

2008-01-01

## On An Integrable Two-Component Camassa-Holm Shallow Water System

Adrian Constantin

*University of Vienna*, [adrian.constantin@univie.ac.at](mailto:adrian.constantin@univie.ac.at)

Rossen Ivanov

*Technological University Dublin*, [rossen.ivanov@tudublin.ie](mailto:rossen.ivanov@tudublin.ie)

Follow this and additional works at: <https://arrow.tudublin.ie/scschmatart>



Part of the [Mathematics Commons](#), [Non-linear Dynamics Commons](#), and the [Partial Differential Equations Commons](#)

---

### Recommended Citation

Constantin, A. & Ivanov, R. (2008). On an integrable two-component Camassa-Holm shallow water system. *Physics Letters A*, vol. 372, pg. 7129-7132. doi:10.1016/j.physleta.2008.10.050

This Article is brought to you for free and open access by the School of Mathematics and Statistics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact [arrow.admin@tudublin.ie](mailto:arrow.admin@tudublin.ie), [aisling.coyne@tudublin.ie](mailto:aisling.coyne@tudublin.ie), [vera.kilshaw@tudublin.ie](mailto:vera.kilshaw@tudublin.ie).

# On an integrable two-component Camassa-Holm shallow water system

Adrian Constantin <sup>1,2</sup> and Rossen I. Ivanov <sup>3,2</sup>

<sup>1</sup>*Faculty of Mathematics, University of Vienna,  
Nordbergstrasse 15, 1090 Vienna, Austria,  
email: adrian.constantin@univie.ac.at*

<sup>2</sup>*Department of Mathematics,  
Lund University, 22100 Lund, Sweden,*

<sup>3</sup>*School of Mathematical Sciences,  
Dublin Institute of Technology,  
Kevin Street, Dublin 8, Ireland,  
e-mail: rivanov@dit.ie*

The interest in the Camassa-Holm equation inspired the search for various generalizations of this equation with interesting properties and applications. In this letter we deal with such a two-component integrable system of coupled equations. First we derive the system in the context of shallow water theory. Then we show that while small initial data develop into global solutions, for some initial data wave breaking occurs. We also discuss the solitary wave solutions. Finally, we present an explicit construction for the peakon solutions in the short wave limit of system.

PACS numbers: 05.45.Yv, 47.35.Bb, 47.35.Fg, 47.35.Jk

In recent years the Camassa-Holm (CH) equation [1]

$$m_t + \omega u_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \quad (1)$$

( $\omega$  being an arbitrary constant) has caught a great deal of attention. It is a nonlinear dispersive wave equation that models the propagation of unidirectional irrotational shallow water waves over a flat bed [1, 2, 3, 4], as well as water waves moving over an underlying shear flow [5]. The CH equation also arises in the study of a certain non-Newtonian fluids [6] and also models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [7]. The CH equation has a bi-Hamiltonian structure [8] (and an infinite number of conservation laws), it is completely integrable (see [1] for the Lax pair formulation and [9] for the direct/inverse scattering approach), and its solitary wave solutions are solitons [1, 10, 11, 12, 13] with stable profiles [11, 12]. The equation attracted a lot of attention in recent years due to two remarkable features. The first is the presence of solutions in the form of peaked solitary waves or 'peakons' [1, 10, 14] for  $\omega = 0$ : the peakon  $u(x, t) = ce^{-|x-ct|}$  travelling at finite speed  $c \neq 0$  is smooth except at its crest, where it is continuous, but has a jump discontinuity in its first derivative. The peakons replicate a characteristic of the travelling waves of greatest height - exact travelling solutions of the governing equations for water waves with a peak at their crest [16, 17, 18] whose capture by simpler approximate shallow water models has eluded researchers until recently [19]. A further remarkable property of the CH equation is the presence of breaking waves (i.e. the

equation has smooth solutions which develop singularities in finite time in the form of breaking waves [1, 20, 21] - the solution remains bounded while its slope becomes unbounded in finite time [19]) as well as that of smooth solutions defined for all times [15]. These two phenomena have always fascinated the fluid mechanics community: 'Although breaking and peaking, as well as criteria for the occurrence of each, are without doubt contained in the equation of the exact potential theory, it is intriguing to know what kind of simpler mathematical equation could include all these phenomena' [19]. The short wave limit of the CH equation is the Hunter-Saxton (HS) equation

