation in [38, 45].

Acknowledgments

The support of the G. Gustafsson Foundation for Research in Natural Sciences and Medicine is gratefully acknowledged. The author is thankful to Prof. A. Constantin and Dr G. Grahovski for many stimulating discussions.

References

- [1] Arnold V and Khesin B, *Topological Methods in Hydrodynamics*, Springer Verlag, New York, 1998.
- [2] Bennewitz C, Brown B M, Weikard R, Inverse Spectral and Scattering Theory for the Half-line Left Definite Sturm-Liouville Problem, *SIAM J. MATH. ANAL.* Vol. 40, (2009) pp. 2105-2131.
- [3] Bressan A and Constantin A, Global Solutions of the Hunter-Saxton Equation, *SIAM J. Math. Anal.* **37** (2005), 996–1026; arXiv:math/0502059
- [4] Bressan A and Constantin A, Global Conservative Solutions of the Camassa-Holm Equation, *Arch. Ration. Mech. Anal.* **183** (2007), 215–239.
- [5] Bressan A and Constantin A, Global Dissipative Solutions of the Camassa-Holm equation, *Anal. Appl. (Singap.)* **5** (2007), 1–27.
- [6] Camassa R and Holm D, An Integrable Shallow Water Equation with Peaked Solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664; arXiv:patt-sol/9305002v1

- [7] Constantin A. On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.* **155** (1998), 352–363.
- [8] Constantin A, On the Scattering Problem for the Camassa-Holm Equation *Proc. R. Soc. Lond.* **A457** (2001), 953–970.
- [9] Constantin A and Escher J, Wave Breaking for Nonlinear Nonlocal Shallow Water Equations, *Acta Mathematica*, **181** (1998), 229–243.
- [10] Constantin A, Gerdjikov V S and Ivanov R I, Inverse Scattering Transform for the Camassa-Holm Equation, *Inv. Problems* **22** (2006), 2197-2207; nlin.SI/0603019.
- [11] Constantin A, Gerdjikov V S and Ivanov R I, Generalised Fourier Transform for the Camassa-Holm Hierarchy, *Inv. Problems* **23** (2007), 1565-1597; nlin.SI/0707.2048.
- [12] Constantin A and Ivanov R I, Poisson Structure and Action-Angle Variables for the Camassa-Holm Equation, *Lett. Math. Phys.* **76** (2006), 93–108; nlin.SI/0602049.
- [13] Constantin A and Ivanov R I, The Camassa-Holm Equation as a Geodesic Flow for the the H^1 -Right- Invariant Metric, in Topics in Contemporary Differential Geometry, Complex Analysis and Mathematical Physics, Editors: Dimiev S and Sekigawa K, World Scientific, London, 2007, 33–41; arXiv:0706.3810 [nlin.SI].
- [14] Constantin A and Kolev B, Geodesic Flow on the Diffeomorphism Group of the Circle, *Comment. Math. Helv.* **78** (2003), 787–804; arXiv:math-ph/0305013
- [15] Constantin A and Kolev B, Integrability of Invariant Metrics on the Diffeomorphism Group of the Circle, *J. Nonlin. Sci.* **16** (2006), 109-122.
- [16] Constantin A and McKean H P, A Shallow Water Equation on the Circle, *Commun. Pure Appl. Math.* **52** (1999), 949–982.
- [17] Boutet de Monvel A and Shepelsky D, Riemann-Hilbert Approach for the Camassa-Holm Equation on the Line, *C.R. Math. Acad. Sci. Paris* **343** (2006), 627-632.
- [18] Dickey L, Soliton Equations and Hamiltonian Systems, World Scientific, New Jersey, 2003.
- [19] Dullin H R, Gottwald G A and Holm D D, Camassa-Holm, Korteweg-de Vries-5 and Other Asymptotically Equivalent Equations for Shallow Water Waves, *Fluid Dynam. Res.* **33** (2003), 73–95.
- [20] Dullin H R, Gottwald G A and Holm D D, On Asymptotically Equivalent Shallow Water Wave Equations, *Physica* **190D**, (2004), 1–14.
- [21] El Dika K and Molinet L, Exponential Decay of H^1 -localized Solutions and Stability of the Train of N Solitary Waves for the Camassa-Holm Equation, *Phil. Trans. R. Soc. A* **365** (2007), 2313–2331.

