Heating Effects Through Harmonic Distortion on Electric Cables in the Built Environment

Kevin O'Connell

Technological University Dublin, Kevin.OConnell@tudublin.ie

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HEATING EFFECTS THROUGH HARMONIC DISTORTION ON ELECTRIC CABLES IN THE BUILT ENVIRONMENT

Kevin O’Connell

Dublin Energy Laboratory
Dublin Institute of Technology
Dublin, Republic of Ireland

A thesis submitted in fulfilment of the requirements for the degree of
Doctor of Philosophy

Lead Supervisor: Prof. Dr. Jonathan Blackledge

Associate Supervisor: Prof. Dr. Eugene Coyle

Advisory Supervisor: Dr. Marek Rebow

July 23, 2013
Abstract

Under ideal circumstances, electric power supply voltage and current waveforms should be sinusoidal. However, this is very seldom the case in the built environment, due to the proliferation of non-linear loads. Examples of non-linear loads are those containing switched mode power supplies, reactors and electronic rectifiers/inverters. Common devices such as personal computers, fluorescent lighting, electric motors, variable speed drives, transformers and reactors and virtually all other electronic equipment are examples of non-linear loads. Non-linear loads are the norm in the built environment rather than the exception. Such loads produce complex current and voltage waves and simple spectral analysis of these complex waves shows that they can be represented by a wave at the fundamental power frequency plus other wave forms at integer and non-integer multiples of this frequency. These harmonics produce an overall effect called ‘Harmonic Distortion’ which can give rise to overheating in plant, equipment and the power cables supplying them, leading to reduced efficiency, operational life and sometimes failure.

Over the last few decades, harmonic distortion in power supplies has increased significantly due to the increasing use of electronic components in industry and elsewhere. Buildings such as modern office blocks, commercial premises, factories, hospitals, etc., contain equipment that generates harmonic loads as described above. Each item of equipment produces a unique harmonic signature and therefore a harmonic distortion which can be predicted if the equipment in use can be determined in advance. This thesis seeks to identify the harmonic signatures of different types of equipment commonly used and to predict the thermal loading effects on distribution cables caused by the skin and proximity effects of harmonic currents.
Declaration

I certify that this thesis, which I now submit for examination for the award of Doctor of Philosophy, is entirely my own work and has not been taken from the work of others, save and to the extent that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for postgraduate study by research of the Dublin Institute of Technology and has not been submitted in whole or in part for another award in any Institute.

The work reported on in this thesis conforms to the principles and requirements of the Institute’s guidelines for ethics in research.

The Institute has permission to keep, lend or copy this thesis in whole or in part, on condition that any such use of the material of the thesis be duly acknowledged.

Kevin O’Connell

Signature: __________________________  Date: July 23, 2013
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I would like to thank my daughters Eleanor, Patricia, Barbara and Catherine for their gentle encouragement. Lastly and not least I must thank my wife Mary for her support and encouragement over the years which is much appreciated.
Publications


Cable Heating Effects due to Harmonic Distortion in Electrical Installations (K O’Connell, J M Blackledge, M Barrett and A Sung), IAENG International Conference of Electrical Engineering, World Congress on Engineering (WCE2012), London 4-6 July, 928-933, 2012.


Evaluation of Proximity Heating Effects in Power Cables (J M Blackledge, E Coyle and K O’Connell), Technology to License, Hothouse: The award winning Innovation and Technology Transfer Centre at Dublin Institute of Technology, April, 2013.
## Glossary of Terms

<table>
<thead>
<tr>
<th>Term</th>
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<tbody>
<tr>
<td>Ampacity</td>
<td>The current carrying capacity of a cable.</td>
</tr>
<tr>
<td>Built Environment</td>
<td>The man made setting for human activity.</td>
</tr>
<tr>
<td>Displacement Power Factor</td>
<td>The ratio of Real Power to Apparent Power caused by the phase displacement between the current and voltage at the fundamental frequency.</td>
</tr>
<tr>
<td>Distortion Power Factor</td>
<td>The ratio of Real Power to Apparent Power caused by the presence of harmonic distortion.</td>
</tr>
<tr>
<td>Distribution Cables</td>
<td>Power cables for distributing electric power in a building or supply system.</td>
</tr>
<tr>
<td>Diversity Factor</td>
<td>A factor which is applied to calculate the overall effect due to the sum of many small inputs which may or may not occur simultaneously.</td>
</tr>
<tr>
<td>Electricity Association</td>
<td>An association of major electricity companies in the UK which closed in 2003 to be replaced by three organisations: Energy Networks Association; Energy Retail Association and the Association of Electricity Producers</td>
</tr>
<tr>
<td>Electromagnetic Compatibility</td>
<td>Electromagnetic Compatibility or (EMC) is a measure of whether an item of equipment or system affects or is affected by other equipment by transmission or absorption of Electromagnetic Interference (EMI).</td>
</tr>
<tr>
<td>Term</td>
<td>Definition</td>
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<tr>
<td>-------------------------</td>
<td>----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Embedded Generation</td>
<td>Small generating sets connected to the supply network provided by consumers of electricity for standby purposes, peak demand management or renewable energy plants designed to meet local authority planning requirements.</td>
</tr>
<tr>
<td>Fourier transform</td>
<td>This is a mathematical transform named after Joseph Fourier commonly used in engineering applications to transform a mathematical function of time into a new function whose argument is, for example, frequency.</td>
</tr>
<tr>
<td>Fundamental Component</td>
<td>Usually the wave of harmonic order 1.</td>
</tr>
<tr>
<td>Harmonic Distortion</td>
<td>Of an AC voltage or current is the presence of harmonic components in the voltage or current wave.</td>
</tr>
<tr>
<td>IEC</td>
<td>The International Electro-technical Commission is an international standards organisation based in Geneva.</td>
</tr>
<tr>
<td>Inter Harmonics</td>
<td>Harmonic components that are non-integer multiples of the fundamental wave &gt; 1.</td>
</tr>
<tr>
<td>Line Conductor</td>
<td>The conductor(s) used to carry electricity to a consumer which have a potential above earth.</td>
</tr>
<tr>
<td>Network Operating Company</td>
<td>The utility company that operates the electricity supply network.</td>
</tr>
<tr>
<td>Term</td>
<td>Definition</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Neutral Conductor</td>
<td>A conductor at or near earth potential used to carry electricity to a consumer.</td>
</tr>
<tr>
<td>Non Inductive Resistor</td>
<td>A pure resistor.</td>
</tr>
<tr>
<td>Power Quality</td>
<td>Power Quality relates to the quality of the electricity supply. It takes into account the variation of frequency, voltage and harmonic distortion from accepted norms.</td>
</tr>
<tr>
<td>Resonance</td>
<td>The electrical phenomenon where inductive reactance = capacitive reactance.</td>
</tr>
<tr>
<td>Sub-Harmonics</td>
<td>These are harmonic components that are non-integer multiples of the fundamental wave &lt; 1.</td>
</tr>
<tr>
<td>Switching Transient</td>
<td>A short term disturbance to the supply voltage caused by an induced voltage or inrush currents arising out of circuit switching operations.</td>
</tr>
<tr>
<td>True Power Factor</td>
<td>The product of Displacement and Distortion power factors.</td>
</tr>
</tbody>
</table>
Glossary of Acronyms and Notation

AC  Alternating Current
C   Capacitance
Ca  Ambient Temperature Rating Factor
Cc  Protective Device Rating Factor
CEN European Committee for Standardisation
CENELEC European Committee for Electrotechnical Standardisation
Cf  Harmonic Rating Factor
CFL Compact Fluorescent Lamp
Cg  Grouping Rating Factor
Ci  Thermal Insulation Rating Factor
\cos\phi  Power Factor
CPU Central Processing Unit
DC  Direct Current
DCT Discrete Cosine Transform
DFT Discrete Fourier Transform
EMC Electromagnetic Compatibility
EMI Electromagnetic Interference
EN European Standard
ER Engineering Regulation
EU European Union
GLS General Lighting Service
HS Harmonic Signature
HV High Voltage
Hz  Frequency
Ib  Design Current of Circuit
IEC International Electrotechnical Commission
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>IEEE</td>
<td>Institute of Electrical and Electronics Engineers</td>
</tr>
<tr>
<td>IET</td>
<td>The Institution of Engineering and Technology</td>
</tr>
<tr>
<td>$I_n$</td>
<td>Nominal Rated Current of Protective Device</td>
</tr>
<tr>
<td>$I_t$</td>
<td>Tabulated Ampacity</td>
</tr>
<tr>
<td>$I_Z$</td>
<td>Continuous Service Ampacity</td>
</tr>
<tr>
<td>$k_d$</td>
<td>Distortion Power Factor</td>
</tr>
<tr>
<td>kV</td>
<td>Kilovolts</td>
</tr>
<tr>
<td>kVA</td>
<td>Kilovoltamps</td>
</tr>
<tr>
<td>kVAR</td>
<td>Kilovoltamp Reactive</td>
</tr>
<tr>
<td>kWhr</td>
<td>Kilowatt Hour</td>
</tr>
<tr>
<td>kW</td>
<td>Kilowatt</td>
</tr>
<tr>
<td>$k_e$</td>
<td>Displacement Power Factor</td>
</tr>
<tr>
<td>L</td>
<td>Inductance</td>
</tr>
<tr>
<td>LV</td>
<td>Low Voltage</td>
</tr>
<tr>
<td>MV</td>
<td>Medium Voltage</td>
</tr>
<tr>
<td>$n$</td>
<td>Harmonic Order</td>
</tr>
<tr>
<td>NEC</td>
<td>National Electrical Code</td>
</tr>
<tr>
<td>NIR</td>
<td>Non Inductive Resistor</td>
</tr>
<tr>
<td>NOC</td>
<td>Network Operating Company</td>
</tr>
<tr>
<td>P</td>
<td>Power (kW)</td>
</tr>
<tr>
<td>p.u.</td>
<td>Per Unit</td>
</tr>
<tr>
<td>PC</td>
<td>Personal Computer</td>
</tr>
<tr>
<td>PCC</td>
<td>Point of Common Connection</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>PF</td>
<td>Power Factor</td>
</tr>
<tr>
<td>PFCC</td>
<td>Power Factor Correction Capacitor</td>
</tr>
</tbody>
</table>
PLC Programmable Logic Controller
PWHD Partially Weighted Harmonic Distortion
$Q$ kVAR
$Q_k$ kVAR of Capacitor Bank
$R$ Resistance
$r_{ac}$ AC Resistance
$r_{dc}$ DC Resistance
RF Radio Frequency
RMS Root Mean Square
$S$ kVA
SMPS Switched Mode Power Supply
T Period of a Signal
TDD Total Demand Distortion
THD Total Harmonic Distortion
THDi Total Harmonic Distortion Current
THDu Total Harmonic Distortion Voltage
TR Technical Report
TS Technical Specification
U Normal Supply Voltage
UK United Kingdom
$U_n$ Normal Supply Voltage at harmonic order $n$
UPS Uninterruptible Power Supply
V Volts
VSD Variable Speed Drive
$X$ Reactance
$X_C$ Capacitive reactance
$X_L$ Inductive reactance
$Y_{\text{CP}}$ Proximity Effect
$Y_{\text{CS}}$ Skin Effect
$Z$ Impedance
$\eta$ Efficiency
$\Omega$ Ohm

JPEG Joint Photographics Experts Group
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Chapter 1

Introduction and Overview of the Thesis

Ideally electric power supply voltage and current waveforms should be sinusoidal, however, this is very seldom the case due to the proliferation of non-linear loads. Examples of non-linear loads are those containing switched mode power supplies, reactors and electronic rectifiers/inverters, thus PCs, fluorescent lighting, electric motors, variable speed drives, transformers and reactors; virtually all electronic equipment are examples of non-linear loads [1]. This demonstrates that non-linear loads are the norm in the built environment rather than the exception. Non-linear loads produce complex current waves that indirectly contribute to the generation of complex voltage waves. A Fourier analysis of these complex waves shows that they can be represented by a wave at the fundamental power frequency plus other waves at integer multiples of this frequency. The wave at the power frequency is referred to as the fundamental and the other waves are called harmonics. Testa et al state that interharmonics, which are harmonics at non-integer multiples of the fundamental are also present in the supply voltage leading to very undesirable side effects [2]. The overall effect can be described as harmonic distortion. The effects of harmonic distortion are many. They include overheating in plant, equipment and the power cables supplying
them, leading to reduced life and sometimes failure. Reduced efficiency in machines and incorrect operation of protective devices is common. The problem can be particularly acute where sensitive medical equipment is being used, harmonics causing errors and possible misdiagnosis.

Over the last number of decades, harmonic distortion in power supplies has increased significantly due to the increasing use of electronic components in industry and elsewhere. Buildings such as modern office blocks, commercial premises, factories, hospitals, etc., contain equipment that generates harmonic loads as described above. Each item of equipment produces a unique harmonic signature and therefore the harmonic distortion for the distribution system can be predicted if the equipment in use can be determined in advance.

This thesis seeks to examine the effects of harmonics in the built environment; to identify the harmonic signatures of different types of equipment commonly used and to predict the thermal loading effects on distribution cables caused by the skin and proximity effects of harmonic currents.

1.1 Statement of the Research Problem

The phenomenon of skin effect and proximity effect although recognized as reducing the ampacity of cables had not evolved into a set of de-rating tables that could be easily applied on a day to day basis in engineering design offices. The problem in quantifying harmonic heating effects on electric cables is that they are a function of frequency. The greater the harmonic distortion present, the larger the number of harmonics present. Each harmonic current generates its own individual heating effect, thus a harmonic rating factor has to take into account a large number of individual elements.
1.2 Research Aims, Objectives and Methodology

1.2.1 Research Aims

The aims of the research are to:

1. Detail the impact of harmonics in the built environment.

2. Identify harmonic signatures for electrical equipment in common use.

3. Propose a new harmonic de-rating algorithm that may be applied to cables supplying harmonic loads.

1.2.2 Research Objectives

The objectives of the research are to:

1. Evaluate the existing international harmonic standards.

2. Present the theoretical background for the harmonic analysis carried out in this thesis.

3. Identify the sources of harmonic distortion in the built environment.

4. Assess the impact of harmonics in the built environment.

5. Quantify the effect of harmonics on individual discrete electrical components.

6. Evaluate existing industry methods for determining the ampacity of cables under harmonic distortion.

7. Develop a mathematical model for predicting the operating temperature of electric cables.

8. Produce an experimental test bed for measuring harmonic signatures.
9. Measure the harmonic signatures of electrical equipment in common use.

10. Produce an experimental test bed for measuring skin and proximity effect.

11. Measure skin and proximity effects for an electric conductor.

12. Analyse the data generated from the experiment test bed.

13. Compare the results of the data generated with published work.

14. Compare the results of the data generated with industry standards on cable sizing.

15. Develop an algorithm for a Harmonic Rating Factor.

16. Develop a simulation model for predicting proximity heating effects in a harmonically rich environment

1.2.3 Research Methodology

In order to realize the research objectives set out in Section 1.2.2 a research methodology was developed which is depicted in the flow diagram shown in in Figure 1.1. Included in this is a full literature search of current and past published papers and journals. Particular focus was placed on the historical context and development of current rating factors to be applied in electrical installations.

1.3 About this Thesis

Chapter 2 of the thesis reviews the international standards for harmonic distortion. It details how harmonic standards have evolved out of the necessity to control and manage harmonics in AC power systems. Poor power quality is estimated to cost billions of euros each year in the EU and much of the problem can be attributed to harmonic distortion. It evaluates and compares the three widely accepted standards that are in use: The
installations together with electromagnetic compatibility and emissions levels for equipment connected to the supply. It discusses the European Directive on Electromagnetic Compatibility [3] and the legislation enacted giving it legal effect.

Chapter 3 discusses the phenomenon of harmonics in electrical power systems and the industrial environment in particular. It examines harmonic sources and typical supply voltage harmonic distortion. It details the harmonic signatures of bridge rectifiers, switched mode power supplies, compact fluorescent lamps, personal computers, variable frequency AC motor drives etc. The chapter also considers the effect of harmonics on AC machines and plant. The heating effect of harmonics on electric cables is examined following on to the measurement, methods of elimination/reduction of harmonics.

Chapter 4 examines the effect of harmonics on individual discrete electrical components such as the resistor, inductor, capacitor and semi-conductor devices. It studies the conditions for parallel and series resonance which lead to power system resonance. Given the range of harmonic frequencies normally present, one or more of them are sure to experience resonance at different load conditions. Chapter 4 also studies voltage distortion due to imported harmonic currents. Power factor is also impacted by the presence of harmonics and this contributes significantly to its magnitude.

Chapter 5 sets out the theoretical background of complex voltage and current waves and the indices used to quantify them. The underlying theory behind cable heating effects in single-phase and three-phase circuits is developed from fundamentals. Reports on the experiments to measure harmonic distortion in electrical equipment in common use are carried. The energy balance equation for a cable is developed from basics leading to the development of a mathematical model for predicting the operating temperature of electric cables under different load conditions. The results of the experiment to measure skin and proximity effects are analysed and used to calculate the AC resistance factors for a specific size of cable. The data provided by IEC 60287-1-1 is used to calculate the AC resistance factors. Finally an algorithm to calculate cable ampacity under harmonic conditions was
developed from fundamental principles and validated by experiment.

Chapter 6 examines the cable heating effects due to harmonic distortion in electrical installations. It reviews the historical situation regarding methods used to determine the appropriate way to determine the ampacity of cables. The mathematical model for heat transfer mechanisms in electric cables is detailed. Since the development of greater awareness of the effects of harmonic distortion, many papers have been written seeking to quantify the problems caused and offering solutions. This chapter critically reviews these papers and the industry methods available for determining cable ampacity under harmonic distortion. Experiments on cable heating effects due to skin and proximity effects carried out by the author are reported on and discussed in this chapter.

Chapter 7 examines proximity effect and heating in electric cables. Through the application of Amperes Law and Maxwells Equations, a simulation model is developed for predicting the heating effect in electric cables.

Chapter 8 carries a summary of the thesis, conclusions and future work.

Appendix A examines the theory of spectral analysis. The following series are derived: the Fourier series, the half range Fourier series, the Fourier series for an arbitrary period leading on to the complex Fourier series and the Fourier transform pair. The coupling of the Fourier Series and Fourier Transform through the Discrete Fourier Transform provides the essential theoretical background to the harmonic analysis considered in the thesis.

Appendix B considers the microscopic and macroscopic forms of Maxwell’s equations. The general solution to Maxwell’s microscopic equations is derived using Lorentz’s Gauge Transformation.

### 1.4 Original Contributions

The thesis set out to examine the heating effects through power system harmonics on electric cables in the built environment and, in particular, how harmonic loads impact on
the current carrying capacity of electric cables. Although, in recent years, some movement has taken place in the standards to offer harmonic de-rating factors, the additional heating in cables due to skin and proximity effects has not been quantified for typical situations.

A novel experimental test bed was developed to enable measurement of harmonic effects in a safe and effective manner. The analysis of these results has led to the development and validation of a new harmonic de-rating algorithm that may be applied to cables supplying harmonic loads. This algorithm can be used to generate a set of new cable rating correction factors for the standards currently in use.

A two- and three-dimensional simulation model for harmonic proximity heating effects is developed based on the application of Maxwells equations. This simulation model can be used to predict hotspots generated by harmonics in cable configurations and in electrical machines and is the basis for a new ‘Technology to License’.
Chapter 2

Industrial Standards in Harmonic Distortion

2.1 Introduction

AC Power Systems are subject to distortion by harmonic and inter harmonic components which affect the supply voltage and load currents. Harmonics can have detrimental effects on the supply system causing a reduction in power quality. Consumers equipment connected to the supply can (i) be adversely affected by existing harmonics, and, (ii) generate harmonic currents that create or add to existing distortion. It is estimated that losses caused by poor power quality cost EU industry and commerce about 10 billion Euros per annum [4]. Harmonic Standards have therefore been created to set acceptable levels of:

- voltage distortion present in supply systems;
- load current distortion in installations connected to the supply system;
- Electromagnetic Compatibility (EMC) for equipment connected to the supply;
- electromagnetic emissions generated by equipment connected to the supply.
Harmonic Standards have been harmonised internationally to promote international trade so that equipment manufactured in one country will comply with emission and immunity limits in force in other countries.

2.2 Planning, Compatibility and Immunity Levels

Electromagnetic compatibility between the user’s equipment and the electricity supply system is essential to ensure that both operate in a satisfactory manner. In the main, this is achieved by (i) limiting the harmonic emissions from user’s non-linear loads and generating plant; (ii) compliance with appropriate harmonic voltage planning levels set for the distribution network.

2.2.1 Harmonic Voltage Planning Levels

The harmonic voltage planning level for systems below 35kV is the maximum harmonic distortion level which the network operating company guarantees to the end user. The user’s equipment must be able to operate satisfactorily in this harmonic environment. For systems below 35kV the harmonic voltage planning levels are set by international standards and for voltages exceeding 35kV, the harmonic voltage planning levels are set taking account of system characteristics appropriate to the local environment. Table 2.1 and Table 2.2 show planning levels used in the UK as quoted in the Engineering Recommendations G5/4-1 [5].

Table 2.3 shows the supply voltage characteristics according to EN 50160 and EN 61000-2-2 [4] and Table 2.4 quotes individual harmonic voltage limits as a percentage of the nominal supply voltage $U_n$. Note that this is in contrast to other standards that quote the individual harmonic voltages as a percentage of the fundamental and makes for greater ease of calculations but can cause confusion when using measuring instruments which are calibrated to measure the latter.
Table 2.1: Summary of THD Planning Levels [5].

<table>
<thead>
<tr>
<th>System Voltage at the PCC</th>
<th>THD Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>400 V</td>
<td>5%</td>
</tr>
<tr>
<td>6.6, 11 and 22 kV</td>
<td>4%</td>
</tr>
<tr>
<td>22 to 400 kV</td>
<td>3%</td>
</tr>
</tbody>
</table>

Table 2.2: Planning Levels for Harmonic Voltages in 400V Systems [5].

<table>
<thead>
<tr>
<th>Odd Harmonics</th>
<th>Odd Harmonics</th>
<th>Even Harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order $n$</td>
<td>Harmonic Voltage %</td>
<td>Order $n$</td>
</tr>
<tr>
<td>5</td>
<td>4.0</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>4.0</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>3.0</td>
<td>15</td>
</tr>
<tr>
<td>13</td>
<td>2.5</td>
<td>21</td>
</tr>
<tr>
<td>17</td>
<td>1.6</td>
<td>&gt;21</td>
</tr>
<tr>
<td>19</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>&gt;25</td>
<td>0.2+0.5</td>
<td></td>
</tr>
</tbody>
</table>
Table 2.3: Supply Voltage Characteristics According to EN 50160 and EN 61000-2-2 [4].

<table>
<thead>
<tr>
<th>No</th>
<th>Parameter</th>
<th>Supply Voltage Characteristics According to EN 50160</th>
<th>Low Voltage Characteristics According to EMC Standard EN 61000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LV, MVB; as detailed in Table 2 EN 50160</td>
<td>EN 61000-2-2</td>
</tr>
<tr>
<td>10</td>
<td>Harmonic Voltage</td>
<td></td>
<td>6%-5th</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3.5%-11th</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>THD&lt;8%</td>
</tr>
</tbody>
</table>

Table 2.4: Values of Individual Harmonic Voltages at the Supply Terminals Given in % $U_n$ [4].

<table>
<thead>
<tr>
<th>Odd Harmonics</th>
<th>Even Harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Multiples of 3</td>
<td>Multiples of 3</td>
</tr>
<tr>
<td>Order h</td>
<td>Relative Voltage %</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>3.5</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>1.5</td>
</tr>
<tr>
<td>23</td>
<td>1.5</td>
</tr>
<tr>
<td>25</td>
<td>1.5</td>
</tr>
</tbody>
</table>
2.2.2 Equipment Compatibility and Immunity Levels

Electromagnetic compatibility (EMC) which is the subject of an EU Directive [3] is defined as the ability of one piece of equipment to operate satisfactorily within a defined electromagnetic environment. It is the responsibility of the user to ensure that his installation is in compliance with the appropriate harmonised EN standards. This is achieved by limiting the harmonic emissions from non-linear loads and generating plant. Equipment emission limits and mitigation measures such as active and passive harmonic filters should be considered together by the user and the supply company when drawing up the connection agreement.

Immunity levels which are equipment test levels, are defined as the level of harmonic distortion above which equipment function and performance are likely to be impaired. Regulations such as G5/4-1 and EN 50160 are designed to limit harmonic voltage distortion in the supply voltage to levels below the immunity levels of the equipment used in the installations. Relevant IEC standards on electromagnetic compatibility such as IEC 61000-2-2, Compatibility Levels for low frequency conducted disturbances and signalling in public low-voltage power supply system, or IEC 61000-2-4, Compatibility levels in industrial plants for low-frequency conducted disturbances should be consulted by users to ensure compliance.

EN 61000-3-2 and ER G5/4-1 specify equipment emission limits for individual and aggregate loads < 16 A per phase which comply with EN 61000-3-2. Individual and aggregate loads > 16 A per phase must comply with IEC Technical Report 61000-3-4. In addition, [5] specifies individual order harmonic emission limits for aggregate loads and equipment > 16A per phase.
2.2.3 Relationship Between Planning, Compatibility and Immunity Levels

Electromagnetic compatibility levels have been set relevant to a given environment so that all equipment used in that environment must have immunity at least to the level of electromagnetic disturbance present. The planning level set for emissions from large installations is below the disturbance levels adopted for equipment intended to be connected to the system. A margin of tolerance is built into the limits to take account of the effects of resonance, the increase in disturbance levels over time due to additional loads being connected and the system impedance which is a variable factor. Figure 2.1 depicts the relationship between compatibility, immunity, planning and emission levels [1].
2.3 Development of Harmonic Standards

The International Electrotechnical Commission (IEC) based in Geneva is the body responsible for electric power quality standards under which harmonics fall. These standards are referred to as the Electromagnetic Compatibility (EMC) Standards and for the most part are covered by the IEC 61000 series [6] applicable in the EU. Other widely accepted international standards are the IEEE Std 519-1992 [7], ER G5/4-1 [5] and EN 50160 [4].

International standards are used as a basis for global coordination but individual countries may make their own adjustments to the international standards to reflect the special characteristics of their distribution systems. For example, in the case of the UK and Ireland, there are many distributed generation centres located close to many load centres. This presents a different system characteristic to the distribution system in New Zealand which is characterised by a small number of generating centres connected by long transmission lines to individual load centres [1].

System voltage distortion is a function of the product of harmonic currents and system impedance. Low impedance systems are referred to as ‘hard systems’ because they are less susceptible to voltage distortion by harmonic currents. The converse is true of high impedance systems which are referred to as ‘soft systems’. In the context of the previous paragraph, the UK and Irish systems are generally classified as ‘hard’ and the New Zealand systems are generally classified as ‘soft’. This is expressed mathematically as the ratio $I_{SC}/I_L$, where $I_{SC}$ is the short circuit current and $I_L$ is the load current. The higher this ratio, the ‘harder’ the system. The Standards quote higher acceptable levels of harmonic currents for ‘hard’ systems.

In setting standards, one must be cognisant of a situation where a large industrial consumer is connected to the supply network together with a number of smaller consumers at a point of common coupling (PCC). Were the limits of harmonic current emissions set in absolute terms rather than as a proportion of the consumers load, this may discriminate
against the large consumer. Whereas if the reverse were the case, this may discriminate against the smaller consumer whose harmonic emissions may be high as a proportion of their load but insignificant in terms of their effect on the system as a whole. It is for this reason and other parallel situations that harmonic standards are issued as guidelines to be applied taking local conditions into account. They are intended to be flexible and applied in a sensible manner.

2.4 The European Union Electromagnetic Compatibility Directive

The EMC requirements in the European Union for electrical and electronic products, are covered by the EMC Directive 89/336/EEC [3]. It came fully into effect on 1st January 1996. This directive has been amended a number of times, the most recent being 93/68/EEC in 2004. The Directive seeks to remove technical barriers to trade by requiring equipment to operate satisfactorily in its specified electromagnetic environment. By limiting these emissions this also serves the public electricity distribution system which must be protected from disturbances emitted by equipment.

ing Recommendation G5/4-1 [5] came into force in October 2005 to ensure compliance for all system voltages from 400 V to 400 kV in the UK. Whereas in Ireland, CENELEC Standard EN 50160 Voltage Characteristics of Electricity Supplied by Public Networks is used as a basis for compliance.

Harmonic distortion limits are not governed by statute. The legally enforcing document is therefore the connection agreement between the network operator and the customer. This agreement lays down connection conditions which will require compliance with ER G5/4-1, IEE Std 519-1992 or IEC 50160 and include any derogation and/or harmonic mitigation measures which may be agreed between the network operator and the customer.

2.5 Existing Harmonic Standards

2.5.1 Scope of Existing Standards

Ong et al [8] presented a comparison of the scope existing harmonic standards which is set out in Table 2.5 and Table 2.6.

2.5.2 Standards Setting Voltage Harmonics Limits


2.5.3 Standards Setting Current Harmonic Limits

IEC 61000-3-2 [6], IEC 61000-3-4 [11] and IEC 61000-3-12 [12] deal with guidelines for limitations of harmonic currents up to the 40th harmonic injected by individual items of equipment into the public supply network. The three standards quoted apply to different
Table 2.5: Scope of Respective Standards [8].

