2011-01-01

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Vladimir Gerdjikov
Bulgarian Academy of Sciences, gerjikov@inrne.bas.bg

Georgi Grahovski
Technological University Dublin, georgi.grahovski@tudublin.ie

Rossen Ivanov
Technological University Dublin, rossen.ivanov@tudublin.ie

Recommended Citation

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On the (Non)-Integrability of KdV Hierarchy with Self-consistent Sources

V. S. Gerdjikov[†], G. G. Grahovski[†,‡] and R. I. Ivanov[‡]

[†] *Institute for Nuclear Research and Nuclear Energy,*
Bulgarian Academy of Sciences, 72 Tsarigradsko chaussee,
1784 Sofia, BULGARIA
[‡] *School of Mathematical Sciences, Dublin Institute of Technology,*
Kevin Street, Dublin 8, IRELAND

Abstract

Non-holonomic deformations of integrable equations of the KdV hierarchy are studied by using the expansions over the so-called "squared solutions" (squared eigenfunctions). Such deformations are equivalent to perturbed models with external (self-consistent) sources. In this regard, the KdV6 equation is viewed as a special perturbation of KdV equation. Applying expansions over the symplectic basis of squared eigenfunctions, the integrability properties of the KdV hierarchy with generic self-consistent sources are analyzed. This allows one to formulate a set of conditions on the perturbation terms that preserve the integrability. The perturbation corrections to the scattering data and to the corresponding action-angle variables are studied. The analysis shows that although many nontrivial solutions of KdV equations with generic self-consistent sources can be obtained by the Inverse Scattering Transform (IST), there are solutions that, in principle, can not be obtained via IST. Examples are considered showing the complete integrability of KdV6 with perturbations that preserve the eigenvalues time-independent. In another type of examples the soliton solutions of the perturbed equations are presented where the perturbed eigenvalue depends explicitly on time. Such equations, however in general, are not completely integrable.

1 Introduction

Nonholonomic deformations of integrable equations attracted the attention of the scientific community in the last few years. In [22], based on the Painlevé test, applied to a class of sixth-order nonlinear wave equations, a list of four equations that pass the test was obtained. Among the three known ones, there was a new equation in the list (later known as sixth-order KdV equation, or just KdV6):

$$\left(-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}\right)(v_t + v_{xxx} - 3v_x^2) = 0. \tag{1}$$

Recognising the potential KdV (PKdV) equation in the second factor in (1) and the Nijenhuis recursion operator for PKdV $\mathcal{N}_{\text{PKdV}} = (-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx})\partial_x^{-1}$, after change of variables $v = v_x$ and $w = v_t + v_{xxx} - 3v_x^2$, one can convert (1) into a "potential" form:

$$u_t + u_{xxx} - 6uu_x - w_x = 0, \tag{2}$$

$$-\frac{1}{4}w_{xxx} + uw_x + \frac{1}{2}wu_x = 0, \tag{3}$$

or equivalently

$$\left(-\frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial^{-1}\right)(u_t + u_{xxx} - 6uu_x) = 0. \tag{4}$$

Here ∂_x^{-1} is a notation for the left-inverse of ∂_x: $\partial_x^{-1}f(x) = \int^x f(y)\,dy$. Note also, that ∂_x^{-1} is a Hamiltonian operator and $-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$ is a symplectic operator. Since both operators $-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$ and $\mathcal{N}_{\text{PKdV}}$ are weakly nonlocal, then eqn. (1) possesses the same recursion operator as the PKdV equation [22, 45] and therefore infinitely many integrals of motion. In the same paper [22], a Lax pair and auto-Bäcklund transformations are obtained for KdV6 equation, (1). In [43], applying Hirota bilinear method to KdV6 equation, another class (much simpler) of auto-Bäcklund transformation is obtained.

In [33] B. Kupershmidt described (2) and (3) as a nonholonomic of the KdV equation, written in a bi-Hamiltonian form:

$$v_t = B^1\left(\frac{\delta H_{n+1}}{\delta v}\right) - B^1(w) = B^2\left(\frac{\delta H_n}{\delta v}\right) - B^1(w), \qquad B^2(w) = 0 \tag{5}$$

where

$$B^1 = \partial_x, \qquad B^2 = -\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$$

are the two standard Hamiltonian operators of the KdV hierarchy, and

$$H_1 = u, \qquad H_2 = \frac{u^2}{2}, \qquad H_3 = \frac{u^3}{3} - \frac{u_x^2}{2}, \dots$$

are the conserved densities for the same hierarchy. The recursion operator for the KdV hierarchy is given (after a suitable rescaling) by $\Lambda = B^2 \circ (B^1)^{-1}$. A more general setup suitable for non-Hamiltonian systems of equations is presented in [45].

Later on, it was shown in [47, 48] that the KdV6 equation is equivalent to a Rosochatius deformation of the KdV equation with self-consistent sources. Soliton equations with self-consistent sources have many physical applications [39, 40, 36, 48]. For example, they describe the interaction of long and short capillary-gravity waves [39]. Different classes of exact solutions of (1) are obtained in [22, 29, 30, 31, 32]. A geometric interpretation of the KdV6 equation is given in [17].

In the present paper, exploiting the potential form of the KdV6 equation (2), we study the class of inhomogeneous equations of KdV type

$$u_t + u_{xxx} - 6uu_x = W_x[u](x), \tag{6}$$

with an inhomogeneity/perturbation that presumably belongs to the same class of functions as the field $u(x)$ (i.e. decreasing fast enough, when $|x| \to \infty$).

Generally speaking, the perturbation, as a rule destroys the integrability of the considered nonlinear evolution equation (NLEE). The idea of perturbation through nonholonomic deformation, however, is to perturb an integrable NLEE with a driving force (deforming function), such that under suitable differential constraints on the perturbing function(s) the integrability of the entire system is preserved. In the case of local NLEE's, having a constraint given through differential relations (not by evolutionary equations) is equivalent to a nonholonomic constraint.

