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Non-Gaussian Analysis of Wind Velocity Data for the Determination of Power Quality Control

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Non-Gaussian Analysis of Wind Velocity Data for the Determination of Power Quality Control

Jonathan Blackledge, Eugene Coyle and Derek Kearney

***Abstract*— The quality of power (i.e. the sustainable power output as a function time) of any wind dependent energy converter (including wind turbines and wave energy converters) is determined by many design and environmental factors but time-dependent variations in the wind speed are arguably the most important. In this paper we consider a non-Gaussian model for analysing and then simulating wind velocity data. In particular, we consider a Lévy distribution for the statistical characteristics of wind velocity and show how this distribution can be used to derive a stochastic fractional diffusion equation for the wind velocity as a function of time whose solution is characterised by the Lévy index. A Lévy index based numerical analysis is then performed on wind velocity data for both rural and urban areas where, in the latter case, the index is shown to have a larger value. Finally, an empirical relationship is derived for the power output from a wind turbine in terms of the Lévy index using Betz law and a similar relationship obtained for a wave energy converter. In both cases, it is shown how the average power output as a function of time is (inversely) related to the Lévy index for the wind velocity. It is concluded that these relationships may have value in determining the optimal geographical locations for the construction of wind and wave farms and for monitoring their performance in terms of power quality control.**

***Index Terms*— Wind turbines, Wave energy conversion, Stochastic wind velocity model, Non-Gaussian statistics, Lévy index, Power quality control.**

I. Introduction

Developing appropriate models for assessing and predicting the *quality of power* for any renewable energy source is important throughout the energy industry. Quality of power modelling is particularly important with regard to wind energy as the construction of new wind farms is growing rapidly compared with other renewable energy systems [2]. By 2030, it is estimated that up to 40% world energy supply will be based on renewable energy sources and in countries with an appropriate disposition to generating energy from wind, wave and tidal power such as the UK and Ireland, the percentage is expected to be much higher. For example, at the end of 2008, the Republic of Ireland had an installed wind power capacity of 1245 MW (MegaWatts) which ranks Ireland 15th in the world in terms of MW installed. This is a marked increase in the level of wind power generation previously available, more than doubling a total of 495 MW in 2005. In 2008 alone, the rate of growth was 54.6%, amongst the highest in the world. On July 31, 2009, the output from the Republic of Ireland's wind turbines peaked at $\sim$1000 MW and during certain times on that day, up to 39% of the countries demand for electricity was met from wind power alone. By 2010 there was 1,161 MW installed wind power generation in the Republic of Ireland, with a further 1,415 MW under contract [1].

Quality of power modelling is often based on a statistical analysis of the available wind velocity data which is used to assess optimum regions for the construction of wind farms, for example [3]. The power generated by a wind turbine is based on a range of design factors, the most important of which is the turbine blade area given that, from Betz law, the power P in Watts is given by [4]

$$P = \frac{1}{2}\alpha\rho A v^3 \tag{1}$$

where v is the upwind speed (i.e. the wind velocity that is incident on the turbine) in metres per second (ms^{-1}), A is the area mapped out by the turbine blades in m^3, ρ is the density of air in kgm^{-3} and $\alpha < 0.593$ is the coefficient of performance. For completeness, a derivation of equation (1) is given in Appendix A which includes the idealised conditions upon which this equation is based. Figure 1 shows a schematic diagram of the increasing size of the blade area (i.e. rotor diameter) that has and continues to evolve in order to produce wind turbines with greater power output.

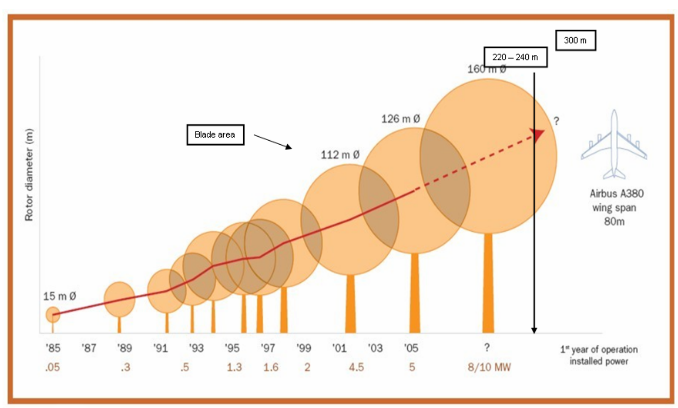

Fig. 1. Schematic diagram of the increasing size in wind turbine blade area (rotor diameter) from 1985 onward required to provide larger power yields according to the Betz law in which the power from a wind turbine is proportional to the blade area (Source: Jos Beurskens ECN).