<table>
<thead>
<tr>
<th>Standard</th>
<th>Area of Focus / Nominal Voltage / Nominal Frequency</th>
<th>Limits Apply At:</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEEE Std 519</td>
<td>Design of power systems with non-linear loads</td>
<td>PCC</td>
</tr>
<tr>
<td>ER G5/4-1</td>
<td>Sets the planning levels for THD’s and harmonic current emissions to be used in the process for connecting non-linear equipment.</td>
<td>PCC</td>
</tr>
<tr>
<td>IEC 61000-2-2</td>
<td>Set the compatibility levels for public LV AC distribution systems. Single-phase: &lt; 420V Three-phase: &lt; 690V Frequency: 50/60Hz</td>
<td>PCC</td>
</tr>
<tr>
<td>IEC 61000-2-4</td>
<td>Set the compatibility levels for industrial and non-public power distribution systems. Three-phase: &lt; 35kV Frequency: 50/60Hz</td>
<td>IPC</td>
</tr>
<tr>
<td>EN 50160</td>
<td>A guide on limits for power quality related phenomena for public LV and MV electricity distribution systems under normal operating conditions. LV: &lt; 1kV MV: between 1kV and 35kV Frequency: 50Hz</td>
<td>PCC</td>
</tr>
<tr>
<td>STC</td>
<td>Set technical requirements to be met by Transmission Licensee, Generation Licensee, Connected Person, Power System Operator. Frequency: 50Hz</td>
<td>PCC</td>
</tr>
</tbody>
</table>

Table 2.6: Comparison of Voltage Distortion Limit at LV [8].

<table>
<thead>
<tr>
<th>Standards</th>
<th>Supply System Voltage at PCC (IPC for IEC61000-2-4)</th>
<th>THDv(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEEE Std 519</td>
<td>&lt;69kV</td>
<td>5</td>
</tr>
<tr>
<td>ER G5/4-1</td>
<td>400V</td>
<td>5</td>
</tr>
<tr>
<td>IEC 61000-2-2</td>
<td>400V &amp; 230V</td>
<td>8</td>
</tr>
<tr>
<td>IEC 61000-2-4</td>
<td>400V &amp; 230V</td>
<td>8</td>
</tr>
<tr>
<td>EN 50160</td>
<td>400V &amp; 230V</td>
<td>8</td>
</tr>
<tr>
<td>STC</td>
<td>400V &amp; 230V</td>
<td>5</td>
</tr>
</tbody>
</table>

where IPC means In-Plant Point of Common Coupling
classes of equipment that are segregated by rated current and application. The harmonic current limits are based on THDi, PWHD and admissible individual harmonic current. IEC 61000 3-2 and 3-4 contain limits for harmonic current emissions by equipment with input currents of less than 16 A and greater than 16 A per phase respectively and IEC 61000 3-12 provides limits for the harmonic currents produced by equipment connected to low-voltage systems with input currents equal to and below 75 A per phase.

For the purpose of harmonic current limitation, equipment is classified under IEC 61000 3-2 and 3-4 in Table 2.7 and Table 2.8.

IEC 61000 4-7 deals with testing and measurement techniques. It gives a general guide on harmonic and inter-harmonic measurements and the necessary instrumentation for connecting to power systems.

### 2.5.4 IEEE Std 519-1992 Recommended Practices and Requirements for Harmonic Control in Electric Power Systems

This standard was prepared by a joint task force sponsored by the Working Group on Power System Harmonics of the Transmission and Distribution Committee of the IEEE Power Engineering Society and the Harmonic and Reactive Compensation Subcommittee of the Industrial Power Conversion Committee of the IEEE Industry Applications Society [7]. It identifies the major sources of harmonics in power system equipment, industrial,
Table 2.8: Limits for Class A Equipment.

<table>
<thead>
<tr>
<th>Harmonic Order n</th>
<th>Maximum permissible harmonic current A</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Odd harmonics</strong></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.30</td>
</tr>
<tr>
<td>5</td>
<td>1.14</td>
</tr>
<tr>
<td>7</td>
<td>0.77</td>
</tr>
<tr>
<td>9</td>
<td>0.40</td>
</tr>
<tr>
<td>11</td>
<td>0.33</td>
</tr>
<tr>
<td>13</td>
<td>0.21</td>
</tr>
<tr>
<td>15 ≤ n ≤ 39</td>
<td>0.15 15/n</td>
</tr>
<tr>
<td><strong>Even harmonics</strong></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.08</td>
</tr>
<tr>
<td>4</td>
<td>0.43</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
</tr>
<tr>
<td>8 ≤ n ≤ 40</td>
<td>0.23 8/n</td>
</tr>
</tbody>
</table>
commercial and residential installations. It also deals with the system frequency responses to the presence of harmonic distortion such as series and parallel resonance, overloading of electric cables and equipment such as motors, generators, transformers, capacitors, electronic equipment, metering equipment, switchgear, relays and static power converters.

Analytical methods for harmonics propagation and measurements required for assessing the level of power system distortion are detailed together with system modelling. The standard also deals with methods of reducing harmonic distortion using passive and active filters, special transformers etc. The special design features and de-rating required for transformers and capacitors to accommodate the presence of harmonics is detailed.

2.5.5 ER G5/4-1 Planning Levels for Harmonic Voltage Distortion

ER G5/4-1 planning levels for harmonic voltage Distortion and the connection of non-linear equipment to transmission systems and distribution networks in the United Kingdom [5] was prepared by the Electricity Association to take account of conditions that apply to them. It provides a standard basis of assessment for use by Network Operating Companies (NOC) and their customers. The aim of the recommendations is to help fulfil the technical objectives of the UK EMC Regulations which implement the EU EMC Directive. The recommendation addresses harmonic voltage planning levels, equipment compatibility and harmonic current emissions, sub-harmonic and inter-harmonic emission levels, assessment procedures, harmonic measurement techniques, short term harmonic distortion.

2.6 Harmonic Indices

The most common index is the Total Harmonic Distortion (THD) this is the ratio of the RMS harmonic content of the wave to the RMS value of the fundamental and is expressed
as follows:

\[ \text{THD} = \sqrt{\sum_{n=2}^{N} \frac{V_n^2}{V_1}} \]

where \( V_1 \) is the RMS value of the fundamental, \( V_n \) is the RMS value of the \( n^{\text{th}} \) harmonic and \( N \) is the maximum harmonic number that needs to be considered (usually 50). The THD can be expressed for a voltage wave as THDu and for a current wave THDi.

The THD can be misleading when used to measure current distortion particularly when the load current is small and the fundamental current is constantly varying. A more practical index is the Total Demand Distortion (TDD) and this is defined as:

\[ \text{TDD} = \sqrt{\frac{\sum_{n=2}^{N} I_n^2}{I_R}} \]

where \( I_R \) is the RMS value of the rated design current, \( I_n \) is the RMS value of the \( n^{\text{th}} \) harmonic and \( N \) is the maximum harmonic number that needs to be considered (usually 50).

Partial Weighted Harmonic Distortion (PWHD) is used to lump together harmonics from the 23\(^{\text{rd}}\) upwards. This has been introduced to cater for loads which have a high level of higher order harmonics such as multi-pulse and active rectifiers. It is defined as

\[ \text{PWHD} = \sqrt{\sum_{n=k}^{N} \left( \frac{I_n}{I_1} \right)^2} \]

### 2.7 Time Varying Harmonics, Interharmonics and Voltage Notching

#### 2.7.1 Time Varying Harmonics

Time varying harmonics are short duration harmonic disturbances caused by switching transients or start up situations for electrical machinery. IEEE Std 519 1992 allows normal
limits to be exceeded by 50% for non-steady state harmonics. Standard measurement methodology recommends using a 95% percentile sample taken over a given time period.

2.7.2 Interharmonics

Interharmonics are harmonics which are not integer multiples of the fundamental frequency. The undesirable side effects include light flicker, overheating of motors and transformers and generation of negative rotational torques in induction motors with a consequent loss of efficiency and reduced life. Standards IEC 61000-2-2, IEC 61000-2-4, ER G5/4-1 and EN 50160 quote a maximum acceptable interharmonic magnitude of 0.2%. Fuchs et al [13] argue that this limit should be as low as 1%.

2.7.3 Voltage Notching

Voltage notching is defined as a severe voltage change of very short duration caused by the commutating action of a rectifier [5]. It is typically produced by three-phase six-pulse thyristor converters and is depicted in Figure 2.2.

Line notching limits are addressed in IEEE Std 519-1992 and ER G5/4-1 (see Table 2.9). The notch depth and area are the two parameters used to quantify this phenomenon.
Table 2.9: Comparison of Line Notching Limits [7].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>IEEE 519-1992 (General System)</th>
<th>ER-G5/4-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notch Depth</td>
<td>20%</td>
<td>15%</td>
</tr>
<tr>
<td>Notch Area</td>
<td>22800Vµs</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.10: Current Distortion Limits for General Distribution Systems (from 120 to 69000 V).

The Engineering Recommendations G5/4-1 and the IEEE Std 519-1992 provide guidance to stakeholders to achieve compliance with the EMC standards. They recognise that there is more than one way of achieving that outcome. In the final analysis, the ultimate decision to approve connection of individual items of equipment lies with the network operating company. Table 2.10 shows the current distortion limits for general distribution systems from 120 to 69000 V.
Chapter 3

Harmonics in the Industrial Environment

3.1 Introduction

This chapter discusses the phenomenon of harmonics in electrical power systems and the industrial environment in particular. The sources and effects of harmonic distortion are identified together with measures that are employed to reduce them. According to Testa et al [2] the harmonic concept is based on Fourier analysis of periodic waves that satisfy the following condition: If \( x(t) \) is a continuous periodic signal with period of \( T \) and it satisfies the Dirichlet condition, it can be represented by a Fourier series

\[
x(t) = \sum_{k=-\infty}^{\infty} X(k\omega_0) \exp(jk\omega_0 t)
\]

where \( \omega_0 = 2\pi/T \) is the fundamental angular frequency and

\[
X(k\omega_0) = \frac{1}{T} \int_{t}^{t+T} x(t) \exp(-ik\omega_0 t) dt
\]

is the Fourier coefficient at the \( k^{th} \) harmonic. Thus a non-sinusoidal periodical signal can be represented by an infinite series of sinusoidal waves, the frequencies of which are
integer multiples of the fundamental frequency.

The Discrete Fourier Transform (DFT) was introduced to implement Fourier analysis carried out by computer where the signal is discrete and has finite length. If more than one fundamental cycle is used to carry out DFT it is possible to obtain components at frequencies that are not integer multiples of the fundamental frequency \[2\]. These components are defined by the IEC as interharmonics \((n > 1)\) and subharmonics \((n < 1)\).

If a signal consists of two frequencies, such as \(x(t) = \sin(2\pi 60t) + 0.5 \sin(2\pi 90t)\), the 90Hz component lies between the fundamental frequency and the second harmonic. The signal will repeat itself every two 60Hz cycle periods. If DFT is performed on this signal using a two 60Hz sampling window, the solution will yield components that are interharmonics.

Testa et al \[2\] states that nonlinear and switched loads and sources can cause distortions of the normal sinusoidal current and voltage waveforms in an AC power system. This waveform distortion may be characterized by a series of sinusoidal components at harmonic frequencies and of sinusoidal components at non-harmonic frequencies. Pure sinusoidal voltage or current waves do not contain harmonics, however, a pure sinusoidal AC wave is very much the ideal situation in electrical power systems and in practice, all AC voltage and current waves are complex sinusoids and can be represented by harmonic, interharmonic and subharmonic components.

The industrial environment is one where power system harmonics proliferate due to the wide scale presence of plant and machinery that generate harmonics. Harmonics fall into two categories:

- voltage harmonics which are present in the voltage waveform supplied by the electrical utility company;

- current harmonics which are generated in the electrical consumer’s installation.

They are made up in part by the electrical loads and components responding to the existing supply voltage harmonics and also by the non-linear loads in the installation
generating current harmonics in their own right. The current harmonics as they flow through the external system impedance will further increase the harmonic content of the supply voltage wave.

The international standards IEC 61000-3-2 (2005) [6] and IEEE 519-(1992) [7] set limits on the amount of harmonic distortion allowable in the supply voltage and also the harmonic current that can be passed into the supply network by the consumer. The electricity supply company has control over the former and the consumer has control over the latter. To an electricity supply company, harmonic distortion in the electrical distribution and transmission system is of great commercial significance as it increases costs across the board and reduces the reliability of the supply. Losses in transmission cables, transformers, generators, etc., are increased and the service life of plant and machinery is reduced. Furthermore, harmonic distortion in one electrical installation can cause additional corruption of the supply voltage leading to a general degradation of the power quality supplied to other consumers.

From an industrial consumer’s point of view, harmonic distortion in the supply voltage can generate harmonic load currents with a greater harmonic distortion due to amplification through electrical plant and equipment. Harmonics in the supply voltage may also give rise to electromagnetic interference which can cause malfunction of telecommunications equipment and sensitive electronic equipment. Nuisance tripping may be experienced with circuit protective devices when no fault is present leading to downtime and loss of production. In applications where electronic variable speed drives (VSD’s) are used, drive motors may run inefficiently, overheat and experience a reduced service life. A host of other undesirable side effects can be experienced such as component failure and in some cases, failure of supply. Harmonic distortion also increases the cost of electricity because it impacts adversely on system power factor causing larger supply currents than would otherwise be the case. This will give rise to greater cable losses and also to a significant increase in electricity tariffs as the utility company seeks to recover the additional
costs of supplying a harmonically distorted current.

3.2 Harmonic Sources

Harmonic distortion can be generated from sources within the electricity supply network, such as transformers, reactors, alternators and by a harmonically distorted load current flowing through the supply impedance. This distortion tends to be steady state for a given load and will change as the system load changes. In addition to this, there are harmonics generated by switching transients and lightning surges. These are of short duration lasting only a fraction of a second but can be very disruptive to the electrical supply and sensitive equipment. Other sources of harmonics are located in the consumers’ installation. These produce non-sinusoidal load currents which can interact with circuit elements within the electrical installation or in the external supply system, corrupting the supply voltage and indirectly affecting other consumers connected to the same system. The greatest source of harmonics in the industrial environment is undoubtedly power electronics. Other sources include electric arc furnaces, discharge lighting, electrical machines and transformers.

3.2.1 Typical Supply Voltage Harmonic Distortion

Measurements of supply voltage harmonic distortion were carried out in DIT Kevin Street laboratories using the experimental method described in Figure 5.3. The supply voltage was found to have a Total Harmonic Distortion (THDu) of 2.8%. This value is comfortably within the limits set by the Voltage Characteristics of Electricity Supplied by Public Distribution Systems, BS EN 50160:2000 [4]. The wave form and harmonic analysis is shown in Figure 3.1.

In general, the harmonic distortion in the supply voltage from a design point of view is an unknown factor. It will vary from location to location and may vary over the full day as the character of the load changes. The THDu of the supply voltage can also be
altered when the supply authority carries out changes to the network or when standby embedded generation switches on. As will be shown later, an electrical installation can import harmonics from a supply that is already distorted. By generating a distorted current of its own it can contribute to distorting the supply voltage further still with undesirable consequences for other users.

### 3.2.2 Bridge Rectifiers

The most common solid state electronic power source is the single-phase rectifier. This device is used to provide power to all types of domestic and office equipment such as television sets, radios, personal computers etc. The basic circuit consists of diodes connected in a bridge configuration as shown in Figure 3.2
In the arrangement shown in Figure 3.2, the smoothing capacitor charges up from the diode output and then discharges into the load. Consequently, the diodes only conduct when the supply voltage is higher than the voltage of the capacitor. This is shown in the wave diagram in Figure 3.3.

The current taken from the supply in this type of circuit is inherently non-sinusoidal in nature and the current wave will be rich in harmonics. The Switched Mode Power Supply (SMPS) is now widely used in the power electronics industry for reasons of weight and cost. It is a development of the bridge rectifier circuit and generates the same harmonic rich output - see Figure 3.4. Personal computers, low energy discharge lighting and virtually every other type of electronic device now use SMPS for their power supplies.

Figure 3.3: Wave Diagram of Bridge Rectifier Circuit.
3.2.3 Experiment for Demonstrating the Harmonic Distortion of Current in Different Devices

A range of different but common devices are considered in terms of the characteristic harmonic distortion that occurs.

The Energy Saver Compact Fluorescent Lamp (CFL)

The CFL lamp is a low energy discharge lamp of the type that is replacing GLS tungsten filament lamps in the drive to reduce energy consumption in lighting schemes. The CFL lamp has a built in switched mode power supply and a power factor correction capacitor. This experiment uses an isolating transformer, a dual beam oscilloscope and a harmonic analyser. The secondary of the isolating transformer is connected directly to the oscilloscope to produce a waveform proportional to the supply voltage. The load current is passed through a non-inductive resistor (NIR) and the voltage across the NIR is connected to the second channel of the oscilloscope. This voltage is in-phase with and proportional to the current flowing in the circuit. Thus the dual wave display can be used to represent the voltage and current in the circuit. The harmonic analyser produces a Fourier analysis.
of the load current. This shows the fundamental and higher order current harmonics in the circuit and provides an automatic calculation of the Total Harmonic Distortion (THDi) of the current. The results of the following experiment show that the current flowing is considerably distorted with a THDi of 78%. This experiment layout is shown in Figure 5.3 and the wave diagram and spectral analysis of CFL current are shown in Figure 3.5.

**Personal Computer (PC)**

A PC also has a switched mode power supply and this accounts for the distorted current wave shown in Figure 3.6. The THDi of the PC current was measured as 70% using the experimental setup shown in Figure 5.3.

**A Twin Fluorescent Lamp**

The twin fluorescent fitting under test was fitted with iron cored chokes. It had a power factor correction capacitor but did not contain a SMPS. The wave diagram and spectral analysis of fluorescent lamp current is shown in Figure 3.7.

In large fluorescent lamp installations there is usually very little diversity. Site tests
conducted by Arrillaga et al [1] have shown 20% voltage distortion at distribution boards with neutral currents that greatly exceeded line currents due to triple-n harmonics. The THDi of the fluorescent lighting load was approximately 100%. Significant harmonic resonance at the third harmonic was experienced between chokes and PFCC which were located within the lighting fittings. It was further recommended that PFCC be centrally located to avoid the above harmonic resonance problem.
Variable Frequency AC Motor Drives

Variable frequency AC motor drives [5] in their simplest form use single or three-phase bridge rectifiers to convert AC to DC. Thyristors are then used to produce a three-phase variable frequency output which is connected to a three-phase motor. The speed of the motor is proportional to the applied frequency. Figure 3.8 [5] shows a basic three-phase frequency converter and Figure 3.9 shows the harmonic spectrum of the current. With this type of converter, the 3rd and 9th harmonic currents are significantly reduced but the 5\textsuperscript{th} and 7\textsuperscript{th} have increased. The THDi in this example was found to be 65.5%. Reduction of the 3rd and 9th harmonics has significance for the current in the neutral of a three-phase system because they are zero phase sequence and add in the neutral. Reduction in the 5\textsuperscript{th} and 7\textsuperscript{th} harmonics has implications for rotating machines because the 5\textsuperscript{th} has a negative phase sequence generating a negative torque, whilst the 7\textsuperscript{th} has a positive phase sequence generating a positive torque.

More sophisticated electronic circuits such as a 12-pulse parallel connected rectifier frequency converter shown in Figure 3.10 can significantly reduce the harmonic content of the load current as can be seen in Figure 3.11. However, slight imbalances in the supply voltages can cause significant harmonics [5].
3.3 Power Transformers

Power transformers consist essentially of insulated windings mounted on silicon sheet steel cores. A transformer which is operated at a higher flux density will require less magnetic material. Therefore from an economic point of view, a transformer core is designed to operate in the saturated region at a high flux density [14]. Figure 3.12 shows the magnetisation curves and magnetising current flowing in such a transformer.
Figure 3.11: Wave Diagram and Harmonic Analysis of the 12 Pulse Parallel Connected Rectifier Frequency Converter.

The non-sinusoidal magnetizing current has a significant third harmonic component [14]. The exciting current will contain the fundamental and all odd harmonics, the third harmonic being the predominant one. At rated voltage, the third harmonic in the exciting current can be 5 - 10% of the fundamental [14] and at 150% rated voltage, the third

Figure 3.12: Transformer Magnetisation Current Waveforms [1].
harmonic can be 30 to 40% of the fundamental.

3.3.1 Rotating Machine Harmonics

Space harmonics are generated both in generators and motors by the magnetic interaction of the different phase windings required to produce the rotating magnetic field. However, they can be reduced by optimising the machine design [15]. In other respects, the excitation of an electrical machine generates harmonic currents in the same way as a transformer. These are not significant as a source of harmonic currents compared to solid state devices previously described.

3.4 The Effect of Harmonics on AC Power Systems

3.4.1 Transmission Systems

The effects on transmission systems can be categorised as follows:

- additional power loss in cables due to the increased RMS value of the current;
- additional power loss in cables due to skin and proximity effects;
- additional power loss in cables due to distortion power factor;
- the creation of harmonic voltage drops across various circuit impedances;
- harmonic voltages increase the dielectric stress on system insulation and reduce service life;
- creation of harmonic resonance in underground feeder cables due to their capacitance.
3.4.2 Transformers

According to IEEE Std C57.110-2008 [16], Transformer losses can be characterised as:

- $I^2R$ winding losses;
- winding eddy current loss;
- stray loss.

Harmonics in transformers can cause an increase in audible noise. In addition, current harmonics cause an increase in copper losses and stray flux losses while voltage harmonics cause an increase in iron losses. The presence of voltage and current harmonics tends to increase the RMS values. IEEE Std C57.110-2008 [16] proposes a THDi limit for transformers of 5% and maximum steady state overvoltages of 5 to 10%.

$I^2R$ Winding Losses

The $I^2R$ winding losses will be increased if the RMS value of the load current is increased due to the presence of harmonic components. In addition, the skin and proximity effects of harmonic currents will increase the AC resistance of the winding with a consequential effect on the winding $I^2R$ losses.

Winding Eddy Current Loss

Winding eddy current loss in power transformers is proportional to the square of the load current and approximately proportional to the square of the frequency. It is therefore affected in a similar way to the $I^2R$ winding losses by an increase in the RMS value of load current and by the presence of harmonic components in the load current. It is this characteristic that can cause excessive winding loss and hence abnormal winding temperature rise and development of winding hot spots in transformers supplying non-sinusoidal currents.
Stray Loss on Non-winding Structural Parts

The stray loss which occurs in non-winding structural parts of the transformer is proportional to the current squared. Therefore a change in the value of Root Mean Square (RMS) current arising from harmonic content will have a significant effect on the heating of the insulating oil. The effect of the harmonic component on heating is proportional to $n^{0.8}$. This effect can be neglected for so called ‘dry’ transformers where oil is not used for insulation.

DC Components of Load Current due to Harmonics

Harmonic load currents are frequently accompanied by a DC component in the load current. Amounts of DC component equal to the magnetising current can be neglected, however, DC components greater than this should be accounted for [16]. If the DC component is significant, it will lead to increased saturation of the magnetic core and thereby increased generation of harmonic distortion in the magnetising current.

Design Considerations for New Transformer Specification

Standard IEEE Std. C57.110-2008 [16] set out design considerations for new transformer specification. Harmonic current filtering is recommended, however, caution should be exercised to avoid current amplification due to resonance with the filtering circuit. When the harmonic current include harmonic orders having multiples of three (3, 6, 9, etc.), zero sequence currents will flow in the neutral. Oversizing of this neutral may be required. A common practice with low-voltage, general-purpose transformers is to double the neutral ampacity. Where power factor correction equipment is installed there is a risk of current amplification at specific harmonic orders due to resonance. This will generate voltage distortion and additional heating effects in the transformer and supply cables. If the harmonic signature of the load is known, a detailed analysis can be carried out to generate
the optimum design, however, if the harmonic signature is not known, appropriate de-rating measures should be taken to avoid damage to equipment or shortened service life.

Procedures for Evaluating the Load Capability of Existing Transformers

Standard IEEE Std C57.110-2008 [16] specifies that the de-rating factor for existing transformers can be calculated by using the following formula:

\[ I_{\text{max}}(pu) = \sqrt{\frac{P_{\text{LL-R}}(pu)}{1 - F_{\text{HL}}P_{\text{EC-R}}(pu)}} \]

where \( I_{\text{max}}(pu) \) is the maximum permissible RMS non-sinusoidal load current under rated conditions;
\( P_{\text{LL-R}}(pu) \) is the per unit load loss under rated conditions;
\( F_{\text{HL}} \) is the harmonic loss factor for winding eddy-currents, and;
\( P_{\text{EC-R}}(pu) \) is the per-unit winding eddy-current loss under rated conditions.

Alternatively, a UL K-factor can be used to determine the suitability of a transformer for a given non-sinusoidal load.

\[ K - \text{factor} = \sum_{h=1}^{\infty} I_h(pu)^2 h^2 \]

where \( I_h(pu) \) is the RMS current at harmonic \( h \) (per unit of rated RMS load current) and \( h \) is the harmonic order.

The K-factor can only be used for loads where the magnitude of the harmonic currents of order \( n > 10 \) is \( < 1/n \) times the magnitude of the fundamental. The relationship between the K-factor and the Harmonic Loss Factor is set out in equation

\[ K - \text{factor} = \sum_{h=1}^{h=\text{max}} \frac{I_h^2}{I_R^2} F_{\text{HL}} \]
3.4.3 Effects of Harmonics on Motors and Generators

One of the main effects of harmonic voltages in induction and synchronous AC machines is increased heating due to iron and copper loss at the harmonic frequencies. The efficiency of the machine is thus adversely affected by being connected to a harmonically distorted supply. Harmonic currents flowing in the windings can also generate audible noise. The torque produced by electric motors is also adversely affected when connected to a distorted supply. This can prevent smooth starting and other undesirable effects such as the motor failing to reach operating speed.

Effect of Harmonic Voltage Distortion on Induction Motor Efficiency

In an experiment conducted by Moulder et al [17], the THDu was varied by adjusting the thyristor firing angles of a variable speed drive whilst keeping the speed and load constant. The losses were seen to increase by 6.7% as the THDu increased from 1 to 15% as shown in Figure 3.13. It was found that a THDu of 7% produced a harmonic loss of 3.8%. Molder et al suggest that harmonic losses account for between 3 and 4% of all motive power energy used in industrial applications.

Effect of Harmonic Voltages on Induction Motor Torque

The torques produced by the 5th and 7th space harmonics can cause a dip in the overall torque generated by the machine when starting and result in an unsatisfactory starting performance. Figure 3.14 shows an intersection between the load torque and the resultant torque due to the effects of the 7th harmonic. This would cause the acceleration of the load to cease at the point of intersection and the motor not to attain full operating speed. The high starting current taken by an induction motor is normally five to eight times the full load current. This has a typical duration of twenty to thirty seconds and the motor is designed to withstand this transient overload. However, in the event of the motor failing to attain full speed quickly, the motor will overheat due to a combination of reduced
self-ventilation and prolonged starting current. The 5th harmonic has a negative phase sequence and therefore the torque produced by the 5th harmonic current will act against the positive torque produced by the fundamental current. This negative torque reduces the efficiency of the motor.

The 5th and 7th harmonic torques can also combine to create mechanical oscillations between a motor and load by exciting the system mechanical resonant frequency thus developing high stress mechanical forces and vibration. The 5th and 7th harmonic currents induced in the rotor can combine to produce a 6th harmonic current [7] and similarly the 12th and 13th to produce a 12th harmonic current and so on for the higher order harmonics. This gives rise to increased heating and reduced or pulsating torques. The net effect can be a reduction in efficiency to between 90 and 95% when motors are exposed to normal supply voltage harmonic distortion of up 5% THDu but can be more severe when the motor is supplied directly by a VSD.

Figure 3.13: Induction Motor Loss .v. THDu
3.4.4 The Effect of Harmonics on Measuring Instruments

Absolute, Average or Peak responding instruments that are calibrated in RMS are not suitable for measuring non-sinusoidal voltages and currents. A reading taken by an averaging meter may indicate up to 40% below the true RMS reading due to a distorted waveform. The standard kWhr induction energy meter commonly used for electricity metering purposes may read high by up to 6% where the harmonic levels are high. This may compensate the electricity supply company for the additional losses they suffer as a result of high harmonics but it does underline the need to reduce or eliminate harmonics where possible.

3.4.5 Electromagnetic Interference

IEEE Std 519-1992 [7] and the IEC Std 61000 [6] series describes the effect of power harmonics on electrical equipment. The operation of electronic solid state switching relies upon accurate determination of voltage zero crossing points. Due to the harmonically distorted wave shape, the firing points of triacs or thyristors may be altered thus causing
the equipment to malfunction. It is well documented that aircraft navigation equipment can be adversely affected by RF electromagnetic interference (EMI) emanating mobile phones or portable computers. All electronic equipment that use central processing units (CPUs) such as computers, programmable logic controllers (PLCs), etc., can be affected to a greater or lesser extent by harmonic distortion imported through power supply units or by magnetic coupling due to the proximity of cables supplying other equipment.

The problem of harmonically induced EMI is particularly acute where medical diagnostic equipment is used, the problem ranging from complete failure of the sensitive equipment to erroneous results. The problem can be alleviated by screening or isolation of the sensitive equipment or by using conditioned power supplies that filter all mains borne harmonics. The harmonic limits required by most electronic equipment is typically $< 5\%$ THDu with the largest single harmonic $< 3\%$ [7]. IEEE Standard 519 1992 [7] also details the adverse impact of harmonics on protective relays used in electrical systems. This impact has been found to be very difficult to predict, in some cases making the equipment over sensitive and in other cases less sensitive. Fuchs et al [13] recommend that very restrictive amplitude limits for inter-harmonics and sub-harmonics should be included in the revision of the above standards due to their detrimental effects on under-frequency relays (malfunctioning), lighting equipment (flicker) and rotating machines (harmonic torques).

### 3.4.6 Cable Heating Effects of Harmonics

Power cables can experience overheating by being exposed to amplified currents caused by harmonic resonance but, in addition to this, the presence of harmonics increases the voltage stress on the insulation leading to reduced service life and in some cases failure. The presence of harmonics currents cause additional heating in cables due to:

- larger values of current flowing in the circuit because of harmonic resonance;
- larger values of current flowing in the circuit due to a reduction in distortion power
factor;

- the skin effect;
- the proximity effect;
- current flowing in the neutral conductor of three-phase systems.

