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Equations of the Camassa-Holm Hierarchy

Rossen I. Ivanov ¹

Abstract

The squared eigenfunctions of the spectral problem associated with the Camassa-Holm (CH) equation represent a complete basis of functions, which helps to describe the inverse scattering transform for the CH hierarchy as a generalized Fourier transform (GFT). All the fundamental properties of the CH equation, such as the integrals of motion, the description of the equations of the whole hierarchy, and their Hamiltonian structures, can be naturally expressed using the completeness relation and the recursion operator, whose eigenfunctions are the squared solutions. Using the GFT, we explicitly describe some members of the CH hierarchy, including integrable deformations for the CH equation. We also show that solutions of some $(1 + 2)$ -dimensional members of the CH hierarchy can be constructed using results for the inverse scattering transform for the CH equation. We give an example of the peakon solution of one such equation.

Keywords: Inverse Scattering, Solitons, Peakons, Integrable systems, Lax Pair

1 Introduction

The Camassa-Holm (CH) equation [1] became famous as a model in the theory of water waves. It is also known that it describes axially symmetric waves in a hyperelastic rod [2, 3]. The most prominent representative of the water-wave equations, the Korteweg-de Vries (KdV) equation does not describe the wave-breaking phenomenon. In addition to the stable soliton solutions, the CH equation, together with another recently derived nonlinear integrable equation, the Degasperis-Procesi equation, has smooth solutions that develop singularities in finite time via a process that captures the features of the breaking waves: the solution remains bounded, but the slope becomes unbounded [4]. For the physical relevance of these two equations as models for the propagation of shallow water waves over a flat bottom one can consult, for example [5, 6, 7, 8, 9, 10, 11]. More about the physical applications, modifications and the type of solutions of the CH equation can also be found in [4, 12, 13, 14, 15, 16, 17, 18, 19, 20].

The CH equation has the form

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (1)$$

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where ω is a real constant. This equation is completely integrable and admits a Lax pair [1]

$$\Psi_{xx} = \left(\frac{1}{4} + \lambda(m + \omega) \right) \Psi, \quad (2)$$

$$\Psi_t = \left(\frac{1}{2\lambda} - u \right) \Psi_x + \frac{u_x}{2} \Psi + \gamma \Psi, \quad (3)$$

where γ is an arbitrary constant and $m = u - u_{xx}$. The CH solitary waves are stable solitons if $\omega > 0$ [6, 12, 13, 21] or peakons if $\omega = 0$ [1, 22, 23].

The KdV and CH equations can also be interpreted as geodesic flow equations for the respective L^2 and H^1 metrics on the Bott-Virasoro group [24, 25, 26, 27, 28, 29].

The CH equation is a bi-Hamiltonian equation, i.e. it admits two compatible hamiltonian structures $J_1 = (2\omega\partial + m\partial + \partial m)$, $J_2 = \partial - \partial^3$ [1, 30]:

$$m_t = -J_2 \frac{\delta H_2[m]}{\delta m} = -J_1 \frac{\delta H_1[m]}{\delta m}, \quad (4)$$

$$H_1 = \frac{1}{2} \int m u dx, \quad (5)$$

$$H_2 = \frac{1}{2} \int (u^3 + u u_x^2 + 2\omega u^2) dx. \quad (6)$$

The infinite sequence of conservation laws (multi-Hamiltonian structure) $H_n[m]$, $n = 0, \pm 1, \pm 2, \dots$, satisfying

$$J_2 \frac{\delta H_n[m]}{\delta m} = J_1 \frac{\delta H_{n-1}[m]}{\delta m} \quad (7)$$

can be computed explicitly [1, 31, 32, 33, 34].

2 Generalized Fourier transform

The so-called recursion operator plays an important role in describing an integrable hierarchy. The recursion operator for the CH hierarchy is $L = J_2^{-1} J_1$. The eigenfunctions of the recursion operator are the squared eigenfunctions of the CH spectral problem. For simplicity we consider the concrete case where m is a Schwartz class function, $\omega > 0$ and $m(x, 0) + \omega > 0$. Then $m(x, t) + \omega > 0$ for all t , e.g. see [35, 15]. It is convenient to introduce the notation: $q \equiv m + \omega$. Let $k^2 = -\frac{1}{4} - \lambda\omega$, i.e.