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Although other physical factors such as air temperature and pressure, angle of attack, etc. are important, given equation (1), it is clear that the scaling law of the output power with regard to wind velocity (i.e. $P \propto v^3$) is the most significant feature for a given design of a wind turbine with a fixed area and coefficient of performance [5]. Moreover, the wind velocity is not something that can be incorporated or controlled in regard to the design of a wind farm other than through an appraisal of the optimum position in which to build a wind farm, i.e. where the average wind velocity is known *a priori* to be relatively high. Thus, an understanding of the time variations in the wind velocity for a given geographical location is of paramount importance with regard to locating a wind farm and monitoring its performance in terms of the power quality. This requires stochastic models to be developed for the power output that are sensitive to the statistical characteristics of wind speeds and accurate over different time scales [6]. Developing such models is of value in both monitoring and predicting power quality for existing wind farms but also in determining the optimal geographical locations where such farms should be constructed. This is the focus for the work reported in this paper.

The acquisition of wind velocity data over different time intervals and localities is a common practice together with a routine statistical analysis of the data. However, the analysis is almost exclusively based on the assumption that time variations in the wind velocity are random Brownian processes and that the rate of change of velocity as a function of time is Gaussian distributed, i.e. the wind velocity conforms to a process of diffusion. However, this is not usually the case as discussed in the following section and in this paper we develop a non-Gaussian stochastic model for the wind velocity that is based on a Lévy distribution and a fractional diffusion equation. This allows us to analyse wind velocity in terms of the Lévy index and thereby yields an approach for assessing the quality of power for a wind turbine in terms of this index. We provide examples of wind velocity data that substantiate this approach and construct an empirical relationship for the power output from a wind turbine based on the Lévy index. We then extend the approach to develop an expression for the power output from a wave energy converter.

II. Statistical Analysis of the Wind Speed

Figure 2 shows a typical example of plots of the wind velocity and wind direction as a function of time together with the associated histograms illustrating a marked difference in their statistical characteristics. This data shows wind velocities (in metres per second) and wind directions (in degrees) and consists of 8000 samples recorded at Dublin Airport, Ireland over intervals of 1 hour from 00:00:00 on 1 January 2008 to 06:00:00 on 29 November 2008. While data taken over shorter time scales clearly provides greater accuracy on the dynamic behaviour of wind speeds, one of the underlying principles associated with the work reported in this paper, is that the statistical characteristics of the wind velocity are self-affine. In other words, the statistical distribution of wind speeds over all time scales is the same, apart from scaling. This concept is fundamental to the application of the non-Gaussian statistical analysis of wind speeds considered in this paper and, in particular, the introduction of a Lévy distribution for characterising the statistics of the wind force (i.e. the gradient of the wind velocity) as discussed in Section III.

The wind velocity $v(t)$ given in Figure 2 has a typical Rayleigh-type distribution with a mode of 5ms^{-1} and a maximum wind velocity of 21.1ms^{-1}. The wind direction has a marked statistical bias toward higher angles with a primary mode of 240 degrees which is characteristic of the prevailing wind direction for the region.

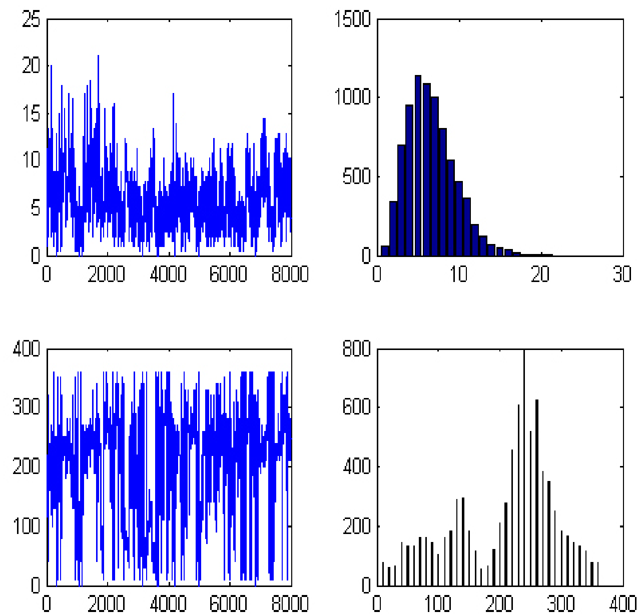

Fig. 2. Plots of the wind velocity (top-left in metres per second) and wind direction (bottom-left in degrees) and the associated 22-bin and 360-bin histograms (top-right and bottom-right), respectively.

Figure 3 compares the velocity gradient $d_t v(t) \equiv dv/dt$ (which represents the force generated by the wind for a unit mass computed using a forward differencing scheme) with the output from a zero-mean Gaussian distributed random number stream. By comparing these signals, it is clear that the statistical characteristics of $d_t v(t)$ are not Gaussian. The plot of $d_t vt$ obtained from the wind velocity data clearly shows that there are a number of rare but extreme events corresponding to short periods of time over which the change in wind velocity is relatively high. This leads to a distribution with a narrow width but longer tail when compared to a normal (Gaussian) distribution. Non-Gaussian distributions of this type are typical of Lévy processes which are discussed in the following section.