To the best of our knowledge, the most natural and efficient way for studying inhomogeneities/perturbations of NLEE integrable by the inverse scattering method is by

using the expansions over the so-called "squared solutions" (squared eigenfunctions). The squared eigenfunctions of the spectral problem associated to an integrable equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform (IST) for the corresponding hierarchy as a Generalized Fourier transform (GFT). The Fourier modes for the GFT are the Scattering data. Thus all the fundamental properties of an integrable equation such as the integrals of motion, the description of the equations of the whole hierarchy and their Hamiltonian structures can be naturally expressed making use of the completeness relation for the squared eigenfunctions and the properties of the corresponding recursion operator.

This approach was developed first for the Zakharov-Shabat system [24, 25, 26, 27, 28, 13, 41]. In particular, in [13] a special combination of squared solutions named the 'symplectic basis' was introduced. Its special property consists in mapping the variation of the potential of the Zakharov-Shabat system into the variations of the action-angle variables.

Later such basis was introduced also for the Sturm-Liouville's operator [28, 1, 18]. Note that the scattering data of the Sturm-Liouville's operator in a generic situation have a pole for $k = 0$ and their variational derivatives with respect to the potential do not vanish when $|x| \to \infty$. This can lead to to nontrivial contributions to the associated symplectic and Poisson structures, coming from the boundary terms [1, 9]. For more details, we refer to [18, 14] and the references therein.

The expansions over the squared solutions have been used for the study of the perturbations of various completely integrable systems, see [26, 41, 27]. As important particular cases these authors analyzed NLEE with singular dispersion relations and have shown that whenever the dispersion relation has a pole located on a given eigenvalue λ of the corresponding Lax operator L, then the corresponding evolution is no longer isospectral. Further generalizations are made in [12, 11]. An alternative approach to the same type of equations has been proposed by Leon in [34, 35, 37, 38]; it is based on the equivalence of the Zakharov-Shabat system to a $\bar{\partial}$-problem.

The structure of the present paper is as follows: In Section 2 we give some background material about the direct scattering problem for KdV hierarchy. In Section 3 we describe briefly the generalized Fourier transform for the equations of KdV hierarchy as a key point for their integrability. This includes the minimal set of scattering data, their time evolution, the expansions over the complete set of squared solutions and over the symplectic basis, action-angle variables, etc. In the next Section 4 we formulate general conditions on the perturbations (driving forces) for preserving integrability, treating separately integrable and non-integrable cases. We provide also a set of nontrivial examples for both cases. Then, in Section 5 we present examples for 1-soliton solutions of some of the perturbed systems from Section 4.

2   Preliminaries: Integrability of the Equations of KdV Hierarchy

The spectral problem for the KdV hierarchy is given by the Sturm-Liouville equation [42, 18]

$$- \Psi_{xx} + u(x)\Psi = k^2\Psi, \tag{7}$$

where $u(x)$ is a real-valued (Schwartz-class) potential on the whole axia and $k \in \mathbb{C}$ is spectral parameter. The continuous spectrum under these conditions corresponds to real k. We assume that the discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \ldots, N$ where κ_n is real.

The direct scattering problem for (7) is based on the so-called "Jost solutions" $f^+(x, k)$ and $\bar{f}^+(x, \bar{k})$,given by their asymptotics: $x \to \infty$ for all real $k \neq 0$ [42]:

$$\lim_{x \to \pm\infty} e^{-ikx} f^{\pm}(x, k) = 1, \qquad k \in \mathbb{R}\backslash\{0\}. \tag{8}$$

From the reality condition for $u(x)$ it follows that $\bar{f}^{\pm}(x, \bar{k}) = f^{\pm}(x, -k)$.

On the continuous spectrum of the problem (7) and the vectors of the two Jost solutions are linearly related [1]:

$$f^-(x, k) = a(k)f^+(x, -k) + b(k)f^+(x, k), \qquad \text{Im } k = 0. \tag{9}$$

In addition, for real $k \neq 0$ we have:

$$\bar{f}^{\pm}(x, k) = f^{\pm}(x, -k). \tag{10}$$

The quantities $\mathcal{R}^{\pm}(k) = b(\pm k)/a(k)$ are known as reflection coefficients (to the right with superscript $(+)$ and to the left with superscript $(-)$ respectively). It is sufficient to know $\mathcal{R}^{\pm}(k)$ only on the half line $k > 0$. Furthermore, $\mathcal{R}^{\pm}(k)$ uniquely determines $a(k)$ [42]. At the points κ_n of the discrete spectrum, $a(k)$ has simple zeroes i.e.:

$$a(k) = (k - i\kappa_n)\dot{a}_n + \frac{1}{2}(k - i\kappa_n)^2\ddot{a}_n + \cdots, \tag{11}$$

Here, the dot stands for a derivative with respect to k and $\dot{a}_n \equiv \dot{a}(i\kappa_n)$, $\ddot{a}_n \equiv \ddot{a}(i\kappa_n)$, etc. At the points of the discrete spectrum f^- and f^+ are again linearly dependent:

$$f^-(x, i\kappa_n) = b_n f^+(x, i\kappa_n). \tag{12}$$

The discrete spectrum is simple, there is only one (real) linearly independent eigenfunction, corresponding to each eigenvalue $i\kappa_n$, say $f_n^-(x) \equiv f^-(x, i\kappa_n)$.

The sets

$$\mathcal{S}^{\pm} \equiv \left\{ \mathcal{R}^{\pm}(k) \quad (k > 0), \quad \kappa_n, \quad R_n^{\pm} \equiv \frac{b_n^{\pm 1}}{i\dot{a}_n}, \quad n = 1, \ldots N \right\} \tag{13}$$

are known as scattering data. Each set – $\mathcal{S}^+$ or $\mathcal{S}^-$ of scattering data uniquely determines the other one and also the potential $u(x)$ [42, 18, 49].

KdV equation appeared initially as models of the propagation of two- dimensional shallow water waves over a flat bottom. More about the physical relevance of the KdV equation can be found e.g. in [4, 19, 21, 7, 20].

3   Generalized Fourier Transforms

A key role in the interpretation of the inverse scattering method as a generalized Fourier transform plays the so-called 'generating' (recursion) operator: for the KdV hierarchy it has the form [1]:

$$L_{\pm} = -\frac{1}{4}\partial^2 + u(x) - \frac{1}{2}\int_{\pm\infty}^{x} d\tilde{x}\, u'(\tilde{x}) \cdot . \tag{14}$$

---

[1]According to the notations used in [42] $f^+(x, k) \equiv \bar{\psi}(x, \bar{k})$, $f^-(x, k) \equiv \varphi(x, k)$.