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0, \quad (2)$$

obtained from (1) by taking  $m = -u_{xx}$ . It describes the propagation of waves in a massive director field of a nematic liquid crystal [22], with the orientation of the molecules described by the field of unit vectors  $(\cos u(x, t), \sin u(x, t))$ , where  $x$  is the space variable in a reference frame moving with the linearized wave velocity, and  $t$  is a 'slow time variable'.

The equations (1), (2) admit many integrable multicomponent generalizations [23], the most popular of which is

$$m_t + 2u_x m + um_x + \sigma \rho \rho_x = 0, \quad (3)$$

$$\rho_t + (u\rho)_x = 0, \quad (4)$$

where  $m = \sigma_1 u - u_{xx}$ ,  $\sigma = \pm 1$  and  $\sigma_1 = 1$  or, in the 'short wave' limit,  $\sigma_1 = 0$ . (The CH equation can be

obtained via the obvious reduction  $\rho \equiv 0$ .) This system appears originally in [24] and its mathematical properties have been studied further in many works, e.g. [23, 25, 26, 27, 28]. The system is integrable – it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter  $\zeta$ :

$$\begin{aligned}\Psi_{xx} &= \left( -\sigma\zeta^2\rho^2 + \zeta m + \frac{\sigma_1}{4} \right) \Psi, \\ \Psi_t &= \left( \frac{1}{2\zeta} - u \right) \Psi_x + \frac{1}{2} u_x \Psi.\end{aligned}$$

It is bi-Hamiltonian, the first Poisson bracket

$$\begin{aligned}\{F_1, F_2\} &= \\ &= -\int \left[ \frac{\delta F_1}{\delta m} (m\partial + \partial m) \frac{\delta F_2}{\delta m} + \frac{\delta F_1}{\delta m} \rho \partial \frac{\delta F_2}{\delta \rho} + \frac{\delta F_1}{\delta \rho} \partial \rho \frac{\delta F_2}{\delta m} \right] dx,\end{aligned}$$

corresponding to the Hamiltonian

$$H = \frac{1}{2} \int (um + \sigma\rho^2) dx$$

and the second Poisson bracket

$$\{F_1, F_2\}_2 = -\int \left[ \frac{\delta F_1}{\delta m} (\partial - \partial^3) \frac{\delta F_2}{\delta m} + \frac{\delta F_1}{\delta \rho} \partial \frac{\delta F_2}{\delta \rho} \right] dx$$

corresponding to the Hamiltonian

$$H_2 = \frac{1}{2} \int (\sigma u \rho^2 + u^3 + uu_x^2) dx.$$

There are two Casimirs:  $\int \rho dx$  and  $\int m dx$ .

In what follows, we are going to demonstrate how the system (3), (4) arises in shallow water theory. We start from the Green-Naghdi (GN) equations [3, 29], which are derived from the Euler's equations under certain assumptions, as follows. Consider the motion of shallow water over a flat surface, which is located at  $z = 0$  with respect to the usual Cartesian reference frame. We assume that the motion is in the  $x$ -direction and the physical variables do not depend on  $y$ . Let  $h$  be the mean level of water,  $a$  – the typical amplitude of the wave and  $\lambda$  – the typical wavelength of the wave. Let us now introduce the dimensionless parameters  $\varepsilon = a/h$  and  $\delta = h/\lambda$ , which are supposed to be small in the shallow water regime. The variable  $u(x, t)$  describes the horizontal velocity of the fluid,  $\eta(x, t)$  describes the horizontal deviation of the surface from equilibrium, all measured in dimensionless units. The GN equations

$$\begin{aligned}u_t + \varepsilon uu_x + \eta_x &= \frac{\delta^2/3}{1+\varepsilon\eta} [(1+\varepsilon\eta)^3 (u_{xt} + \varepsilon uu_{xx} - \varepsilon u_x^2)]_x, \\ \eta_t + [(u(1+\varepsilon\eta))]_x &= 0.\end{aligned}$$