III. Lévy Processes

Lévy processes are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems are usually concerned with stochastic fields that have Probability Density Functions (PDFs) where (at least) the first two moments (the mean and variance) are well defined and

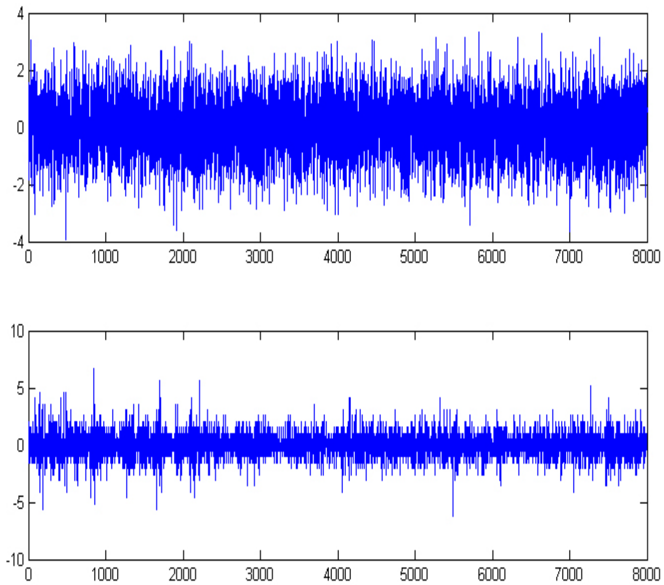

Fig. 3. Plots of a zero-mean Gaussian distributed stochastic signal obtained using MATLAB V7 *randn* function (above) and the gradient of the wind velocity $d_t v(t)$ given in Figure 2 (below).

finite. Lévy statistics is concerned with stochastic processes where all the moments (starting with the mean) are infinite. Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions [7]. Lévy's original approach to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only) [7]

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2 \tag{2}$$

where a is a constant and γ is the Lévy index. For $\gamma \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed.

If a stochastic process is characterised by a random walk a distribution governed by $p(x)$ with $\gamma = 2$, then the result is normal (Gaussian) diffusion, i.e. a Brownian random walk process. For $\gamma < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. For values of γ between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form (see Appendix B)

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \rightarrow \infty$$

If we compare this PDF with a Gaussian distribution given by (ignoring scaling normalisation constants)

$$p(x) = \exp(-ax^2)$$

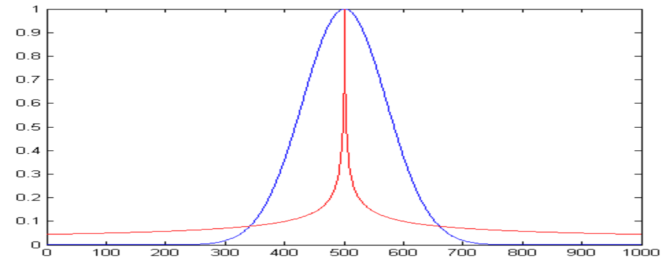

Fig. 4. Comparison between a Gaussian distribution (blue) for $a = 0.0001$ and a Lévy distribution (red) for $\gamma = 0.5$ and $p(0) = 1$.

which is the case when $\gamma = 2$ then it is clear that a Lévy distribution has a longer tail. This is illustrated in Figure 4. The long tail Lévy distribution represents a stochastic process in which extreme events are more likely when compared to a Gaussian process as illustrated in Figure 3. Moreover, the length of the tails of a Lévy distribution is determined by the value of the Lévy index such that the larger the value of the index is, the shorter the tail becomes. Unlike the Gaussian distribution which has finite statistical moments, the Lévy distribution has infinite moments and 'long tails'. Furthermore, Lévy processes characterised by a PDF of this type conform to a fractional diffusion equation [8] as shown in the following section.

IV. Derivation of the Fractional Diffusion Equation for the Wind Velocity

Let $p(x)$ denote the Probability Density Function (PDF) associated with the position in a one-dimensional space x where a particle can exist as a result of a 'random walk' generated by a sequence of 'elastic scattering' processes (with other like particles). Also, assume that the random walk takes place over a time scale where the random walk 'environment' does not change (i.e. the statistical processes are 'stationary' and do not change with time). Suppose we consider an infinite concentration of particles at a time $t = 0$ to be located at the origin $x = 0$ and described by a perfect spatial impulse, i.e. a delta function $\delta(x)$. Then the characteristic Impulse Response Function f of the 'random walk system' at a short time later $t = \tau$ is given by

$$f(x, \tau) = \delta(x) \otimes_x p(x) = p(x)$$

where $\otimes_x$ denotes the convolution integral over x. Thus, if $f(x, t)$ denotes a macroscopic field at a time t which describes the concentration of a canonical assemble of particles all undergoing the same random walk process, then the field at $t + \tau$ will be given by

$$f(x, t + \tau) = f(x, t) \otimes_x p(x)$$

In terms of the application considered in this paper f represents the space-time varying force of the wind that is incident on a wind turbine.