The eigenfunctions of the recursion operator are the squared eigenfunctions of the spectral problem (7):

$$F^{\pm}(x,k) \equiv (f^{\pm}(x,k))^2, \qquad F_n^{\pm}(x) \equiv F(x,i\kappa_n), \tag{15}$$

From the completeness of the squared eigenfunctions it follows that every function $g(x)$, belonging to the same class of functions as the potential $u(x)$ of the Lax operator, can be expanded over the two complete sets of "squared solutions" [18]:

$$g(x) = \pm\frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{g}^{\pm}(k) F_x^{\pm}(x,k) \mathrm{d}k \mp \sum_{j=1}^{N} \left(g_{1,j}^{\pm} \dot{F}_{j,x}^{\pm}(x) + g_{2,j}^{\pm} F_{j,x}^{\pm}(x) \right). \tag{16}$$

where $\dot{F}_j^{\pm}(x) \equiv [\frac{\partial}{\partial k} F^{\pm}(x,k)]_{k=k_j}$ and the Fourier coefficients are

$$\tilde{g}^{\pm}(k) = \frac{1}{ka^2(k)} \left(g, F^{\mp} \right), \quad \text{where} \quad (g,F) \equiv \int_{-\infty}^{\infty} g(x) F(x) \mathrm{d}x,$$

$$g_{1,j}^{\pm} = \frac{1}{k_j \dot{a}_j^2} \left(g, F_j^{\mp} \right), \qquad g_{2,j}^{\pm} = \frac{1}{k_j \dot{a}_j^2} \left[\left(g, \dot{F}_j^{\mp} \right) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j} \right) \left(g, F_j^{\mp} \right) \right].$$

Here we assume that in addition f^+ and f^- are not linearly dependent at $x = 0$. The details of the derivation can be found e.g. in [8, 18].

In particular one can expand the potential $u(x)$, the coefficients are given through the scattering data [8, 18]:

$$u(x) = \pm\frac{2}{\pi i} \int_{-\infty}^{\infty} k\mathcal{R}^{\pm}(k) F^{\pm}(x,k) \mathrm{d}k + 4i \sum_{j=1}^{N} k_j R_j^{\pm} F_j^{\pm}(x), \tag{17}$$

The variation $\delta u(x)$ under the assumption that the number of the discrete eigenvalues is conserved is

$$\delta u(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \delta\mathcal{R}^{\pm}(k) F_x^{\pm}(x,k) \mathrm{d}k \pm 2 \sum_{j=1}^{N} \left[R_j^{\pm} \delta k_j \dot{F}_{j,x}^{\pm}(x) + \delta R_j^{\pm} F_{j,x}^{\pm} \right]. \tag{18}$$

An important subclass of variations are due to the time-evolution of u, i.e. effectively we consider a one-parametric family of spectral problems, allowing a dependence of an additional parameter t (time). Then $\delta u(x,t) = u_t \delta t + Q((\delta t)^2)$, etc. The equations of the KdV hierarchy can be written as

$$u_t + \partial_x \Omega(L_{\pm}) u(x,t) = 0, \tag{19}$$

where $\Omega(k^2)$ is a rational function specifying the dispersion law of the equation. The substitution of (18) and (17) in (19) gives a system of trivial linear ordinary differential equations for the scattering data:

$$\mathcal{R}_t^{\pm} \pm 2ik\Omega(k^2)\mathcal{R}^{\pm} = 0, \tag{20}$$
$$R_{j,t}^{\pm} \pm 2ik_j\Omega(k_j^2)R_j^{\pm} = 0, \tag{21}$$
$$k_{j,t} = 0. \tag{22}$$

The KdV equation $u_t - 6uu_x + u_{xxx} = 0$ can be obtained for $\Omega(k^2) = -4k^2$.

Once the scattering data are determined from (20) – (22) one can recover the solution from (18). Thus the Inverse Scattering Transform can be viewed as a GFT.

For our purposes, it is more convenient to adopt a special set of "squared solutions", called symplectic basis [13, 18]. It has the property that the expansion coefficients of the potential $u(x)$ over the symplectic basis are the so-called action-angle variables for the corresponding NLEE.

For the KdV hierarchy, the symplectic basis is given by:

$$\mathcal{P}(x,k) = \mp \left(\mathcal{R}^{\pm}(k)F^{\pm}(x,k) - \mathcal{R}^{\pm}(-k)F^{\pm}(x,-k) \right), \tag{23}$$

$$\mathcal{Q}(x,k) = \mathcal{R}^{-}(k)F^{-}(x,k) + \mathcal{R}^{+}(k)F^{+}(x,k), \tag{24}$$

$$P_n(x) = -R_n^{\pm} F_n^{\pm}(x), \qquad Q_n(x) = -\frac{1}{2k_n}\left(R_n^{+}\dot{F}_n^{-}(x) - R_n^{-}\dot{F}_n^{+}(x) \right). \tag{25}$$

Its elements satisfy the following canonical relations:

$$[\![\mathcal{P}(k_1),\mathcal{Q}(k_2)]\!] = \delta(k_1 - k_2), \qquad [\![\mathcal{P}(k_1),\mathcal{P}(k_2)]\!] = [\![\mathcal{Q}(k_1),\mathcal{Q}(k_2)]\!] = 0, \tag{26}$$

$$[\![P_m, Q_n]\!] = \delta_{mn}, \qquad [\![P_m, P_n]\!] = [\![Q_m, Q_n]\!] = 0, \tag{27}$$

$(k_1 > 0, k_2 > 0)$ with respect to the skew-symmetric product

$$[\![f,g]\!] \equiv \frac{1}{2}\int_{-\infty}^{\infty}(f(x)g_x(x) - g(x)f_x(x))\mathrm{d}x = \int_{-\infty}^{\infty}f(x)g_x(x)\mathrm{d}x, \tag{28}$$