Current amplification due to harmonic resonance and the increase in current due to a reduction in power factor is dealt with in later sections. The skin effect is an electromagnetic phenomenon which is dependent on frequency and cable size. The current density in the current carrying conductor becomes non-uniform due to magnetic flux linkages within the conductor. The impedance of the conductor core increases as a result and displaces current from the centre toward the surface of the conductor. This has the effect of increasing the effective resistance of the cable and thereby increasing the cable $I^2R$ losses.

The proximity effect is also dependent on frequency, the proximity of cables to each other and to other conducting materials such as structural steel, cable tray and ductwork. When current is flowing in two parallel conductors, a force of attraction or repulsion is experienced by the currents. When the current is flowing in opposite directions, the current paths are pulled closer together thus altering the current density in the cable. The reverse is the case when the currents are flowing in the same direction. In a similar way to skin effect, the effective resistance of the cable is increased thereby increasing the cable $I^2R$ losses.

Triple-$n$ harmonic currents have zero phase sequence and consequently are additive in the neutral conductor. Three-phase cables are normally sized on the basis that the supply current is sinusoidal and there will be no load current in the neutral conductor for balanced loads. With harmonic distortion, the current in the neutral can equal or exceed the line conductors leading to a very significant increase in the cable temperature.
If the problem was anticipated, the cable rating can be reduced; otherwise the cable may experience a greatly reduced life or may fail in service due to overheating.

3.4.7 Harmonic Load Diversity Factors

Harmonic currents can vary in phase angle and amplitude. Thus it is possible for two harmonic currents of order $n$ originating from two different loads to be in phase opposition to each other. If these currents are supplied from a common busbar, they will either partially or fully cancel each other, thus reducing the harmonic distortion in the current flowing from the ‘busbar’ to the supply. A study based on the selection of random vectors, was carried out by Mansoor et al [18]. The study indicated that, owing to the phase angle dispersion, a significant amount of harmonic cancellation can take place. Further studies were carried out by Hegazy et al [19] showing that the calculated deterministic diversity factors decrease when the number of working loads and the line reactance to resistance ratio both increase. The diversity factor varies from one type of load to another and also from one installation to another.

Other factors include: the harmonic order, the type of load and the number of harmonic loads connected in parallel. The graph shown in Figure 3.15 was extracted from data presented by Hegazy et al [19]. It sets out the relationship between harmonic diversity factor and harmonic order for Group A classified loads for a particular ratio of line reactance $X$ to line resistance $R$. The main elements of this group are the three-phase, six-pulse bridge rectifier power converters commonly used for industrial loads. The paper provides data on other load types used in defined conditions. It is common industry practice to use the algebraic sum of the harmonic currents when assessing the total harmonic loading, however, this will tend to overestimate the cumulative effect of harmonic loads supplied from a common point.
3.5 Elimination/Reduction of Harmonics

The benefits of reducing harmonics include: reduction of neutral currents; reduction of transformer loading; protection of electrical systems; reduction of fire hazard due to cables overheating; reduction of system voltage distortion; reduction of local neutral to ground voltage; reduction of peak phase current/average phase current; increased system capacity; decreased system losses; improves power factor on non-linear loads. However, the reduction in THDi by the use of filters applies only to the supply side. The cables supplying the load still carry the uncorrected harmonic currents.

3.5.1 Measurement at Point of Common Coupling (PCC)

The PCC is the point at which the public electricity supply company connects to the electricity consumer. The harmonic distortion measured at the PCC must conform to the THD limits set out in G5/4-1, IEEE Std 519 and IEC 61000-3-4. In some instances, phase-shift transformers may be used to reduce the level of 5th and 7th harmonics present [20]. However, if this is not an option, passive or active harmonic filters may be used.
3.5.2 Passive Harmonic Filters

The values of $L$ and $C$ are selected in Figure 3.16 to generate resonance at the selected harmonic frequency and thereby provide a low impedance path for the harmonic current to circulate between the filter and the harmonic load and not pass into the supply system. The impedance at frequency $f_n$ is equal to the resistance $R$, thus the value of $R$ can be selected so that the amount of harmonics that passes into the supply is controlled to an acceptable limit. This filter is designed to target one specific harmonic frequency by selecting values of $L$ and $C$ that satisfy the equation

$$f_n = \frac{1}{2\pi\sqrt{LC}}$$

The filter shown in Figure 3.17 presents low impedance to a wide range of higher order harmonics, usually from $17^{th}$ upwards. It is used in conjunction with a number of single shunt filters as shown in Figure 3.18 designed to meet the requirements of a specific installation.

Values of $L, C$ and $R$ are selected for each branch to generate resonance at the target harmonic frequencies. Thus the $5^{th}$, $7^{th}$, $11^{th}$ and $13^{th}$ harmonics are shunted through branches 1 to 4 preventing them from flowing back to the supply point. Harmonics of order 17 and upwards are shunted through the high pass series / parallel filter.
3.5.3 Active Harmonic Filters

Active harmonic filters generate currents which are anti-phase to the harmonic currents required by the load. In this way the non-linear load is supplied with harmonic currents whilst the current taken from the supply is sinusoidal. The active filters fall into three
categories: Series; Parallel and Hybrid.

**Series Filters**

This type of filter is connected in line with the load and corrects the voltage distortion already in the supply as well as the harmonic currents in the load. The filter must be rated to match the full load current.

**Parallel Filters**

These filters are connected in parallel with the load in the same way as the passive shunt filters.

**Hybrid Filters**

This filter uses a combination of active and passive filters with the passive filters targeting a specific harmonic and the active filter cancels the other harmonics.
Chapter 4

The Effect of Harmonics on Individual Electrical Components and System Power Factor

This chapter considers the effect of harmonics on individual electrical components such as resistors, inductors, capacitors. It examines the impact of harmonic distortion on system power factor.

4.1 RLC Circuits

Resistors, Inductors and Capacitors are the building blocks of most electric and electronic circuits and each of them react differently in the presence of harmonics. In order to achieve a better understanding of the overall effects of harmonics on RLC circuits, we shall first examine the effects on individual components.
4.1.1 The Non-Inductive Resistor (NIR)

An NIR is generally unaffected by the presence of harmonics and does not itself generate harmonics when connected to a supply and is considered a linear load. As it is used in conjunction with other harmonic sensitive components, it will indirectly experience harmonic effects.

4.1.2 The Inductor

This device has an electrical property called self-inductance \(L\). The Inductive Reactance \(X_L\) of the inductor is given by the formula \(X_L = 2\pi fL\). In general, the reactance of a reactor at a harmonic order \(n\) is given by \(nX_L\) where \(X_L\) is the reactance at \(n = 1\). The inductor will thus offer high impedance to high frequency AC and will therefore tend to limit the flow of high order harmonic currents. Many inductors have magnetic cores and these are susceptible to magnetic saturation. When this occurs, the inductor becomes non-linear and will generate harmonic currents directly.

4.1.3 The Capacitor

The capacitor has an electrical property called capacitance \(C\). The Capacitive Reactance \(X_C\) of the capacitor is given by the formula \(X_C = 1/(2\pi fC)\). The capacitor is a linear device whose reactance is inversely proportional to the supply frequency. The capacitive reactance at a harmonic order \(n\) is given by \(X_c/n\), where \(X_c\) is the capacitive reactance at \(n = 1\).

Whereas a capacitor does not generate harmonic currents directly, the higher order harmonic voltages experience lower impedance and consequently the THDi of the associated current may be many times that of the applied voltage. Therefore, a capacitor carries a larger RMS current when the supply voltage is distorted than it would with a pure sinusoidal supply. This increased current can cause overheating of the capacitor and
in extreme cases, lead to breakdown of the dielectric and system failure.

4.2 The $LC$ Series or Parallel Combination

An $LC$ combination gives rise to a phenomenon called resonance which occurs when $X_L = X_C$, i.e. when

$$2\pi f L = \frac{1}{2\pi f C}$$

or

$$f = \frac{1}{2\pi \sqrt{LC}}$$

In general

$$f_0 = \frac{1}{2\pi \sqrt{LC}}$$

where $f_0$ is the resonant frequency for any combination of $L$ and $C$.

4.2.1 Series Resonance

Figure 4.1 shows a $RLC$ series circuit. The current ($I$) is given by the following formula:

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}}$$

At resonance when $X_L = X_C$ and $I = V/\sqrt{R^2} = V/R$. This will rise to a maximum value of $V/R$. This value could be very large if the value of $R$ is very small. If $V$ is the mains supply voltage, then the voltage appearing across the capacitor is $VX_L/R$ which can be many times greater than the supply voltage.

In a harmonic environment,

$$I_n = \frac{V_n}{R_{AC_n}}$$

where $I_n$ is the $n^{th}$ order harmonic current, $V_n$ is the $n^{th}$ order harmonic voltage and $R_{AC_n}$ is the AC resistance at harmonic order $n$. 

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The harmonic voltage present in the supply, \( V_n = I_n R_{AC_n} \), the harmonic voltage across the reactor and capacitor will be \( V_{nXL} = I_n X_L \) and \( V_{nXC} = I_n X_C \). respectively. If the values of \( X_L \) and \( X_C \) are large compared to \( R_{AC_n} \), as is often the case, the harmonic voltages across L and C will be multiples of the harmonic voltage in the supply.

### 4.2.2 Parallel Resonance

For a parallel combination as shown in Figure 4.2, the impedance is given by the product of branch impedances divided by the sum of branch impedances, i.e.

\[
Z = \frac{iX_L(-iX_C)}{iX_L - iX_C} = \frac{-iX_LX_C}{X_L - X_C}
\]
When $X_L = X_C$, $Z = \infty$, and the current taken from the supply is zero. The current taken in each branch is given by

$$I = \frac{V}{X_L} = \frac{V}{X_C}$$

When the resistance $R$ of the reactor is taken into account, the resulting impedance of the parallel combination is known as the Dynamic Impedance and is given by $L/CR$. Figure 4.3 shows that resonance occurs at approximately harmonic order 7.2 at which point the circuit impedance is maximum.

4.3 Power System Resonance

System resonant conditions are the most important factors affecting system harmonic levels [7]. Parallel resonance causes high impedance to the flow of harmonic current, while series resonance causes a low impedance to the flow of harmonic current. It is
when harmonic currents meet high impedances that significant voltage distortion and current amplification occur. It is necessary to analyse the systems frequency response characteristics in order to avoid system resonance problems.

4.3.1 Distribution Systems

In an AC power system where harmonic distortion is present, there is a risk that resonance will be created between system $L$ and $C$ components at one or more of the harmonic frequencies present. Most severe resonant conditions occur when there is a single large shunt capacitor bank connected in the system. This resonance commonly occurs near the 5th harmonic [7]. The resistive component of the load is most important in providing damping for resonance conditions and thus reducing their severity.

There is usually a number of large capacitor banks present in a distribution system which serves a number of consumers. Some of these banks may be automatically switched to match load conditions. Thus the system characteristics change as the load profile changes. This naturally affects the system resonant frequency so that resonance problems may occur at full load or part-load. There may be resonance at different harmonic frequencies over the full range from no-load to full load. A full and detailed analysis is necessary to cover all the permutations. Where capacitors are placed at the extremity of feeders, the additional impedance of the feeder can create the conditions for resonance and give rise to significant voltage distortion as the harmonic currents now flow for a greater distance.

4.3.2 Industrial Systems

Industrial systems resemble compact distribution systems as shown in Figure 4.4. Due to the industrial nature of the loads, there are significant differences as follows:

- There will usually be one or more large capacitor banks matched with low system
impedance. The power factor of the loads and the combination of the above factors mean that resonance occurs mainly at the low order harmonic frequencies.

- There will be a large number of non-linear loads such as uninterruptible power supplies (UPS), variable speed drives, switched mode power supplies, discharge lighting, etc.

- There are fewer resistive loads to provide harmonic damping, thus the effects of resonance are more severe. Motor loads are seen by the harmonic sources as purely inductive and in parallel with the supply impedance, thus the resonant frequency changes as the motors are switched in and out of the circuit.

This makes it more difficult to design harmonic filters as the resonant frequency is constantly changing. The problems outlined above are exemplified in the case study given in the following section.
Table 4.1: Harmonic Load Profile

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>10</td>
<td>7.14</td>
<td>5.13</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Figure 4.5: Electrical Supply System Connected to a Non-linear Load and Capacitor Bank.

4.3.3 Case Study: Parallel and Series Harmonic Resonance

Consider the case of a 400V 400kVA three-phase transformer with 4.5% impedance supplying an industrial load of 208 kW at a power factor of 0.62. The harmonic profile of the load is as given in Table 4.1.

Power factor correction capacitors (PFCC) are installed to improve the power factor to 0.92 and it is now necessary to analyse the system characteristics to check if the harmonic currents in the load will produce resonance. The non-linear load can be considered to be a harmonic current source in the system as shown in Figure 4.5
To Carry out an Analysis of System Characteristics

Figure 4.6 shows the equivalent circuit where load harmonic currents are represented as a harmonic current source supplying the system impedance and the electrical installation power factor correction capacitors. The supply impedance is mainly reactive and is represented by $X_{tr}$ and the PFCCs are represented by $X_C/n$ where $X_C$ is the capacitive reactance at $n = 1$ and $n$ is the harmonic order of the current source. Each individual harmonic current in the load acts as a current source supplying the parallel circuit. The impedance of the parallel circuit using complex notation may be derived as follows:

$$Z_{S_n} = \frac{inX_{tr}(-iX_c/n)}{inX_{tr} - oX_c/n} = \frac{i(-nX_cX_{tr})}{(n^2X_{tr} - X_c)}$$

At resonance conditions, the impedance of the above circuit will be infinity. Therefore the denominator will be zero, i.e. $n^2X_{tr} - X_c = 0$ and

$$n = \sqrt{\frac{X_c}{X_{tr}}} \quad (5.1)$$

The kVAR rating of the capacitor bank required is given by

$$Q_k = P(\tan \phi_1 - \tan \phi_2)$$

Where $P$ is the power consumed by the load, $\phi_1$ is the phase displacement before PF correction, $\phi_2$ is the phase displacement after PF correction so that

$$Q_k = 208(1.265 \times 0.426) = 174.61 \text{kVAR}$$
Neglecting the resistance in the circuit which will be very small in comparison to the reactance, the supply transformer reactance is given by

\[ X_{tr} = \frac{e_u U^2}{100S_{tr}} = \frac{0.045 \times 400^2}{400} = 0.018\Omega \]

The capacitive reactance of the capacitor bank can be determined by

\[ X_c = \frac{U^2}{Q_k} = \frac{400^2}{174.61 \times 10^3} = 0.916\Omega \]

From equation (5.1),

\[ n = \sqrt{\frac{0.916}{0.018}} = \sqrt{50.88} = 7.13 \]

Since there is a significant 7th harmonic current in the load, this current will generate parallel resonance between the PFCCs and the supply transformer at 350 Hz (which is the 7th harmonic). In theory, the magnitude of the resonant current flowing at this frequency will be infinite but when the damping effect of resistance in the supply cables and transformer windings, etc., is factored in, the current magnification, given by \( X_{tr}/R \), will be in the order often to twenty times the value prior to power factor correction. In this example, the 7th harmonic current will typically increase from 7.14 to 7.14 \times 15 = 107.1A. This is very significant given that the supply current is approximately 330A. The load current will therefore breach the limits set down under the harmonic standards G5/4-1 [5]. It will also probably bring about an immediate failure of the power factor capacitors due to overheating and subsequent breakdown of dielectric.

For conditions approaching resonance, significant magnification of the harmonic load currents will also occur. With automatic power factor correction systems, where additional stages of the capacitor bank are switched in and out by power factor relays at the main switchboard as the load changes, the resonant frequency in the system will change as the load changes. Therefore, it is possible that the conditions for harmonic parallel resonance can occur at low, medium or high load conditions. In a situation such as the above, it is necessary after analysis, to introduce a series reactor with the capacitor bank such
Table 4.2: Harmonic Voltage Distortion.

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(A)$</td>
<td>10</td>
<td>107.1</td>
<td>5.13</td>
<td>4.1</td>
</tr>
<tr>
<td>$X_{tr_n}$</td>
<td>0.09</td>
<td>0.126</td>
<td>0.162</td>
<td>0.234</td>
</tr>
<tr>
<td>$U_n(V)$</td>
<td>0.9</td>
<td>13.49</td>
<td>0.83</td>
<td>0.96</td>
</tr>
</tbody>
</table>

that resonance does not occur at a frequency that matches any of the harmonic currents present in the load current.

### 4.3.4 Voltage Distortion due to Imported Harmonic Currents

When a supply system of impedance $Z_S$ imports harmonically distorted currents from an industrial load, there is an associated voltage drop in the supply voltage of magnitude $I_nZ_S$. Supply systems with higher values of $Z_S$ will be worse affected. Once a system experiences voltage distortion, all consumers connected to the system, even those previously without a harmonics problem, may now experience one. In the previous example calculation of voltage distortion can be accomplished as follows: Since $X_{tr_n} = n.X_{tr}$. Therefore substituting these values in the Table 4.2 allows one to calculate the resulting individual harmonic voltage distortion in the supply voltage, thus

$$THD = \sqrt{\frac{\sum_{2}^{n} U_n^2}{U_1}} = \frac{\sqrt{0.9^2 + 13.49^2 + 0.83^2 + 0.96^2}}{230} = 0.059$$

or 5.9%.

Although this complies with the global THD limit of 8% set by BS EN 50160 [4] it exceeds the limit of 5% set for the seventh harmonic, therefore this level of distortion does not comply with [4]. The level set by G5/4-1 [UK] = 5% THD limit and an individual 7th harmonic of 4%.

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4.3.5 De-tuning the Power Factor Capacitor to Avoid Parallel Harmonic Resonance

In order to prevent parallel harmonic resonance occurring between the PFCCs and the supply impedance at one of the harmonic frequencies present, a reactor ($X_d$), is connected in series with the PFCC. The value of this reactor is chosen so that the impedance of the parallel circuit shown below does not create parallel resonance at one of the other harmonic current orders present in the load. However, as a consequence of this, series resonance may be created between $X_d$ and $X_c$.

Figure 4.7 shows a de-tuned capacitor which gives rise to risk of series resonance.

The impedance of this parallel combination can be calculated as follows:

$$Z_{S_n} = \frac{inX_{tr}i(nX_d - X_c/n)}{inX_{tr} + i(nX_d - X_c/n)} = \frac{inX_{tr}(n^2X_d - X_c)}{n^2(X_{tr} + X_d) - X_c}$$

The impedance $Z_{S_n} = \infty$ when

$$n^2(X_{tr} + X_d - X_c) = 0$$

which is the condition for parallel resonance. The harmonic order at which parallel resonance occurs is given by

$$n_p = \sqrt{\frac{X_c}{X_{tr} + X_d}}$$  \hspace{1cm} (5.2)

The value of $X_d$ should be selected so that $n_p$ does not match any of the harmonic currents present in the load.
We can examine the circuit to check for occurrence of series resonance. Looking at the parallel combination again, the numerator is zero when \( n^2X_d - X_c = 0 \), i.e. when \( n^2X_d = X_c \). This gives a condition for series resonance in this branch of the parallel circuit at a harmonic order

\[
n_s = \sqrt{\frac{X_c}{X_d}}
\]

Therefore if there is a harmonic current in the load of order \( n_s \) or a harmonic voltage in the supply voltage of order \( n_s \), these will give rise to magnified voltages of order \( n_s \) appearing across the terminals of both the capacitor and the de-tuning reactor.

In this case, assuming negligible resistance in the branch, the voltage magnification in theory will be infinite, however, in practice, the voltage magnification given by \( X_d/R \), will be typically in the range of 30-50 and both the PFC capacitor and the inductor will experience elevated voltage levels. The capacitor should be designed to withstand 110% nominal voltage according to IEEE standard 18-2002 [22].

As has been stated above, the condition when \( X_L = X_C \) produces perfect resonance, however, the resonance effect will be experienced to a greater or lesser degree at every harmonic frequency leading to the undesirable side effects described above.

### 4.4 Power Factors

The power factor is important in assessing the losses associated with all circuits. We conclude this chapter with a short overview of power factor definitions.

#### 4.4.1 Displacement Power Factor

For an environment where there is no harmonic distortion and the voltage and current are given by

\[ v(t) = V_1 \sin(\omega_0 t + \delta_1) \]
and
\[ i(t) = I_1 \sin(\omega_0 t + \theta_1) \]

then, for a sinusoidal power supply with no harmonic distortion the power is given by
\[
P = \frac{1}{T} \int_0^T v i dt = VI/2 \cos(\delta_1 - \theta_1) = V_{\text{RMS}} I_{\text{RMS}} \cos(\delta_1 - \theta_1)
\]

Power Factor is defined as
\[
\frac{\text{Real Power}}{\text{Apparent Power}} = \frac{V_{\text{RMS}} I_{\text{RMS}} \cos \phi}{V_{\text{RMS}} I_{\text{RMS}}} = \cos \phi = \frac{P}{S}
\]

The ‘Power Triangle’ shown in Figure 4.8 can be used to represent Power $P$, Apparent Power $S$ and Reactive Power $Q$ where
\[
Q = V_{\text{RMS}} I_{\text{RMS}} \sin \phi
\]

### 4.4.2 Distortion Power Factor

For a sinusoidal voltage supply with harmonic current distortion, the complex voltage wave can be represented thus:
\[
v(t) = V_1 \sin(\omega_0 t + \delta_1) + V_2 \sin(2\omega_0 t + \delta_2) + ... V_n \sin(n\omega_0 t + \delta_n)
\]

where $V_1$ is the amplitude of the 1st harmonic or fundamental voltage wave, $V_2$ is the amplitude of the 2nd harmonic voltage wave and $V_n$ is the amplitude of the $n^{\text{th}}$ harmonic.
voltage wave. The complex current wave is given by

\[ i(t) = I_1 \sin(\omega_0 t + \phi_1) + I_2 \sin(2\omega_0 t + \phi_2) + \ldots I_n \sin(n\omega_0 t + \phi_n) \]

where \( I_1 \) is the amplitude of the 1\textsuperscript{st} harmonic or fundamental current wave, \( I_2 \) is the amplitude of the 2\textsuperscript{nd} harmonic current wave and \( I_n \) is the amplitude of the \( n \textsuperscript{th} \) harmonic current wave.

For a complex wave, it can be shown that only the product of voltages and currents of the same frequency deliver power in the circuit [23]. Therefore for a circuit with harmonics, the power factor is given by

\[
\frac{\sum_{n=1}^{N} V_n I_n \cos \theta_n}{V_{\text{RMS}} I_{\text{RMS}}} = \frac{V_1 I_1 \cos \theta_1}{V_{\text{RMS}} I_{\text{RMS}}} + \sum_{n=2}^{N} \frac{V_n I_n \cos \theta_n}{V_{\text{RMS}} I_{\text{RMS}}}
\]

The supply voltage typically has a THD < 5\%. A useful approximation can be developed by assuming that the supply voltage has 0\% THD. In this case there will be no harmonic voltages present and therefore the power delivered by the harmonic currents which is the product of the \( v_n \) and \( i_n \) will be zero.

Assuming that the voltage wave has no harmonic distortion, it can be represented thus:

\[ v(t) = V_1 \sin(\omega_0 t) \]

The current wave can thus be represented as

\[ i(t) = \sum_{n=1}^{\infty} I_n \sin(n\omega_0 t + \theta_n) = I_1 \sin(\omega_0 t + \theta_1) + \sum_{n=2}^{\infty} I_n \sin(n\omega_0 t + \theta_n) \]

Therefore since power is only delivered by the product of the supply voltage and the fundamental current wave

\[
P = \frac{1}{T} \int_{0}^{T} v i dt = \frac{V_1}{2} \cos \theta_1 = V_{\text{RMS}} I_{\text{RMS}} \cos \theta_1
\]
Multiplying by $I_{\text{RMS}}/I_{\text{RMS}}$

$$P = V_{\text{RMS}}I_{\text{RMS}} \frac{I_{\text{RMS}}}{I_{\text{RMS}}} \cos \theta_1 = V_{\text{RMS}}I_{\text{RMS}} \cos \theta_1 \frac{I_{\text{RMS}}}{I_{\text{RMS}}} = S \cos \theta_1 k_d$$

where

$$S = V_{\text{RMS}}I_{\text{RMS}}$$

$\cos \theta_1$ is the displacement power factor, $I_{\text{1RMS}}/I_{\text{RMS}}$ is the distortion power factor $k_d$ and $\cos \theta_1$ is the displacement power factor $k_\theta$.

Expressing $k_d$ in terms of THD, we obtain

$$\text{THD} = \sqrt{\frac{I_{2\text{RMS}}^2 + I_{3\text{RMS}}^2 + \ldots + I_{n\text{RMS}}^2}{I_{\text{1RMS}}^2}} = \sqrt{\frac{I_{2\text{RMS}}^2 - I_{\text{1RMS}}^2}{I_{\text{1RMS}}^2}}$$

and

$$\text{THD}^2 = \frac{I_{\text{RMS}}^2 - I_{\text{1RMS}}^2}{I_{\text{1RMS}}^2} = \frac{I_{2\text{RMS}}^2}{I_{\text{1RMS}}^2} - 1$$

so that we can write

$$\text{THD}^2 = \frac{1}{k_d^2} - 1$$

or

$$\text{THD}^2 + 1 = \frac{1}{k_d^2}$$

and

$$k_d = \sqrt{\frac{1}{1 + \text{THD}^2}}$$

Note that if the THD = 100% then

$$k_d = \sqrt{\frac{1}{1 + 1^2}} = 0.707$$

Displacement power factor can be corrected by the use of PFC capacitors, however, distortion power factor cannot be corrected unless the harmonics are eliminated. This is achieved only by using passive or active harmonic filters.
4.4.3 Effects of Low Power Factor

Grady et al [24] established that, for a given power delivery, the system losses increased significantly as the power factor reduces. With harmonic distortion present, power factor needs to be redefined to include the harmonic effects. True power factor is given by the product of displacement factor $k_\theta$ and the distortion factor $k_d$ - see Figure 4.9.

Figure 4.10 shows that even with displacement power factor at 1, the true power factor reduces as the THDi increases. The only way of dealing with distortion power factor is to eliminate harmonics using passive and/or active harmonic filters. The supply authority have significant charges for power factors that fall below 0.92 so the commercial incentive for addressing harmonic distortion is considerable.
Chapter 5

Electric Cable Heating Effects
Through Harmonic Distortion

Electrical power systems are invariably supplied with electricity by non-sinusoidal AC waves which contain harmonics. The international harmonic standards seek to limit the harmonic distortion in these supplies so that electrical machinery and sensitive electronic equipment will operate correctly. Similarly electricity consumers are required to limit the harmonic current emissions from their electrical installations to the public network and also create an electromagnetic environment within their installation so that their own equipment will operate correctly. This chapter examines the underlying theory of harmonics as applied to AC power systems, with particular emphasis on the thermal heating effects on electrical cables by harmonic currents.

5.1 Complex Voltage and Current Waves

A complex voltage wave can be represented thus:

\[ v = V_{1m} \sin(\omega_0 t) + V_{2m} \sin(2\omega_0 t) + \ldots + V_{nm} \sin(n\omega_0 t) \]
where $V_{1m}$ is the amplitude of the 1st harmonic or fundamental voltage wave; $V_{2m}$ is the amplitude of the 2nd harmonic voltage wave and $V_{nm}$ is the amplitude of the $n$th harmonic voltage wave, etc.

Similarly, a complex current wave can be represented thus:

$$i = I_{1m} \sin(\omega_0 t - \phi_1) + I_{2m} \sin(2\omega_0 t - \phi_2) + ... + I_{nm} \sin(n\omega_0 t - \phi_n)$$

where $I_{1m}$ is the amplitude of the 1st harmonic or fundamental voltage wave; $I_{2m}$ is the amplitude of the 2nd harmonic voltage wave and $I_{nm}$ is the amplitude of the $n$th harmonic current wave, etc.; $\phi_n$ is the phase angle at harmonic order $n$ and $\omega_0$ is the angular velocity in rad/s (radian per second).

5.2 Power Factor, RMS Value and Total Harmonic Distortion

The determination of power factor of the system is complicated by the presence of harmonics because the true power delivered is the sum of the power delivered by each harmonic voltage and its respective harmonic current. For a complex wave, it can be shown that only the product of voltages and currents of the same frequency deliver power in the circuit [1].

Therefore for a circuit with harmonics, the power factor is equal to the Real Power divided by the Apparent Power, i.e.

$$\text{Power Factor} = \frac{\sum_{n=1}^{N} V_n I_n \cos \phi_n}{V_{\text{RMS}} I_{\text{RMS}}}$$

where the Root Mean Square (RMS) values for a complex wave are given by

$$V_{\text{RMS}} = \sqrt{V_{1\text{RMS}}^2 + V_{2\text{RMS}}^2 + ... + V_{N\text{RMS}}^2}$$
\[ I_{\text{RMS}} = \sqrt{I_{1\text{RMS}}^2 + I_{2\text{RMS}}^2 + \ldots + I_{N\text{RMS}}^2} \]

The Total Harmonic Distortion (THD) of a complex wave is given by (as a percentage)

\[ \text{THD} = \sqrt{\frac{I_{2\text{RMS}}^2 + I_{3\text{RMS}}^2 + \ldots + I_{N\text{RMS}}^2}{I_{1\text{RMS}}^2}} \times 100\% \]

For a current this is expressed as THDi and for the voltage as THDu.