From the convolution theorem, in Fourier space, this equation for $f(x, t + \tau)$ becomes

$$F(k, t + \tau) = F(k, t)P(k)$$

where F and P are the Fourier transforms of f and p, respectively. From equation (2), we note that

$$P(k) = 1 - a \mid k \mid^\gamma, \quad a \to 0$$

so that we can write

$$\frac{F(k, t+\tau) - F(k,t)}{\tau} \simeq -\frac{a}{\tau} \mid k \mid^\gamma F(k,t)$$

which for $\tau \to 0$ gives the fractional diffusion equation

$$\sigma \frac{\partial}{\partial t} f(x,t) = \frac{\partial^\gamma}{\partial x^\gamma} f(x,t), \quad \gamma \in (0, 2]$$

where $\sigma = \tau / a$ and we have used the result

$$\frac{\partial^\gamma}{\partial x^\gamma} f(x,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mid k \mid^\gamma F(k,t) \exp(ikx) dk$$

However, since, for unit mass

$$f(x,t) = \frac{\partial}{\partial t} v(x,t)$$

where v denotes the wind velocity, we can consider the equation

$$\sigma \frac{\partial}{\partial t} v(x,t) = \frac{\partial^\gamma}{\partial x^\gamma} v(x,t), \quad \gamma \in (0, 2] \tag{3}$$

The solution to this equation with the singular initial condition $v(x, 0) = \delta(x)$ is given by

$$v(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx - t \mid k \mid^\gamma / \sigma) dk$$

which is itself Lévy distributed. This derivation of the fractional diffusion equation reveals its physical origin in terms of Lévy statistics.

For normalized units $\sigma = 1$ we consider equation (3) for a 'white noise' source function $n(t)$ and a spatial impulse function $-\delta(x)$ so that

$$\frac{\partial^\gamma}{\partial x^\gamma} v(x,t) - \frac{\partial}{\partial t} v(x,t) = -\delta(x) n(t), \quad \gamma \in (0, 2]$$

which, ignoring (complex) scaling constants, has the Green's function solution [9]

$$v(t) = \frac{1}{t^{1-1/\gamma}} \otimes_t n(t) \tag{4}$$

where $\otimes_t$ denotes the convolution integral over t and $v(t) \equiv v(0,t)$. The function $v(t)$ has a Power Spectral Density Function (PSDF) given by (for scaling constant c)

$$\mid V(\omega) \mid^2 = \frac{c}{\mid \omega \mid^{2/\gamma}}$$

where

$$V(\omega) = \int_{-\infty}^{\infty} v(t) \exp(-i\omega t) dt$$

and a self-affine scaling relationship

$$\Pr[v(at)] = a^{1/\gamma} \Pr[v(t)]$$

for scaling parameter $a > 0$ where $\Pr[v(t)]$ denotes the PDF of $v(t)$. This scaling relationship means that the statistical characteristics of $v(t)$ are invariant of time except for scaling factor $a^{1/\gamma}$. Thus, if $v(t)$ is taken to be the wind velocity as a function of time, then the statistical distribution of this function will be the same over different time scales whether, in practice, it is sampled in hours or seconds, for example.

V. Lévy Index Analysis

The PSDF $\mid V(\omega) \mid^2$ provides a method of computing γ using the least squares method based on minimizing the error function

$$e(c, \gamma) = \| 2 \ln \mid V(\omega) \mid - \ln c - 2\gamma^{-1} \ln \mid \omega \mid \|_2^2, \quad \omega > 0$$

Figures 5 and 6 show the computation of $\gamma(t)$ for a moving window of size 1024 elements. The accompanying tables (Table I and Table II) provide some basic statistical information with regard to $\gamma(t)$ for these data sets. Application of the Bera-Jarque parametric hypothesis test of composite normality is rejected (i.e. 'Composite Normality' is of type 'Reject') and thus $\gamma(t)$ is not normally distributed.