The symplectic basis satisfies the completeness relation [18]:

$$\frac{\theta(x-y) - \theta(y-x)}{2} = \frac{1}{2\pi}\int_0^{\infty}\left(\mathcal{P}(x,k)\mathcal{Q}(y,k) - \mathcal{Q}(x,k),\mathcal{P}(y,k) \right)\frac{\mathrm{d}k}{\beta(k)}$$

$$- \sum_{n=1}^{N}\left(P_n(x)Q_n(y) - Q_n(x)P_n(y) \right), \tag{29}$$

where $\beta(k) = 2ikb(k)b(-k)$. Notice that the integration over k is from 0 to ∞. It follows that every function $X(x)$ from the same class as the potential $u(x)$ (i.e. smooth and vanishing fast enough when $x \to \pm\infty$) can be expanded over the symplectic basis:

$$X(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{\mathrm{d}k}{\beta(k)}\left(\mathcal{P}(x,k)\phi_X(k) - \mathcal{Q}(x,k)\rho_X(k) \right)$$

$$- \sum_{n=1}^{N}\left(P_n(x)\phi_{n,X} - Q_n(x)\rho_{n,X} \right). \tag{30}$$

The expansion coefficients can be recovered from the so-called inversion formulas:

$$\phi_X(k) = [\![\mathcal{Q}(y,k), X(y)]\!], \qquad \rho_X(k) = [\![\mathcal{P}(y,k), X(y)]\!],$$
$$\phi_{n,X} = [\![Q_n(y), X(y)]\!], \qquad \rho_{n,X} = [\![P_n(y), X(y)]\!]. \tag{31}$$

In particular, if $X(x) = u(x)$ is a solution of the spectral problem, one can compute [18]:

$$[\![\mathcal{P}(y,k), u(y)]\!] = 0, \qquad [\![\mathcal{Q}(y,k), u(y)]\!] = -4ik\beta(k), \tag{32}$$
$$[\![P_n(y), u(y)]\!] = 0, \qquad [\![Q_n(y), u(y)]\!] = 4ik_n. \tag{33}$$

Thus, from (32) one gets:

$$u(x) = \frac{2}{\pi i} \int_0^\infty \mathcal{P}(x,k)\mathrm{d}k - \sum_{n=1}^N 4ik_n P_n(x). \tag{34}$$

The expression for the variation of the potential is

$$\delta u(x) = \frac{1}{2\pi} \int_0^\infty \frac{\mathrm{d}k}{\beta(k)} \left(\mathcal{P}_x(x,k)\delta\phi(k) - \mathcal{Q}_x(x,k)\delta\rho(k) \right)$$

$$- \sum_{n=1}^N \left(P_{n,x}\delta\phi_n - Q_{n,x}\delta\rho_n \right) \tag{35}$$

with expansion coefficients

$$\rho(k) \equiv -2ik \ln|a(k)| = -2ik \ln(1 - \mathcal{R}^-(k)\mathcal{R}^-(-k)), \quad k > 0, \tag{36}$$

$$\phi(k) \equiv 2i\beta(k) \arg b(k) = \beta(k) \ln \frac{\mathcal{R}^-(k)}{\mathcal{R}^+(k)}, \tag{37}$$

$$\rho_n = -\lambda_n = -k_n^2, \qquad \phi_n = 2\ln b_n = \ln \frac{R_n^-}{R_n^+}. \tag{38}$$

These are known as action $(\rho(k))$ - angle $(\phi(k))$ variables for KdV equation [49, 42]. They satisfy the canonical relations

$$\{\rho(k_1), \phi(k_2)\} = \delta(k_1 - k_2) \qquad \{\rho(k_1), \rho(k_2)\} = \{\phi(k_1), \phi(k_2)\} = 0, \tag{39}$$

$$\{\rho_m, \phi_n\} = \delta_{mn}, \qquad \{\rho_m, \rho_n\} = \{\phi_m, \phi_n\} = 0. \tag{40}$$

$(k_1 > 0, k_2 > 0)$ with respect to the Poisson bracket:

$$\{A, B\} = \left[\!\!\left[\frac{\delta A}{\delta u}, \frac{\delta B}{\delta u} \right]\!\!\right].$$

One can verify that the scattering data determines the canonical variables and vice-versa. The symplectic basis consists of the eigenfunctions of the operator $\Lambda = \frac{1}{2}(L_+ + L_-)$:

$$\Lambda \mathcal{P}(x,k) = k^2 \mathcal{P}(x,k), \tag{41}$$
$$\Lambda \mathcal{Q}(x,k) = k^2 \mathcal{Q}(x,k), \tag{42}$$
$$\Lambda P_n(x) = k_n^2 P_n(x), \tag{43}$$
$$\Lambda Q_n(x) = k_n^2 Q_n(x). \tag{44}$$

Again, for variations due to the time-evolution of u, $\delta u(x,t) = u_t \delta t + Q((\delta t)^2)$, etc. the equations of the KdV hierarchy

$$u_t + \partial_x \Omega(\Lambda) u(x,t) = 0, \tag{45}$$

with (34) and (35) are equivalent to a system of trivial linear ordinary differential equations for the canonical variables (which can be considered as scattering data):

$$\begin{aligned} \phi_t &= 4i\beta(k)\Omega(k^2), & \rho_t(k) &= 0, \\ \phi_{n,t} &= 4ik_n\Omega(k_n^2), & \rho_{n,t} &= 0. \end{aligned} \tag{46}$$

7

4 Perturbations to the equations of the KdV hierarchy

4.1 General perturbations

Let us consider a general perturbation $W_x[u]$ to an equation from the KdV hierarchy:

$$u_t + \partial_x \Omega(\Lambda) u(x,t) = W_x[u]. \tag{47}$$

The function $W_x[u]$ is assumed to belong to the class of admissible potentials for the associated spectral problem (7) (Schwartz class functions, in our case).