$$\lambda(k) = -\frac{1}{\omega} \left(k^2 + \frac{1}{4} \right). \quad (8)$$

A basis in the space of solutions of (2) can be introduced: $f^+(x, k)$ and $\bar{f}^+(x, \bar{k})$. For all real $k \neq 0$ it is fixed by its asymptotic when $x \rightarrow \infty$ [35], (also see [32, 36, 37]):

$$\lim_{x \rightarrow \infty} e^{-ikx} f^+(x, k) = 1, \quad (9)$$

We can introduce another basis, $f^-(x, k)$ and $\bar{f}^-(x, \bar{k})$ fixed by its asymptotic when $x \rightarrow -\infty$ for all real $k \neq 0$:

$$\lim_{x \rightarrow -\infty} e^{ikx} f^-(x, k) = 1, \quad (10)$$

Because $m(x)$ and ω are real we find that if $f^+(x, k)$ and $f^-(x, k)$ are solutions of (2) then

$$\bar{f}^+(x, \bar{k}) = f^+(x, -k), \quad \text{and} \quad \bar{f}^-(x, \bar{k}) = f^-(x, -k), \quad (11)$$

are also solutions of (2). The squared solutions are

$$F^\pm(x, k) \equiv (f^\pm(x, k))^2, \quad F_n^\pm(x) \equiv F(x, i\kappa_n), \quad (12)$$

where $F_n^\pm(x)$ are related to the discrete spectrum $k = i\kappa_n$, where $0 < \kappa_1 < \dots < \kappa_n < 1/2$.

Using the asymptotics (9), (10) and the Lax equation (2) one can show that

$$L_\pm F^\pm(x, k) = \frac{1}{\lambda} F^\pm(x, k). \quad (13)$$

where

$$L_\pm = (\partial^2 - 1)^{-1} \left[4q(x) - 2 \int_{\pm\infty}^x d\tilde{x} m'(\tilde{x}) \right] \quad (14)$$

is the recursion operator. The inverse of this operator is also well defined.

We introduce the notation $\partial_\pm^{-1} \equiv \int_{\pm\infty}^x d\tilde{x}$. The squared solutions (12) form a complete basis in the space of the Schwartz class functions $m(x)$, and y, t , can be treated as some additional parameters. Also, the Generalised Fourier Transform (GFT) for q and its variation over this basis is [34]

$$\sqrt{\frac{\omega}{q(x)}} - 1 = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2k\mathcal{R}^\pm(k)}{\omega\lambda(k)} F^\pm(x, k) dk + \sum_{n=1}^N \frac{2\kappa_n}{\omega\lambda_n} R_n^\pm F_n^\pm(x), \quad (15)$$

$$\begin{aligned} \frac{\partial_\pm^{-1} \delta(\sqrt{q})}{\sqrt{q}} &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{i\delta\mathcal{R}^\pm(k)}{\omega\lambda(k)} F^\pm(x, k) dk \\ &\pm \sum_{n=1}^N \left[\frac{\delta R_n^\pm - R_n^\pm \delta\lambda_n}{\omega\lambda_n} F_n^\pm(x) + \frac{R_n^\pm}{i\omega\lambda_n} \delta\kappa_n \tilde{F}_n^\pm(x) \right], \end{aligned} \quad (16)$$

where $\tilde{F}_n^\pm(x) \equiv \frac{\partial}{\partial k} F^\pm(x, k)|_{k=i\kappa_n}$. The generalized Fourier coefficients $\mathcal{R}^\pm(k)$, R_n^\pm , together with the set of discrete eigenvalues, are called *scattering data*. The variation is with respect to any additional parameter, e.g. y, t .

The equations of the CH Hierarchy can be written as

$$P_2(L_\pm) \frac{2\partial_\pm^{-1}(\sqrt{q})t}{\sqrt{q}} + P_1(L_\pm) \left(\sqrt{\frac{\omega}{q}} - 1 \right) = 0, \quad (17)$$

where $P_1(z)$ and $P_2(z)$ are two polynomials. If $\Omega(z) = \frac{P_1(z)}{P_2(z)}$ is a ratio of these two polynomials one can define $\Omega(L_\pm) \equiv P_1(L_\pm)P_2^{-1}(L_\pm)$ (provided $P_2(L_\pm)$ is an invertible operator). Then (17) can be written in the equivalent form

$$q_t + 2q\tilde{u}_x + q_x\tilde{u} = 0, \quad \tilde{u} = \frac{1}{2}\Omega(L_\pm)\left(\sqrt{\frac{\omega}{q}} - 1\right). \quad (18)$$

Due to the completeness of the squared eigenfunctions basis, from (17), (15) and (16) we obtain linear differential equations for the scattering data:

$$\mathcal{R}_t^\pm \mp ik\Omega(\lambda^{-1})\mathcal{R}^\pm(k) = 0, \quad (19)$$

$$R_{n,t}^\pm \pm \kappa_n\Omega(\lambda_n^{-1})R_n^\pm = 0, \quad (20)$$

$$\lambda_{n,t} = 0. \quad (21)$$

The GFT for other integrable systems is derived e.g. in [38, 39, 40, 41, 42, 43].