are obtained under the assumption that at leading order  $u$  is not a function of  $z$ . The leading order expansion with respect to the parameters  $\varepsilon$  and  $\delta^2$  gives the system

$$\left( u - \frac{\delta^2}{3} u_{xx} \right)_t + \varepsilon uu_x + \eta_x = 0, \quad (5)$$

$$\eta_t + [(u(1+\varepsilon\eta))]_x = 0. \quad (6)$$

One can demonstrate that the system (5), (6) can be related to the system (3), (4) in the first order with respect to  $\varepsilon$  and  $\delta^2$ . Indeed, let us define

$$\rho = 1 + \frac{1}{2}\varepsilon\eta - \frac{1}{8}\varepsilon^2(u^2 + \eta^2).$$

The expansion of  $\rho^2$  in the same order of  $\varepsilon$  is

$$\rho^2 = 1 + \varepsilon\eta - \frac{1}{4}\varepsilon^2 u^2.$$

With this definition it is straightforward to write (5) in the form

$$\left( u - \frac{\delta^2}{3} u_{xx} \right)_t + \frac{3}{2}\varepsilon uu_x + \frac{1}{\varepsilon}(\rho^2)_x = 0,$$

or, introducing the variable  $m = u - \frac{1}{3}\delta^2 u_{xx}$ , at the same order (i.e. neglecting terms of order  $\varepsilon\delta^2$ )

$$m_t + \varepsilon m u_x + \frac{1}{2}\varepsilon u m_x + \frac{1}{\varepsilon}(\rho^2)_x = 0. \quad (7)$$

Next, using the fact that  $u_t \approx -\eta_x$ ,  $\eta_t \approx -u_x$ , from the definition of  $\rho$  we get  $\rho_t = \frac{1}{2}\varepsilon\eta_t + \frac{1}{4}\varepsilon^2(\eta u)_x$ . With this expression for  $\rho_t$  and with  $\rho \approx 1 + \varepsilon u$ , equation (6) can be written as

$$\rho_t + \frac{\varepsilon}{2}(\rho u)_x = 0. \quad (8)$$

The rescaling  $u \rightarrow \frac{2}{\varepsilon}u$ ,  $x \rightarrow \frac{\delta}{\sqrt{3}}x$ ,  $t \rightarrow \frac{\delta}{\sqrt{3}}t$  in (7), (8) gives (3), (4) with  $\sigma = \sigma_1 = 1$ . The case  $\sigma = -1$ , which is often considered, corresponds to the situation in which the gravity acceleration points upwards. We mention also that the Kaup - Boussinesq system [30] is another integrable system matching the GN equation to the same order of the parameters  $\varepsilon, \delta$  [19, 31]. Notice that in the hydrodynamical derivation of (3)-(4) we require that  $u(x, t) \rightarrow 0$  and  $\rho(x, t) \rightarrow 1$  as  $|x| \rightarrow \infty$ , at any instant  $t$ .

We will now show that for the system (3), (4) in the hydrodynamically relevant case  $\sigma = \sigma_1 = 1$  wave breaking is the only way that singularities arise in smooth solutions. The system admits breaking wave solutions as well as solutions defined for all times. In particular, we will analyze the traveling wave solutions

The well-posedness (existence, uniqueness, and continuous dependence on data) follows by Kato's semigroup theory [32] for initial data  $u_0 = u(\cdot, 0) \in H^3$  and  $\rho_0 = \rho(\cdot, 0)$  such that  $(\rho_0 - 1) \in H^2$  [33]. If  $T = T(u_0, \rho_0) > 0$  is the maximal existence time, then the integral of motion

$$\int [u^2 + u_x^2 + (\rho - 1)^2] dx \quad (9)$$

ensures that  $u(\cdot, t)$  is uniformly bounded (i.e. for all values of  $x \in \mathbb{R}$  and all  $0 \leq t < T$ ) in view of the inequality

$$\sup_{x \in \mathbb{R}} |u(t, x)|^2 \leq \frac{1}{2} \int (u^2 + u_x^2) dx. \quad (10)$$

Considerations analogous to those made in [27] for a similar system show that the solution blows up in finite time (i.e.  $T < \infty$ ) if and only if

$$\liminf_{t \uparrow T} \{u_x(t, x)\} = -\infty, \quad (11)$$

which, in light of the uniform boundedness of  $u$ , is interpreted as wave breaking.