The expansion of the perturbation over the symplectic basis is:

$$W_x[u] = \frac{1}{2\pi} \int_0^\infty \frac{\mathrm{d}k}{\beta(k)} \left(\mathcal{P}_x(x,k)\phi_W(k) - \mathcal{Q}_x(x,k)\rho_W(k) \right) \tag{48}$$

$$+ \phi_0 \mathcal{P}_x(x,0) - \sum_{n=1}^N \left(P_{n,x}(x)\phi_{n,W} - Q_{n,x}(x)\rho_{n,W} \right).$$

The substitution of the above expansion (48) in (47) together with (34) and (35) leads to a modification of the time evolution (46) of the scattering data as follows:

$$\phi_t = 4i\beta(k)\Omega(k^2) + \phi_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)) + \phi_0\delta(k), \tag{49}$$

$$\rho_t(k) = \rho_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), \tag{50}$$

$$\phi_{n,t} = 4ik_n\Omega(k_n^2) + \phi_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), \tag{51}$$

$$\rho_{n,t} = \rho_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)). \tag{52}$$

Since $W = W[u]$ and u depend on the scattering data, we observe that the expansion coefficients of the perturbation ($\phi_W(k) = [\![\mathcal{Q}(y,k),W(y)]\!]$ etc.) also depend on the scattering data. Thus for generic W the new dynamical system (49) – (52) for the scattering data can be extremely complicated and non-integrable in general. This reflects the obvious fact that the perturbed integrable equations are, in general, not integrable.

Below we consider several important particular cases.

4.2 Self-consistent sources – integrable case

It is widely spread that KdV with self-consistent sources is a special case of perturbed KdV with

$$W_x[u] = \sum_{n=1}^N c_n \mathcal{P}_{n,x}(x) + c_0 \mathcal{P}_x(x,0), \tag{53}$$

where c_n are some constants and the constant c_0 is usually set to 0. The last term in (53) affects only the end-point of the continuous spectrum of L. Such choice of the perturbation substantially simplifies the evolution of the scattering data:

$$\begin{aligned} \phi_t &= 4i\beta(k)\Omega(k^2) - c_0\delta(k), & \rho_t(k) &= 0 \\ \phi_{n,t} &= 4ik_n\Omega(k_n^2) - c_n, & \rho_{n,t} &= 0. \end{aligned} \tag{54}$$

It is obvious that perturbations (53) preserve integrability. Indeed, from eqs. (54) one finds that such perturbations are isospectral, since $k_{n,t} = 0$. The only effect of the perturbation (53) consists in changing the time-dependence of the angle variables, related to the discrete spectrum of L.

Example 1. *Let $c_0 \neq 0$ and all other $c_n = 0$. Then $W_x[u] = c_0 P(x, 0)$. Since $\Lambda P(x, 0) = 0$ (see eq. (37)) we easily conclude, that*

$$\Lambda(v_t + v_{xxx} - 3(v^2)_x - c_0 \mathcal{P}(x, 0)) = \Lambda(v_t + v_{xxx} - 3(v^2)_x) = 0, \tag{55}$$

i.e such perturbed KdV is equivalent to the KdV6 equation. Thus from the above considerations we confirm the result in [33], that KdV6 is integrable.

4.3 Self-consistent sources – non-integrable case

Here we consider a more general type of self-consistent sources given by:

$$W_x[u] = \sum_{n=1}^{N}(c_n \mathcal{P}_{n,x}(x) + \tilde{c}_n \mathcal{Q}_{n,x}(x)) + c_0 \mathcal{P}_x(x, 0), \tag{56}$$

Again we consider c_n, $\tilde{c}_n$ and c_0 as constants. Note, that $\mathcal{Q}_x(x, k)$ is an odd function of k and therefore $\mathcal{Q}_x(x, 0) = 0$. The evolution of the scattering data under such perturbation is:

$$\begin{aligned} \phi_t &= 4i\beta(k)\Omega(k^2) - c_0\delta(k), & \rho_t(k) &= 0 \\ \phi_{n,t} &= 4ik_n\Omega(k_n^2) - c_n, & \rho_{n,t} &= \tilde{c}_n. \end{aligned} \tag{57}$$

i.e both the action variables ρ_n and the discrete eigenvalues become time-dependent. Such perturbation is not isospectral and obviously violates integrability. In addition the angle variables become nonlinear functions of t.

Let us investigate the integrability of the following equation:

$$\Lambda^*(u_t + \partial_x \Omega(\Lambda)u(x, t)) = 0, \tag{58}$$

where the star is a notation for a Hermitian conjugation. KdV6 in (4) is a particular case of this equation with $\Omega(\Lambda) = -4\Lambda$. In order to simplify our further analysis, instead of the equation (58) we study the following one:

$$(\Lambda^* - \lambda_1)(u_t + \partial_x \Omega(\Lambda)u(x, t)) = 0, \tag{59}$$

where λ_1 is a constant. The corresponding analogue for KdV6 is

$$v_{6x} + v_{txxx} - 2v_t v_{xx} - 4v_x v_{xt} - 10v_x v_{4x} - 20v_{xx}v_{xxx} + 30v_x^2 v_{xx}$$
$$+ 4\lambda_1(v_{xt} + v_{xxxx} - 6v_x v_{xx}) = 0.$$

Due to the identity $\partial \Lambda = \Lambda^* \partial$ we can represent (59) in the form

$$\partial(\Lambda - \lambda_1)((\partial_x^{-1} u_t) + \Omega(\Lambda)u(x, t)) = 0. \tag{60}$$

Since the operator ∂ does not have a kernel when u is Schwartz class, (60) is equivalent to

$$u_t + \partial_x \Omega(\Lambda)u(x, t) = \begin{cases} (c_1 P_1(x, t) + \tilde{c}_1 Q_1(x, t))_x & \text{for} \quad \lambda_1 = k_1^2 < 0, \\ (c_1 \mathcal{P}(x, k_1, t) + \tilde{c}_1 \mathcal{Q}(x, k_1, t))_x & \text{for} \quad \lambda_1 = k_1^2 > 0, \end{cases} \tag{61}$$

where

$$\begin{aligned} c_1 &= c_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t)), \\ \tilde{c}_1 &= \tilde{c}_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t)) \end{aligned}$$

are x-independent functions, but the important observation is that the time-dependence could be implicit through the scattering data of the potential $u(x,t)$. Equation (61) is a perturbed equation from the KdV hierarchy. The perturbation in the right-hand side of (61) is in the eigenspace of the recursion operator corresponding to the eigenvalue λ_1, i.e. it is given by 'squared' eigenfunctions of the spectral problem (7) at λ_1.