Example: We now consider the case $\Omega(z) = a_{-1}z^{-1} + a_0 + a_1z$ (where a_j are constants). The (17) equation can then be rewritten as

$$L_\pm \frac{2\partial_\pm^{-1}(\sqrt{q})_t}{\sqrt{q}} + (a_{-1} + a_0L_\pm + a_1L_\pm^2)\left(\sqrt{\frac{\omega}{q}} - 1\right) = 0, \quad (22)$$

Taking the identities

$$\begin{aligned} L_\pm \frac{2\partial_\pm^{-1}(\sqrt{q})_t}{\sqrt{q}} &= -4\partial_\pm^{-1}u_t, & \frac{1}{2}L_\pm\left(\sqrt{\frac{\omega}{q}} - 1\right) &= u, \\ L_\pm u &= -2(1 - \partial^2)^{-1}\partial_\pm^{-1}(uq_x + 2qu_x) \end{aligned}$$

into account, we obtain an integrable equation

$$q_t + a_1(2qu_x + q_xu) - \frac{a_0}{2}q_x - \frac{a_{-1}}{4}(\partial - \partial^3)\sqrt{\frac{\omega}{q}} = 0, \quad (23)$$

which becomes the Camassa-Holm equation (1) with the choice $a_1 = 1$, $a_0 = a_{-1} = 0$. Therefore, (23) can be considered as an integrable 'deformed' version of the CH equation. Another choice for the constants, $a_{-1} = 1$, $a_0 = a_1 = 0$, leads to the extended Dym equation [1, 34, 44]. If $a_1 = 1$, $a_{-1} = 0$ but $a_0 \neq 0$ the equation is usually called Dullin-Gottwald-Holm Equation [7, 8, 18, 19].

The Hamiltonian of (18) with respect to the Poisson bracket related to the Hamiltonian operator J_1 ,

$$\{A, B\} = - \int (\omega + m) \left(\frac{\delta A}{\delta m} \partial \frac{\delta B}{\delta m} - \frac{\delta B}{\delta m} \partial \frac{\delta A}{\delta m} \right) dx, \quad (24)$$

is (see [36])

$$H^\Omega = \int_0^\infty \frac{k^2 \Omega(\lambda^{-1})}{\pi \omega \lambda(k)^2} \ln \left(1 - \mathcal{R}^\pm(k) \mathcal{R}^\pm(-k) \right) dk - \frac{2}{\omega} \sum_{n=1}^N \int \frac{\kappa_n^2}{\lambda_n^2} \Omega(\lambda_n^{-1}) d\kappa_n. \quad (25)$$

In (25) the Hamiltonian is given in terms of the Scattering data. In general, it is not straightforward to find the corresponding expressions in terms of the field variable $q(x)$ (or $m(x)$). For example, the Hamiltonian of (23) with respect to (24) is

$$H^\Omega = a_1 H_1^{CH} - \frac{a_0}{2} I_0 - \frac{a_{-1}}{4} H_{-1}^{CH},$$

where $H_1^{CH} = \frac{1}{2} \int_{-\infty}^\infty m u dx$ is the first CH Hamiltonian (5),

$$H_{-1}^{CH} = \frac{1}{2} \int_{-\infty}^\infty \left[\left(\sqrt[4]{\frac{\omega}{q}} - \sqrt[4]{\frac{q}{\omega}} \right)^2 + \frac{\sqrt{\omega} q_x^2}{4q^{5/2}} \right] dx, \quad (26)$$

is the (-1)-st Hamiltonian for the CH equation, and the integral

$$I_0 = \int_{-\infty}^\infty m dx = H_0^{CH} + 2\omega\alpha, \quad (27)$$

is related to the other two CH integrals [34]

$$H_0^{CH} = \int_{-\infty}^\infty (\sqrt{q} - \sqrt{\omega})^2 dx, \quad \alpha = \int_{-\infty}^\infty \left(\sqrt{\frac{\omega}{q}} - 1 \right) dx.$$