To show that wave breaking occurs, we introduce the family  $\{\varphi(\cdot, t)\}_{t \in [0, T]}$  of diffeomorphisms  $\varphi(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\partial_t \varphi(x, t) = u(\varphi(x, t), t), \quad \varphi(x, 0) = x, \quad (12)$$

and we denote

$$M(x, t) = u_x(\varphi(x, t), t), \quad \gamma(x, t) = \rho(\varphi(x, t), t).$$

Consider now initial data satisfying  $\rho_0(0) = 0$  and

$$u'_0(0) < -2 \left( \|u_0\|_1^2 + \|\rho - 1\|_0^2 \right)^{1/2}. \quad (13)$$

Noticing that  $(1 - \partial_x^2)^{-1} f = p * f$  (convolution) with  $p(x) = \frac{1}{2} e^{-|x|}$ , and applying the operator  $(1 - \partial_x^2)^{-1}$  to (3), we get

$$u_t + uu_x + p * (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2) = 0.$$

Applying now  $\partial_x$  and using the identity  $\partial_x^2 p * f = p * f - f$ , we obtain

$$u_{tx} + uu_{xx} + \frac{1}{2} u_x^2 = \frac{1}{2} (u^2 + \rho^2) - p * (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2). \quad (14)$$

This equation in combination with (12) yields

$$\partial_t M(t, x) + \frac{1}{2} M^2(t, x) \leq \frac{1}{2} \left( u^2(\varphi(x, t), t) + \gamma^2(x, t) \right). \quad (15)$$

On the other hand, from (12) and (4), we obtain

$$\partial_t \gamma = -\gamma M. \quad (16)$$

Since  $\gamma(0, 0) = 0$  we infer that  $\gamma(0, t) = 0$  for  $0 \leq t < T$ . The relation (13) together with (9), (10) ensure that  $4u^2(\varphi(x, t), t) \leq M^2(0, 0)$ . But then (15) yields

$$\partial_t M(0, t) \leq -\frac{1}{4} M^2(t, 0) \quad \text{for} \quad 0 \leq t < T.$$

As  $M(0, 0) = u'_0(0) < 0$ , this implies

$$M(0, t) \leq \frac{4u'_0(0)}{4 + u'_0(0)t} \rightarrow -\infty$$

in finite time.

However, not all solutions develop singularities in finite time. For example, if the initial data is sufficiently small, then the solution evolving from it is defined for all times. More precisely, let  $\alpha \in (0, 1)$  and

assume that  $|1 - \rho_0(x)| \leq \alpha$  for all  $x \in \mathbb{R}$ , while  $\|u_0\|_2 + \|\rho_0 - 1\|_0 \leq \alpha$ . Then the corresponding solution to (3), (4) is global in time. Indeed, if the maximal existence time were  $T < \infty$ , then for some  $x_0 \in \mathbb{R}$  we would have  $\liminf_{t \uparrow T} M(t, x_0) = -\infty$ . We now show that this is impossible. First, notice that

$$u^2(x, t) \leq \alpha^2, \quad (x, t) \in \mathbb{R} \times [0, T]. \quad (17)$$

by (9), (10). The integral of motion (9) also ensures that  $\|\rho(\cdot, t) - 1\|_0 \leq \alpha$  on  $[0, T]$ . Using this and (4), (9), (17), we get

$$\begin{aligned} 0 &\leq p * (u^2 + \frac{1}{2} [u_x^2 + \rho^2]) \\ &\leq \frac{1}{2} p * u^2 + \frac{1}{2} (u^2 + u_x^2 + (\rho - 1)^2) + p * (\rho - 1) + \frac{1}{2} p * 1 \\ &\leq \alpha^2 + \alpha + 1 \end{aligned}$$

on  $\mathbb{R} \times [0, T]$ . We now infer from (12), (14) that at  $x = x_0$ ,