If $\lambda_1 < 0$ is a discrete eigenvalue, the corresponding dynamical system (49) – (52) for the scattering data is

$$\phi_t = 4i\beta(k)\Omega(k^2), \tag{62}$$

$$\rho_t(k) = 0, \tag{63}$$

$$\phi_{n,t} = 4ik_n\Omega(k_n^2) - c_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t))\delta_{n,1}, \tag{64}$$

$$\rho_{n,t} = \tilde{c}_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t))\delta_{n,1}. \tag{65}$$

If $\lambda_1 > 0$ is a continuous spectrum eigenvalue, the dynamical system (49) – (52) has the form

$$\phi_t = 4i\beta(k)\Omega(k^2)$$
$$+ 2\pi\beta(k)c_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t))\delta(k - k_1), \tag{66}$$

$$\rho_t(k) = -2\pi\beta(k)\tilde{c}_1(t, k_1; \rho(k_1, t), \phi(k_1, t), \rho_n(t), \phi_n(t))\delta(k - k_1), \tag{67}$$

$$\phi_{n,t} = 4ik_n\Omega(k_n^2), \tag{68}$$

$$\rho_{n,t} = 0. \tag{69}$$

It is clear, that dynamical systems like (62) – (65) or (66) – (69) can not be integrable for a general functional dependence of c_1 and $\tilde{c}_1$ on the scattering data. Thus the generic perturbed equations from KdV Hierarchy(58), are not *completely* integrable. In other words, there are solutions, which can not be obtained via the Inverse Scattering Method, since the aforementioned dynamical systems for the scattering data are not always integrable.

Example 2. *Let us assume that $\lambda_1 = -\kappa_1^2 < 0$ is a discrete eigenvalue of L, and that c_1 and $\tilde{c}_1$ in eq. (61) are constants. Then the corresponding perturbed KdV will be equivalent to the following equations for the scattering data of L:*

$$\begin{aligned} \phi_t &= 4i\beta(k)\Omega(k^2), & \rho_t(k) &= 0, \\ \phi_{n,t} &= 4ik_n\Omega(k_n^2) - c_1\delta_{n,1}, & \rho_{n,t} &= \tilde{c}_1\delta_{n,1}. \end{aligned} \tag{70}$$

From the last of these equations we find

$$\kappa_1^2(t) = \tilde{c}_1 t + \kappa_1^2(0), \tag{71}$$

i.e the eigenvalue $\kappa_1(t)$ explicitly depends on t. Clearly (71) makes sense for those t for which $\tilde{c}_1 t + \kappa_1^2(0) > 0$. Then (i) if $\tilde{c}_1 t + \kappa_1^2(0) < 0$ the eigenvalue overlaps with the continuous spectrum of L and the solution develops singularities; (ii) if for some t κ_1 crosses another discrete eigenvalue the spectrum cease to be simple and the solution is not in the assumed Schwartz class, i.e. again may develop singularities.

Example 3. *Let us assume that $\lambda_1 = -\kappa_1^2 < 0$ is the discrete eigenvalue of L, and that c_1 is constant and $\tilde{c}_1$ is a function of time, for example:*

$$\tilde{c}_1(t) = \epsilon_0\kappa_1^2(0)\omega\cos(\omega t) \tag{72}$$

where $\epsilon_0 \ll 1$ *and* ω_0 *are constants. Then the eigenvalue* κ_1^2 *depends explicitly on* t *according to*

$$\kappa_1^2(t) = \kappa_1^2(0)(1 + \epsilon_0 \sin(\omega t)). \tag{73}$$

If ϵ_0 *is chosen small enough the corresponding eigenvalue will not overlap with the continuous spectrum, nor with other discrete eigenvalues. Otherwise - as above and again we may have singularities.*

5   Examples of exact solutions

As we have seen, KdV6 in general is not completely integrable system. However, there are many solutions of (58) or (60) that can be written explicitly (apart from the KdV solutions, which are an obvious subclass of solutions).

One such example is when c_1 and $\tilde{c}_1$ depend only on t. Then the system (62) – (65) is integrable, with new canonical variables

$$\tilde{\phi}_n = \phi_n(0) + 4ik_n\Omega(k_n^2)t - \delta_{n,1}\int_0^t c_1(\tau)d\tau,$$

$$\tilde{\rho}_n = \rho_n(0) + \delta_{n,1}\int_0^t \tilde{c}_1(\tau)d\tau.$$

Since $c_1(t)$ and $\tilde{c}_1$ do not depend on the scattering data and therefore on $u(x,t)$, the Poisson brackets between $\tilde{\phi}_n$ and $\tilde{\rho}_n$ are canonical.

This is most often the situation that many authors assume about KdV6 (i.e. $c_1 = c_1(t)$ and $\tilde{c}_1 = \tilde{c}_1(t)$ simply given functions of t) when discuss its integrability.

There are, however other exactly solvable cases. Indeed, if

$$c_1 = \sum_{k=1}^N (\alpha_{1k}\phi_k + \beta_{1k}\rho_k),$$

$$\tilde{c}_1 = \sum_{k=1}^N (\alpha_{2k}\phi_k + \beta_{2k}\rho_k)$$

are linear combinations of the scattering data with some constants $\alpha_{p,q}$, $\beta_{p,q}$ the system (62) – (65) becomes a system of linear ordinary ODEs and therefore is integrable.

To illustrate the effect of non-holonomic deformations on the soliton solutions, let us consider the 1-soliton solutions corresponding to the specific choice of perturbation functions c_1 and $\tilde{c}_1$ considered in examples 2 and 3 in the previous Section. The one-soliton solution for the KdV equation (corresponding to a discrete eigenvalue $\lambda_1 = -\kappa_1^2$) is given by [42]:

$$u_{1s}(x,t) = -\frac{2\kappa_1^2}{\text{ch}^2 \kappa_1 \left(x - 4\kappa_1^2 t - \frac{\phi_1}{\kappa_1}\right)}, \tag{74}$$

where ϕ_1 is the angle variable that corresponds to the prescribed eigenvalue $\lambda_1 = -\kappa_1^2$. Here, also, the quantity $v = 4\kappa_1^2$ can be interpreted as a velocity of the soliton, ϕ_1 – as a position of soliton's mass-center at $t = 0$ (it is often called a soliton phase).