### 5.3 Cable Heating Effects of Harmonics in Single-phase Circuits

For a given amount of power delivered, the presence of harmonics has the effect of increasing the RMS current and reducing the power factor. In addition to this, the high frequency harmonic currents tend to flow on the surface of the cables due to ‘skin effect’. This will cause additional heating in the cable because the ‘AC resistance’ of the cable is increased for the higher harmonic currents. The ‘proximity effect’ also arises due to mutual inductance of cables in close proximity to each other. This is discussed in Chapter 7. There is also a ‘proximity effect’ between cables and adjacent conducting materials such as cable tray and structural steelwork. This effect is more marked for higher frequency harmonic currents.

### 5.4 Cable Heating Effects of Harmonics in Three-phase Circuits

Three-phase circuits experience the same effects as single-phase circuits when under the influence of harmonic currents. In addition to the skin and proximity effects, they experience additional problems with currents of harmonic order \(3n\) such as 3, 6, 9, 12 etc.
These currents are called triple-$n$ harmonics and in a three-phase system, are in time-phase with each other. Their phase sequence is zero and, as a result, are additive in the neutral conductor of three-phase 4-wire systems instead of cancelling out in the normal way. Therefore the neutral conductor in a system with significant harmonic distortion may have a current that exceeds the individual phase currents. Three-phase transformers with delta windings are significantly loaded by circulating triple-$n$ harmonic currents.

5.4.1 Calculation of the Neutral Current in a Three-phase Four-wire System

With reference to Figure 5.1, when harmonic distortion is present, the three-phase currents are as follows:

\[ i_R = I_{1m} \sin(\omega_0 t + \phi_1) + I_{3m} \sin(3\omega_0 t + \phi_3) + I_{5m} \sin(5\omega_0 t + \phi_5) + \ldots \]

\[ i_Y = I_{1m} \sin(\omega_0 t - 2\pi/3 + \phi_1) + I_{3m} \sin(3\omega_0 t - 2\pi/3 + \phi_3) + I_{5m} \sin(5\omega_0 t - 2\pi/3 + \phi_5) + \ldots \]

\[ = I_{1m} \sin(\omega_0 t - 2\pi/3 + \phi_1) + I_{3m} \sin(3\omega_0 t - 6\pi/3 + \phi_3) + I_{5m} \sin(5\omega_0 t - 10\pi/3 + \phi_5) + \ldots \]

\[ = I_{1m} \sin(\omega_0 t - 2\pi/3 + \phi_1) + I_{3m} \sin(3\omega_0 t + \phi_3) + I_{5m} \sin(5\omega_0 t - \pi/3 + \phi_5) + \ldots \]
Figure 5.2: Demonstrating the Overloading Effect of Triple-\(n\) Harmonic Current Flowing in the Neutral Conductor.

and

\[
i_B = I_{1m}\sin(\omega_0t + 2\pi/3 + \phi_1) + I_{3m}\sin[3(\omega_0t + 2\pi/3) + \phi_3] + I_{5m}\sin[5(\omega_0t + 2\pi/3) + \phi_5)] + ... \\
= I_{1m}\sin(\omega_0t + 2\pi/3 + \phi_1) + I_{3m}\sin[(3\omega_0t + 6\pi/3) + \phi_3] + I_{5m}\sin[(5\omega_0t + 10\pi/3) + \phi_5)] + ... \\
= I_{1m}\sin(\omega_0t + 2\pi/3 + \phi_1) + I_{3m}\sin(3\omega_0t + \phi_3) + I_{5m}\sin(5\omega_0t + \pi/3 + \phi_5) + ... 
\]

The current in the neutral is given by \(i_n = i_R + i_Y + i_B\). Assuming the currents are balanced and adding the individual terms of the same harmonic order:

\[
i_R + i_Y + i_B = 0 + 3I_{3m}\sin(3\omega_0t + \phi_3) + 0
\]

This demonstrates that triple-\(n\) harmonics are additive in the neutral and are therefore described as having zero phase sequence whereas all other balanced harmonic currents will cancel each other out in the neutral and have either a positive or negative phase sequence. Figure 5.2 demonstrates the overloading effect of triple-\(n\) harmonic current flowing in the neutral conductor.
5.4.2 Typical Harmonic Loading in an Office Block

One hundred and twenty PCs and monitors are supplied by a three-phase distribution board which in turn is supplied by a four-core three-phase cable. Forty PCs are supplied per phase at 1.5 amps per PC giving a total load per phase of 60 amps. For a single-phase circuit, the only effect on the supply cable will be the skin effects that cause currents to flow on the surface of cables. The third harmonic for this type of load will be 0.6 times the RMS current.

The loading on the three-phase cable supplying the distribution board may be found as follows: Phase current = 60 amps. Since the third harmonic is 0.6 x 60 = 36A then the neutral current = 36 x 3 = 108 A. The need for de-rating the cable is clear because cable manufacturers rate four-core cables on the basis that only three cores are loaded. In the above situation all four cores are loaded and in fact the neutral conductor will be carrying 60 x 1.8 = 108 amps when each phase conductor is carrying 60 amps. In the above situation BS 7671 Appendix 11 recommends that the cable should be rated to carry 108 amps instead of 60 amps carried by each line conductor due to the fact that all four cores are carrying current and therefore emitting heat. In addition to the above, the cables will be subjected to harmonic skin and proximity effects which will increase the heat generated in the cable further.

5.5 Measurement of Supply Voltage Harmonic Distortion

An experiment was set up in Dublin Institute of Technology at the Kevin Street Laboratories to determine the harmonic content of a supply voltage. The experimental layout is as shown in Figure 5.3.

The equipment used included an isolating transformer; a non-inductive resistor (NIR);
Figure 5.3: Experiment to Measure Harmonic Distortion in the Supply Voltage.

A twin channel oscilloscope; a harmonic analyser and a 230V load. The volt drop across this NIR is in phase with the supply voltage and therefore an analysis of this volt drop will reflect the harmonic distortion of the supply voltage. Using this method, the supply voltage was found to have a THD of 2.8%. An oscilloscope was used to display the supply waveform a Fourier analysis being used to determine the individual harmonic frequencies in the supply voltage wave as discussed in Chapter 4.

In general, the harmonic distortion in the supply voltage is unknown because it is a function of external supply conditions as well as internal loading. It will therefore vary from location to location and may vary over the full day as the character of the load changes. The THDu of the supply voltage can also be altered when the supply authority carry out changes to the network such as switching arrangements to cater for repairs or maintenance or on a permanent basis when additional generating capacity is added to the system. An electrical installation can import harmonics from a supply that is already distorted and by exporting a distorted current can contribute to distorting the supply voltage further still with undesirable consequences for other users.
5.6 Harmonic Distortion of Current Flowing in Different Components

5.6.1 The Capacitor

Capacitors have very low impedance at high frequencies because the reactance of a capacitor with capacitance \( C \) at a given harmonic order \( n \) is given by

\[
X_{C_n} = \frac{X_{C_1}}{n}
\]

because

\[
X_C = \frac{1}{2\pi fC}
\]

that is, it is inversely proportional to the frequency. A capacitor will therefore import a disproportionately large amount of high order harmonic currents from a distorted AC supply voltage. The capacitors so affected will draw a larger current that would be the case from an undistorted supply and this may result in overheating and, in extreme cases, the dielectric will fail leading to a short-circuit. The experimental set up introduced in the previous section was used to produce the waveforms and spectral analysis shown in Figure 5.4.

The supply voltage has a modest THDu of 2.8%. Notwithstanding that a capacitor is a linear device, the capacitor current has a THDi of 15.4%. This demonstrates the importance of reducing harmonic distortion in the supply voltage because any voltage distortion in the supply is naturally amplified by capacitors in the installation.

5.6.2 The Energy Saver Compact Fluorescent Lamp (CFL)

The CFL lamp is a low energy lamp of the type replacing GLS tungsten filament lamps in the drive to reduce energy consumption in lighting schemes. The CFL lamp has a built
in switched mode power supply and a power factor correction capacitor. The current was found to have a THDi of 78%.

5.6.3 Current Flowing in a Personal Computer

A PC has a switched mode power supply and this accounts for the distorted current wave. The THDi of the current was measured as 70%.

5.7 Skin and Proximity effects due to Harmonic Currents

5.7.1 Skin Effect

AC flowing in a conductor generates eddy currents in the conductor due to electromagnetic induction. These eddy currents cancel the main current in the centre and assist the main current at the surface thereby reducing the current density in the centre of the conductor and increasing it toward the surface. This effect, known as the skin effect, is a function
of frequency and increases as the frequency of the AC increases. The effective resistance of the conductor is increased due to the skin effect and this increases the $I^2R$ losses in the conductor leading to a higher operating temperature in the conductor for a given RMS current. Conductors which carry harmonically distorted AC are exposed to the high frequency harmonic currents and therefore tend to operate at higher temperatures (see Figure 5.5).

5.7.2 Skin Depth

The skin depth is defined as the distance from the surface over which the current falls to $1/e$ times its surface value. This relationship is depicted in Figure 5.6.

For a good conductor, the skin depth is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

where $\pi \simeq 3.142$, $f$ is the frequency of propagating wave, $\mu$ is the permeability of the conductor and $\sigma$ is the electrical conductivity. Since electrical resistivity $\rho$ is defined as $1/\sigma$, relative permeability $\mu_r = \mu/\mu_0$ where $\mu_0$ is permittivity of free space, the skin depth can be written in the form

$$\delta = \frac{1}{\sqrt{\pi \mu_0} \sqrt{\mu_r f}}$$
Table 5.1 shows the skin depth for some common conductors.

It follows from Table 5.1 that only conductors having a diameter equal to or exceeding twice the skin depth will experience significant additional heating effects. At 50Hz, the skin depth for copper conductor is 9.3mm and therefore only copper conductors > 150² mm will be affected. However at 500Hz, the skin depth is 1.19mm and therefore all cables down to and including 16² mm will be affected by higher order harmonic currents. Cable sizes smaller than 16² mm are not normally considered to be affected by skin effect for typical industrial loads (see Table 5.2).
Table 5.2: Conductor Diameter for Standard Conductor Sizes.

<table>
<thead>
<tr>
<th>Conductor CSA mm$^2$</th>
<th>16</th>
<th>25</th>
<th>35</th>
<th>50</th>
<th>70</th>
<th>95</th>
<th>120</th>
<th>150</th>
<th>185</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conductor Diameter (mm)</td>
<td>4.51</td>
<td>5.64</td>
<td>6.68</td>
<td>7.98</td>
<td>9.44</td>
<td>11</td>
<td>12.36</td>
<td>13.82</td>
<td>15.35</td>
<td>17.49</td>
</tr>
</tbody>
</table>

5.7.3 Proximity Effect

In a similar manner, when cables run in proximity to each other or in proximity to conducting metallic objects such as cable tray, ducting or structural steel, eddy currents are induced in the conducting materials which subsequently interact with the cable conductor changing the current density and thereby increasing the effective resistance of the cable conductor. Whereas it is relatively easy to quantify the proximity effect for trefoil cables, it is much more complex to quantify the influence of other conducting materials that are located in varying degrees of proximity to the cable. Neher and McGrath [25] developed 66 equations to cover most configurations anticipated in practice but these prove very unwieldy and difficult to apply in everyday applications. The equations applied to specific frequencies and did not address a situation where a number of harmonic currents of different order were simultaneously applied. The proximity effect increases the cable operating temperature in a similar manner to the skin effect - see Figure 5.7.

5.7.4 The Additional Cable Heating Caused by Harmonic Skin and Proximity Effects

As seen in the above experiments which measured the harmonic content of various items of equipment commonly used, each load will have its own individual harmonic signature. Each load can be broken down into individual harmonic components as shown by the spectral analysis and each of these components will have a greater or lesser skin effect...
depending on its harmonic number, the conducting material and the size of conductor. It follows therefore that the AC resistance of a cable will increase as the harmonic content of the load increases. As the heat generated in a cable = $I^2R$, it follows that the heating effect in a cable will increase as the harmonic distortion in the cable increases. The next series of experiments is designed to measure how this affects the heating of the cable and thus its current rating.

5.8 The Energy Balance Equation for a Cable

5.8.1 Mathematical Model

The following theoretical model predicts the build-up of temperature in a cable carrying current. The following assumptions have been made:

- The cable is made from a homogenous material.
- The heat transfer from the cable is proportional to the surface area of the cable.
- The emissivity factor of the cable surface is stable over the operating range.
- The heat loss from the cable is directly proportional to the difference between the temperature of the cable surface and the ambient air temperature.

- The heat gained by the cable is proportional to $I^2R$.

- There is a negligible temperature gradient between the cable core and its surface.

When current flows in a cable, heat is generated, stored and lost to the surroundings. According to Newton’s Law of Cooling: Energy generated = Energy stored + Energy Lost which we can write as

$$P\delta t = mc\delta \theta + \alpha \theta A\delta t$$

(5.1)

where $P$ is the $I^2R$ power loss of the conductor, $m$ is the mass of the cable, $c$ is the specific heat capacity of the cable, $\delta \theta$ is the temperature rise above ambient temperature during time $\delta t$, $A$ is the surface area of cable and $\alpha$ is the emissivity factor of cable surface.

The solution of equation (5.1) takes the form:

$$\theta(t) = \theta_{\text{Final}}[1 - \exp(-t/\tau)]$$

(5.2)

where $t$ is the time in seconds from the start, $\theta$ is the conductor temperature above ambient and $\tau$ is the time constant of the cable. The solution of this is shown graphically in Figure 5.8 for $\theta_{\text{ Ambient}} = 30^\circ C$.

From equation (5.1) it is clear that

$$P\delta t = mc\delta \theta + \alpha \theta A\delta t$$

In the steady state, the heat stored is $mc\delta \theta = 0$ and thus, the heat gained is equal to the heat lost and $P\delta t = \alpha \theta A\delta t$ so that $\theta = \frac{P}{\alpha A}$ and

$$\theta \propto P \propto I^2R$$

(5.3)

where $\theta$ is the steady state temperature above ambient of the conductor.
Note that the temperature coefficient of resistance for aluminium at 20\degree C is 0.00403\Omega K^{-1}.

The cable resistance is now given by \( r_{DC} = R_0[1 + \alpha_{20}(\theta - 20)] \) \[36\] where \( R_0 \) is the DC resistance at 20\degree C, \( \alpha_{20} \) is the constant mass temperature coefficient at 20\degree C per Kelvin and \( \theta \) is the actual operating temperature in \degree C. Equation (5.3) now becomes

\[
\theta \propto I^2 R_0[1 + \alpha_{20}(\theta - 20)]
\]

(5.4)

The build-up of temperature in a PVC insulated cable to a final operating temperature of 70\degree C with an ambient temperature of 30\degree C is depicted in Figure 5.8. This is the standard operating condition for general purpose PVC cables quoted in BS 7671.

### 5.8.2 Calculation of Conductor Operating Temperature Under Overload Conditions

BS 7671 uses an ambient temperature of 30\degree C in the standard rating tables. For general purpose PVC insulation, the full load operating temperature is 70\degree C. Equation (5.3) states that \( \theta \propto P \propto I^2 R \) where \( \theta \) is the steady state temperature above ambient of the conductor. Thus the temperature rise above ambient is \( \propto I^2 \). Therefore for a conductor
with a full load design current of $I_b$ operating in an ambient temperature of $\theta_A$ carrying a current of $I_x$ will run at operating temperature $\theta_{\text{op}}$ which is given by

$$\theta_{\text{op}} = \theta_A + \left( \frac{I_{\text{op}}}{I_b} \right)^2 \times (70 - 30)$$

Thus, for example, calculate the operating temperature of a conductor which is operating in an ambient temperature of $30^\circ C$ and is carrying a 10% overload,

$$\theta_{\text{op}} = 30 + \left( \frac{1.1}{1} \right)^2 \times (70 - 30) = 30 + (1.21 \times 40) = 78.4^\circ C$$

The same conductor carrying 80% full load in an ambient temperature of $20^\circ C$ would have an operating temperature of $\theta_{\text{op}} = 20 + 0.8^2 \times [70 - 30] = 45.6^\circ C$.

### 5.9 Changes in AC Resistance Due to Skin and Proximity Effects

Skin and Proximity effects in harmonic circuits cause the effective resistance of the circuit to increase. This effective resistance at harmonic frequencies of order $> 1$ is called the AC resistance $r_{\text{AC}}$. The effective resistance at the fundamental frequency (order $= 1$) is called the DC resistance $r_{\text{DC}}$. The AC resistance factor is defined as the ratio of $r_{\text{AC}}$ to $r_{\text{DC}}$. The AC resistance factor is a function of frequency, increasing as the frequency increases. Meliopoulos et al. [26] described The AC resistance $r_{\text{AC}}$ for a given harmonic order as follows:

$$r_{\text{AC}}(h) = r_{\text{DC}}[1 + x_s(h) + x_{\text{sp}}(h) + x_{\text{cp}}(h)]$$  \hspace{1cm} (5.6)

where $x_s(h)$ is the increase in resistance due to skin effect, $x_{\text{sp}}(h)$ is the increase in resistance due to conductor proximity effect, $x_{\text{cp}}(h)$ is the increase in resistance due to adjacent steel pipe or metallic conduit.

The Standard IEC 60287-1-1 Electric Cables Calculation of the Current Rating [36] defines the DC resistance per unit length at its maximum operating temperature $\theta = \ldots$
$R_0[1 + \alpha_{20}(\theta - 20)]$ where $R_0$ is the DC resistance at $20^\circ C$, $\alpha_{20}$ is the constant mass temperature coefficient at $20^\circ C$ per Kelvin and $\theta$ is the maximum operating temperature in $^\circ C$. The AC resistance is defined by the Standard is $r_{AC} = r_{DC}(1 + y_s + y_p)$ where $y_s$ is the skin effect factor and is given by

$$y_s = \frac{x_s^4}{(192 + 0.8x_s^4)}, \quad x_s^2 = \frac{8\pi f}{r_{DC}}10^{-7}k_s$$

for frequency $f$ in Hz and where $k_s$ is an empirical value given in Table 2 of the Standard [36], which relates to the shape of the conductor, its construction and the material from which it is made. Here, $y_p$ is the proximity effect factor which, for two single-core cables, is given by

$$y_p = \frac{x_p^4}{(192 + 0.8x_p^4)} \left( \frac{d_c}{s} \right)^2 \times 2.9, \quad x_p^2 = \frac{8\pi f}{r_{DC}}k_p$$

where $d_c$ is the diameter of the conductor in mm, $s$ is the space between conductor axis in mm and $k_p$ is an empirical value as given in Table 2 of [36].

The proximity factor for three single core cables is given by

$$y_p = \frac{x_p^4}{192 + 0.8x_p^4} \left( \frac{d_c}{s} \right)^2 \left[ 0.312 \left( \frac{d_c}{s} \right)^2 + \frac{1.18}{(192 + 0.8x_p^4)} + 0.27 \right] \frac{x_p^2}{r_{DC}k_p}$$

### 5.9.1 Calculation of AC Resistance Factor for an Aluminium Conductor According to IEC60287

The AC Resistance Factors $r_{AC}/r_{DC} = 1 + y_s + y_p$ may be calculated using empirical values provided in the Standard 60287-1-1. They are frequency dependent and therefore vary across the harmonic spectrum. Therefore a family of AC resistance factors are applicable to an AC current with harmonic distortion. They may be calculated as follows:

$$R_0 = \frac{\rho \ell}{A} = \frac{2.8264 \times 10^{-8} \times 1}{185 \times 10^{-6}} = 0.1528 \times 10^{-3} \Omega/m$$

and

$$r_{DC} = R_0[1 + \alpha_{20}(\theta - 20)] = 0.1528 \times 10^{-3}[1 + 4.03 \times 10^{-3}(70 - 20)] = 0.1836 \times 10^{-3}$$
Table 5.3: Skin Effect for a Single Conductor.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{AC}$</td>
<td>1.00</td>
<td>1.02</td>
<td>1.06</td>
<td>1.11</td>
<td>1.17</td>
<td>1.24</td>
<td>1.31</td>
<td>1.38</td>
<td>1.45</td>
<td>1.52</td>
<td>1.58</td>
</tr>
</tbody>
</table>

Table 5.4: Skin Effect and Proximity Effect of Two Conductors.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{AC}$</td>
<td>1.01</td>
<td>1.05</td>
<td>1.12</td>
<td>1.23</td>
<td>1.37</td>
<td>1.52</td>
<td>1.68</td>
<td>1.85</td>
<td>2.02</td>
<td>2.18</td>
<td>2.34</td>
</tr>
</tbody>
</table>

The AC Resistance Factors due to Skin Effect for a Single Conductor

These factors are given by

$$\frac{r_{AC}}{r_{DC}} = 1 + y_s$$

where

$$y_s = \frac{x_s^4}{(192 + 0.8x_s^4)}$$

$$x_s^2 = \frac{8\pi f}{r_{DC} \times 10^{-7}} k_s$$

and $k_s$ is an empirical value extracted from the Standard. Using the above formula for different harmonic frequency orders yields the data given in Table 5.3.

Skin and Proximity Effects for Two Parallel Conductors

We have

$$\frac{r_{AC}}{r_{DC}} = 1 + y_s + y_p$$

where

$$y_p = \frac{x_p^4}{192 + 0.8x_p^4} \left( \frac{d_c}{s} \right)^2 \times 2.9$$

d_c is the diameter of the conductor (mm) = 15.35mm and $s$ is the distance between conductor axis (mm) = 20mm which yields the results given in Table 5.4.
Table 5.5: Skin Effect and Proximity Effect of Three Conductors.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{AC}$</td>
<td>1.01</td>
<td>1.06</td>
<td>1.15</td>
<td>1.26</td>
<td>1.39</td>
<td>1.52</td>
<td>1.64</td>
<td>1.76</td>
<td>1.87</td>
<td>1.97</td>
<td>2.06</td>
</tr>
</tbody>
</table>

Skin and Proximity Effect for Three Parallel Conductors

In this case

$$y_p = \frac{x_p^4}{192 + 0.8x_p^4} \left( \frac{d_c}{s} \right)^2 \left[ 0.312 \left( \frac{d_c}{s} \right)^2 + \frac{1.18}{\frac{x_p^4}{192 + 0.8x_p^4} + 0.27} \right]$$

From the calculations given in the previous two section and using the same methodology yields the following the results given in Table 5.5

Discussion

The graphs displayed in Figure 5.9 demonstrate that the proximity effect is more dominant than the skin effect for two cables running parallel according to the IEC 60287 formulae. Two-core cables or two single-core conductors are normally supplying single-phase loads where the currents are flowing in phase with each other. For the three-core cable supplying three-phase, the currents are not in phase and the proximity effects on currents flowing in the three are influenced by the instantaneous values of each. The skin and proximity effect is seen to be less marked for higher harmonic orders with the three conductor configuration compared to the two core cable. This is counter intuitive and perhaps points to a limitation in the formula used.

5.9.2 Determination of AC Resistance Factors from Experimental Data

From equation (6.3),

$$\Delta \theta \propto I^2 r_{AC_n}$$
where $\theta$ is the conductor temperature rise above ambient in $^\circ$C, $I$ is the load current and $r_{AC_n}$ is the AC resistance at harmonic order $n$. Thus

$$\frac{r_{AC_1}}{r_{AC_n}} = \frac{\Delta \theta_1}{I_1^2} \frac{I_n^2}{\Delta \theta_n}$$

and if $I_n^2 = I_1^2$, then

$$r_{AC_n} = r_{AC_1} \frac{\Delta \theta_n}{\Delta \theta_1}$$

and hence

$$\frac{r_{AC_n}}{r_{AC_1}} = \frac{\Delta \theta_n}{\Delta \theta_1}$$  \hspace{1cm} (5.7)

Using this relationship, the resistance ratios $r_{AC_n}/r_{AC_1}$ for harmonic orders 1 to 21 can be calculated from experimental data generated by the experiment described in Section 5.5. From equation (5.7), the ratio $r_{AC_n}/r_{AC_1}$ has been shown to equal $\Delta \theta_n/\Delta \theta_1$. This can be quantified from the data used to construct the graph shown in Figure 5.9.
Table 5.6: Harmonic Correction Factors for Heating Effects.

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>Heating effect generated</th>
<th>Expressed as P.U. of overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st harmonic</td>
<td>$I_1^2 r_{AC_1}$</td>
<td>$(I_1/I_{RMS})^2 r_{AC_1}$</td>
</tr>
<tr>
<td>n\textsuperscript{th} harmonic</td>
<td>$I_{n}^2 r_{AC_n}$</td>
<td>$(I_n/I_{RMS})^2 r_{AC_n}$</td>
</tr>
</tbody>
</table>

5.10 Development of an Algorithm to Calculate Harmonic Correction Factors

For a complex current, $I_1 + I_2 + I_3 + \ldots + I_n$ with an RMS current equal to the design current, the contribution to the overall heating effect by using the super-position theorem is compounded in Table 5.5.

The total P.U. heating effect $H$ for a harmonic load is given by

$$H = \sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n}$$

and the total P.U. heating effect for non-harmonic load is 1. Hence the fractional increase in heating effect is given by equation (5.8).

From equation (5.3), $\theta_f \propto P \propto I^2 R$ and the effective load current is therefore determined by

$$I_{RMS} \left( \sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n} \right)^{\frac{1}{2}}$$

and since the correction factor $C_f$ should be applied to $I_{RMS}$ we obtain

$$C_f = \frac{1}{\sqrt{\sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n}}}$$
5.10.1 PC Loading

A three-phase load is made up of approximately 300 personal computers with a harmonic signature as shown in Figure 5.10.

\[ I_{\text{RMS}} = 170.75 \text{ (Line current)} \]

Applying equation (5.8) we obtain the results given in Table 5.7.

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_n ) (Amps)</td>
<td>68</td>
<td>66</td>
<td>58</td>
<td>54</td>
<td>52</td>
<td>40</td>
<td>43</td>
<td>45</td>
</tr>
<tr>
<td>( (I_n/I_{\text{RMS}})^2 )</td>
<td>0.1586</td>
<td>0.1494</td>
<td>0.1405</td>
<td>0.1154</td>
<td>0.0927</td>
<td>0.0549</td>
<td>0.0634</td>
<td>0.0695</td>
</tr>
<tr>
<td>( r_{AC_n}/r_{AC_1} )</td>
<td>1.00</td>
<td>1.05</td>
<td>1.10</td>
<td>1.16</td>
<td>1.24</td>
<td>1.31</td>
<td>1.39</td>
<td>1.47</td>
</tr>
<tr>
<td>( (I_n/I_{\text{RMS}})^2r_{AC_n} )</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
<td>0.07</td>
<td>0.09</td>
<td>0.10</td>
</tr>
</tbody>
</table>
Table 5.8: Calculation of Per Unit Heating Effect (continued from Table 5.7)

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n$(Amps)</td>
<td>44</td>
<td>38</td>
<td>34</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2$</td>
<td>0.0664</td>
<td>0.0495</td>
<td>0.0396</td>
</tr>
<tr>
<td>$r_{AC_n}/r_{AC_1}$</td>
<td>1.55</td>
<td>1.64</td>
<td>1.71</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2r_{AC_n}$</td>
<td>0.10</td>
<td>0.08</td>
<td>0.07</td>
</tr>
</tbody>
</table>

The per unit heating effect is then given by

$$\sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n} = 1.23$$

Without the application of a de-rating factor the conductor would overheat due to the 23% increase in heat generated in the conductor by harmonic skin effects. The value of this conductor temperature can be calculated as follows: Final operating temperature $= 30 + (40 \times 1.23) = 79.2^\circ C$. This is a projected temperature based on the harmonic signature of the PC.

This increased temperature of $9.2^\circ C$ would reduce to service life of the conductor by more than 50%. In order to reduce the operating temperature of the cable to the value for which it was designed, it will be necessary to apply the harmonic skin effect correction factor

$$\sqrt{\frac{1}{\sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n}}} = \frac{1}{1.23} = 0.901$$
Table 5.9: Calculation of Per Unit Heating Effect

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triple-$n$ Line Current (Amps)</td>
<td>66</td>
<td>52</td>
<td>45</td>
<td>34</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_n$ (Amps)</td>
<td>198</td>
<td>156</td>
<td>135</td>
<td>102</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cable $I_{RMS}$ = 170.75 Amps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2$</td>
<td>1.34</td>
<td>0.83</td>
<td>0.63</td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{AC_n}/r_{AC_1}$</td>
<td>1.05</td>
<td>1.24</td>
<td>1.47</td>
<td>1.71</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2 r_{AC_n}/r_{AC_1}$</td>
<td>1.41</td>
<td>1.03</td>
<td>0.92</td>
<td>0.61</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.10.2 Calculation of the Effect of Triple-$n$ Harmonics in the Neutral

For a three-phase balanced load with a harmonic load similar to that shown in Figure 5.10, the neutral current will be as shown in Table 5.10 where the triple-$n$ harmonics will be 3 times the line values.

Thus, with

$$\sum_{n}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n} = 3.97$$

and the final operating temperature of the neutral conductor = $30 + (40 \times 3.97) = 188.8^\circ C$.

Applying the harmonic skin effect correction factor correction factor to limit the conductor temperature to $70^\circ C$ with

$$\sqrt{\frac{1}{\sum_{n=1}^{N} \left( \frac{I_n}{I_{RMS}} \right)^2 r_{AC_n}}} = \frac{1}{3.97} = 0.5$$

This result shows that a neutral cable should be selected using a correction factor of
Table 5.10: Calculation of Per Unit Heating Effect

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n (\text{Amps})$</td>
<td>38.1</td>
<td>131.70</td>
<td>110.9</td>
<td>53.80</td>
<td>30.7</td>
<td>14.60</td>
<td>8.7</td>
<td>4.90</td>
<td>1.3</td>
<td>1.80</td>
<td>1.4</td>
</tr>
</tbody>
</table>

0.5. This means in effect that the neutral conductor then selected will be twice the size of the line conductor. Looking at the cable as a whole and having applied a correction factor of 0.901 to the line conductors and a correction factor of 0.5 to the neutral conductor, it will still be necessary to further de-rate the cable because there are now four heat emitters compared to the normal three, the heat emitted by the cable will be 1.33 times the normal. The cable correction factor can now be calculated using the formula $1 / \sqrt{1.33} = 1 / 1.153 = 0.867$. The overall correction factors are now for the line conductors $0.901 \times 0.867 = 0.78$ and for the neutral conductor $0.5 \times 0.867 = 0.43$.