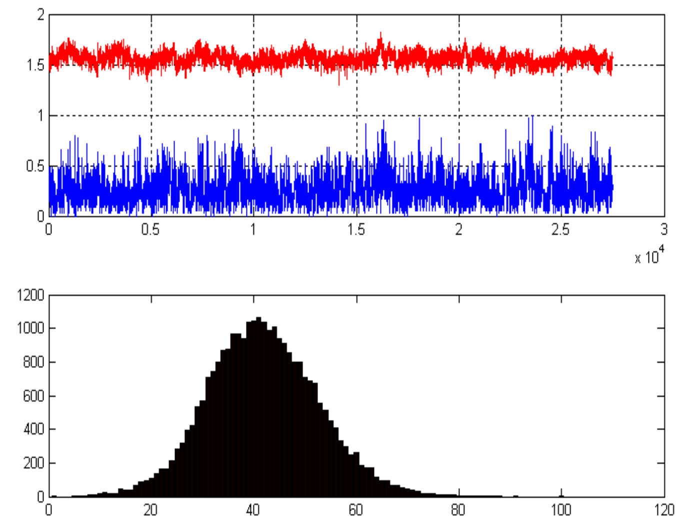

Fig. 5. Cork Airport (12/11/2003-1/1/2007) for hourly (averaged) sampled data. Above: Normalised wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

TABLE I

Statistical parameters associated with the Lévy index function given in Figure 5.

Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3001
Maximum value	1.8142
Range	0.5141
Mean	1.5615
Median	1.5613
Standard Deviation	0.0569
Variance	0.0032
Skewness	0.0759
Kertosis	3.1966
Composite Normality	'Reject'

These result illustrates that the wind velocity function appears to be a self-affine stochastic function with a mean

Lévy index of ~ 1.5. Figure 7 shows a simulation of the wind velocity based on the computation of $v(t)$ in equation (4) for $\gamma = 1.5$. The simulation is based on transforming equation (4) into Fourier space and using a Discrete Fourier Transform. The function $n(t)$ is computed using the MATLAB (V7) uniform random number generator *rand* with $seed = 1$.

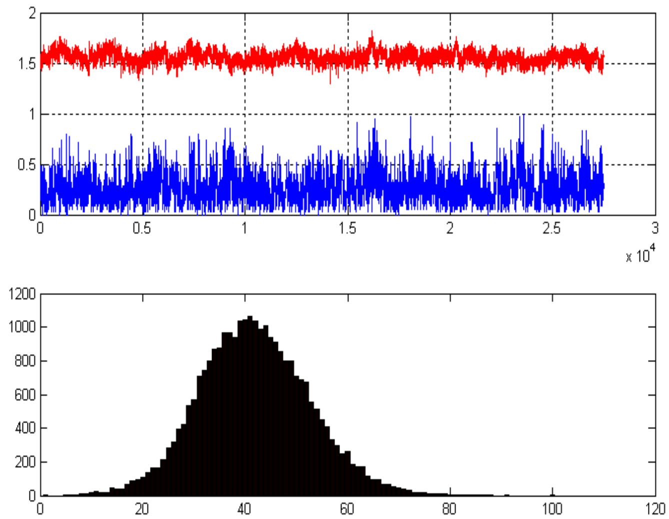

Fig. 6. Knock Airport (12/11/2003-1/1/2005) for hourly (averaged) sampled data. Above: Normalised wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

TABLE II

STATISTICAL PARAMETERS ASSOCIATED WITH THE LÉVY INDEX FUNCTION GIVEN IN FIGURE 6.

Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3846
Maximum value	1.7600
Range	0.3754
Mean	1.5777
Median	1.5788
Standard Deviation	0.0510
Variance	0.0026
Skewness	-0.1538
Kertosis	3.0764
Composite Normality	'Reject'

The results given in Figure 5 and Figure 6 are for wind velocity data obtained in rural areas, i.e. at Cork and Knock airports, respectively. It is interesting to note that, in urban areas, the Lévy index may be expected to increase as a result of the further 'diffusion' of the wind velocity through 'random scattering' of the wind from buildings in the local vicinity when, according the model being considered, $\gamma \to 2$. An example of this is given in Figure 8 and Table III in which the average Lévy index is ~ 1.72 thereby confirming this expectation.

VI. Power Quality Estimation for Wind Energy Generation

Given equation (1) and equation (4), we can obtain an expression for the power output by a (ideal) wind turbine in

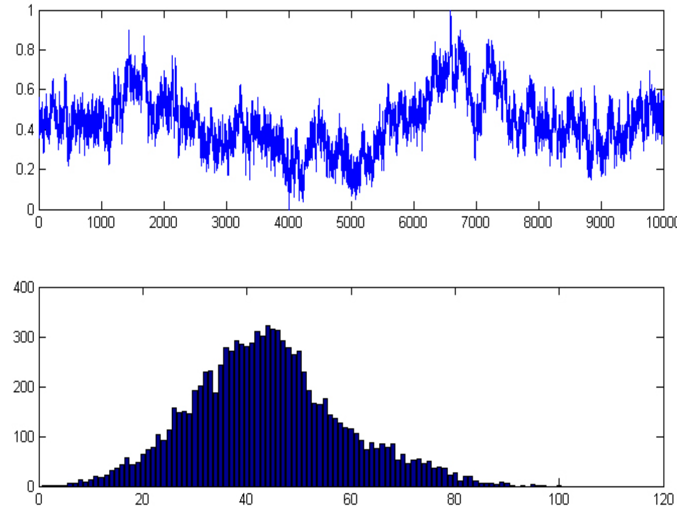

Fig. 7. Simulated normalised wind velocities computed for a Lévy index $\gamma = 1.5$ (above) and the corresponding 100-bine histogram (below)