For the special choice of constants in *Example 2* one can write down the corresponding perturbed 1-soliton solutionin the following form:

$$\tilde{u}_{1\text{s}}(x,t) = -\frac{2(\tilde{c}_1 t + \kappa_1(0)^2)}{\text{ch}^2\left\{\sqrt{\tilde{c}_1 t + \kappa_1(0)^2}\left[x - \left(4\kappa_1^2 - \frac{\tilde{c}_1}{\kappa_1}\right)t - \frac{\phi_1}{\kappa_1}\right]\right\}}. \tag{75}$$

Here the correction in the velocity of the soliton $\tilde{v} = v + \frac{\tilde{c}_1}{\kappa_1}$ is due to shift in the angle variable ϕ_1.

For the case of *Example 3* the corresponding perturbed 1-soliton solution reads:

$$\tilde{u}_{1\text{s}}(x,t) = \frac{2\kappa_1(0)^2(1 + \epsilon_0 \sin \omega t)}{\text{ch}^2\left\{\kappa_1(0)\sqrt{1 + \epsilon_0 \sin \omega t}\left[x + 4\kappa_1^2(0)(1 + \epsilon_0 \sin \omega t)t - \frac{\phi_1}{\kappa_1}\right]\right\}}. \tag{76}$$

Here the soliton velocity depends on time, which is typical for the so-called boomerons [3]. Note however, that the boomeron type behavior is characteristic for multi-component KdV equations, while here we get it as an effect of the perturbation. Moreover, the time dependence of the velocity is controlled by the time-depending coefficient in the perturbation $\tilde{c}_1(t)$.

There are of course many other nontrivial integrable examples that one can construct by various choices of the functions c_1 and $\tilde{c}_1$ – when $(62) - (65)$ is an integrable system of ODEs. In general, however this system is not integrable and therefore there exist solutions that can not be constructed by the means of IST.

6   Discussions and Conclusions

Here we have used the expansion over the eigenfunctions of the recursion operator for the KdV hierarchy for studying nonholonomic deformations of the corresponding NLEE from the hierarchy. We have shown, that in the case of self-consistent sources, the corresponding perturbed NLEE is integrable, but not completely integrable.

The perturbation results for the Zakharov-Shabat (ZS) type spectral problems have been obtained firstly in [24] and for KdV in [23]. As it has been explained, the perturbation theory is based on the completeness relations for the squared eigenfunctions. For the Sturm-Liouville spectral problem such relations apparently have been studied as early as in 1946 [2] and then by other authors, e.g. [28, 18]. The completeness relation for the eigenfunctions of the ZS spectral problem is derived in [25] and generalisations are studied further in [10, 12, 13, 11, 15, 44], see also [14] (and the references therein).

The approach presented in this article can be applied also to the study of inhomogeneous versions of NLEE, related to other linear spectral problems, e.g. the Camassa-Holm equation [5, 6], various difference and matrix generalizations of KdV-like and Zakharov - Shabat spectral problems, various non-Hamiltonian systems [45], etc.

Acknowledgments

References

[1] V. A. Arkad'ev, A. K. Pogrebkov and M. K. Polivanov, *Expansions with respect to squares, symplectic and Poisson structures associated with the Sturm-Liouville problem. I*, Theor. Math. Phys. **72** (1987) No. 3, 909–920;
V. A. Arkad'ev, A. K. Pogrebkov and M. K. Polivanov, *Expansions with respect to squares, symplectic and Poisson structures associated with the Sturm-Liouville problem. II*, Theor. Math. Phys. **75** (1988) No. 2, 448–460.

[2] G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Math. **78** (1946), 1–96, (German).

[3] F. Calogero, A. Degasperis, *Spectral Transform and Solitons. Vol 1. Tools to Solve and Investigate Nonlinear Evolution Equations* Studies in Mathematics and its Applications **13** (Lecture Notes in Computer Science vol 144) (Amsterdam: North-Holland) p. 516.

[4] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664 (E-print: `patt-sol/9305002`).

[5] A. Constantin, V. Gerdjikov and R. Ivanov, *Inverse scattering transform for the Camassa-Holm equation*, Inv. Problems **22** (2006), 2197–2207 (E-print: `nlin/0603019`).

[6] A. Constantin, V. Gerdjikov and R. Ivanov, *Generalized Fourier transform for the Camassa-Holm hierarchy*, Inverse Problems **23** (2007), 1565–1597 (E-print: `arXiv:0707.2048`).

[7] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi Equations*, Arch. Rat. Mech. Anal. **192** (2009), 165–186 (E-print: `arXiv:0709.0905`).

[8] G. Eilenberger, *Solitons: Mathematical Methods for Physicists*; Springer Series in Solid-State Sciences. vol. **19**; Springer-Verlag, Berlin, 1981.

[9] L. D. Faddeev and L. A. Takhtajan, *Poisson structure for the KdV equation*, Lett. MAth. Phys. **10** (1985), 183–188.

[10] V.S. Gerdjikov, *Generalised Fourier transforms for the soliton equations. Gauge-covariant formulation*, Inv. Problems **2** (1986), 51–74.

[11] V.S. Gerdjikov, *The generalized Zakharov-Shabat system and the soliton perturbations*, Theoret. and Math. Phys. **99** (1994), 593–598.

[12] V.S. Gerdjikov and M.I. Ivanov, *Expansions over the "squared" solutions and the inhomogeneous nonlinear Schrödinger equation*, Inv. Problems **8** (1992), 831–847.

[13] V. S. Gerdjikov and E. Kh. Khristov, *Evolution equations solvable by the inverse-scattering method. I. Spectral theory*, Bulgarian J. Phys. **7** No.1, 28–41, (1980). (In Russian);
On evolution equations solvable by the inverse scattering method. II. Hamiltonian structure and Bäcklund transformations Bulgarian J. Phys. **7** No.2, 119–133, (1980) (In Russian).

[14] V.S. Gerdjikov, G. Vilasi and A.B. Yanovski, *Integrable Hamiltonian hierarchies. Spectral and geometric methods.* Lecture Notes in Physics, **748**. Springer-Verlag, Berlin, 2008.

[15] V.S. Gerdjikov and A.B. Yanovski, *Completeness of the eigenfunctions for the Caudrey-Beals-Coifman system*, J. Math. Phys. **35** (1994), 3687–3725.