$$M_t = -\frac{1}{2} M^2 + \frac{1}{2} \gamma^2 - f(t), \quad t \in [0, T], \quad (18)$$

with the continuous function  $f : [0, T] \rightarrow [0, \infty)$  bounded, i.e. there is a constant  $k_0 > 0$  such that  $0 \leq \gamma(t) \leq k_0$  on  $[0, T]$ . The solution  $(\gamma(t), M(t))$  of the nonlinear system (16), (18) with initial data  $\gamma(0) = \rho_0(x_0) > 0$  and  $M(0) = u'_0(x_0)$  supposedly blows up in finite time as  $\liminf_{t \uparrow T} M(t) = -\infty$ . However, notice that (16) and  $\gamma(0) > 0$  ensure  $\gamma(t) > 0$  for all  $t \in [0, T]$ . Thus we may consider the positive function

$$w(t) = \gamma(0)\gamma(t) + \frac{\gamma(0)}{\gamma(t)} [1 + M^2(t)]$$

for  $t \in [0, T]$ . Using (16), (18) we get

$$\begin{aligned} w'(t) &= 2 \frac{\gamma(0)}{\gamma(t)} M(t) [f(t) + \frac{1}{2}] \\ &\leq [k_0 + 1] \frac{\gamma(0)}{\gamma(t)} (1 + M^2) \leq [k_0 + 1] w(t) \end{aligned}$$

for all  $t \in (0, T)$ . Thus

$$w(t) \leq w(0) \exp([k_0 + 1]t)$$

on  $[0, T]$  and this prevents blowup. The obtained contradiction shows that the solution is defined globally in time.

Global existence is however not confined to small amplitude waves, as we shall see now by establishing the existence of traveling waves of large amplitude: solutions  $u(x, t) = \psi(x - ct)$ ,  $\rho(x, t) = \xi(x - ct)$  traveling with constant wave speed  $c > 0$ . To find whether such solutions exist, notice that with the previous Ansatz equation (4) becomes  $\xi'(c - \psi) = \xi\psi'$  and the asymptotic limits  $\psi(x) \rightarrow 0$  and  $\xi(x) \rightarrow 1$  as  $|x| \rightarrow \infty$  yield

$$\xi = \frac{c}{c - \psi}.$$

Thus (3) becomes a differential equation solely for the unknown  $\psi$ . Integrating this equation on  $(-\infty, x]$  and taking into account the asymptotic behaviour of  $\psi$ , we get the equation

$$-c\psi + c\psi'' + \frac{3}{2}\psi^2 - \psi\psi'' - \frac{1}{2}(\psi')^2 + \frac{c^2}{2(c-\psi)^2} = \frac{1}{2}.$$

Multiplication by  $\psi'$  and another integration on  $(-\infty, x]$  leads to

$$\left((\psi')^2 - \psi^2\right)(c - \psi) + \frac{c^2}{c - \psi} = \psi + c,$$

recalling the decay of  $\psi$  far out. Thus

$$(\psi')^2 = \frac{\psi^2}{(c - \psi)^2} (c - \psi - 1)(c - \psi + 1). \quad (19)$$

The asymptotic behaviour  $\psi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  yields now the necessary condition  $c \geq 1$  for the existence of traveling waves. For  $c \geq 1$ , a qualitative analysis of (19) shows that  $0 \leq \psi \leq c - 1$ . Thus nontrivial traveling waves exist only for  $c > 1$ , in which case both  $\psi$  and  $\xi$  are smooth waves of elevation with a single crest profile of maximal amplitude  $c - 1$ , respectively  $c$ . It is possible to find explicit formulas for the traveling waves in terms of elliptic functions cf. [34]. Due to the integrability of the system, we expect the solitary waves to interact like solitons.

Notice the absence of peakons among the solitary wave solutions. However, there are peakon solutions of the 'short wave limit' equation  $\sigma_1 = 0$ . Although this limit is not covered by the presented hydrodynamical derivation, we will describe briefly the construction of the peakon solutions, since these are interesting by themselves. The limit  $\sigma_1 = 0$  is a two component analog of the Hunter-Saxton equation. Such system is a particular case of the Gurevich-Zybin system [35], which describes the dynamics in a model of nondissipative dark matter [36].