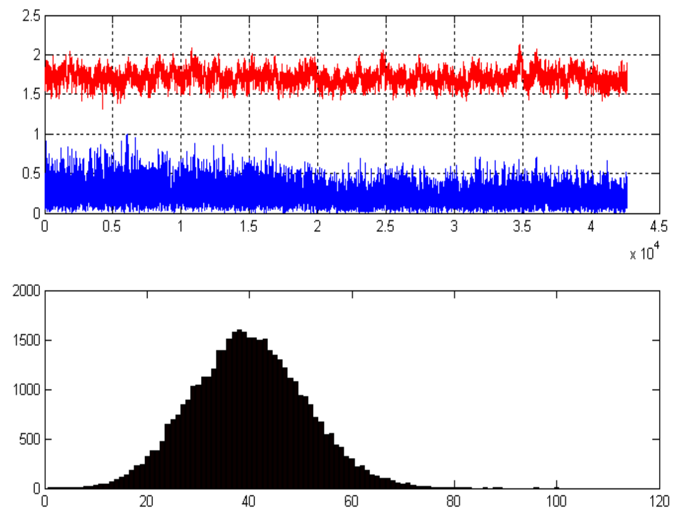

Fig. 8. Example of urban data analysis using wind velocities recorded at Dublin Institute of Technology, Kevin Street, Dublin 8 from 14 September 2010 at 22:20:44 to 15 September 2010 at 10:11:51 and sampled in seconds. Above: Normalised wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

TABLE III

STATISTICAL PARAMETERS ASSOCIATED WITH THE LÉVY INDEX FUNCTION GIVEN IN FIGURE 8.

Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3209
Maximum value	2.1358
Range	0.8149
Mean	1.7236
Median	1.7204
Standard Deviation	0.0944
Variance	0.0089
Skewness	0.1939
Kertosis	3.0374
Composite Normality	'Reject'

terms of the Lévy index γ as a function of time. Let the noise function in equation (4) be a simple impulse at an instant in time so that $n(t) = \delta(t)$. Then

$$v(t) = \frac{1}{t^{1-1/\gamma}}$$

and, from equation (1),

$$P(t) = \frac{\beta}{t^{3(1-1/\gamma)}}$$

where $\beta = \alpha\rho A/2$ so that

$$\ln P(t) = \ln \beta - 3 \ln t + \frac{3}{\gamma} \ln t$$

Given that β is a constant, it is then clear that, for any time t, the magnitude of $\ln P$ is determined by γ^{-1}. In this sense, γ^{-1} is a coefficient of power quality as a function of time and we see that, according to this model, power output increases as γ decreases. Thus, the signal $\gamma(t)$ given in Figure 5 and Figure 6, for example, represents a time varying measure of the average output power at a time τ according to the scaling law

$$\langle \ln P(t) \rangle_\tau = a + \frac{b}{\gamma(\tau)}$$

where $\langle \ln P(t) \rangle_\tau$ denotes the (moving) average value of $\ln P(t)$ at a time τ given that

$$\langle \ln P(t) \rangle = \frac{1}{T} \int^{T} \ln P(t) dt$$

where T is the period of time over which the average value is computed. The constants a and b are given by

$$a = \ln \beta - \frac{3}{T} \int^{T} \ln t dt$$

and

$$b = \frac{3}{T} \int^{T} \ln t dt$$

respectively.

VII. Energy Quality Estimation for Wave Power Generation

We can use a similar approach to that adopted in the last section to derive a scaling law for the power output by a (ideal) wave energy converter. However, to do this, we must derive a solution to the wave equation where the source function (i.e. the source of wave generation) is determined by the wind force. From equation (4), the force generated for a unit mass is given by

$$f(t) = d_t v(t) = \frac{1}{t^{1-1/\gamma}} \otimes_t d_t n(t) \tag{5}$$

Working in a one-dimensional space, consider the wave equation (for unit wave speed)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u(x,t) = -\delta(x) f(t)$$

where u denotes the wave amplitude and the source function is taken to have a spatial impulse $\delta(x)$. This allows us to develop a solution in terms for the time evolution of the wave amplitude. Let

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x,\omega) \exp(i\omega t) d\omega$$

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega$$

so that we can write

$$\left(\frac{\partial^2}{\partial x^2} + \omega^2 \right) u(x,\omega) = -\delta(x) F(\omega)$$

The Green's function solution to this wave equation is given by [10]

$$\begin{aligned} U(x,\omega) &= \frac{i}{2\omega} \exp(i\omega \mid x \mid) \otimes_x \delta(x) F(\omega) \\ &= \frac{i}{2\omega} \exp(i\omega \mid x \mid) F(\omega) \end{aligned}$$

so that

$$U(\omega) \sim \frac{i}{2\omega} F(\omega), \quad \omega \in [-\Omega, \Omega], \quad \Omega \to 0$$