[16] G. G. Grahovski and R.I. Ivanov, *Generalised Fourier Transform and Perturbations to Soliton Equations*, Discr. Cont. Dyn. Syst. B **12** (2009), No.3, 579 – 595 (E-print: `arXiv:0907.2062`).

[17] P. Guha, *Nonholonomic deformation of generalized KdV-type equations*, J. Phys. A: Math. Theor. **42**, 345201, 2009.

[18] I. Iliev, E. Khristov and K. Kirchev, *Spectral Methods in Soliton Equations*, Pitman Monographs and Surveys in Pure and Appl. Math. vol. **73**, Pitman, London, 1994.

[19] R.S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid. Mech. **457** (2002), 63–82.

[20] R.I. Ivanov, *Water waves and integrability*, Philos. Trans. R. Soc. Lond. Ser. A: Math. Phys. Eng. Sci. **365** (2007), 2267–2280 (E-print: `arXiv:0707.1839`).

[21] R.S. Johnson, *On solutions of the Camassa-Holm equation*, Proc. Roy. Soc. London A **459** (2003), 1687–1708.

[22] A. Karasu-Kalkantli, A. Karasu, A. Sakovich, S. Sakovich, R. Turhan, *A new integrable generalization of the KdV equation*, J. Math. Phys. **49**, 073516, 2008.

[23] V.I. Karpman and E.M. Maslov, *Perturbation theory for solitons*, Soviet Phys. JETP, **46** (1977), 537 – 559.

[24] D.J. Kaup, *A perturbation expansion for the Zakharov-Shabat inverse scattering transform*, SIAM J. Appl. Math. **31** (1976), 121–133.

[25] D.J. Kaup, *Closure of the squared Zakharov-Shabat eigenstates*, J. Math. Anal. Appl. **54** (1976), 849–864.

[26] D. J. Kaup. In *"Significance of Nonlinearity in the Natural Science"*, eds. Kursunoglu, A. Perlmutter, L. F. Scott, Plenum Press, p. 97, (1977).

[27] D. J. Kaup, A. C. Newell. Proc. Roy. Soc. **A361,** 413, (1978).

[28] K.P. Kirchev and E. Kh. Hristov, *Expansions connected with the products of the solutions of two regular Sturm-Liouville problems*, Sibirsk. Mat. Zh. **21** (1980), 98–109 (Russian).

[29] A. Kundu, *Exact accelerating soliton in nonholonomic deformation of the KdV equation with two-fold integrable hierarchy*, J. Phys. A: Math. Theor. **41**, 495201, 2008.

[30] A. Kundu, R. Sahadevan, L. Nalinidevi, *Nonholonomic deformation of KdV and mKdV equations and their symmetries, hierarchies and integrability*, J. Phys. A: Math. Theor. **42**, 115213, 2009.

[31] A. Kundu, *Nonlinearizing linear equations to integrable systems including new hierarchies with nonholonomic deformations*, J. Math. Phys. **50**, 102702, 2009.

[32] A. Kundu, *Two-fold integrable hierarchy of nonholonomic deformation of the derivative nonlinear Schrdinger and the Lenells-Fokas equation*, J. Math. Phys. **51**, 022901, 2010.

[33] B. A. Kupershmidt, *KdV6: an integrable system*, Phys. Lett. A **372**, 2634–2639, 2008.

[34] J. P. Leon, *General evolution of the spectral transform from the Image $\bar{\partial}$-approach*, Phys. Lett **123A** 65–70, (1987).

[35] C. Claude, A. Latifi, and J. P. Leon, *Nonlinear resonant scattering and plasma instability: an integrable model*, J. Math Phys. **32** 3321–3330, (1991).

[36] J. P. Leon and A. Latifi, *Solution of an initial-boundary value problem for coupled nonlinear waves*, J. Phys. A: Math. Gen. **23** 1385–1403, (1990).

[37] J. P. Leon, *Nonlinear evolutions with singular dispersion laws and forced systems*, Phys. Lett **144A,** 444–452, (1990).

[38] J. Leon, *Spectral transform and solitons for generalized coupled Bloch systems*, J. Math. Phys. **29,** 2012–2019 (1988)

[39] V. K. Melnikov, *Integration of the Korteweg-de Vries eqtion with source*, Inverse Probl. **6**, 233–246, 1990.

[40] V. K. Melnikov, *Creation and annihilation of solitons in the system described by the Kortweg-de Vries equation with a self-consistent source*, Inverse Probl. **6**, 809–823, 1990.

[41] A. C. Newell. In *"Solitons",* eds. R. K. Bullough, P. L. Caudrey, Springer Verlag, (1980).

[42] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, *Theory of solitons: the inverse scattering method*, Plenum, New York, 1984.

[43] A. Ramani, B. Grammaticos, R. Willox, *Bilinearization and solutions of the KdV6 equation* Anal. Appl. **6** 401-412, 2008.

[44] T. Valchev, *On the Kaup-Kupershmidt equation. Completeness relations for the squared solutions*, in " Nineth International Conference on Geometry, Integrability and Quantization, June 8-13 2007, Varna, Bulgaria" (eds. I. Mladenov and M. de Leon), SOFTEX, Sofia (2008), 308–319.

[45] Jing Ping Wang, *Extension of integrable equations*, J. Phys. A: Math. Theor. **42** 362204 (2009).

[46] A-M. Wazwaz, *The integrable KdV6 equations: Multiple soliton solutions and multiple singular soliton solutions*, Applied Mathematics and Computation **204**, 963-972, 2008.

[47] Y. Q. Yao, Y. B. Zeng, *Integrable Rosochatius deformations of higher-order constrained flows and the soliton hierachy with self-consistent source*, J. Phys. A: Math. Theor. **41**, 295205, 2008.

[48] Y. Q. Yao, Y. B. Zeng, *The bi-Hamiltonian structure and new solutions of KdV6 equation*, Lett. Math. Phys. **86**, 193–208, 2008.

[49] V. Zakharov and L. Faddeev, *Korteweg-de Vries equation is a completely integrable Hamiltonian system*, Func. Anal. Appl. **5** (1971), 280-287 (English).

[50] Zeng, Y.B., Shao, Y.J., Ma, W.X., *Negaton and positon solutions of the soliton equation with sself-consistent sources*, J. Phys. A Math. Gen. **36**, 5035–5043 (2003).