Appendix 11 of BS 7671 sets out recommended de-rating factors based on neutral conductor loading due to triple-n harmonics covering a number of scenarios but based on the above relationship. However there is not a table of correction factors provided for electrical engineers to take account of the heating effects due to harmonically distorted currents. A reference is made however to the formula provided in IEC 60287-1-1. In practice, these calculations are very detailed and require specialist knowledge to apply.

5.11 Experiment to Validate the De-rating Algorithm

The same test rig as that shown in Figure 5.3 was used to inject four different complex test currents into the cable. For complex wave 1, RMS = 188.5A and THDi = 495% the harmonics and given in Table 5.10 and Figure 5.11.

The correction factor can be calculated as shown in Table 5.11
Figure 5.11: Harmonics for Complex Wave 1.

Table 5.11: Calculation of Per Unit Heating Effect

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n$ (Amps)</td>
<td>38.1</td>
<td>131.70</td>
<td>110.9</td>
<td>53.80</td>
<td>30.7</td>
<td>14.60</td>
<td>8.7</td>
<td>4.90</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2$</td>
<td>0.04</td>
<td>0.49</td>
<td>0.35</td>
<td>0.08</td>
<td>0.03</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$r_{AC_n}/r_{AC_1}$</td>
<td>1.00</td>
<td>1.05</td>
<td>1.10</td>
<td>1.16</td>
<td>1.24</td>
<td>1.31</td>
<td>1.39</td>
<td>1.47</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2 r_{AC_n}$</td>
<td>0.04</td>
<td>0.52</td>
<td>0.38</td>
<td>0.10</td>
<td>0.03</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Given that

$$\sum_{n=1}^{N} (I_n/I_{RMS})^2 r_{AC_n} = 1.08$$

this predicts that the final operating temperature = $30 + (40 \times 1.08) = 73.2^\circ C$.

The harmonics associated with complex wave 2 are shown in Figure 5.12 from which the following can be computed:

$$\sum_{n=1}^{N} (I_n/I_{RMS})^2 r_{AC_n} = 1.06$$

yielding a final operating temperature of $30 + (40 \times 1.06) = 72.4^\circ C$.

Similarly, using the spectra given in Figures 5.13 and 5.14, the following conductor
Table 5.12: Calculation of Per Unit Heating Effect (continued from Table 5.11)

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>17</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n (Amps)$</td>
<td>1.3</td>
<td>1.80</td>
<td>1.4</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$r_{AC_n}/r_{AC_1}$</td>
<td>1.55</td>
<td>1.64</td>
<td>1.71</td>
</tr>
<tr>
<td>$(I_n/I_{RMS})^2r_{AC_n}$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 5.12: Harmonics for Complex Wave 2.

Temperatures can be computed, yielding operating temperatures of $75.6^\circ C$ and $76^\circ C$, respectively.

$$\sum_{n=1}^{N} (I_n/I_{RMS})^2r_{AC_n} = 1.14$$
$$\sum_{n=1}^{N} (I_n/I_{RMS})^2r_{AC_n} = 1.15$$
Figure 5.13: Harmonics for Complex Wave 3.

Figure 5.14: Harmonics for Complex Wave 4.
Chapter 6

Harmonic Distortion in Electrical Installations

The increasing use of non-linear loads in electrical installations has exacerbated the problems of harmonic distortion in industrial and commercial electrical systems. In the UK the current practice to determine the cable size for an electric circuit is to use BS7671 [27]. However, previously the 16th edition IEE Wiring Regulations only dealt with situations where cables attain the conductor temperature generated by sinusoidal currents at the fundamental power frequency. This chapter outlines the methods available to determine the minimum size of line conductors for protection against overload currents, taking into account the harmonic content of the load current, and explains the harmonic rating factor Cf introduced in 2008 for cables that are under significant harmonic influences. Since the effect of harmonic currents is to increase the Joule losses in a cable, the ampacity of the cable will need to be corrected to ensure the maximum conductor operating temperature is not exceeded. An experiment on how cable temperature can be measured under harmonic influence is described, and several sets of measurements taken on a typical cable are analysed.
6.1 BS7671 Method for Determining Cable Size for Protection Against Overload

Harmonic distortion in low voltage electrical installations is now a common occurrence in the built environment. It is caused by non-linear loads and historically was only associated with industrial power systems that used large static power converters. The increased usage of information technology equipment and low energy devices in buildings over the past twenty years, which result in non-linear electrical loads, has introduced a high level of harmonic distortion into the LV electrical system. As a result, it has become necessary to establish criteria for limiting problems from system quality degradation. E. W. Fuchs et al [13] reported that the present versions of IEEE Std 519-1992 [7] and IEC61000-3-2 [6] harmonic standards are too restrictive for low-frequency voltage and current harmonics, as they apply to residential power systems.

In the UK the general method used to determine the size of a line conductor is based on the method described in BS7671 (17th edition IEE Wiring Regulations).

In appendix 4 of BS7671, tables of ampacity and the associated impedance drop of common types of cables can be found. However, the ampacity published are based on the assumption that there is no harmonics present in the cabling system. Clearly the three-phase four wire ampacity rating column in the tables does not count the neutral conductor as a current carrying conductor hence it has no heat emission.

The basic method to determine the size of a line conductor for protection against overload is given in BS7671 Appendix 4, Section 5 and is as follows:

$$I_Z = I_t C_a C_g C_i C_c \geq I_n \geq I_b$$  \hspace{5cm} (6.1)

where $I_Z$ is the continuous service ampacity in amperes of a cable having taken all the applicable rating factors into account under defined installation conditions; $I_t$ is the tabulated ampacity in amperes of a cable (BS7671 Table 4A2, gives a schedule of appropriate
ampacity tables included in BS7671); $C_a$ is the rating factor for ambient temperature and is given in BS7671 Tables 4B1 and 4B2; $C_g$ is the rating factor for conductors that are grouped in defined installation arrangements (given in BS7671 Tables 4C1, 4C2, 4C3, 4C4 and 4C5); $C_i$ is the rating factor for conductors embedded within thermal insulation, (given in BS7671 Part 5 Regulation 523.7 and Table 52.2); $C_c$ is the rating factor for the type of protective device or under defined installation conditions (given in BS7671 appendix 4, section 5.1 and Tables 4B3); $I_n$ is the nominal rated current or current setting in amperes of the over-current device (its value can be selected from either BS7671 Appendix 3 or device manufacturers technical data literatures); $I_b$ is the design current in amperes of the circuit under normal steady state operating conditions and calculated using the declared nominal voltage level.

For single-phase loads and for single-phase motors

$$I_b = \frac{P}{U_0 \cos \phi} \quad \text{and} \quad I_b = \frac{P_m}{U_0 \cos \phi \eta}$$

respectively, where $P$ is the total active power of the load in W, $P_m$ is the total mechanical power of the load in W, $U_0$ is the nominal $AC_{RMS}$ line to an earthed neutral voltage in V, $\cos \phi$ is the displacement power factor without harmonic contents and $\eta$ is the mechanical efficiency of the motor. Similarly for three-phase loads and motors,

$$I_b = \frac{P}{\sqrt{3} U \cos \phi} \quad \text{and} \quad I_b = \frac{P_m}{\sqrt{3} U \cos \phi \eta}$$

respectively, where $U$ is the line-to-line voltage in V, and $\sqrt{3} U_0 = U$.

Depending on the actual installation circuit arrangement, not all rating factors $C_a, C_g, C_i$ or $C_c$ need to be applied. For example, if the circuit is not buried and an approved type of circuit breaker (BS EN 60898) is being used, which is usually the case, $C_c$ should be omitted from the equation (6.1). Further, if the cable is not totally surrounded by thermal insulations equation (6.1) can be reduced to

$$I_Z = I_n C_a C_g \geq I_n \geq I_b$$
Figure 6.1: Current Waveforms of (a) Linear Load vs (b) Non-linear Load for a Sinusoidal AC Supply at Fundamental Frequency of 50 or 60 Hz.

The ampacity tables found in BS7671 and other international standards such as the ET 101 [28] and IEC 60364 [29] generally assume a balanced three-phase linear load when the current in the neutral conductor of a three-phase four-wire circuit will be negligible.

Most commercial and industrial establishments tend to employ balanced three-phase four-wire distribution systems with a reduced size neutral conductor. However, with the proliferation of non-linear loads in this type of electrical design, the triple-n (multiples of 3rd order harmonics) harmonic current from each phase sums in the neutral. P. Cook et. al. [30] report that neutral currents can be higher than the phase currents where high harmonic distortion exists. The line and neutral currents that were taken by two separate but balanced three-phase four-wire linear loads and non-linear loads are shown in Figure 6.1(a) and Figure 6.1(b), respectively.

In Figure 6.1(a), with a linear load, all three phases draw perfect sinusoidal current
waves. Since they are balanced they cancel out in the neutral, thus the neutral conductor carries negligible current. Although in practice, even where load currents are at the standard fundamental power frequency of 50Hz or 60Hz there is rarely zero current in the neutral. Any value of current that exists in the neutral simply reflects an out-of-balance in the three-phase load, assuming that no phase is overloaded and the cable was sized using the basic method outlined above. The effect of an out of balance three phase load causing the neutral conductor to become a heat emitter should not cause any overheating to the group of three-phase four-wire cable. However, when harmonics are present as shown in Figure 6.1(b), the line currents in the three phases are no longer balanced sine waves and, if they are in triple-n order (i.e. $3n$), they will be additive in the neutral. Now the neutral conductor becomes a fourth and additional current carrying conductor. As a result, it is an additional heat-emitting source in the group of four conductors.

In view of the fact that the neutral conductor is now a current carrying conductor, hence a heat emitter, steps need to be taken to take account of the extra heat that is produced by the neutral conductor in a three-phase circuit. The update published by the IEE (now the IET) [34] states that for every $8^\circ$C increase above the maximum core conductor continuous operating temperature the life of the cable will be halved (e.g. 25 years reduced to 12.5 years). A method is thus required to size the cable accordingly to dissipate the extra heat that is being generated within a group of three-phase four-wire conductors to ensure that the group of four conductors does not overheat.

6.2 Heat Transfer Mechanisms in Electric Cables

A comprehensive review and research on the heat transfer mechanisms of electrical cables, which focused mainly on overhead line cables is given in [31]. In general, the heat balance of any cable can be considered by the law of conservation of energy and on a rate basis,
we have
\[ \frac{\partial}{\partial t} q_{\text{in}} + \frac{\partial}{\partial t} q_{\text{gen}} = \frac{\partial}{\partial t} q_{\text{stored}} + \frac{\partial}{\partial t} q_{\text{out}} \]

where \( \partial q/\partial t \) is the rate of heat energy per unit volume, \( q_{\text{in}} \) is the heat input, \( q_{\text{gen}} \) is heat generated internally, \( q_{\text{stored}} \) is heat stored by the medium and \( q_{\text{out}} \) is heat loss to the external environment.

In [31] it is shown that the heat transfer mechanisms associated with an electric cable immersed in air can be approximated using the following assumptions: (i) using cylindrical coordinates, it is a one-dimensional radial conduction, convection and radiation system; (ii) it has uniform volumetric heat generation; (iii) the thermal contact resistance between the conductor material and electrical insulation material is negligible; (iv) the electrical and thermal properties of the conductor and insulation materials are constant (i.e. homogeneous); (v) the surroundings are large compared to the cable; (vi) the analysis is for steady state conditions.

An energy balance rate basis analytic technique can be applied to an electrical cable (see Figure 6.2) to evaluate the surface temperature of the conductor or the cable. For the control surface (see Figure 6.3) placed around the inner and outer surfaces of the insulation material:

\[ E'_{\text{in}} - E'_{\text{out}} = 0 \]
Figure 6.3: Energy Exchanged at the Inner and Outer Surfaces of the Electrical Insulation of a Conductor.

since energy in is taken to be equal to energy out,

\[ E'_{in} = E'_{out} = q'_r = q'_1 \pi r_1^2 \] (6.2)

\[ q'_r - q'_{cv} - q'_{rad} = 0 \] (6.3)

hence, from equation (6.2) and (6.3), at the outer surface

\[ \pi q'_1 r_1^2 - h(2\pi r_2)(T_{s,2} - T_\infty) - \epsilon(2\pi r_2)\sigma(T_{s,2}^4 - T_{sur}^4) = 0 \]

This general equation can be used to determine the surface temperature \( T_{s,2} \) of a cable in terms of \( q' \), \( r_1 \), \( r_2 \), \( h \), \( T \) and \( \epsilon \) where \( h \) is the convection coefficient in W/m²K, \( \epsilon \) is the dimensionless emissivity of the cable surface and \( \sigma \) is the Stefan-Boltzmann constant in W/m⁻²K⁻⁴.
6.3 Industry Methods Available for Determining the Size of a Cable Under Harmonic Distortion

6.3.1 The Neher-McGrath Method

In 1957 Neher and McGrath [25] derived a set of Neher and McGrath (NM) cable rating equations to predict the resulting ampacity of a group of four single core cables. They are a more complex version of the Fourier heat transfer equations. There are many variables in the 66 equations used to account for the number of conductors, number and size of adjacent conduits, number and size of adjacent duct banks, coefficient of surface emissivity, number of cables, axial spacing between cables, extraneous heat sources, and wind velocity. All these factors and more, effect the calculation of ampacity. Two of the factors affecting the final ampacity value of a cable under harmonic conditions are the 'skin and proximity' effects. The NM equation is given by

\[ I = \sqrt{\frac{T_c - (T_a - \Delta T_d)}{r_{DC}(1 + Y_c)r_{ca}}} \]  

(6.4)

where \( I \) is the conductor current in kA, \( T_c \) is the conductor temperature in degree °C, \( T_a \) is the ambient temperature in degree °C, \( \Delta T_d \) is the temperature difference due to dielectric loss in degree °C, \( r_{DC} \) is the direct current resistance of the conductors in Ω at the conductors operating temperature per unit length, \( r_{ca} \) is the effective resistance between the conductor and ambient for a conductor loss in Ω at the conductors operating temperature per unit length. The parameter \( Y_c \) is the increment of AC/DC ratio in p.u. due to losses originating in the conductor, having components \( Y_{cs} \) (the skin effect) and \( Y_{cp} \) (the proximity effect) where

\[ Y_c = 1 + Y_{cs} + Y_{cp} \]

\[ Y_{cs} = 0.875 \sqrt{\frac{f k_s}{r_{DC}}} \]
\[ Y_{cp} = x_p \left( \frac{D_c}{S} \right)^2 \left[ \left( \frac{1.18}{x_p + 0.27} \right) + 0.312 \left( \frac{D_c}{S} \right)^2 \right] \]

and

\[ x_p = \frac{6.80}{\sqrt{r_{DC}/k_p}} \text{ at 60Hz} \]

where \( f \) is the frequency in Hz, \( D_c \) is the conductors outer diameter (in inches), \( S \) is the axial spacing between cables (in inches) and \( k_s \) and \( k_p \) are the skin and proximity effect factor respectively (with recommended values of 60Hz [25]).

The NM method does not cater for the inclusion of a range of harmonic components in the generalised equation (6.4). The NM equation is very similar to the IEC60287 method [36] as they are both based on the same principle.

### 6.3.2 The Meliopoulos and Martin Method

The Meliopoulos and Martin method [26] provides an extension of the Neher-McGrath equation using power losses in the cable under harmonic conditions to derive a de-rating factor for cables given in the NEC [32]. For single-phase circuits

\[ \kappa = \sqrt{\frac{\alpha_1^2 I_B^2 r_{AC}(1)}{P_{loss}}} \]

and for three phase circuits

\[ \kappa = \sqrt{\frac{\alpha_1^2 I_B^2 [r_{AC,A}(1) + r_{AC,B}(1) + r_{AC,C}(1)]}{P_{loss}}} \]

where \( \kappa \) is the desired harmonic de-rating factor \( r_{AC,A}(1), r_{AC,B}(1) \) and \( r_{AC,C}(1) \) are the AC resistance of phase A, B and C conductors at fundamental frequency, \( P_{loss} \) is the total ohmic losses of the cable including harmonic effects, \( I_B \) is the base RMS value of the design current and \( \alpha_1 \) is the p.u. value of the fundamental with respect to the base \( I_B \).

For harmonics at a frequency of \( h \times f_{fundamental} \), additional values can be found by the equations for \( x_s(h) \), \( x_{sp}(h) \) and \( x_{cp}(h) \) as given below:

\[
x_s(h) = \left( \frac{ka M_0(ka)}{2 M_1(ka)} \sin[\theta_1(ka) - \theta_0(ka) - \pi/4] \right) - 1 \tag{6.5}
\]
where \( k = \sqrt{2\pi f h \mu \sigma} \), \( a \) is the conductor radius in metres, \( f \) is the fundamental power frequency in Hz, \( \mu \) is the relative permittivity of the conductor, \( \sigma \) is the conductivity (of the conductor), \( h \) is the harmonic order and \( M_0(ka), M_1(ka), \theta_0(ka) \) and \( \theta_1(ka) \) are Bessel functions obtained from [26].

\[
x_{sp}(h) = F(x_p) \left( \frac{D_c}{S} \right)^2 \left[ \left( \frac{1.18}{F(x_p) + 0.27} \right) + 0.312 \sqrt{h} \left( \frac{D_c}{S} \right)^2 \right]
\]  
(6.6)

where \( x_p = k \sqrt{k_p/(\pi \sigma r_{DC})} \) at the \( h \)th harmonic and \( k_p \) is the empirical factor at fundamental power frequency from [25].

\[
F(x_p) = \left( \frac{x_p M_0(x_p)}{2 M_1(x_p)} \sin[\theta_1(x_p) - \theta_0(x_p) - \pi/4] \right) - 1
\]  
(6.7)

The contribution to the increase of conductor AC resistance due to proximity to a steel pipe or magnetic conduit is given by the following expressions: For a trefoil arrangement

\[
x_{cp}(h) = \alpha \sqrt{n} \left( \frac{0.89 S - 0.115 D_p}{r_{DC}} \right) \times 0.3048 \times 10^6
\]  
(6.8)

and for a flat cradled arrangement

\[
x_{cp}(h) = \alpha \sqrt{n} \left( \frac{0.89 S - 0.175 D_p}{r_{DC}} \right) \times 0.3048 \times 10^6
\]  
(6.9)

where \( \alpha=1.7 \) for a steel pipe and 0.8 for an iron conduit, \( D_p \) is the inside diameter of the pipe or conduit in metres and

\[
r_{AC}(h) = r_{DC}[1 + x_s(h) + x_{sp}(h) + x_{cp}(h)]
\]  
(6.10)

Two examples were given by Meliopoulos and Martin [26] to illustrate the full computation procedures to use the above equations. It will not be repeated here and readers who wish to consider these equations should refer to the original paper [26].

However, if the designer finds that the Meliopoulos and Martin approximation de-rates the cable leading to an inaccuracy or when significant zero sequence harmonic currents are present in the neutral, then the Neher-McGrath equation should be used to re-rate the cable.
6.3.3 The AH Generalised Ampacity Model

Ajit Hiranandani [33] develops a simple general equation that can be used to evaluate separate harmonic de-rating factors for line and neutral conductors. The AH method, named after the author, for calculating a cables ampacity in the presence of harmonics for NEC cables, can be summarized as follows:

1. Determine the harmonic signature of the line and neutral conductors by either calculation or measurement. The Harmonic Signature (HS) is then determined by the equation

\[
HS = \left( I_1, \ \alpha_y = \frac{I_y}{I_1}, \ \alpha_{y+1} = \frac{I_{y+1}}{I_1}, \ \alpha_{y+2} = \frac{I_{y+2}}{I_1}, \ldots \right) \tag{6.11}
\]

where \( y = 2 \). For example, a three-phase distribution circuit with a THD=41.9% has a phase current \( I_{RMS} = 99.12\text{A} \); \( I_{50Hz} = I_1 = 90\text{A} \); \( I_{150Hz} = I_3 = 35\text{A} \); \( I_{250Hz} = I_5 = 20\text{A} \); \( I_{350Hz} = I_7 = 10\text{A} \) and a neutral current \( I_{150Hz} = I_3 = 3 \times 35 = 105\text{A} \). Hence from equation (6.11), the harmonic signatures are: Line HS=(90, \( \alpha_3=0.39 \), \( \alpha_5=0.222 \), \( \alpha_7=0.111 \)) and Neutral HS=(90, \( \alpha_3=1.17 \)).

2. Determine the total AC resistance \( r_{AC} \) of the line and neutral conductors including skin effect and proximity effect using equation (6.10).

\[
r_{AC} = r_{DC} + x_s \cdot r_{AC} + x_{sp} \cdot r_{AC} + x_{cp} \cdot r_{AC}
\]

where \( h \) is the order of the harmonic, \( r_{DC} \) is the DC resistance of the conductor in \( \Omega \) at the conductors operating temperature per unit length, \( x_s \) is the contribution factor to AC resistance due to skin effect, \( x_{sp} \) is the contribution factor to AC resistance due to proximity effects of neighbouring conductors, \( x_{cp} \) is the contribution factor to AC resistance due to proximity effect of a metallic pipe or conduit, \( r_{AC}(h) \) is the AC resistance, and \( x_s(h) \), \( x_{sp}(h) \) and \( x_{cp}(h) \) are skin effect and proximity effect factors calculated for each harmonic order \( h \) as given in [26] from equations (5.5)-(5.9).

3. The de-rating factor for line and neutral conductors in the presence of harmonics
can be evaluated using equation (6.12):

\[
\text{HDF} = \left( 1 + \sum_{h=2}^{n} \alpha_h^2 \beta_h \right)^{-\frac{1}{2}}
\]

where \( \alpha_h \) is the harmonic distribution factor per unit harmonic content due to each harmonic with respect to base load current (i.e. \( \alpha_h = I_h/I_1 \)) and \( \beta_h \) is the normalized harmonic AC resistance factor, i.e. the ratio of conductor resistance at \( n^{th} \) harmonic frequency to resistance at fundamental power frequency - \( \beta_h = r_{AC}(h)/r_{AC}(1) \).

Similar to the approach proposed by Meliopoulos and Martin, it is necessary to assume a certain cable size and de-rate it by the factor HDF accordingly. Hiranandani presents a worked example similar to the one given by Meliopoulos and Martin but giving a different set of results for the derived de-rating factor.

### 6.3.4 The BS7671 Appendix 11 Method

In 2004 the International Electrotechnical Commission (IEC) published a set of harmonic rating factors so that allowance can be made for 3\(^{rd}\) harmonic currents in 4 and 5 core cables, where all the cores have the same conductor size. It is now included in BS7671 Appendix 11 as informative guidance - see Table 6.1.

Cook and Coate [30] explained the guidance given by the IEC such that if the third harmonic content of the current in each phase is between 33\% and 45\%, i.e. the neutral current is greater than the fundamental phase current, then selection of the conductor size should be based on the current in the neutral conductor divided by the given factor. If the harmonic content is greater than 45\% then the size of the line conductor chosen is based on the neutral conductor current. In this case the line conductors will be larger than that required to carry the line current and this ‘spare’ capacity allows the factor of 0.86 to be omitted.

Three examples are given in BS7671 Appendix 11 illustrating how to apply the rating factor in practice. It will not be repeated here and readers who wish to consider the rating
Table 6.1: Rating Factors for Triple Harmonic Currents in Four- and Five-core Cables
(Source: BS7671). *The 3rd harmonic content is expressed in terms of the total harmonic distortion.

<table>
<thead>
<tr>
<th>3rd harmonic content of line current (%)</th>
<th>Rating Factor</th>
<th>Rating Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size selection based on the line current</td>
<td>Size selection based on the neutral current</td>
<td></td>
</tr>
<tr>
<td>0-15</td>
<td>1.0</td>
<td>-</td>
</tr>
<tr>
<td>15-33</td>
<td>0.86</td>
<td>-</td>
</tr>
<tr>
<td>33-45</td>
<td>-</td>
<td>0.86</td>
</tr>
<tr>
<td>&gt;45</td>
<td>-</td>
<td>1.0</td>
</tr>
</tbody>
</table>

factor should refer BS7671.

### 6.4 Experimental Determination

An experiment was set up to inject discrete harmonic currents into a 185 sq.mm solid core aluminium cable. A schematic diagram of the equipment used is as shown in Figure 6.4. A signal generator was connected to the input of a 400W power amplifier, which, in turn, was connected to a current transformer as shown. The cable under test was connected to the primary of a current transformer. This arrangement allowed the full rated current of 400A to be injected into the cable at discrete frequencies, which could be set by the signal generator and measured by the grip ammeter. Thermocouples embedded in the cable at points B and C (see Figure 6.4) accurately measure the conductor temperature. Thermocouple A measures the ambient air temperature. The cable was shaped as shown so that thermocouple B would indicate the temperature of a single conductor suspended in
free air. Thermocouple C would indicate the temperature of conductors in close proximity running parallel to each other also suspended in free air. The thermocouples B and C can thus measure the additional heating effect due to the skin and proximity effects respectively as the cable is run at full load at frequencies of n, (50Hz), 3n, 5n, 7n ... up to 21n.

Initially, the cable was fully loaded at 50 Hz and the result compared with the value of 70°C quoted in BS 7671. Adjustments to the readings were made to compensate for the actual ambient temperature in the laboratory at the time of the test. This test validated the accuracy of the measuring system used as shown in the graph in Figure 6.5. Further tests were carried out by injecting the full rated current value (400 A) at various harmonic frequencies. and the results are shown in Figure 6.6. It can be seen that the conductor temperature at 50Hz is approximately 70°C, which agrees with BS7671. However as the frequency is increased whilst maintaining the current at 400A full rated value, it can be seen that the temperature of the conductor increases significantly. The temperature of
the parallel conductors has increased by a larger amount reflecting the combined skin and proximity effects.
Load currents that have significant harmonic distortion such as those supplying personal computers (see Figure 6.7) will therefore experience additional heating due to both the skin and proximity effects. Arising from this, cables will operate at a higher temperature than would be the case without harmonic distortion. If one can predict the degree of harmonic distortion in the load current then it is possible to determine the degree of additional heating that will occur and apply a suitable de-rating correction factor.

The experimental data collected and analysed in the above experiment proves that there are indeed significant heating effects in a conductor carrying harmonic currents as predicted by Meliopoulos and Martin [26] and in the AH Generalised Ampacity Model by Hiranandani [33]. Those effects must be taken into consideration and it is likely that the oversizing of conductors by the BS7671 Appendix 11 method can adequately cover the excess heating caused by the harmonic content of the currents.

6.5 Discussion

There are a number of ways to reduce and combat the detrimental effects of high levels of harmonic distortion in an electrical installation, e.g. by the application of filters etc.
Active filters are devices which actively inject opposite harmonics into a system to cancel out the harmonics created by the non-linear loads. Passive filters trap or resist the flow of harmonics through them. They do this through various capacitors or reactors. Harmonics rated transformers known as K factor transformers are specifically designed in order to cope with the excess heating problem caused by the presence of high level circulating harmonic currents. The neutral connections are sometimes being sized at around 200% of the size required in order to accommodate the harmonic loads [6]. Most of the remedial systems that are put into place do work, but are usually quite costly especially if they are not initially included at the design stage. Also, filters and transformers may require maintenance or could suffer failures if not designed and installed properly. The dire consequence of which will render the system unprotected and the harmonics present may cause damage in this time period, especially if there is a failure in the equipment as there may be a certain length of time until the fault is found and rectified.

Reducing the temperature of the conductors is one of the most important cabling design aspects in an electrical installation. It is has been clearly demonstrated in this chapter, that in order to maintain the operating temperature of the cable within the specified maximum tolerable temperature, an increase in the cross-sectional area of the conductor is required. With a larger cross-sectional area, even if the filter or the transformer is faulty, the cables will be sized to cope with the extra currents, reducing the damage that can be caused. Another fact to consider is that these calculations have been carried out on the assumption that the neutral conductor is carrying 100% third harmonic load. However, at certain times of the day, if the equipment that causes harmonic distortion is not operating, the harmonic load will be reduced and, as a result of this, the voltage drop will reduce further, making the circuit much more energy efficient.

There are several advantages to increasing the cable size as a result of harmonic de-rating, in most cases, only up to the next size. They include:

- harmonic loads are accounted for and even if preventative measures (e.g. filters,
transformers) fail, the cables are adequately sized to carry the load;

- temperature rise of the cable is reduced, reducing losses, maintenance and running costs (increasing the life expectancy of the cable);

- larger cross-sectional areas can reduce the voltage drop along the circuit, proving more efficient by delivering close to the declared voltage to the current using equipment.