The peakon solutions have the form

$$m(x, t) = \sum_{k=1}^N m_k(t) \delta(x - x_k(t)), \quad (20)$$

$$u(x, t) = -\frac{1}{2} \sum_{k=1}^N m_k(t) |x - x_k(t)|, \quad (21)$$

$$\rho(x, t) = \sum_{k=1}^N \rho_k(t) \theta(x - x_k(t)), \quad (22)$$

where  $\theta$  is the Heaviside unit step function. The asymptotic behaviour  $\rho(x, t) \rightarrow 0$  for  $x \rightarrow \infty$  and the condition  $\int m dx = 0$  (recall that  $m = -u_{xx}$ ) lead to

$$\sum_{l=1}^N m_l = \sum_{l=1}^N \rho_l = 0,$$

or

$$\sum_{l=1}^N \mu_l = 0$$

in terms of the new complex variable  $\mu_k \equiv m_k + i\rho_k$ . The substitution of the Ansatz (20) – (22) into (3), (4), under the assumption that  $x_1(t) < x_2(t) < \dots < x_N(t)$  for all  $t$ , (a condition holding for the peakons of (2) cf. [37]) gives the following dynamical system for the time-dependent variables:

$$\frac{dx_k}{dt} = -\frac{1}{2} \sum_{l=1}^N m_l |x_l - x_k|, \quad (23)$$

$$\frac{d\mu_k}{dt} = \frac{\mu_k}{2} \sum_{l=1}^N \mu_l \operatorname{sgn}(k-l), \quad (24)$$

with the convention  $\operatorname{sgn}(0) = 0$ . The integrals for this system can be obtained from the integrals of (3), (4) (available in [23]) by substituting the expressions (20), (21), (22). It is convenient to write the system in terms of the new independent variables

$$\Delta_k \equiv x_{k+1} - x_k, \quad M_k \equiv \mu_1 + \dots + \mu_k,$$

with  $k = 1, 2, \dots, N - 1$ . The Hamiltonian of the new system is

$$H = \frac{1}{2} \sum_{l=1}^{N-1} |M_k|^2 \Delta_k,$$

the equations

$$\frac{d\Delta_k}{dt} = -\operatorname{Re}(M_k) \Delta_k, \quad \frac{dM_k}{dt} = \frac{1}{2} M_k^2,$$

are Hamiltonian with respect to the bracket

$$\{\Delta_k, M_l\} = -\frac{M_k}{\bar{M}_k} \delta_{lk},$$

in which the bar stands for complex conjugation. These equations integrate immediately:

$$M_k(t) = -\frac{1}{t/2 + c_k},$$

$$\Delta_k(t) = \Delta_k(0) \frac{(t/2 + c_{k,1})^2 + c_{k,2}^2}{c_{k,1}^2 + c_{k,2}^2},$$

where  $c_k \equiv c_{k,1} + ic_{k,2} = -M_k^{-1}(0)$  is a complex constant with real and imaginary parts  $c_{k,1}$  and  $c_{k,2}$  respectively. Notice that the large time asymptotics

$$M_k \sim t^{-1}, \quad \Delta_k \sim t^2,$$

are the same as those for the peakons of the Hunter-Saxton equation (2) when  $\rho_k \equiv 0$  (see [37]).

### Acknowledgements

The support of the G. Gustafsson Foundation for Research in Natural Sciences and Medicine is gratefully acknowledged. R.I. is thankful to Prof. R. Camassa, Prof. D. D. Holm and Dr. G. Grahovski for discussions.