This result is based on exploiting the low frequency limit for sea surface waves in the locality of $\delta(x)$ which are taken to have a bandwidth Ω. Noting that, from equation (5), in Fourier space,

$$F(\omega) = i\omega V(\omega)$$

where

$$V(\omega) = \int_{-\infty}^{\infty} v(t) \exp(-i\omega t) dt$$

we then have (ignoring scaling)

$$U(\omega) \sim -\frac{1}{2} V(\omega) = -\frac{N(\omega)}{2(i\omega)^{1/\gamma}}, \quad \omega \in [-\Omega, \Omega], \quad \Omega \to 0$$

where

$$N(\omega) = \int_{-\infty}^{\infty} n(t) \exp(-i\omega t) dt$$

so that in real space, we can write (ignoring scaling)

$$u(t) = -\frac{\sin(\Omega t)}{\Omega t} \otimes_t v(t), \quad \Omega \to 0 \tag{6}$$

The Power Spectral Density Function of $u(t)$ is given by (for scaling constant c)

$$P(\omega) = \mid U(\omega) \mid^2 = \frac{c}{\mid \omega \mid^{2/\gamma}}, \quad \mid \omega \mid \leq \Omega$$

and, for a fixed bandwidth Ω, it is clear that the power output depends upon γ associated with the wind velocity according to the model compounded in equation (6). Thus we can consider a time dependent wave power scaling relationship of the form

$$\langle \ln P(\omega) \rangle_\tau = a - \frac{b}{\gamma(\tau)}$$

where

$$\langle \ln P(\omega) \rangle = \frac{1}{\Omega} \int^{\Omega} \ln P(\omega) d\omega,$$

$$a = \ln c \quad \text{and} \quad b = \frac{2}{\Omega} \int^{\Omega} \ln \omega d\omega$$

VIII. Summary

We have considered a Lévy distributed model and constructed a fractional diffusion equation for the wind velocity whose temporal solution is characterised by the Lévy index. Analysis of wind velocity data (some examples of which have been provided in this paper) according to this model shows that the Lévy index is a time varying non-Gaussian stochastic function. Based on the data analysed to date, the index appears to be larger ~ 1.7 for urban areas compared to rural areas when $\gamma \sim 1.5$. These results are consistent with the underlying rationale associated with the model, where, as $\gamma \rightarrow 2$, the stochastic processes become increasingly diffusive. The model presented allows times series for wind velocity to be simulated whose statistical properties are consistent with experimental data as illustrated in Figure 3, for example. Moreover, based on the calculations performed in Sections VI and VII, the Lévy index may provide a useful measure on the power quality of wind turbines and wave energy generators, respectively. Further investigations are required to ascertain whether it may be possible to use the signal $\gamma(t)$ for short and possibly long term predictive analysis on power quality following the methods developed for financial risk management reported in [11], for example.

Appendix A: Derivation of Betz Law for a Wind Turbine

Consider the case where a wind turbine is driven by a change in the wind velocity from v_1 to v_2 where v_1 and v_2 are the upstream and downstream wind velocities respectively. We consider the ideal case where: (i) the turbine is an ideal rotor consisting of an infinite number of blades that do not have drag; (ii) the direction of the wind velocity on the blades is perfectly axial rather than at an angle thereby giving maximum possible performance; (iii) the flow is incompressible; (iv) the density of air remains constant; (v) there is no heat transfer from the air to the rotors; (vi) the rotors are mass-less so that no effect occurs through the angular momentum imparted to the rotor or flow of air. Under these conditions we can consider two approaches to generating an expression for the output power of such an idealized wind turbine. The first is based on considering the change of energy generated from the change in the upstream and downstream velocities which is given by

$$E = \frac{1}{2} m (v_1^2 - v_2^2), \quad v_1 > v_2$$

where m is the mass of air flowing through turbine. The power P is then determined by the rate of change of this mass since

$$P = \frac{dE}{dt} = \frac{1}{2}(v_1^2 - v_2^2)\frac{dm}{dt}$$

But

$$\frac{dm}{dt} = \rho A v$$

where ρ is the density of air, A is the area of the turbine and v is the velocity of wind through the turbine. Thus the power is given by

$$P = \frac{1}{2}\rho A v (v_1^2 - v_2^2) \qquad (A.1)$$

Another way of deriving an expression for the power is by considering the force that is generated by the rate of change of mass. This is given by

$$F = (v_1 - v_2)\frac{dm}{dt} = (v_1 - v_2)\rho A v$$

The power is then given by

$$P = \frac{dE}{dt}$$

where $dE = F dx$ and hence,

$$P = Fv = \rho A v^2 (v_1 - v_2) \qquad (A.2)$$

By comparing these two equivalent expressions for the power P, i.e. equations (A.1) and (A.2) it is clear that