Harmonic distortion in electrical installations of tomorrow is likely to get worse as the rise in use of low energy electrical equipment in the built environment increases. Steps need to be taken by electrical designers and installers to minimise its detrimental influence on the interconnecting cables, busbars, energy sources and neighbouring equipment. With the newly published BS7671 (17th edition IEE Wiring Regulations) in 2008, at long last designers and installers now have a set of harmonic rating factors which can be used in the initial design calculations to determine reasonably accurately the size of cables to allow for conditions when harmonic distortion is present in a system. Alternatively, they can use the other calculation methods given in this chapter to calculate the heating effect of harmonic currents from first principles. Often the BS7671 method may result in an oversized cable, however, it was discussed earlier that this process is only beneficial as it can reduce the operating temperature of the cable and as a result the voltage drop in the cable is also reduced, thereby decreasing losses and increasing transmission efficiency.
Chapter 7

Proximity Heating Effects in Power Cables

The material presented in this chapter relates to the study of power system harmonics in the built environment and in particular, cable heating caused by proximity effects due to harmonic distortion. Although in recent years, some movement has taken place in the standards to offer harmonic de-rating factors, heating in cables due to skin and proximity effects has not been quantified effectively. Thus, in this chapter we present a model for proximity heating and consider numerical simulations to assess this effect in both two- and three-dimensions for different harmonics. Example results are presented to illustrate the model developed which is based on a general solution for the Magnetic Vector Potential in the Fresnel zone. The model provides the basis for using voxel modelling systems to investigate proximity effects for a range of configurations and complex topologies with applications to the design of power cables, cable trays and ducts, inter-connections, busbar junctions and transformers, for example.
7.1 Introduction

Being able to accurately model harmonic proximity effects in the design of cables, junctions, transformers and electrical appliances in general is particularly important in the design of electrical installations with regard to simulating heating effects. It is important to be able to simulate potential ‘hotspots’ in the built environment and check that heating effects conform to international standards especially with regard to the effect of higher order harmonics. This is because the heat generated is proportional to the square frequency of the harmonic. The two-dimensional cross sectional geometry of modern cables (e.g. see Figure 7.1) necessitates the accurate simulation of harmonic proximity effects in two-dimensions. The three-dimensional topological complexity of high current busbar interconnections (e.g. see Figure 7.2) contained inside switchgear, panel boards or busbars and carrying tens of thousands of amperes used in electrochemical production, for example, necessitate the use of full three-dimensional simulation. Flat and hollow topologies are used that allow heat to dissipate more efficiently due to the high surface to cross-sectional area ratio. The skin depth at 50 Hz for copper conductors is approximately 8 mm but at 500 Hz is 0.8 mm, therefore high frequency harmonics can lead to excessive heating in these situations which can be identified by a 3-D simulation.

7.2 The Proximity Effect in Electric Cables

Proximity heating effects occur in AC power systems. When alternating current flows in adjacent electric conductors, eddy currents are induced in both conductors by electromagnetic induction. The eddy currents in each conductor are the sum of the self-induced eddy currents and the eddy currents induced by the adjacent conductor current. The eddy currents cause an alteration to the distribution of the main current flowing in each conductor. In the case of currents flowing in opposite directions in the conductors, as would be the case for a single-phase circuit, the current tends to concentrate on the adjacent sides.
Figure 7.1: Cross-section of a Typical Multi-core Armoured Power Cable.

Figure 7.2: Example Interconnections in a High Current Busbar Junction.
When the current flowing in both conductors is in the same direction, the current tends to concentrate on opposite sides of the conductors. This effect is known as the proximity effect and causes the main current to flow in a restricted higher resistance path giving rise to the generation of additional square current heating losses, that, in turn, causes the operating temperature of the conductor to rise, the apparent increased resistance being referred to as the ‘AC resistance’. This same interaction also occurs between a current carrying conductor and an adjacent conducting material such as structural metal work, a cable tray or metal cladding on cables. In this case the presence of a ferrous material greatly increases the electromagnetic effect. The distribution of current in the conductor is altered in a similar manner to that previously described, leading to an increased operating conductor temperature. The induced currents in the adjacent metalwork give rise to a power loss but they also cause the metal work to increase in temperature. This reduces the heat dissipation of the electric cable and further increases its operating temperature.

7.2.1 The Proximity Effect on Electrical Equipment

The proximity effect is frequency dependent, increasing as the frequency increases. It also depends on the conductor material and its diameter. Thus in a harmonically rich environment, the higher order harmonics will generate significant proximity effects with other conductors and adjacent conducting materials. In the case of power transformers, the windings of the transformer are wound as compactly as possible to reduce size; however, the proximity of the conductors to each other and to the magnetic core tend to increase the associated proximity loss. In addition, significant eddy currents are induced in the magnetic core. This leads to a power loss and heat generation in the core. The transformer core is constructed from high electrical resistance silicon steel lamina in order to minimise the eddy current power loss. This loss is proportional to the square of the frequency so higher order harmonics have a very significant heating effect on the core. In a similar manner to transformers, all electrical machines such as electric motors, gener-
tors, reactors etc., whose design requires the use of magnetic cores, experience proximity effect losses. These losses increase significantly when harmonic distortion is present.

7.2.2 The Skin Effect

The skin effect was first described in 1883 by Sir Horace Lamb, a British applied mathematician who described the effect as it related to a spherical conductor. In 1885 the model was generalised and applied to conductors of any shape by Oliver Heaviside, an English engineer and mathematician. The skin effect is similar to the proximity effect insofar as it is a consequence of electromagnetic induction. It applies to a single conductor carrying AC current. In this case, eddy currents are self-induced and interact with the main current in such a way that the current reduces in the centre of the conductor and tends to flow near the surface of the conductor in a skin region, hence the name skin effect. In a similar manner to that described for the proximity effect, the current is forced to flow through a restricted cross-section thus increasing the resistance of the current path. The net effect is the same as for the proximity effect which is to increase the operating temperature of the conductor. The skin depth is defined as the distance from the conductor surface by which the current density reduces by $e^{-1}$. The effect is a function of:

1. the conductor material;
2. the diameter of the conductor;
3. the frequency of the current.

In a similar manner to the proximity effect, the higher order harmonics are affected more than the lower order harmonics, therefore the skin depth is different for each harmonic.
7.2.3 Proximity and Skin Effects

These effects are usually observed together in the built environment. The skin effect is always present to a greater or lesser extent. The proximity effect comes into play whenever two conductors or more run parallel to each other or when a single or double conductor runs close to metal work be it a cable tray, cable protective steel cladding or the metal tank of a power transformer or switchgear, for example.

7.3 Proximity Heating Model

It is well known from Ampere’s law that a current generates a magnetic field in the plane through which the current flows. In general, Ampere’s law is expressed in differential form by (see Appendix B)

$$\nabla \times \mathbf{B} = \mathbf{j}$$

where $\mathbf{B}$ is the Magnetic Field Vector and $\mathbf{j}$ is the current density. This is Maxwell’s equation without the inclusion of the displacement current. When an alternating current passes through a cable an alternating magnetic field is produced within and beyond the extent of the conductor. Any conductor within the immediate vicinity or ‘proximity’ of the cable will have a current induced by the presence of the alternating magnetic field. This includes currents induced in the cable that is generating the magnetic field - self-induced currents. In turn, this produces further heating of the cable(s) which is proportional to the square of the induced current. This effect can become significantly complex especially when the topology of the cable configuration is non-trivial such as with braided cables. For this reason, it requires the problem to be solved in a three-dimensional geometry and necessitates a numerical approach to solving the problem. The proximity effect is an iterative effect in that it depends on a current generating a magnetic field which induces another current in the proximity of the first which then goes on to generate another magnetic field, each cycle becoming weaker and weaker but each cycle inducing further
heating effects. To model such an effect, the magnetic field must first be computed.

In Appendix B, an overview of Maxwell’s equations is provided for completeness. Using the Lorentz Gauge transformation discussed in Section B.3 it can be shown that the Magnetic Vector Field Potential $A$ is given by - see the derivation of equation (B.13)

$$ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -j $$  \hspace{1cm} (1)

where $c$ is the speed of light in a vacuum. This equation is, in effect, an exact solution to Maxwell’s equations subject to the Lorentz Gauge Transform Condition (as discussed further in Appendix B). The relationship between the Electric Field $E$ and $A$ is

$$ E = -\nabla U - \mu \frac{\partial A}{\partial t} $$  \hspace{1cm} (2)

where $U$ is the Electric Scalar Field Potential which is the solution to (as shown in Appendix B)

$$ \nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = -\frac{\rho}{\epsilon} $$

where $\rho$ is the charge density. For a cable composed of a highly conductive material with conductivity $\sigma$, the charge density is effectively zero over the macro-time scales of interest since

$$ \rho(t) = \rho_0 \exp(-\sigma t/\epsilon), \text{ where } \rho_0 = \rho(t = 0) $$

where $\epsilon$ is the permittivity of the material. Hence, the Electric Scalar Potential can be taken to be given by the solution of the homogeneous wave equation

$$ \nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0 $$  \hspace{1cm} (3)

Let $j_p$ be some induced proximity current density generated in a conductor with conductivity $\sigma_p$ in the proximity of the magnetic field associated with the Magnetic Vector Potential $A$. From Ohms Law

$$ j_p = \sigma_p E $$  \hspace{1cm} (4)
and any proximity heating effect can be taken to be proportional to $|\mathbf{j}_p|^2$ - a measure of the proximity temperature effect. Thus, we consider a proximity effect model based on solving equation (1) subject to a solution to equation (3) from which the electric field can be computed from equation (2) thereby providing an estimate of the induced proximity current obtained via equation (4).

### 7.4 Harmonic Solution

Consider the case for a single harmonic when, for angular frequency $\omega$, all scalar or vector functions ($F$ and $\mathbf{F}$, respectively) of time $t$ are taken to be of the form

$$F(r, t) := F(r, \omega) \exp(-i\omega t), \quad \mathbf{F}(r, t) := \mathbf{F}(r, \omega) \exp(-i\omega t)$$

Equations (1) and (3) then reduce to

$$(\nabla^2 + k^2) A = -j$$  \hspace{1cm} (5)$$

and

$$(\nabla^2 + k^2) U = 0$$  \hspace{1cm} (6)$$

respectively, and equation (2) becomes

$$\mathbf{E} = -\nabla U + ik\mu A$$  \hspace{1cm} (7)$$

where $k = \omega/c = 2\pi/\lambda$ and $\lambda$ is the wavelength. Note that these equations are the same for all vector components of the vector function and thus can be evaluated in terms of a scalar field for each component. Thus, with regard to equations (5) and (6), the problem reduces to solving the scalar equation

$$(\nabla^2 + k^2) u(r, k) = -f(r, k), \quad r \in \mathbb{R}^3$$  \hspace{1cm} (8)$$

for scalar field $u$ and scalar function $f$ which can be zero, thereby satisfying equation (6).
7.5 Green’s Function Solution

The general solutions to equation (8) using the free space Green’s function method is well known and given by

\[ u(r, k) = \oint_S (g \nabla u - u \nabla g) \cdot \mathbf{n} d^2 r + g(r, k) \otimes_3 f(r, k) \]  

(9)

where \( g \) is the Green’s function

\[ g(r, k) = \frac{\exp(ikr)}{4\pi r}, \quad r \equiv |r| \]

which is the solution to

\[ (\nabla^2 + k^2)g(r, k) = -\delta^3(r) \]

and \( \otimes_3 \) denotes the three-dimensional convolution integral

\[ g(r) \otimes_3 f(r) = \int_{\mathbb{R}^3} g(|r - s|, k) f(s, k) d^3 s \]

The surface integral (obtained through application of Green’s Theorem) represents the effect of the field \( u \) generated by a boundary defining the surface \( S \). This field, together with its respective gradients, need to be specified - the ‘Boundary Conditions’. The surface integral determines the effect of the surface of the source when it is taken to be of compact support, e.g. the conductive material from which a cable is composed.

7.5.1 Homogeneous Boundary Conditions

In the context of the model considered here, the surface integral is taken to be zero so that volume effects are considered alone. Formally, this requires that we invoke the ‘Homogeneous Boundary Conditions’

\[ u = 0, \quad \nabla u = 0 \quad \forall r \in S \]

so that equation (9) is reduced to

\[ u(r, k) = g(r, k) \otimes_3 f(r, k) \]  

(10)
which is a solution to equation (8) since

\[(\nabla^2 + k^2)u(r, k) = (\nabla^2 + k^2)[g(r, k) \otimes_3 f(r, k)]
\]

\[= -\delta^3(r) \otimes_3 f(r, k) = -f(r, k)
\]

However, self-consistency requires that the ‘Homogenous Boundary Conditions’ also apply in the solution of equation (6) so that \(U = 0\) and equation (7) reduces to (for any vector component of the Magnetic Vector Potential \(A\) and Electric Field \(E\))

\[E = ik\mu A\]

Combining the results, we then obtain the following solution for proximity temperature effect \(T_p\)

\[T_p(r, k) = T_0 k^2 \mu^2 |\sigma_p(r) | g(r, k) \otimes_3 j(r, k) |\]^2 \hspace{1cm} (11)

where \(T_0\) is a scaling constant determined by the resistivity of the proximity conductor and it is noted that the proximity temperature scales as the square of the frequency (since \(k = \omega/c\)) and the square of the magnetic permeability \(\mu\).

### 7.5.2 Skin Depth and Absorption Characteristics

For \(\rho = 0\) equation (B.5) reduces to

\[\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{j}}{\partial t}
\]

so that, using Ohm’s law \(\mathbf{j} = \sigma \mathbf{E}\), for any vector component of the Electric Field \(\mathbf{E}\), the Scalar Electric Field \(u\) satisfies the harmonic equation

\[\nabla^2 + k^2 + ikz\sigma)u(r, k) = 0
\]

where \(z = \mu c\) defines the impedance of a material with conductivity \(\sigma\) and permeability \(\mu\). For a unit vector \(\hat{n}\), this equation has a simple ‘plane wave solution’ of the form

\[u(r, k) = \exp(iK \hat{n} \cdot r), \hspace{1cm} K = k\sqrt{1 + iz\sigma/k}
\]
Thus, when $\sigma/k >> 1$, and noting that

$$K \simeq \sqrt{ikz\sigma} = (1 + i)\sqrt{kz\sigma/2}$$

we obtain the physically significant result (i.e. the wave amplitude cannot increase indefinitely)

$$u(r, k) = \exp(i\sqrt{kz\sigma/2}\hat{n} \cdot r) \exp(-\sqrt{kz\sigma/2}\hat{n} \cdot r)$$

which yields a solution with a negative exponential decay, i.e. an absorption of the Electric Field characterised by the skin depth

$$\delta = \sqrt{\frac{2}{kz\sigma}} = \sqrt{\frac{\lambda}{\pi z\sigma}}$$

Further, if the absorption characteristics of the medium are taken to be determined from the solution (for real $K$)

$$u(r, k) = \exp(iK\hat{n} \cdot r) \exp(-\alpha\hat{n} \cdot r)$$

then the wave equation for the scalar electric field becomes

$$-(K + i\alpha)^2 + k^2 + ikz\sigma = 0$$

so that, upon equating real and imaginary parts, the real component of $\alpha$ is given by

$$\alpha = \frac{k}{\sqrt{2}} \left[ \left(1 + \frac{z^2\sigma^2}{k^2} \right)^{\frac{1}{2}} - 1 \right]^{\frac{1}{2}}$$

Since we can write, upon binomial expansion,

$$\alpha = \frac{k}{\sqrt{2}} \left( \frac{z^2\sigma^2}{2k^2} + ... \right)^{\frac{1}{2}}$$

when $z^2\sigma^2/k^2 << 1$, $\alpha = z\sigma/2$ and the absorption characteristics are independent of wavelength.
7.6 Fresnel Zone Analysis

Proximity effects occur in the near field which is determined by the form of the Green’s function given in equation (11). However, for computational reasons, it is useful to consider a solution in the intermediate or Fresnel zone. It is well known that the key to evaluating the Green’s function in the Fresnel zones relies on a binomial expansion of \(|r - s|\) in the exponential component of the Green’s function and considering the relative magnitudes of the vectors \(r\) and \(s\) (given by \(r\) and \(s\), respectively).

In the Fresnel zone, the Green’s function is given by

\[
g(r, s, k) = \frac{1}{4\pi s} \exp(iks) \exp(-ik\hat{n} \cdot r) \exp(i\alpha r^2)
\]

where

\[
\hat{n} = \frac{s}{s}, \quad \alpha = \frac{k}{2s} = \frac{\pi}{s\lambda} \quad \text{and} \quad s \equiv |s|
\]

This result is based on relaxing the condition \(r/s << 1\) and ignoring all terms with higher order powers greater than 2 in the binomial expansion of \(|r - s|\). In this case, equation (10) becomes

\[
u(r, s, k) = \frac{\exp(iks)}{4\pi s} \int_{\mathbb{R}^3} f(r, k) \exp(-ik\hat{n} \cdot r) \exp(i\alpha r^2) d^3r
\]

However, noting that

\[
\frac{ik}{2s} |s - r|^2 = \frac{ik}{2s} (s^2 + r^2 - 2s \cdot r)
\]

\[
= \frac{iks}{2} + \frac{ikr^2}{2s} - ik\hat{n} \cdot r
\]

we can write \(u\) in the form

\[
u(r, s, k) = \frac{\exp(iks/2)}{4\pi s} f(r, k) \otimes_3 \exp(i\alpha r^2)
\]

the function \(\exp(i\alpha r^2)\) being the (three-dimensional) Fresnel Point Spread Function (PSF).

From equation (11) the (Fresnel zone) proximity temperature effect can now by written as

\[
T_p(r, k) = T_0 k^2 \mu^2 \left| \sigma_p(r) [j(r, k) \otimes_3 \exp(i\alpha r^2)] \right|^2
\]

(12)
where $T_0 := T_0/4\pi s$.

7.7 Two-Dimensional Simulations

Two-dimensional simulations are appropriate for cable configurations in which the axial geometry is uniform. In this case we can consider a computation using a regular two-dimensional Cartesian mesh of size $N^2$ where the cable cross-section is taken to be in the $(x, y)$-plane at $z = 0$. Computation of the (two-dimensional) Fresnel PSF is also undertaken on a Cartesian grid. For a given value of $N$, the scaling of this function (i.e. the range of values of $\alpha$ that can be applied) is important in order to avoid aliasing. If $\Delta$ defines the spatial resolution of a mesh (the length of each pixel being taken to be given by $\Delta$ for both coordinates), then we can write

$$\alpha(x^2 + y^2) = \frac{\pi}{s\lambda} \Delta^2 (n_x^2 + n_y^2)$$

where $n_x$ and $n_y$ are array indices running from $-N/2$ through 0 to $N/2 - 1$, the Fresnel PSF being computed over all space and not just a positive (two-dimensional) half-space. Thus, if we consider a scaling relationship based on multiples $h$ of the wavelength such that

$$h\lambda = \Delta N \text{ and } \frac{\Delta}{s} = \frac{1}{N}$$

where $h = 1, 2, \ldots$ (for integer harmonics) then we have

$$\alpha(x^2 + y^2) = \frac{h\pi}{N^2} (n_x^2 + n_y^2)$$

The purpose of developing this result is that it allows us to investigate the proximity temperature as a function of multiples of the wavelength (i.e. harmonics). Given equation (12), we consider a simulation based on the normalised proximity temperature

$$T(x, y) = \frac{T_p(x, y, k)}{T_0 k^2 \mu^2}$$

$$= |\sigma_p(x, y) \{ j(x, y) \otimes_2 \exp[i\alpha(x^2 + y^2)] \} |^2$$

(13)
where

\[ \|T(x,y)\|_\infty = 1 \]

and \( \otimes_2 \) denotes the two-dimensional convolution sum, the current density being taken to be independent of the frequency \( k \).

Numerical evaluation of equation (13) is undertaken using MATLAB, the convolution sum being computed using the MATLAB function \texttt{conv2} with the option ‘same’ which returns the central part of the convolution that is the same size as the input arrays. Colour coding of the two-dimensional scalar function \( T(x,y) \) is used to display the spatial distribution of the temperature in the proximity conductor(s) defined by the function \( \sigma_p(x,y) \), the amplitude of the ‘Magnetic Scalar Potential’ being given by

\[
M(x,y) = |j(x,y) \otimes_2 \exp[i\alpha(x^2 + y^2)]|
\] (14)

Figure 7.3 shows an example of a proximity simulation for a simple three-cable based configuration and a uniform cross-sectional current density for the lowest harmonic \( h = 1 \) and where the current is taken to flow in the same direction (in or out of the plane). Figure 7.3 also shows the PSF, the Magnetic Field Potential and the proximity temperature based on equations (14) and (13), respectively. Each cable is taken to ‘radiate’ a Magnetic Potential Field in the plane which induces a secondary Electric Field in the proximity cable. This secondary Electric Field induces a proximity current density and thus a (square current) proximity temperature effect whose field pattern is determined by equation (13). Figure 7.4 shows the proximity temperatures for higher harmonics \( h = 3, 5, 7, 9 \) and illustrates the surface heating that occurs when higher harmonics are considered in terms of the proximity effects generated by the influence of the magnetic field generated by one cable with another. Three results emerge from these simulations that are notable:

1. Lower harmonic proximity temperatures are biased toward the cable (outer) surfaces in local proximity to each other.
Figure 7.3: A three conductor configuration in the plane (top-left for a grey scale colour map) for a uniform current density using a $500 \times 500$ regular Cartesian Grid, the Fresnel Point Spread Function (top-right for a grey scale colour map), the Magnetic Field Potential (bottom-left) given by equation (14) and the proximity temperature (bottom-right) given by equation (13) for $h = 1$ using the MATLAB ‘Hot’ colour map. In each case, the fields shown are normalised and therefore represents a two-dimensional numerical field with values between 0 and 1 inclusively. Thus in both the colour maps used, 0 corresponds to Black and 1 corresponds to White.
2. The proximity temperature exhibits an increased asymmetry as the harmonic order increases, the region of maximum temperature being skewed toward the surfaces in closer proximity to each other.

The last point in the above list is explained in terms of the skin effect. However, as the harmonics increase further the heat is distributed on both the surface and the interior of the cable. This effect is due to the nature of the PSF which generates quadratic phase wavefronts as the value of the harmonic increases. This is not a feature of the conventional skin effect that is based on a model for the penetration on plane wave (for the Electric Field) in a conductor, which is consistent only with regard to a far-field analysis of the Green’s function used to obtain a solution to equations (5).

Figure 7.4: Proximity temperature distributions associated with the cable configuration given in Figure 7.3 for harmonics \( h = 3, 5, 7, 9 \) (from top-left to bottom-right, respectively) using the MATLAB ‘Hot’ colour map. In each case, the fields shown are normalised and therefore represents a two-dimensional numerical field with values between 0 and 1 inclusively.

The effect of multiple cable arrays and the induction temperatures induced in neutral
proximity conductors (i.e. conductors with zero primary current density) is easily simulated using equation (13) where the proximity conductivity is taken to be the combination of conductors with and without a primary current density $j(x, y)$. This is a common problem with regard to the assembly of power cables in cable trays, an example of which is shown in Figure 7.5. An example is shown in Figure 7.6 for a non-symmetrical array of nine cables supported by a semi-rectangular metal cable tray and where it is clear that:

1. proximity temperature effects are induced in the tray which have proximity temperatures of the order of the cables closest to the tray;

2. the inner cable of the array reaches the highest proximity temperature (for all harmonics) due to its proximity to largest number of surrounding cables.

Figure 7.5: Example of a Power Cable Assembly in a Conductive Cable Tray.
Figure 7.6: Proximity temperature distributions associated with a stack of cables all with the same flow of alternating current (in or out of the plane) housed in a rectangular cable tray which is taken to have the same conductivity as the cables but with no primary (only an induced) current (top-left). The proximity temperature generated by the induced current (in both the cables and tray) are shown for harmonics $h = 11, 13, 15$ (from top-right to bottom-right, respectively) using the MATLAB ‘Hot’ colour map. In each case the fields shown are normalised and therefore represents a two-dimensional numerical field with values between 0 and 1 inclusively.
7.8 Three-Dimensional Simulations

For the three-dimensional case, equations (14) and (13) become

\[
M(x, y, z) = | j(x, y, z) \otimes_3 \exp[i\alpha(x^2 + y^2 + z^2)] |
\]

and

\[
T(x, y, z) = \frac{T_p(x, y, z, k)}{T_0 k^2 \mu^2} = \sigma_p(x, y, z) \{ j(x, y, z) \otimes_3 \exp[i\alpha(x^2 + y^2 + z^2)] \}^2
\]

respectively, where

\[
\alpha(x^2 + y^2 + z^2) = \frac{h\pi}{N^2}(n_x^2 + n_y^2 + n_z^2)
\]

\( \Delta \) being taken to define the spatial resolution of a voxel. Three-dimensional simulations are necessary with regard to computing proximity heating effects in high current loading inter-connectors such as illustrated in Figure 7.2 when radial symmetry can not be assumed. The same principles apply as those used to generate two-dimensional simulations. However, working with a three-dimensional regular grid requires significantly greater CPU time. Further, voxel modelling system are required to generate the input arrays representing the functions \( j(x, y, z) \) and \( \sigma_p(x, y, z) \) together with voxel graphical representations to visualise the three-dimensional output numerical fields generated.

Voxel modelling systems such as **Voxelogic** [37] and **Voxel Sculpting** [38] allow designers to sculpt without any topological constraints. These systems include open source produce such as **VoxCAD** [39] that provides for the inclusion of multiple materials and is therefore ideal for introducing designs based on variations in the conductivity and current density. Systems such as **Pendix**, [40] operate like 2D graphics software while producing 3D models, [41] and are therefore ideal for designing structures with a single value of the conductivity, for example, although the application of these systems lies beyond the scope of this work.

By way of an example, we consider a simplified voxel model of the connecting bars and their (square plate) root given in Figure 7.2. Figure 7.7 shows a simple visualisation of
this model together with an iso-surface of the corresponding Fresnel PSF for $h = 10$ using a regular Cartesian grid of $100^3$. Figure 7.7 also shows the two-dimensional proximity temperature field in the $(x,y)$-planes for $z = 1$ and $z = 100$ based on equation (15), the three-dimensional convolution process being undertaken using the MATLAB function \textit{convn}. The results are illustrative of the effect of induction currents generated by a three-dimensional magnetic field associated with a uniform current flowing in a relatively simple but three-dimensional structure.

Figure 7.7: Simplified voxel model of the connecting bars and root given in Figure 7.2 (top-left) for a uniform current density using $100 \times 100 \times 100$ regular Cartesian grid and an iso-surface of the three-dimensional Fresnel PSF for $h = 10$ (top-right). The proximity temperature fields are shown for the $(x,y)$-plane at $z = 100$ and $z = 1$ in the bottom-left and bottom-right images, respectively, using the MATLAB ‘Hot’ colour map.
7.9 Thermal Diffusion Effects

The total induced temperature is the sum of the proximity temperature effects for all harmonics which is proportional to the square of the frequency. This temperature profile is the source for a temperature field that will thermally diffuse throughout the regions of compact support. Thus, referring to equation (12), the diffusion equation becomes

$$\left( D \nabla^2 - \frac{\partial}{\partial t} \right) T(r, t) = -S(r), \quad T(r, t = 0) = T_i$$

(16)

where $D$ is the Thermal Diffusivity of the conductor(s), $T_i$ is the initial temperature and the Source Function $S(r, t)$ is given by

$$S(r) = T_0 \sum_{h=1}^{N} T_0 P_h k_h^2 \mu^2 [\sigma_p(r)[j(r, k) \otimes_n \exp(i\alpha r^2)]]^2$$

(17)

for dimensions $n = 1, 2$ and $3$, where $P_h$ is the Power Spectrum and it is noted that $\alpha$ is a function of $h$. The general solution to equation (16) can be obtained using the Green’s function for the diffusion equation and is given by [42]

$$T(r, t) = T_i(r) \otimes_n G_n(r, t) + S(r) \otimes_n G_n(r, t)$$

where

$$G_n(r, t) = \left( \frac{1}{4\pi Dt} \right)^\frac{n}{2} \exp\left[ -\left( \frac{r^2}{4Dt} \right) \right] H(t)$$

and

$$H(t) = \begin{cases} 1, & t > 0; \\ 0, & t < 0. \end{cases}$$

For the steady state case, the time independent diffusion problem applies for which we are required to solve the Poisson equation

$$D \nabla^2 T(r) = -S(r)$$

the two- and three-dimensional solutions to this equation being [42]

$$T(r) = \begin{cases} -D^{-1} \ln r \otimes_2 S(r), & r \in (0, 1]; \\ +D^{-1} \ln r \otimes_2 S(r), & r \in [1, \infty). \end{cases}$$
and

\[ T(r) = \frac{1}{4\pi D_r} \otimes_3 S(r, k) \]

respectively.

Given that the Diffusivity \( D \) of conductive materials used in the construction of power cables is readily available, the principal unknown with regard to the evaluation of equation (17) is the characteristic Power Spectrum. There is no unifying scaling law for the power spectrum associated with harmonic distortion in power cables and the power spectrum can change from one built environment to another depending on the loads and appliances. For example, Figure 7.8 shows the AC wave form and the corresponding harmonic distortion associated with a single-phase rectifier [5]. Power spectra of this type generate additional heating due to both the skin and proximity effects causing cables to operate at a higher temperature than would be the case without harmonic distortion. The degree of harmonic distortion in the load current determine the degree of additional heating that can occur to which suitable de-rating corrections factor can be applied [43].

![Figure 7.8: Wave Form and Harmonic Current Spectrum for a Single-phase Rectifier.](image)

Figure 7.8: Wave Form and Harmonic Current Spectrum for a Single-phase Rectifier.
7.10 Discussion

The phenomenon of both the skin and proximity effect, although recognised as reducing the ampacity of cables, has not yet evolved into a set of de-rating tables that can be easily applied on a day to day basis in engineering design. The problem in quantifying harmonic heating effects is that they are a function of frequency. The greater the harmonic distortion present, the larger the number of harmonics present. Each harmonic current generates its own individual heating effect and thus, a harmonic rating factor has to be taken into account for a large number of individual elements. Further, in general, proximity effects tend to be understated because the effect on extraneous metalwork including metal enclosures such as cable trays and metal cladding on cables, has, to date, not been fully considered either experimentally or in the Standard International Electrotechnical Commission 60287-1-1. It is for this reason that, the model considered in the chapter has been developed.

7.11 Conclusions

The model developed in this chapter is based on decoupling Maxwell’s equations through application of a Gauge transformation as detailed in Appendix B. The method of solution is based on a Green’s function solution for the Magnetic Vector Potential given the current density and the application of Ohm’s law. Under this condition the charge density is zero and application of the homogenous boundary conditions on the surface of a conductor allows equation (11) to be derived which shows that the proximity temperature is proportional to the square frequency. With regard to the development of a proximity temperature simulation, the key to the approach taken is to consider the Green’s function solution in the Fresnel zone which yields a convolution based model compounded in equation (12).