- 
- [1] R. Camassa and D.D. Holm, Phys. Rev. Lett. **71**(1993) 1661.
- [2] H.R. Dullin, G.A. Gottwald and D.D. Holm, Fluid Dynam. Res. **33** (2003) 73–95; Physica **190D** (2004) 1–14.
- [3] R.S. Johnson, J. Fluid. Mech. **457** (2002) 63.
- [4] A. Constantin and D. Lannes, Arch. Rat. Mech. Anal., in print.
- [5] R.S. Johnson, Fluid Dynam. Res. **33** (2003) 97.
- [6] V. Busuioc, C. R. Acad. Sci. Paris Ser. I **328** (1999) 1241.
- [7] H.-H. Dai, Wave Motion **28** (1998) 367.
- [8] A. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento **28** (1980) 299.
- [9] A. Constantin, V.S. Gerdjikov and R.I. Ivanov, Inv. Probl. **22** (2006) 2197–2207.
- [10] R. Beals, D. Sattinger and J. Szmigielski, Inv. Probl. **15** (1999) L1–L4.
- [11] A. Constantin and W. Strauss, Comm. Pure Appl. Math. **53** (2000) 603–610.
- [12] A. Constantin and W. Strauss, J. Nonlinear Sci. **12** (2002) 415–422.
- [13] A. Parker, Chaos, Solitons and Fractals **35** (2008) 220–237.
- [14] J. Lenells, J. Diff. Eq. **217** (2005) 393–430.
- [15] A. Constantin, Ann. Inst. Fourier (Grenoble) **50** (2000) 321–362.
- [16] A. Constantin, Invent. Math. **166** (2006) 523–535.
- [17] A. Constantin and J. Escher, Bull. Amer. Math. Soc. **44** (2007) 423–431.
- [18] J. F. Toland, Topol. Meth. Nonlin. Anal. **7** (1996) 1–48.
- [19] G. B. Whitham, *Linear and nonlinear waves*, J. Wiley & Sons Inc. (1999).
- [20] A. Constantin and J. Escher, Acta Mathematica **181** (1998) 229–243.
- [21] H. P. McKean, Asian J. Math. **2** (1998) 867–874.
- [22] J. Hunter and R. Saxton, SIAM J. Appl. Math. **51** (1991) 1498.
- [23] R. Ivanov, Z. Naturforsch. **61a** (2006) 133; nlin/0601066.
- [24] P. Olver and P. Rosenau, Phys. Rev. E **53** (1996) 1900.
- [25] A. Shabat and L. Martínez Alonso, On the prolongation of a hierarchy of hydrodynamic chains, in: *New trends in integrability and partial solvability* (ed. A.B. Shabat et al.), Proceedings of the NATO advanced research workshop, Cadiz, Spain 2002, NATO Science Series, Kluwer Academic Publishers, Dordrecht: 2004, pp. 263–280.
- [26] M. Chen, S.-Q. Liu and Y. Zhang, Lett. Math. Phys. **75** (2006) 1–15.
- [27] J. Escher, O. Lechtenfeld and Z. Yin, Discrete Contin. Dyn. Syst. **19** (2007) 493–513.
- [28] H. Aratyn, J.F. Gomes and A.H. Zimerman, SIGMA **2** (2006) paper 070.
- [29] A. Green and P. Naghdi, J. Fluid Mech. **78** (1976) 237–246.
- [30] D.J. Kaup, Progr. Theor. Phys. **64** (1976) 396–408.
- [31] G.A. El, R.H.J. Grimshaw, M.V. Pavlov, Stud. Appl. Math. **106** (2001) 157–186.
- [32] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in *Spectral theory and differential equations*, pp. 25–70. Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975.
- [33] For  $n \geq 0$  integer  $H^n$  is the Hilbert space of square integrable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with square integrable distributional derivatives up to order  $n$ , endowed with the norm  $\|f\|_n = \left( \int \sum_{j=0}^n (\partial_x^j f)^2(x) dx \right)^{1/2}$ . If  $f \in H^n$  with  $n \geq 1$ , then  $f$  has continuous derivatives up to order  $(n - 1)$  and  $f$  decays to zero at infinity.
- [34] H. Hancock, *Lectures on the theory of elliptic functions*, Dover Publ., Inc., New York, 1958.
- [35] A.V. Gurevich and K.P. Zybin, Soviet Phys. JETP **67** (1988) 1–12; Soviet Phys. Usp **38** (1995) 687–622.
- [36] M. Pavlov, J. Phys. A: Math. Gen. **38** (2005) 3823–3840.
- [37] R. Beals, D. Sattinger and J. Szmigielski, Appl. Anal. **78** (2001) 255–269.