- [22] Escher J and Yin Z, Initial Boundary Value Problems of the Camassa-Holm Equation, *Commun. Partial Diff. Eq.* **33** (2008), 377–395.
- [23] Golovko V, Kersten P, Krasil'shchik I and Verbovetsky A, On Integrability of the Camassa-Holm Equation and Its Invariants. A Geometrical Approach, *Acta Appl. Math.* **101** (2008), 59–83.
- [24] Harnad J and Kupershmidt B A, Super Loop Groups, Hamiltonian Actions and Super Virasoro Algebras, *Commun. Math. Phys.* **132** (1990), 315–347.
- [25] Holm D, Marsden J and Ratiu T, The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories, *Adv. Math.* **137** (1998), 1–81.
- [26] Henry D, Compactly Supported Solutions of the Camassa-Holm Equation. *J. Nonlin. Math. Phys.* **12** (2005), 342–347.
- [27] Hunter J K and Saxton R A, Dynamics of Director Fields, *SIAM J. Appl. Math.* **51** (1991), 1498–1521.
- [28] Hunter J and Zheng Y, On a Completely Integrable Nonlinear Hyperbolic Variational Equation, *Physica D* **79** (1994), 361–386.
- [29] Ionescu-Kruse D, Variational Derivation of the Camassa-Holm Shallow Water Equation with Non-zero Vorticity, *Discr. Cont. Dyn. Syst.* **19** (2007), 531–543.
- [30] Ivanov R I, Conformal Properties and Bäcklund Transform for the Associated Camassa-Holm Equation, *Phys. Lett. A* **345** (2005), 235–243; nlin.SI/0507005.
- [31] Ivanov R I, Extended Camassa-Holm hierarchy and conserved quantities, *Z. Naturforsch.*, **61a** (2006) pp. 133–138; nlin.SI/0601066.
- [32] Ivanov R I, Water Waves and Integrability, *Philos. Trans. Roy. Soc.: Ser. A* **365** (2007), 2267–2280; arXiv:0707.1839 [nlin.SI].
- [33] Ivanov R I, Conformal and Geometric Properties of the Camassa-Holm Hierarchy, *Discr. Cont. Dyn. Syst.* **19** (2007), 545–554; arXiv:0907.1107v1 [nlin.SI]
- [34] Johnson R S, Camassa-Holm, Korteweg-de Vries and Related Models for Water Waves, *J. Fluid. Mech.* **457** (2002), 63–82.
- [35] Johnson R S, The Camassa-Holm Equation for Water Waves Moving over a Shear Flow. *Fluid Dyn. Res.* **33** (2003), 97–111.
- [36] Kaup D J, Evolution of the Scattering Data of the Camassa-Holm Equation for General Initial Data, *Stud. Appl. Math.* **117** (2006), 149–164.
- [37] Khesin B, and Misiołek G, Euler Equations on Homogeneous Spaces and Virasoro Orbits, *Adv. Math.* **176** (2003), 116–144.
- [38] Kupershmidt B, Oscillator Representations of the 2D Conformal Algebra and Superalgebra, *J. Phys. A: Math. Gen.* **18** (1985), L1123–L1126.

- [39] Kolev B, Lie Groups and Mechanics: An Introduction, *J. Nonlinear Math. Phys.* **11** (2004) 480; arXiv:math-ph/0402052
- [40] Lenells J, The Hunter-Saxton Equation Describes the Geodesic Flow on a Sphere, *J. Geom. Phys.* **57** (2007), 2049–2064.
- [41] Liu Z and Ouyang Z, A Note on Solitary Waves for Modified forms of Camassa-Holm and Degasperis-Procesi Equations, *Phys. Lett. A* **366** (2007), 377–381.
- [42] Misiołek G, A Shallow Water Equation as a Geodesic Flow on the Bott-Virasoro Group, *J. Geom. Phys.* **24** (1998), 203–208.
- [43] Mustafa O, Existence and Uniqueness of Low Regularity Solutions for the Dullin-Gottwald-Holm Equation, *Commun. Math. Phys.* **265** (2006), 189–200.
- [44] Mustafa O, Global Conservative Solutions of the Dullin-Gottwald-Holm Equation *Discr. Cont. Dyn. Syst.* **19** (2007), 575–594.
- [45] Ovsienko V and Khesin B, Korteweg-de Vries Superequation as an Euler Equation, *Funct. Analysis and Appl.* **21** (1987), 329–331.
- [46] Qiao Z J and Zhang G, On Peaked and Smooth Solitons for the Camassa-Holm Equation. *Europhys. Lett.* **73** (2006), 657–663.
- [47] Stanislavova M and Stefanov A, Attractors for the Viscous Camassa-Holm Equation *Discr. Cont. Dyn. Syst.* **18** (2007), 159–186; math.DS/0612321.
- [48] Vaninsky K L, Equations of Camassa-Holm Type and Jacobi Ellipsoidal Coordinates, *Comm. Pure Appl. Math.* **58** (2005), 1149–1187.
- [49] Zhou Y, Blow-up of Solutions to the DGH Equation, *J. Funct. Analysis* **250** (2007), 227–248.
- [50] Zeitlin V, Finite-mode Analogs of 2D Ideal Hydrodynamics: Coadjoint Orbits and Local Canonical Structure, *Phys. D* **49** (1991), 353–362.