The simulations are similar in both two- and three-dimensions and in both cases can be
used to investigate the proximity heating effects generated at different harmonics. Higher harmonic effects are particularly important in the design of cable arrays and component structures but relies on knowledge of the spectrum of the harmonic components present in the wave form of an AC load current which must be determined experimentally, \textit{a priori}. Note that in the example simulations presented, the primary current density \( j \) is taken to be uniformly distributed which, in general, will not be the case due to (self-induced) skin effects in the primary conductor(s). Furthermore, the proximity conductivity \( \sigma_p \) may be non-uniform and the magnetic permeability \( \mu \) may also be expected to change for different elements of a power cable and proximity components. However, within the context of the model presented, these physical aspects are readily incorporated.

One of the goals of the simulator developed is to evaluate the possible presence of ‘hotspots’ in the design of cable configurations and proximal structures for harmonics that are present in the load. Although proximity temperature effects are subject to thermal diffusion, proximity related hotspots represent a continuous source of heat and can therefore be evaluated independently of thermal diffusion processes. As these effects are also present in electrical machines and equipment used in the built environment, the simulator in addition to determining harmonic de-rating factors for power cables arranged in different layouts and combinations will provide a powerful design tool for a diverse range electrical equipment and machines which contain non-trivial conductor topologies and magnetic circuits.
Chapter 8

Conclusion and Future Research

8.1 Summary

The thesis set out to examine the heating effects through harmonics on electric cables in the built environment and in particular how harmonic loads impact on the current carrying capacity of electric cables. The phenomenon of skin effect and proximity effect although recognized as reducing the ampacity of cables had not evolved into a set of de-rating tables that could be easily applied on a day to day basis in engineering design offices. The research examined existing methods of determining the ampacity of cables and proposes a new harmonic de-rating algorithm that may be applied to cables supplying harmonic loads.

8.1.1 Harmonic Standards

Harmonic standards have evolved out of the necessity to control and manage harmonics in AC power systems. Poor power quality is estimated to cost billions of euros each year in the EU and much of the problem can be attributed to harmonic distortion. There are three widely accepted standards in use: The IEC 61000 series produced by the International Electrotechnical Commission based in Geneva; the Engineering Regulations G5/4-1...
produced in the UK and the Standard 519-1992 which originated in the US. The standards set limits on supply voltage harmonic distortion; the load current harmonic distortion in electrical installations together with electromagnetic compatibility and emissions levels for equipment connected to the supply. They also set out a methodology for measuring harmonic distortion and identifying sources in a system.

Planning, compatibility and immunity levels are set so that designers of electrical installations can seek to achieve an electromagnetic environment within which electrical equipment with the appropriate compatibility level will be able to operate correctly. There is equally a requirement placed on manufacturers of equipment to ensure that the emission levels of their equipment are such that electromagnetic interference with other equipment used in the environment will not occur. The existence of these standards is also intended to promote the free movement of goods between countries worldwide.

The harmonic standards are issued as guidelines and should be interpreted taking local conditions into account. The final decision for connecting an installation to a network is taken by the Network Operator who in negotiation with the user may require the implementation of harmonic mitigation measures as a condition for supply. Depending on the characteristics of the network, derogation from some aspects of the standards may be granted by mutual agreement between the user and the network operator.

The EU issued an Electromagnetic Compatibility Directive which came fully into effect in 1996. The Directive seeks to remove technical barriers to trade by requiring equipment to operate satisfactorily in its specified electromagnetic environment. National governments are required by the Directive to enact laws to have harmonized standards at European level. SI No 22/1998 European Communities (Electromagnetic Compatibility) Regulations 1998 gave legal effect to the Directive in Ireland and Engineering Recommendation G5/4-1 came into force in the UK in 2005. The legally enforcing document is the connection agreement between the Network Operator and the customer. The agreement lays down connection conditions which require compliance with the Standards and include
any derogation and/or harmonic mitigation measures which may be agreed between the network operator and the customer.

8.1.2 Spectral Analysis

Spectral analysis is the analytic tool of choice for assessing harmonic distortion. The Fourier series and Fourier Transform are coupled via the Discrete Fourier Transform which forms the basis for the harmonic analysis carried out in the thesis. The object is to represent a function in terms of an infinite series such that the representation of the function can be manipulated and analysed and the coefficients of the series evaluated easily. The basic (complex) Fourier series

\[ f(t) = \sum_n c_n \exp(int) \]

provides one of the most versatile representations of signals with a finite period.

The Discrete Fourier Transform (DFT) is commonly used for processing digital signals and harmonic analysis in power systems. It is used for Fourier analysis of finite-domain (or periodic) discrete-time functions. It can be implemented in computers by using the fast Fourier transform algorithms. The DFT was introduced to implement Fourier analysis carried out by computer where the signal is discrete and has finite length.

8.1.3 Harmonics in the Industrial Environment

Harmonics were seen to be present in virtually every item of plant and equipment that used electronic components. Solid state power electronics such as pulse rectifiers, variable speed drives, un-interruptible power supplies and equipment using switched mode power supplies were major sources of harmonics. Discharge lighting, electric motors, transformers and reactors were also significant contributors to harmonic distortion.

Harmonics were seen to have detrimental effects on the performance of electric cables, plant and machinery invariably causing loss of efficiency, overheating, reduced reliability
and service life. Sensitive measuring instruments were seen to produce erroneous readings and protective electronic relays produce nuisance tripping in circuit breakers, etc. These effects could be limited by the use of passive and active harmonic filters.

8.1.4 The Effect of Harmonics on Individual Electrical Components and System Power Factor

Harmonics react with discrete electrical components such as the capacitor and inductor to produce power system resonance at specific harmonic frequencies. This resonance could lead to energy oscillations within the system causing energy loss, overheating of components and ultimately failure of supply. In addition power system resonance can cause severe voltage distortion in the supply and breach the electromagnetic compatibility limits for equipment used in the system giving rise to equipment malfunction in certain cases. The presence of harmonics causes the power factor to reduce significantly. This may trigger an increase in electricity tariffs leading to increased operating costs for the consumer.

8.1.5 Cable Heating Effects due to Harmonic Distortion in Electrical Installations

Electric cables are subject to overheating due to the presence of triple-n harmonics, skin and proximity effects. A variety of correction factors are available to electrical designers to take account of variables such as ambient temperature; grouping of cables; embedded in thermal insulation, type of protective device, etc. In 2008 BS7671 included correction factors for the presence of triple-n harmonics however there is not a single correction factor offered yet to take account of skin and proximity effects. A mathematical model is developed for predicting the final operating temperatures for cables under overload conditions together with an algorithm for calculating a harmonic de-rating factor.
8.1.6 Harmonic Distortion in Electrical Installations

The increasing use of non-linear loads in electrical installations has exacerbated the problems of harmonic distortion in industrial and commercial electrical systems.

Quantifying the heating effects due to proximity and skin effects in a harmonically rich environment has proved to be complex and to a large extent has been ignored because of the difficulty in determining a suitable de-rating factor. The problem is usually side-stepped by oversizing the power cables on an ad hoc basis which has the added positive side effects of the circuit remaining safe in the event of a failure of the harmonic mitigation measures; reducing voltage drop, reducing cable losses, reducing running costs, increasing cable life expectancy, etc.

8.1.7 Proximity Heating Effects in Power Cables

A model was developed based on decoupling Maxwell’s equations through application of a Lorentz gauge transformation. The method of solution is based on a Green’s function solution for the Magnetic Vector Potential given the current density and the application of Ohm’s law. The key to the approach taken is to consider the Green’s function solution in the Fresnel zone.

The simulator developed works on two- and three-dimensions and in both cases can be used to investigate the proximity heating effects generated at different harmonics. One of the goals of the simulator developed is to evaluate the possible presence of ‘hotspots’ in the design of cable configurations and proximal structures for harmonics that are present in the load.

As these effects are also present in electrical machines and equipment used in the built environment, the simulator in addition to determining harmonic de-rating factors for power cables arranged in different layouts and combinations will provide a powerful design tool for a diverse range electrical equipment and machines which contain non-trivial
8.2 Review of Experimental Work

8.2.1 Harmonic Signatures

The harmonic signatures for a number of common devices showed that the harmonic content was very significant with personal computers 90% THD; Energy Saver lamps 80% THD; fluorescent lamps 60% THD; 6-pulse bridge rectifiers 90% THD.

8.2.2 Principal Experimental Results

Development of the Experimental Test Bed

The traditional difficulty in measuring the thermal effects of harmonic currents was mainly that of access. Mains power systems by their very nature carried a lot of danger with them due to risk of electrocution or injury through short-circuit. The conductors operating in live systems were naturally operating at live mains voltage and so could not be touched by humans. Equally, the conductors were surrounded by electrical insulation and mechanical protection so it was difficult if not impossible to get access to the live conductor in order to measure its temperature. This was one of the problems to be overcome in designing the experimental test bed to carry out the necessary measurements.

The experiment made use of an industrial current transformer operating at a ratio of 100 : 1 i.e. one hundred amps in the primary producing one amp in the secondary. By reversing the normal configuration and injecting current into the secondary, a primary current of 100 times the secondary current could be induced. As this did not involve the use of high voltage, the health and safety issues were now removed.

A signal generator connected to a 400 W stereo power amplifier was now used to inject current at discrete harmonic frequencies into the secondary of the current transformer.
A 2m length of 185$^2$ mm aluminium cable connected to the primary now carried the amplified current flowing at a frequency set by the signal generator. Bead thermocouples were inserted into the conductor to directly measure the conductor temperature thus overcoming the difficulty of the insulation masking the temperature of the conductor. The thermocouples were connected to PicoLog TC-08 data logger. Four channels were used to measure and log ambient temperature, single conductor, double conductor and sheath temperature. The resolution on the data logger was to two decimal places. The data logger was connected to a laptop and the data stored on a spreadsheet.

**Validation of the Experimental Test Bed**

BS 7671 quote a rated current for 185$^2$ mm aluminium cable of 394 Amps with a design temperature of 70°C when the conductor is suspended in free air with an ambient temperature of 30°C. The mathematical model developed in this thesis predicted an increase in conductor temperature of 40°C above ambient for full load current.

The results required normalization to allow for the ambient temperature of the laboratory where the test was conducted which was approximately 22°C. This required an adjustment of approximately 8°C which, when factored into the results, produced the graph of conductor temperature Vs load current that is displayed in Figure 6.5. The results demonstrate that (a) the mathematical model was correct and (b) the operating temperature of the conductor coincided with the tabulated value given by BS 7671.

**Measurement of Heating Effect due to Skin and Proximity Effects**

Full load rated current was now injected into the test conductor at frequencies of 50Hz, 150Hz, 250Hz and so on up to 1050Hz and the temperature of the conductor was recorded by the data logger. It was discovered that, at the higher frequencies, there was difficulty in generating the full load rated current. This was due to the impedance of the current transformer increasing as a function of frequency. This difficulty was overcome by intro-
ducing different size capacitors in series with the current transformer in order to generate resonant conditions and counteract the increase in inductive reactance. The data was normalized to adjust for variation in ambient temperature and small variations in the load current. The relationship between conductor temperature and frequency was seen to be linear for both skin and skin + proximity effects. The graph depicting this relationship is shown in Figure 6.6. Given that the temperature rise is proportional to AC resistance it is possible to compute the per unit variation in AC resistance over the range of frequencies tested. The Standard IEC 60287-1-1 1994 Electric Cables Calculation of the Current Rating also provides data from which the AC resistance factors for skin and proximity effects can be calculated. A comparison of the AC resistance factors calculated from experimental data and those calculated from data provided by IEC 60287-1-1 is available in Figure 8.1. It can be seen that there a reasonable close correlation between the skin effect computed using experimental data and the data provided by IEC 60287-1-1. However, there is a large difference between the two when the proximity effect is added in.

Figure 8.1: Comparison of Resistance Factors Calculated from Experimental Data and Data Provided by IEC 60287-1-1.
Validation of the Harmonic De-rating Algorithm

The experimental test bed was used to introduce a complex wave into the test conductor. In this set-up, a harmonically rich load comprising of a bank of low energy CFL lamps was supplied through a mains isolating transformer. The load was supplied through a non-inductive resistor and the potential difference across this NIR was applied to the input of the power amplifier. Thus the complex current wave was injected into the test cable. A second set of data was collected by adding a set of capacitors in series with the secondary of the current transformer which had the effect of allowing higher order harmonics in the load to be amplified. It was possible to achieve several sets of readings in this way. The harmonic de-rating algorithm was applied and the temperature rise in the test conductor was seen to coincide with the predicted value. It was thus concluded that the proposed algorithm was accurate and could be used to calculate Harmonic Derating Factors for cables.

8.3 Future Work and Research Directions

8.3.1 Investigation of Economic Sizing of Cables under Harmonic Distortion

The normal standard requires that cables are selected on the basis of the maximum thermal limits of the cable. If an economic model was applied taking into account the lifetime costs Vs the installation costs it would be found that the optimum size of a cable would be several sizes larger. Harmonic distortion has a very significant effect on power factor which in the case of a 6-pulse diode rectifier can cause an increase of 41% in the RMS current. The consequent increase in the heating effect $\propto I^2$ which in this case is $1.41^2 = 2$. Added to this the additional heating effect of current flowing in the neutral conductor and the skin and proximity effects the energy losses could be in excess of three
times the normal cable losses. Thus the economic size of the cable would be larger still.

8.3.2 Open Problems

Open problem associated with future directions stimulated by the research reported in this thesis include the following:

- Investigation of the phenomenon of Inter and Sub Harmonics in Electrical Power Systems Causes and Effects
- Investigation of Electromagnetic Interference in the Built Environment
- Further Verification of the Proximity Effect

In particular, further research is required to experimentally verify the proximity effect and how it impacts on the ampacity of cables. There is a disparity between the IEC projection and that of the experimental data generated for the specific cable studied in this thesis. The overall proximity effect would appear to be understated because the proximity effect due to extraneous metalwork such as structural steel, metal enclosures and metal cladding on cables was not considered either in the experiments carried out or in the Standard IEC 60287-1-1.
Appendix A: Overview of the Theory of Spectral Analysis

In this Appendix, the Fourier series and Fourier transform are considered in an integrated form. Both the Fourier series and the Fourier transform are intimately related and coupled via the Discrete Fourier Transform which is also presented in this appendix, thereby providing the essential theoretical background to the harmonic analysis considered in this thesis\(^1\).

The Fourier series provides a way of representing a signal in terms of its Fourier coefficients and after discussing the method of representing a signal as a Fourier series, the ‘classical’ Fourier transform is derived which is then explored further (in terms of the generalized Fourier transform).

Many signals are periodic but the period may approach infinity (when the signal is, in effect, non-periodic). Approximations to such signals are important for their analysis and many possibilities exist. The principal idea is to represent a function in terms of some (infinite) series. The problem is to find a series that provides:

- a useful representation of the function that can manipulated and analysed accord-

ingly;

- a series whose component parts (coefficients) can be evaluated relatively easily.

We could, for example, consider a simple power series

\[ f(t) = \sum_{n} a_n t^n, \]

or a Taylor series

\[ f(t) = \sum_{n} \frac{(t-a)^n}{n!} f^{(n)}(a), \]

but the Fourier series

\[ f(t) = \sum_{n} c_n \exp(int), \]

which is written here in complex form, provides one of the most versatile representations of signals with a finite period.

### A.1 Derivation of the Fourier Series

The Fourier series is a trigonometrical series that can be used to represent almost any function \( f(t) \) in the range \( -\pi \leq t \leq \pi \). Outside this range, the series gives a periodic extension of \( f(t) \) with period \( 2\pi \), i.e.

\[ f(t + 2\pi) = f(t). \]

Such a series is useful when:

1. \( f(t) \) is already periodic;

2. \( f(t) \) is not periodic but only exists in the range \( [-\pi, \pi] \).

Let us consider the trigonometrical Fourier series, given by

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt). \quad (3.1.1) \]
Our problem now, is to find expressions for the coefficients $a_n$ and $b_n$.

**Computation of $a_0$:** Integrate both sides of equation (3.1) between $-\pi$ and $\pi$ giving

$$\int_{-\pi}^{\pi} f(t) dt = \int_{-\pi}^{\pi} \frac{a_0}{2} dt = \pi a_0,$$

since all integrals of the type $\int_{-\pi}^{\pi} \sin(nt) dt$ and $\int_{-\pi}^{\pi} \cos(nt) dt$ are zero. Hence

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt.$$

**Computation of $a_n$:** Multiply both sides of equation (3.1) by $\cos(kt)$ where $k$ is an integer ($k = 1, 2, 3, ...$) and integrate between $-\pi$ and $\pi$ giving

$$\int_{-\pi}^{\pi} f(t) \cos(kt) dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(kt) dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nt) \cos(kt) dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nt) \cos(kt) dt.$$

Now,

$$\int_{-\pi}^{\pi} \sin(nt) \cos(kt) dt = 0 \quad \forall \quad n \text{ and } k.$$

Hence, all terms involving $b_n$ are zero. Also,

$$\int_{-\pi}^{\pi} \cos(nt) \cos(kt) dt = \begin{cases} 0, & n \neq k; \\ \pi, & n = k. \end{cases}$$

This property illustrates that the functions $\cos(nt)$ and $\cos(kt)$ are ‘orthogonal’. Also, since

$$\int_{-\pi}^{\pi} \cos(kt) dt = 0, \quad \int_{-\pi}^{\pi} f(t) \cos(nt) dt = a_n \pi$$

150
or

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt. \]

**Computation of** \( b_n \): Multiply both sides of equation (3.1) by \( \sin(kt) \) and integrate over \( t \) between \( -\pi \) and \( \pi \). Using the results

\[ \int_{-\pi}^{\pi} \sin(kt) dt = 0, \quad \int_{-\pi}^{\pi} \cos(nt) \sin(kt) dt = 0 \]

and

\[ \int_{-\pi}^{\pi} \sin(nt) \sin(kt) dx = \begin{cases} 0, & n \neq k; \\ \pi, & n = k. \end{cases} \]

we obtain

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt. \]

Note that these results have been obtained by exploiting the orthogonality of the trigonometrical functions \( \sin \) and \( \cos \). This provides a Fourier series representation of a function in the range \( -\pi \leq t \leq \pi \) that can be written out as

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \]

where

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad n = 1, 2, 3, \ldots, \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, \quad n = 1, 2, 3, \ldots. \]
A standard but important example of the Fourier series is the Fourier series representation of a ‘square wave’ signal given by

$$f(t) = \begin{cases} 
-1, & -\pi \leq t < 0; \\
1, & 0 \leq t \leq \pi.
\end{cases}$$

where \(f(t + 2\pi) = f(t)\). In this case,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \int_{0}^{0} (-1) dt + \frac{1}{\pi} \int_{0}^{\pi} 1 dt = 0,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \int_{-\pi}^{\pi} (-1) \cos(nt) dt + \int_{0}^{\pi} \cos(nt) dt$$

$$= \left[ -\frac{\sin(nt)}{n} \right]_{-\pi}^{0} + \left[ \frac{\sin(nt)}{n} \right]_{0}^{\pi} = 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = \int_{-\pi}^{\pi} (-1) \sin(nt) dt + \int_{0}^{\pi} \sin(nt) dt$$

$$= \left[ -\frac{\cos(nt)}{n} \right]_{-\pi}^{0} + \left[ -\frac{\cos(nt)}{n} \right]_{0}^{\pi} = \frac{1}{n} [1 - \cos(-n\pi)] - \frac{1}{n} [\cos(n\pi) - 1] = \frac{2}{n} [1 - \cos(n\pi)].$$

Now, since

$$\cos(n\pi) = \begin{cases} 
1, & n \text{ even}; \\
-1, & n \text{ odd}.
\end{cases} \quad \text{or} \quad \cos(n\pi) = (-1)^n, \quad n = 1, 2, \ldots$$

we can write

$$b_n = \frac{2}{\pi n} [1 - (-1)^n]$$

and hence, \(f(t)\) for the square wave signal in which \(f(t) = f(t + 2\pi)\) is given by

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(nt) = \frac{4}{\pi} \left( \frac{\sin t}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots \right).$$

Like any series representation of a function, we may need many terms to get a good approximation, particularly if the function it describes has discontinuities - ‘jump’ type and other
‘sharp’ and/or ‘spiky’ features. Note that the series goes to zero at \( t = 0, \pm \pi, \pm 2\pi, \ldots \) where \( f(t) \) has discontinuities. The term \( \frac{4}{\pi} \sin t \) is the ‘fundamental’ frequency and the other terms are the harmonics. Each term of the series represents the harmonic components required to describe a square wave. The values \( 1, 3, 5, \ldots \) determine the frequency at which these harmonics oscillate (which becomes increasingly large) and the values \( 1, \frac{1}{3}, \frac{1}{5}, \ldots \) determine the amplitudes of these oscillations (which become increasingly small). Note that this square wave is ‘odd’, i.e. \( f(-t) = -f(t) \) and that \( \sin(nt) \) is also odd, so only sine terms occur in the Fourier series. Observing this result saves the inconvenience of finding that the cosine terms are all zero. Similarly, an even function, where \( f(-t) = f(t) \), only has cosine terms. Thus, if \( f(-t) = f(t) \) we need only compute \( a_n \) and if \( f(-t) = -f(t) \) we need only compute \( b_n \).

**The Half Range Fourier Series**

If we require a series representation for \( f(t) \) in the range \( 0 \leq t \leq \pi \) rather than in the range \( -\pi \leq t \leq \pi \), then we can choose either a sine or a cosine Fourier series.

**Cosine Series**

Define a new function \( g(t) \) where \( g(t) = f(t) \), \( 0 \leq t \leq \pi \) and \( g(t) = f(-t) \), \( -\pi \leq t \leq 0 \). Since \( g(-t) = g(t) \), \( g(t) \) is even. Hence, the Fourier series has only cosine terms and

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt) dt = \frac{1}{\pi} \int_{-\pi}^{0} f(-t) \cos(nt) dt + \int_{0}^{\pi} f(t) \cos(nt) dt
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} f(t) \cos(nt) dt + \int_{0}^{\pi} f(t) \cos(nt) dt = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos(nt) dt, \quad n = 0, 1, 2, \ldots
\]

**Sine Series**

Define \( g(t) \) so that \( g(t) = f(t) \), \( 0 \leq t \leq \pi \) and \( g(t) = -f(-t) \), \( -\pi \leq t \leq 0 \). In this case, \( g(-t) = -f(t) = -g(t) \) so \( g(t) \) is odd. Thus, the series has only sine terms, the
coefficients being given by
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt) dt = -\frac{1}{\pi} \int_{-\pi}^{0} f(-t) \sin(nt) dt + \int_{0}^{\pi} f(t) \sin(nt) dt \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} f(t) \sin(nt) dt + \int_{0}^{\pi} f(t) \sin(nt) dt = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin(nt) dt \]

A.2 Fourier Series for an Arbitrary Period

The analysis in the previous section was based on a function with a period of 2\(\pi\) (or \(\pi\) in the case of the half. If we consider an arbitrary value for this period of 2\(T\) say, then
\[ f(t) = f(x + 2T) \]
and by induction, based on the results and approach covered previously, we have
\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi nt/T) + \sum_{n=1}^{\infty} b_n \sin(\pi nt/T) \]
where
\[ a_n = \frac{1}{T} \int_{-T}^{T} f(t) \cos(n\pi t/T) dt \]
and
\[ b_n = \frac{1}{T} \int_{-T}^{T} f(t) \sin(n\pi t/T) dt. \]

A.3 The Complex Fourier Series

A complex Fourier series performs a similar role to the trigonometrical Fourier series although its implementation is easier and more general. It is just one of a number of linear polynomials which can be used to ‘model’ a piecewise continuous function \(f(t)\). In general, we can consider
\[ f(t) = \sum_{n} c_n B_n(t) \]
where $B_n(t)$ are the basis function and $c_n$ are the coefficients (complex or otherwise). A complex Fourier series is one in which the basis functions are of the form

$$B_n(t) = \exp(int).$$

This series is basic to all Fourier theory and is used to model signals that are periodic. The problem is then reduced to finding the coefficients $c_n$.

Consider the complex Fourier series for an arbitrary period $2T$ given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(i n t \pi / T).$$

Observe that the summation is now over $(-\infty, \infty)$. To find the coefficients $c_n$, we multiply both sides of the equation above by $\exp(-im t \pi / T)$ and integrate from $-T$ to $T$, thus,

$$\int_{-T}^{T} f(t) \exp(-im t \pi / T) dt = \sum_{n=-\infty}^{\infty} c_n \int_{-T}^{T} \exp[i(n-m)t \pi / T] dt.$$

Now, the integral on the right hand side is given by

$$\frac{2T \sin \pi(n-m)}{\pi(n-m)} = \begin{cases} 2T, & n = m; \\ 0, & n \neq m. \end{cases}$$

Note that,

$$\frac{\sin t}{t} = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots$$

so that

$$\left[ \frac{\sin t}{t} \right]_{t=0} = 1.$$

Thus, all terms on the right hand side vanish except for the case when $n = m$ and we can therefore write

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(t) \exp(-int \pi / T) dt.$$
By way of an example, suppose we require the complex Fourier series for a square wave signal with period of $2\pi$ given by

$$f(t) = \begin{cases} 
-1, & -\pi \leq t < 0; \\
1, & 0 \leq x \leq \pi.
\end{cases}$$

where \(f(t + 2\pi) = f(t)\), then

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-int) dt = -\frac{1}{2\pi} \int_{-\pi}^{0} \exp(-int) dt + \frac{1}{2\pi} \int_{0}^{\pi} \exp(-int) dt$$

$$= \frac{1}{in\pi} [1 - \cos(n\pi)]$$

and

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{in\pi} [1 - \cos(n\pi)] \exp(in\pi)$$

$$= \sum_{n=0}^{\infty} \frac{1}{in\pi} [1 - (-1)^n] \exp(in\pi) - \sum_{n=0}^{\infty} \frac{1}{in\pi} [1 - (-1)^n] \exp(-in\pi)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(nt).$$

which recovers the result obtained using the trigonometrical Fourier series (but with less computation). As mentioned before, with either the trigonometrical or complex Fourier series, we need many terms to get a good fit to the sharp features and discontinuities. The Fourier series representation of an ‘on-off’ type signal such as a square wave requires many terms to represent it accurately. Truncation of the series leads to truncation errors which in a Fourier series is generally referred to as ‘ringing’. The generalization of this effect is called the Gibbs’ phenomenon. This leads to a general rule of thumb which is an important aspect of all signal processing, namely, that ‘sharp’ features in a signal require many Fourier coefficients to be represented accurately whereas smooth features in a signal require fewer Fourier coefficients. Hence, one way of ‘smoothing’ a signal is to reduce the number of Fourier coefficients used to represent the signal. This is the basis of lowpass
filtering. Moreover, if a signal is relatively smooth, it may require relatively few Fourier coefficients to reconstruct it accurately. In such cases, storing the coefficients $c_n$ instead of the (digital) signal itself can lead to a method of reducing the amount of data required to store the (digital) signal. This approach to data compression is actually applied in practice using the Discrete Cosine Transform (DCT) and is the basis for the Joint Photographics Experts Group or JPEG compression scheme. The DCT is used because it has properties that are optimal in terms of using it to design a data compression algorithm, i.e. in expressing a digital signal in terms of its DCT coefficients and reproducing it from them. However, the principal ‘philosophy’ behind this approach is the same as that discussed above. Actually, the Discrete Fourier Transform or DFT, which is discussed shortly, can also be used for this purpose. It is not such a good compressor as the DCT (because it is a complex transform with both real and imaginary parts), but it does provide the option of processing the data in compression space using the Fourier amplitude and phase which the DCT does not provide.

A.4 The Fourier Transform Pair

It is prudent at this stage to derive the Fourier transform pair (i.e. the forward and inverse Fourier transforms) from the complex Fourier series. To do this in a way that is notationally consistent, we let $c_n = F_n/2T$ so that

$$f(t) = \frac{1}{2T} \sum_n F_n \exp(int\pi/T)$$

and

$$F_n = \int_{-T}^{T} f(t) \exp(-int\pi/T) dt$$

where

$$\sum_n \equiv \sum_{n=-\infty}^{\infty}.$$
Now, let $\omega_n = n\pi/T$ and $\Delta\omega = \pi/T$. We can then write

$$f(t) = \frac{1}{2\pi} \sum_n F_n \exp(i\omega_n t) \Delta\omega_n$$

and

$$F_n = \int_{-T}^{T} f(t) \exp(-i\omega_n t) dt.$$

Then, in the limit as $T \to \infty$, we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega,$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt.$$

Here, $F(\omega)$ is the Fourier transform of $f(t)$ and $f(t)$ is the inverse Fourier transform of $F(\omega)$. Taken together, these integral transforms form the ‘Fourier transform pair’. Note that $f(t)$ is a non-periodic function, since we have used $T \to \infty$ to obtain this result.

### A.5 The Discrete Fourier Transform

The discrete Fourier transform or DFT is the ‘work horse’ for so many of the routine algorithms used for processing digital signals and is the basis of a fast algorithm for computing the DFT. For now, it is useful and informative to demonstrate the derivation of the DFT from the complex Fourier series.