3 Peakon solutions of a (1+2) - dimensional equation from the CH hierarchy

We consider an integrable member of the CH hierarchy with two 'time' variables - t and y (cf. [45])

$$q_t + 2(U_{xy} + \eta U_{xx})q + (U_y + \eta U_x + \gamma)q_x = 0, \quad q = U_x - U_{xxx} + \omega, \quad (28)$$

where ω , γ and η are arbitrary constants. The Lax pair for (28) is

$$\begin{aligned} \Psi_{xx} &= \left(\frac{1}{4} + \lambda(m + \omega) \right) \Psi, \\ \left(\partial_t - \frac{1}{2\lambda} \partial_y \right) \Psi &= - \left(U_y + \eta U_x + \gamma - \frac{\eta}{2\lambda} \right) \Psi_x + \frac{1}{2} (U_{xy} + \eta U_{xx}) \Psi. \end{aligned}$$

The Lax pair represents a non-isospectral problem. Indeed, the equation (28) can be written as a compatibility condition

$$\left(\partial_t - \frac{1}{2\lambda} \partial_y \right) (\Psi_{xx}) = \partial_x^2 \left(\partial_t - \frac{1}{2\lambda} \partial_y \right) \Psi,$$

where λ satisfies the relaxed condition $\lambda_t - \frac{1}{2\lambda}\lambda_y = 0$. But we assume that the spectrum is t - and y -independent in what follows, and we can generalise the solutions obtained for the CH equation.

We can also write (28) as

$$(\sqrt{q})_t + [(U_y + \eta U_x + \gamma)\sqrt{q}]_x = 0. \quad (29)$$

Then

$$\partial_{\pm}^{-1}(\sqrt{q})_t + (U_y + \eta U_x + \gamma)\sqrt{q} + \beta = 0, \quad (30)$$

where β is an integration constant. Further, choosing $\beta = -\gamma\sqrt{\omega}$ and using the identities

$$U_y = -\frac{1}{2}L_{\pm}\left(\frac{\partial_{\pm}^{-1}(\sqrt{q})_y}{\sqrt{q}}\right), \quad U_x = \frac{1}{2}L_{\pm}\left(\sqrt{\frac{\omega}{q}} - 1\right) \quad (31)$$

we can write (28) in the form

$$\frac{\partial_{\pm}^{-1}(\sqrt{q})_t}{\sqrt{q}} - \frac{1}{2}L_{\pm}\left(\frac{\partial_{\pm}^{-1}(\sqrt{q})_y}{\sqrt{q}}\right) + \left(\frac{\eta}{2}L_{\pm} - \gamma\right)\left(\sqrt{\frac{\omega}{q}} - 1\right) = 0. \quad (32)$$

Taking (32), (15) and (16) into account and considering variations with respect to y and t we obtain linear equations for the scattering data:

$$\mathcal{R}_t^{\pm} - \frac{1}{2\lambda}\mathcal{R}_y^{\pm} \pm 2ik\left(\gamma - \frac{\eta}{2\lambda}\right)\mathcal{R}^{\pm} = 0, \quad (33)$$

$$R_{n,t}^{\pm} - \frac{1}{2\lambda_n}R_{n,y}^{\pm} \mp 2\left(\gamma - \frac{\eta}{2\lambda_n}\right)\kappa_n R_n^{\pm} = 0, \quad (34)$$

assuming $\lambda_{n,t} = 0$. For example, if $\gamma = \eta = 0$, then the solution is any function of $t + 2\lambda y$ (with appropriate decaying properties):

$$\mathcal{R}^{\pm}(y, t) = \mathcal{R}^{\pm}(t + 2\lambda y), \quad R_n^{\pm}(y, t) = R_n^{\pm}(t + 2\lambda_n y). \quad (35)$$

We can obtain CH equation itself (1) for $x = y$, $u = U_x$, $\gamma = \eta = 0$.

We demonstrate how to write explicit peakon solutions for this equation ($\gamma = \eta = \omega = 0$). Until now, ω was strictly positive in our considerations, but we can take the limit $\omega \rightarrow 0$ [46] which produces 'peaked' solitons and the equations for the scattering data should also hold in this case. We assume that $x_k = x_k(t, y)$, $p_k = p_k(t, y)$ and introduce the notations $\varepsilon(x) \equiv \text{sign}(x)$, $p'_k(t, y) = \frac{\partial p_k}{\partial y}$ etc. The ansatz that produces the N -peakon solution for the CH equation can be generalised as

$$q(x; t, y) = \sum_{k=1}^N p_k \delta(x - x_k).$$

We hence obtain

$$\begin{aligned}
U(x; t, y) &= \frac{1}{2} \sum_{k=1}^N p_k \varepsilon(x - x_k) (1 - e^{-|x-x_k|}), \\
U_y(x; t, y) &= \frac{1}{2} \sum_{k=1}^N [p'_k \varepsilon(x - x_k) (1 - e^{-|x-x_k|}) - p_k e^{-|x-x_k|} x'_k], \\
U_{xy}(x; t, y) &= \frac{1}{2} \sum_{k=1}^N [p'_k e^{-|x-x_k|} + p_k \varepsilon(x - x_k) e^{-|x-x_k|} x'_k].
\end{aligned}$$