$$v = \frac{1}{2}(v_1 + v_2)$$

which is an expression for the average velocity that 'drives' the wind turbine in terms of the upstream and downstream velocities. Using this expression for v we can now write equation (A.1) as

$$P(V) = \frac{1}{4}\rho A (v_1 + v_2)(v_1^2 - v_2^2)$$
$$= \frac{1}{4}\rho A v_1^3 (1 - V^2 + V - V^3)$$

where $V = v_2/v_1$. The maximum value of P (denoted by $P_{\max}$) is then given when

$$\frac{dP}{dV} = 0$$

or

$$-1 + 2V - 3V^2 = 0$$

for which $\text{Re}[V] = 1/3$ so that

$$P_{\max} = \frac{1}{2}\alpha \rho A v_1$$

where $\alpha = 16/27 = 0.593$ is defined as the 'coefficient of performance'.

Appendix B: Evaluation of the Lévy Distribution

We wish to show that the Characteristic Function

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2$$

is equivalent to a Probability Density Function given by

$$p(x) \sim x^{-(1+\gamma)}, \quad x \rightarrow \infty$$

i.e. we wish to prove the following:

Theorem

$$\frac{1}{x^{1+\gamma}} \leftrightarrow \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2, \quad x \rightarrow \infty$$

where $\leftrightarrow$ denotes transformation from real to Fourier space[1].

Proof of Theorem

For $0 < \gamma < 1$, and since the characteristic function is symmetric, we have

$$p(x) = \text{Re}[f(x)]$$

where

$$f(x) = \frac{1}{\pi}\int_0^\infty e^{ikx}e^{-k^\gamma}dk$$

$$= \frac{1}{\pi}\left(\left[\frac{1}{ix}e^{ikx}e^{-k^\gamma}\right]_{k=0}^\infty - \frac{1}{ix}\int_0^\infty e^{ikx}(-\gamma k^{\gamma-1}e^{-k^\gamma})dk\right)$$

$$= \frac{\gamma}{2\pi ix}\int_{-\infty}^\infty dkH(k)k^{\gamma-1}e^{-k^\gamma}e^{ikx}, \quad x \to \infty$$

where

$$H(k) = \begin{cases} 1, & k > 0 \\ 0, & k < 0 \end{cases}$$

For $0 < \gamma < 1$, $f(x)$ is singular at $k = 0$ and the greatest contribution to this integral is the inverse Fourier transform of $H(k)k^{\gamma-1}$. Noting that

$$\mathcal{F}^{-1}\left[\frac{1}{(ik)^\gamma}\right] \sim \frac{1}{x^{1-\gamma}}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and that

$$H(k) \leftrightarrow \delta(x) + \frac{i}{\pi x} \sim \delta(x), \quad x \to \infty$$

then, using the convolution theorem, we have

$$f(x) \sim \frac{\gamma}{i\pi x}\frac{i^{1-\gamma}}{x^\gamma}$$

and thus

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

For $1 < \gamma < 2$, we can integrate by parts twice to obtain

$$f(x) = \frac{\gamma}{i\pi x}\int_0^\infty dkk^{\gamma-1}e^{-k^\gamma}e^{ikx}$$

$$= \frac{\gamma}{i\pi x}\left[\frac{1}{ix}k^{\gamma-1}e^{-k^\gamma}e^{ikx}\right]_{k=0}^\infty$$

$$+\frac{\gamma}{\pi x^2}\int_0^\infty dke^{ikx}[(\gamma-1)k^{\gamma-2}e^{-k^\gamma} - \gamma(k^{\gamma-1})^2e^{-k^\gamma}]$$

$$= \frac{\gamma}{\pi x^2}\int_0^\infty dke^{ikx}[(\gamma-1)k^{\gamma-2}e^{-k^\gamma} - \gamma(k^{\gamma-1})^2e^{-k^\gamma}], \quad x \to \infty$$

[1]The authors acknowledge Dr K I Hopcraft, School of Mathematical Sciences, Nottingham University, England, for his advice in respect of this result.

The first term of this result is singular and therefore provides the greatest contribution and thus we can write,

$$f(x) \simeq \frac{\gamma(\gamma-1)}{2\pi x^2}\int_{-\infty}^\infty H(k)e^{ikx}(k^{\gamma-2}e^{-k^\gamma})dk$$

In this case, for $1 < \gamma < 2$, the greatest contribution to this integral is the inverse Fourier transform of $k^{\gamma-2}$ and hence,

$$f(x) \sim \frac{\gamma(\gamma-1)}{\pi x^2}\frac{i^{2-\gamma}}{x^{\gamma-1}}$$

so that

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

which maps onto the previous asymptotic as $\gamma \to 1$ from the above.

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