Using this ansatz and the identity

$$f(x)\delta'(x - x_0) = f(x_0)\delta'(x - x_0) - f'(x_0)\delta(x - x_0)$$

we obtain the following system of PDEs for the quantities $x_k(t, y)$, $p_k(t, y)$ from (28):

$$\dot{x}_l = \frac{1}{2} \sum_{k=1}^N [p'_k \varepsilon(x_l - x_k) (1 - e^{-|x_l-x_k|}) - p_k e^{-|x_l-x_k|} x'_k], \quad (36)$$

$$\dot{p}_l = -\frac{1}{2} p_l \sum_{k=1}^N [p'_k e^{-|x_l-x_k|} + p_k \varepsilon(x_l - x_k) e^{-|x_l-x_k|} x'_k], \quad (37)$$

where $\dot{x}_k(t, y) = \frac{\partial x_k}{\partial t}$ etc. The solutions of this system can be obtained from the N -peakon solution for the CH equation [22, 23], where the scattering data are now arbitrary functions of their argument (with appropriate decaying properties): $R_n \equiv R_n^+(t + 2\lambda_n y)$. For example, if $N = 1$, then the system has the form

$$\dot{x}_1 + \frac{1}{2} p_1 x'_1 = 0, \quad \dot{p}_1 + \frac{1}{2} p_1 p'_1 = 0,$$

with a solution $p_1 = -\frac{1}{\lambda_1} = \text{const}$, $x_1 = \ln R_1(t + 2\lambda_1 y)$, cf. [22, 23]. The system for $N = 2$ peakons is (we assume that $x_1 < x_2$ for all y and t , i.e. the case for which the N -peakon solution for the CH equation is obtained in [22, 23])

$$\dot{x}_1 = \frac{1}{2} [-p'_2 (1 - e^{-|x_1-x_2|}) - p_1 x'_1 - p_2 e^{-|x_1-x_2|} x'_2], \quad (38)$$

$$\dot{x}_2 = \frac{1}{2} [p'_1 (1 - e^{-|x_1-x_2|}) - p_1 e^{-|x_1-x_2|} x'_1 - p_2 x'_2], \quad (39)$$

$$\dot{p}_1 = -\frac{1}{2} p_1 [p'_1 + p'_2 e^{-|x_1-x_2|} - p_2 e^{-|x_1-x_2|} x'_2], \quad (40)$$

$$\dot{p}_2 = -\frac{1}{2} p_2 [p'_1 e^{-|x_1-x_2|} + p'_2 + p_1 e^{-|x_1-x_2|} x'_1], \quad (41)$$

with solutions [22, 23]

$$\begin{aligned}
 p_1 &= -\frac{\lambda_1^2 R_1 + \lambda_2^2 R_2}{\lambda_1 \lambda_2 (\lambda_1 R_1 + \lambda_2 R_2)}, & p_2 &= -\frac{R_1 + R_2}{\lambda_1 R_1 + \lambda_2 R_2} \\
 x_1 &= \ln \frac{(\lambda_1 - \lambda_2)^2 R_1 R_2}{\lambda_1^2 R_1 + \lambda_2^2 R_2}, & x_2 &= \ln(R_1 + R_2),
 \end{aligned}$$

where $R_k = R_k(t + 2\lambda_k y)$ (this can be easily verified). We note that the total momentum $p_1 + p_2 = -\frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}$ is conserved. Of course, in general, for an arbitrary N , the N -peakon solution for the CH equation obtained in [22, 23] can be used because the inverse scattering method for hierarchy (18) is the same as for the CH equation [34, 36]. The only difference is the time dependence of the scattering data (and/or the additional y -dependence, etc.). In this example, y has the meaning of a second 'time' variable. Clearly, the conserved quantities in terms of x_k and p_k have the same form as those for the CH peakons and can be expressed in terms of the quantities λ_k , which we already assumed to be independent of y and t . But the Hamiltonian formulation is problematic because formally $\Omega(z) \equiv 0$ for the peakon solution and the Hamiltonian with respect to (24) is degenerate because of (25). Moreover, the right-hand side of system (38) – (41) involves not only the quantities x_k and p_k but also their y -derivatives.

The explicit dependence on the scattering data given in [36] can be used in the same way for the N -soliton solutions of the CH hierarchy. The situation where the initial data condition $q(x, 0) \equiv m(x, 0) + \omega > 0$ does not hold is more complicated and requires a separate analysis [4, 35, 47].

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