The complex Fourier series can be written as

$$f(t) = \frac{1}{2T} \sum_n F_n \exp(\pi nt/T)$$

where

$$F_n = \int_{-T}^{T} f(t) \exp(-\pi nt/T) dt.$$
over the range $[-T, T]$. The DFT can now be derived by considering a discretized form of the function $f(t)$ with uniform sampling at points $t_0, t_1, t_2, ..., t_{N-1}$ giving the discrete function or vector

$$f_m \equiv f(t_m); \ m = 0, 2, 3, ..., N - 1$$

with sampling interval $\Delta t$. Now, $t_m = m\Delta t$ and if we let $N = T/\Delta t$, we have

$$f_m = \frac{1}{N} \sum_n F_n \exp(i2\pi nm/N),$$

$$F_n = \sum_m f_m \exp(-i2\pi nm/N).$$

Note that the summations are now finite with $n$ and $m$ running from $-N/2$ to $(N/2) - 1$ or alternatively from $n = 0$ to $N - 1$.

**Relationship between the DFT and the Fourier Transform**

Consider the Fourier transform pair given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega$$

and the DFT pair, i.e.

$$F_n = \sum_m f_m \exp(-i2\pi nm/N),$$

$$f_m = \frac{1}{N} \sum_n F_n \exp(i2\pi nm/N).$$

To study the relationship between these two results, we can consider the following discretization of the Fourier transform pair:

$$F(\omega_n) = \sum_m f(t_m) \exp(-i\omega_n t_m) \Delta t,$$
\[ f(t_m) = \frac{1}{2\pi} \sum_n F(\omega_n) \exp(i\omega_n t_m) \Delta \omega \]

where \( \Delta t \) and \( \Delta \omega \) are the sampling intervals. Writing \( \omega_n = n \Delta \omega \) and \( t_m = m \Delta t \), by inspection (i.e. comparing the results above with the DFT pair) we see that

\[ \Delta \omega \Delta t = \frac{2\pi}{N}. \]

This result provides the relationship between the (sampling) interval \( \Delta \omega \) of the numbers \( F_n \) and the (sampling) interval \( \Delta t \) of the numbers \( f_m \).

‘Standard’ and ‘Optical’ Forms of the DFT

In the previous section, the limits on the sums defining the DFT have been assumed to run from \(-N/2\) to \((N/2) - 1\) assuming the data is composed of \(N - 1\) elements. When we consider the case where

\[ \sum_n \equiv \sum_{n=-N/2}^{(N/2)-1} \]

the DC (Direct Current) or zero frequency level is taken to occur at the centre of array \( F_m \) giving what is termed the optical form of the DFT. In the case when

\[ \sum_n \equiv \sum_{n=0}^{N-1} \]

the DC level is taken to occur at \( F_0 \) - first value of array \( F_m \). This is known as the standard form of the DFT.

The optical form has some valuable advantages as it provides results that are compatible with those associated with Fourier theory in which the spectrum \( F(\omega) \) is taken to have its DC component at the centre of the function. The reason for calling this form of the DFT ‘optical’ is that there is an analogy between this form and that of the 2D DFT in which the DC component occurs at the centre of a 2D array. In turn the 2D Fourier transform can be used to model the process of a well corrected lens focusing light on to the ‘focal plane’ in which the zero frequency occurs in the centre of this plane. The
standard form of the DFT is often useful for undertaking analytical work with the DFT and, in particular, developing the Fast Fourier Transform algorithm.

A.6 The Fourier Transform

The Fourier transform is used extensively in many branches of science and engineering. It is particularly important in signal processing and forms the ‘work horse’ of many methods and algorithms. This appendix provides an introduction to this transform and presents most of the fundamental ideas, results and theorems that are needed to use Fourier theory for the analysis of signals.

Notation

The Fourier transform of a function $f$ is usually denoted by the upper case $F$ but many authors prefer to use a tilde above this function, i.e. to denote the Fourier transform of $f$ by $\tilde{f}$. In this work, the former notation is used throughout. Thus, the Fourier transform of $f$ can be written in the form

$$F(\omega) \equiv \hat{F}_1 f(t) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

where $\hat{F}_1$ denotes the one-dimensional Fourier transform operator. Here, $F(\omega)$ is referred to as the Fourier transform of $f(t)$ where $f(t)$ is a non-periodic function.

The sufficient condition for the existence of the Fourier transform is that $f$ is square integrable, i.e.

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty.$$  

Physical Interpretation

Physically, the Fourier transform of a function provides a quantitative picture of the frequency content of the function which is important in a wide range of physical problems.
and is fundamental to the processing and analysis of signals and images. The variable $\omega$ has dimensions that are reciprocal to those of the variable $t$. There are two important cases which arise:

1. $t$ is time in seconds and $\omega$ is the angular velocity which is given by $2\pi \times \nu$ where $\nu$ is the frequency.

2. $t$ is distance in metres (usually denoted by $x$) and $\omega$ and the spatial frequency in cycles per metre (usually denoted by $k$). Here, $k$ is known as the wavenumber and is given by

$$k = \frac{2\pi}{\lambda}$$

where $\lambda$ is the wavelength and we note that

$$c = \frac{\omega}{k} = \nu \lambda$$

where $c$ is the wavespeed.

The Fourier transform is just one of a variety of integral transforms but it has certain properties which make it particularly versatile and easy to work with. This was expressed eloquently by Lord Kelvin, who stated that:

*Fourier’s theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.*

As discussed at the beginning of this appendix, it is interesting to note that this important transform arose from a scientist having to contemplate a technical problem that was directly related to a military issue - such is the nature of so many aspects of modern science and engineering!
The Spectrum

The Fourier transform of a function is called its ‘spectrum’ or frequency distribution - a
term that should not be confused with that used in statistics. It is generally a complex
function which can be written in the form

\[ F(\omega) = F_r(\omega) + iF_i(\omega) \]

where

\[ F_r = \text{Re}[F] \quad \text{and} \quad F_i = \text{Im}[F], \]

i.e. the real and imaginary parts of the spectrum respectively. Note that if \( f(t) \) is a real
valued function, then the real and imaginary parts of its Fourier transform are given by

\[ F_r(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) \, dt \]

and

\[ F_i(\omega) = \int_{-\infty}^{\infty} f(t) \sin(\omega t) \, dt \]

respectively. An alternative and often more informative (Argand diagram) representation
of the spectrum is based on writing it in the form

\[ F(\omega) = A(\omega) \exp[i\theta(\omega)] \]

where

\[ A(\omega) = |F(\omega)| = \sqrt{F_r^2(\omega) + F_i^2(\omega)} \]

and

\[ \theta(\omega) = \tan^{-1} \left( \frac{F_i(\omega)}{F_r(\omega)} \right). \]

The functions \( F, A \) and \( \theta \) are known as the complex spectrum, the amplitude spectrum
and the phase spectrum respectively. In addition to these functions, the function

\[ A^2(\omega) = |F(\omega)|^2 \]
is also important in Fourier analysis. This function is known as the Power Spectral Density Function (PSDF) or just the power spectrum. Finally, the value of the spectrum at $\omega = 0$ is called the zero frequency or DC (after Direct Current) level and is given by the integral of $f$, i.e.

$$F(0) = \int_{-\infty}^{\infty} f(t)dt.$$ 

Note that this value provides a measure of the scale of the Fourier transform and is proportional to its mean value.

### A.7 The Inverse Fourier Transform

The function $f(t)$ can be recovered from $F(\omega)$ by employing the inverse Fourier transform which is given by

$$f(t) = \hat{F}^{-1} \hat{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega.$$ 

The operator $\hat{F}^{-1}$ is used to denote the inverse Fourier transform. The superscript $-1$ is used to denote that this operator is an inverse operator (N.B. it does not mean $1/\hat{F}_1$).

Previously in this appendix, the inverse Fourier transform was derived from the complex Fourier series in a way that is often referred to as the ‘classical approach’ and is informative in terms of detailing the relationships between the complex Fourier series, the Fourier transform pair and the discrete Fourier transform pair. However, ‘armed’ with the ‘power’ of the $\delta$-function, we can derive this result in an arguably more elegant way; thus, multiplying $F(\omega)$ by $\exp(i\omega t')$ and integrating over $\omega$ from $-\infty$ to $\infty$ we can write

$$\int_{-\infty}^{\infty} F(\omega) \exp(i\omega t') d\omega = \int_{-\infty}^{\infty} dt f(t) \int_{-\infty}^{\infty} \exp[i\omega(t' - t)] d\omega.$$ 

We now employ the integral representation for the delta function discussed, i.e.

$$\int_{-\infty}^{\infty} \exp[i\omega(t' - t)] d\omega = 2\pi \delta(t' - t).$$
By substituting this result into the previous equation and using the sampling property of
the delta function, we get

\[ \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t') d\omega = \int_{-\infty}^{\infty} dt f(t) 2\pi \delta(t' - t) = 2\pi f(t') \]

or

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega. \]

The inverse Fourier transform is essentially the same as the forward Fourier transform
(ignoring scaling) except for a change from \(-i\) to \(+i\). This is one of the most unique and
important features of the Fourier transform; essentially, computing the inverse Fourier
transform is the same as computing a forward Fourier transform which is not the case
with other integral transforms such as the Laplace transform, for example

A.8 Useful Notation

To avoid constantly having to write integral signs and specify the forward or inverse
Fourier transform in full, we can make use of the symbolic form

\[ f(t) \leftrightarrow F(\omega) \]

which means that \( F \) is the Fourier transform of \( f \) and \( f \) is the inverse Fourier transform
of \( F \). This notation is useful when we want to indicate the relationship between a mathe-
matical operation on \( f \) and its effect on \( F \). Mathematical operations on \( f \) are referred to
as operations in real or \( t \)-space. Similarly, operations on \( F \) are referred to as operations
in Fourier space or \( \omega \)-space.

A.9 Bandlimited Functions

A bandlimited function is characterized by a complex spectrum \( F(\omega) \) such that

\[ F(\omega) = 0, \quad |\omega| > \Omega. \]
In this case, the inverse Fourier transform is given by

\[ f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) \exp(i\omega t) d\omega. \]

Here, \( f \) is known as a bandlimited function and \( 2\Omega \) is referred to as the bandwidth. A bandlimited function is therefore a function that is composed of frequencies which are limited to a particular finite band.

If \( f(t) \) is such that \( f(t) = 0, \ |t| > T \)
then its complex spectrum is given by

\[ F(\omega) = \int_{-T}^{T} f(t) \exp(-i\omega t) dt. \]

In this case, \( f \) is referred to as a time (\( t \) - time in seconds) or space (\( t \) - length in meters) limited signal. Note that in practice, all signals are of a finite duration and are therefore space/time limited; in the latter case, the function is said to be of ‘compact support’. They are also nearly always bandlimited for a variety of different physical reasons.

### A.10 The Amplitude and Phase Spectra

Given that the amplitude spectrum and the phase spectrum, taken together, uniquely describe the time signature of a signal, it is pertinent to ask how these spectra, taken separately, contribute to the signal. As a general rule of thumb, the phase spectrum is more important than the amplitude spectrum in that, if the amplitude spectrum is perturbed but the phase spectrum remains intact, then the signal can be reconstructed relatively accurately (via the inverse Fourier transform) particularly in terms of the positions at which the signal is zero. For a real valued signal with \( A(\omega) > 0, \ \forall \omega \), its zeros occur
when $\omega + \theta(\omega) = \pm n\pi/2$, $n = 1, 2, 3...$ since

$$f(t) = \text{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp[i\theta(\omega)] \exp(i\omega t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \cos[\omega + \theta(\omega)] d\omega.$$  

This can be observed by taking an amplitude only and a phase only reconstruction of a signal, i.e. computing

$$f_A(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp(i\omega t) d\omega$$

and

$$f_\theta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\theta(\omega)] \exp(i\omega t) d\omega.$$  

An amplitude only reconstruction basically yields ‘rubbish’ compared to a phase only reconstruction which yields features that are recognisable in terms of the original signal. Thus, the Fourier phase of a signal is ‘more important’ than the Fourier amplitude and is less robust to error. In some special applications, only the amplitude or power spectrum can be measured and it is necessary to recover the phase. A well known example is X-ray crystal developing the double helix model for DNA in 1953 for example. Here, X-rays with a constant wavelength of $\lambda$ are diffracted by the three dimensional molecular structure (in a crystal defined by an object function $O(x, y, z)$ which is of compact support. To a first order (single or weak scattering) approximation, the diffracted field $F(x_0, y_0)$ generated by a plane wave travelling along the $z$-axis and recorded at a point $z_0$ in the far field is given by (ignoring scaling)

$$F(x_0, y_0) = \int \int f(x, y) \exp(-i2\pi x_0 x/\lambda z_0) \exp(-i2\pi y_0 y/\lambda z_0) dx dy$$

where

$$f(x, y) = \int O(x, y, z) dz.$$  

Now, the X-ray image that is recorded is not $F$ but the intensity $|F|^2$ and so our problem becomes: given $|F(u, v)|^2$ find $f(x, y)$ where

$$F(u, v) = \int \int f(x, y) \exp(-ixu) \exp(-ivy) dx dy, \quad u = 2\pi x_0/\lambda z_0, \quad v = 2\pi y_0/\lambda z_0.$$
This is equivalent to computing the two-dimensional Fourier transform of the function \( f(x, y) \), deleting the phase spectrum and then having to recover it from the amplitude spectrum alone together with any available \textit{a priori} information on the diffractor itself such as its spatial extent (because the diffractor will be of compact support). This is an ill-posed problem and, like so many ‘solutions’ to such problems in signal processing, relies on the application of \textit{a priori} knowledge on the object function coupled with an information theoretic criterion upon which a conditional solution is developed.

### A.11 Differentiation and Integration

**Differentiation**

Given that 
\[
  f(t) \iff F(\omega)
\]
we have
\[
  \frac{d}{dt} f(t) \iff i\omega F(\omega).
\]

The proof of this result is easily established by making use of the inverse Fourier transform, thus:
\[
  \frac{d}{dt} f(t) = \frac{1}{2\pi} \frac{d}{dt} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega F(\omega) \exp(i\omega t) d\omega.
\]

Differentiating \( n \) times, by induction, we have
\[
  \frac{d^n}{dt^n} f(t) \iff (i\omega)^n F(\omega).
\]

This simple and elegant result is the basis for an important and versatile generalization which stems from the question as to the meaning of a fractional differential. In this case, we can consider a definition for a fractional differential based on a generalization of the result above to
\[
  \frac{d^n}{dt^n} f(t) \iff (i\omega)^n F(\omega)
\]
where \( q > 0 \) and can be non-integer. Although many other definitions for a fractional differential exist, the one considered here is arguably one of the simplest and most versatile of them. Moreover, this definition can be used to define a fractal signal.

**Integration**

If we consider integration to be the inverse of differentiation, then it is clear that

\[
\int f(t) \, dt \Leftrightarrow \frac{1}{i\omega} F(\omega)
\]

and that

\[
\int ... \int f(t) \, dt ... \, dt \Leftrightarrow \frac{1}{(i\omega)^n} F(\omega)
\]

where \( n \) is the number of times that the integration is performed. Similarly, we can define a fractional integral to be one that is characterized by \( (i\omega)^{-q} \) where \( q > 0 \) is the (fractional) order of integration.

**A.12 Important Theorems**

**Addition Theorem**

The Fourier transform of the sum of two functions \( f \) and \( g \) is equal to the sum of their Fourier transforms \( F \) and \( G \) respectively.

Proof:

\[
\int_{-\infty}^{\infty} [f(t) + g(t)] \exp(-i\omega t) \, dt = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) \, dt + \int_{-\infty}^{\infty} g(t) \exp(-i\omega t) \, dt
\]

\[= F(\omega) + G(\omega).\]
Similarity Theorem

The Fourier transform of \( f(at) \) is \( \frac{1}{a}F\left(\frac{\omega}{a}\right) \) where \( a \) is a constant.

Proof:
\[
\int_{-\infty}^{\infty} f(at) \exp(-i\omega t) dt = \frac{1}{a} \int_{-\infty}^{\infty} f(at) \exp \left( i \frac{\omega}{a} at \right) d(at) = \frac{1}{a} F\left( \frac{\omega}{a} \right).
\]

Shift Theorem

The Fourier transform of \( f(t-a) \) is given by \( \exp(-i\omega a)F(\omega) \).

Proof:
\[
\int_{-\infty}^{\infty} f(t-a) \exp(-i\omega t) dt = \int_{-\infty}^{\infty} f(t-a) \exp[-i\omega(t-a)] \exp(-i\omega a) d(t-a) = \exp(-i\omega a)F(\omega).
\]

Parseval’s Theorem

If \( f \) and \( g \) have Fourier transforms \( F \) and \( G \) respectively, then
\[
\int_{-\infty}^{\infty} f(t)g^\ast(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G^\ast(\omega) d\omega
\]
where \( g^\ast \) is the complex conjugate of \( g \) and \( G^\ast \) is the complex conjugate of \( G \).

Proof:
\[
\int_{-\infty}^{\infty} f(t)g^\ast(t) dt = \int_{-\infty}^{\infty} g^\ast(t) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega \right) dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left( \int_{-\infty}^{\infty} g^\ast(t) \exp(i\omega t) dt \right) d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left( \int_{-\infty}^{\infty} g(t) \exp(-i\omega t) dt \right)^\ast d\omega.
\]
Rayleigh’s Theorem (also known as the energy theorem)

If \( f \Leftrightarrow F \), then

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega.
\]

Proof:

The proof follows directly from setting \( g = f \) in Parseval’s theorem.

A.13 Convolution and Correlation

Convolution is fundamental to the mathematical models and methods used in signal analysis. The process of correlation is very similar to that of convolution but it has certain properties that are critically different. In both cases, these processes are fundamentally associated with the Fourier transform, an association that is compounded in the convolution and correlation theorems.

Notation

Because the convolution and correlation integrals are of such importance and occur regularly, they are usually given a convenient notation. Throughout this work, convolution shall be denoted by the symbol \( \otimes \) and correlation by the symbol \( \odot \).

Convolution

The convolution of two functions \( f \) and \( g \) in one dimension is defined by the operation

\[
f \otimes g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau) \, d\tau
\]
where \( f \otimes g \) is taken to be a function of \( t \). This is a convolution over the interval \((-\infty, \infty)\) and is sometime written in the form \( f(t) \otimes g(t) \) or \( (f \otimes g)(t) \) to emphasize the fact that the operation is a function of the independent variable \( t \).

If \( f \) and \( g \) are of finite extent \(| t | \leq T\), then the convolution is finite and given by

\[
f \otimes g = \int_{-T}^{T} f(\tau)g(t - \tau) \, d\tau.
\]

Note that if \( g(t) = \delta(t), \ t \in (-\infty, \infty) \), then

\[
f(t) \otimes g(t) = f(t) \otimes \delta(t) = f(t).
\]

In other words, the convolution of a function with the delta function replicates the function. Also, note that if we let \( t' = t - \tau \) then we can write

\[
f \otimes g = \int_{-\infty}^{\infty} f(t - t')g(t') \, dt'
\]

and hence, convolution is commutative, i.e.

\[
f \otimes g = g \otimes f.
\]

If \( f \) and \( g \) are both zero for \( t < 0 \), then

\[
f \otimes g = \int_{0}^{\infty} f(\tau)g(t - \tau) \, d\tau.
\]

The equation

\[
h(t) = \int_{0}^{\infty} f(\tau)g(t - \tau) \, d\tau
\]

is known as the Wiener-Hopf equation and is important in solving problems which are causal, i.e. problems that are governed by an initial condition at \( t = 0 \).
Correlation

The correlation (also known as cross-correlation) of two functions $f$ and $g$ in one dimension is defined by the operation

$$f \odot g = \int_{-\infty}^{\infty} f(\tau)g(\tau - t)d\tau.$$  

This is very similar to convolution except that the function $g$ is a function of $\tau - t$ and not $t - \tau$, a seemingly small but very important difference. When the functions are complex, it is often useful to define the complex correlation operation

$$f^* \odot g = \int_{-\infty}^{\infty} f^*(\tau)g(\tau - t)d\tau.$$  

The important difference between correlation and convolution is that, in correlation, the function $g$ is not reversed about the origin as in convolution. Note that for real functions

$$f(t) \otimes g(t) = f(t) \odot g(-t).$$  

Also, note that if we let $t' = \tau - t$ then the correlation integral can be written as

$$f(t) \odot g(t) = \int_{-\infty}^{\infty} f(t + t')g(t')dt'.$$

Some authors prefer to define the correlation integral in this way where the independent variable of one of the functions is expressed in terms of an addition rather than a subtraction. However, note that the correlation integral, (unlike the convolution integral) is not generally commutative, i.e.

$$f \odot g \neq g \odot f.$$  

Physical Interpretation

Physically, convolution can be thought of as a ‘blurring’ or ‘smearing’ of one function by another. Convolution integrals occur in a wide variety of physical problems. They occur
as a natural consequence of solving linear inhomogeneous partial differential equations using the Green’s function method, i.e. if \( f \) is some source term for a linear PDE, then a solution of the form \( f \otimes g \) can be developed where \( g \) is the Green’s function. Convolution type processes are important in all aspects of signal and image analysis where convolution equations are used to describe signals and images of many different types. If we describe some system in terms of an input signal \( f(t) \) and some output signal \( s(t) \) and this system performs some process that operates on \( f(t) \), then we can write

\[
s(t) = \hat{P} f(t)
\]

where \( \hat{P} \) is the process operator. In many cases, this operator can be expressed in terms of a convolution with a so called Impulse Response Function \( p \) and we can write

\[
s(t) = p(t) \otimes f(t).
\]

This result is used routinely to model a signal (typically a recorded signal that is output from some passive or active ‘system’) when it can be assumed that the system processes are linear and stationary (i.e. \( p \) does not change with time). Such systems are referred to as time invariant linear systems.

**Autoconvolution and Autocorrelation**

Two other definitions which are important in the context of convolution and correlation are:

**Autoconvolution**

\[
f \otimes f = \int_{-\infty}^{\infty} f(t)f(\tau - t)dt
\]

and
Autocorrelation

\[ f \circ f = \int_{-\infty}^{\infty} f(t) f(t - \tau) \, dt. \]

**A.14 The Convolution Theorem**

The convolution theorem is one of the most important results of Fourier theory. It can be stated thus:

The convolution of two functions in real space is the same as the product of their Fourier transforms in Fourier space.

The proof of this result can be obtained in a variety of ways. Here, we give a proof that is relatively ‘short and sweet’ and based on the definition for an inverse Fourier transform. Thus, writing

\[
 f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) \, d\omega, \\
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp(i\omega t) \, d\omega
\]

we have

\[
 f \otimes g = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) \, d\omega \int_{-\infty}^{\infty} G(\omega') \exp[i\omega'(t - t)] \, d\omega' \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\omega) \int_{-\infty}^{\infty} d\omega' G(\omega') \exp(i\omega'\tau) \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[it(\omega - \omega')] \, dt \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\omega) \int_{-\infty}^{\infty} d\omega' G(\omega') \exp(i\omega'\tau) \delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G(\omega') \exp(i\omega\tau) \, d\omega
\]

or

\[
f(t) \otimes g(t) \Leftrightarrow F(\omega)G(\omega).
\]
The Product Theorem

The product theorem states that the product of two functions in real space is the same (ignoring scaling) as the convolution of their Fourier transforms in Fourier space. To prove this result, let

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \]

and

\[ G(\omega) = \int_{-\infty}^{\infty} g(t) \exp(-i\omega t) dt. \]

Then

\[ F \otimes G = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \int_{-\infty}^{\infty} g(\tau) \exp(-i\tau(\omega' - \omega)) d\tau \]

\[ = \int_{-\infty}^{\infty} dt f(t) \int_{-\infty}^{\infty} d\tau g(\tau) \exp(-i\omega'\tau) \int \exp(-i\omega(t - \tau)) d\omega \]

\[ = \int_{-\infty}^{\infty} dt f(t) g(t) \exp(-i\omega't) \]

or

\[ f(t) g(t) \iff \frac{1}{2\pi} F(\omega) \otimes G(\omega). \]

The Correlation Theorem

The correlation theorem follows from the convolution theorem and can be written in the form

\[ f(t) \odot g(t) \iff F(\omega)G(-\omega) \]

for \( f \) and \( g \) real and

\[ f(t) \odot g^*(t) \iff F(\omega)G^*(\omega) \]
for $f$ and $g$ complex. Note that if $g$ is a purely real function, then the real part of its Fourier transform $G(\omega)$ is symmetric and its imaginary part is asymmetric, i.e.

$$G_r(-\omega) = G_r(\omega)$$

and

$$G_i(-\omega) = -G_i(\omega).$$

In this case,

$$G(-\omega) = G_r(-\omega) + iG_i(-\omega) = G_r(\omega) - iG_i(\omega) = G^*(\omega)$$

and thus, for real functions $f$ and $g$, we can write

$$f(t) \circ g(t) \iff F(\omega)G^*(\omega).$$

The Autoconvolution and Autocorrelation Theorems

From the convolution theorem, we have

$$f(t) \otimes f(t) \iff [F(\omega)]^2$$

and from the correlation theorem, we have

$$f(t) \circ f(t) \iff |F(\omega)|^2.$$

The last result has a unique feature which is that information about the phase of $F$ is entirely missing from $|F|^2$ in contrast to the autoconvolution theorem where information about the phase of the spectrum is retained, i.e.

$$[F(\omega)]^2 = A_F^2(\omega) \exp[2i\theta_F(\omega)]$$

where $A_F$ and $\theta_F$ are the amplitude and phase spectra of $F$ respectively. Hence, the autocorrelation function $f \odot f$ contains no information about the phase of the Fourier components of $f$ and is consequently unchanged if the phase changes.
Important Properties

1. Convolution is commutative, i.e.

\[ f \otimes g = g \otimes f. \]

2. Convolution is associative, namely,

\[ f \otimes (g \otimes h) = (f \otimes g) \otimes h. \]

Multiple convolutions can therefore be carried out in any order.

3. Convolution is distributive, or

\[ f \otimes (g + h) = f \otimes g + f \otimes h. \]

4. The derivative of a convolution can be written as

\[ \frac{d}{dx}[f(x) \otimes g(x)] = f(x) \otimes \frac{d}{dx}g(x) = g(x) \otimes \frac{d}{dx}f(x). \]

5. Correlation does not in general commute, i.e.

\[ f \circ g \neq g \circ f \]

unless both \( f \) or \( g \) are symmetric functions.

A.15 The Discrete Fourier Transform

Given the analysis provided in the previous parts of this appendix and the results discussed and coupled with the fact that many of the results and ideas developed using the Fourier transform can be implemented using the Discrete Fourier Transform or DFT, it is pertinent to ask how the results developed here can be rigorously justified using the DFT. In other words, how can we develop results (such as the convolution theorem for
example) that are based explicitly on the use of the DFT. If we consider a generalized
approach, then it is imperative that we consider a discrete version of the delta function
since nearly all of the results discussed in this appendix have been based on using the
generalized Fourier transform (compared with those previously discussed which are based
on a classical approach) which in turn have been dependent on the application of the delta
function and its properties. The solution to this problem is based on the introduction of
the so called Kronecker delta function which can be defined as

\[
\delta_{nm} = \begin{cases} 
1, & n = m; \\
0, & n \neq m.
\end{cases}
\]

to which we can extend the following properties:

\[
\sum_n f_n \delta_{nm} = f_m
\]

and

\[
\delta_{nm} = \frac{1}{N} \sum_k \exp[2\pi i k (n - m)/N]
\]

where \(N\) is the array size and the limits of the sums are taken to run from \(-\infty\) to \(\infty\).

The first of these properties is the definition of this discrete delta function in terms of its
fundamental sampling property. The second of these properties can be derived directly
from a discretization of the integral representation of the delta function, i.e.

\[
\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i \omega t) d\omega.
\]

Thus, consider

\[
\delta(t_n) = \frac{1}{2\pi} \sum_m \exp(i \omega_m t_n) \Delta \omega
\]

which can be written in the form

\[
\delta(t_n) = \frac{1}{2\pi} \sum_m \exp(i m n \Delta \omega \Delta t) \Delta \omega
\]
where $\Delta t$ and $\Delta \omega$ denote the sampling intervals between the elements $t_n$ and $\omega_m$ respectively. Now, with

$$\Delta \omega \Delta t = \frac{2\pi}{N},$$

$$\delta(t_n) = \frac{1}{\Delta t N} \sum_m \exp(2\pi im/N)$$

or

$$\Delta t \delta(t_n) = \frac{1}{N} \sum_m \exp(2\pi im/N).$$

Hence, defining the Kronecker delta function as $\delta_n \equiv \delta(t_n) \Delta t$ we obtain the desired result.

With these results, we can derive a number of important results in a way that is directly analogous to those derived for the Fourier transform. For example, suppose we want to prove the convolution theorem for the DFT, i.e. show that

$$f_n \otimes g_n \iff F_n G_n$$

where

$$f_n \otimes g_n = \sum_n f_n g_{m-n}$$

and $F_n$ and $G_n$ are the DFTs of $f_n$ and $g_n$ respectively. Let

$$f_n = \frac{1}{N} \sum_m F_m \exp(2\pi i mn/N)$$

and

$$g_n = \frac{1}{N} \sum_m G_m \exp(2\pi i mn/N).$$

Then

$$\sum_n f_n g_{m-n} = \frac{1}{N^2} \sum_n \sum_k F_k \exp(2\pi i kn/N) \sum_\ell G_\ell \exp[2\pi i \ell (m - n)/N]$$

$$= \frac{1}{N} \sum_k F_k \sum_\ell G_\ell \exp(2\pi i \ell m/N) \sum_n \exp[2\pi i n (k - \ell)/N]$$

$$= \frac{1}{N} \sum_k F_k \sum_\ell G_\ell \exp(2\pi i \ell m/N) \delta_{k\ell}$$
\[
= \frac{1}{N} \sum_k F_k G_k \exp(2\pi ikm/N).
\]

Hence,

\[f_n \otimes g_n \iff F_n G_n.\]

**Deriving Filters using the DFT**

Digital filters can be derived from application of the DFT in much the same way as analogue filters can from application of the Fourier transform. For example, consider the differentiation of a digital signal using forward differencing and whose digital gradient is given by (ignoring scaling by the ‘step length’)

\[g_n = f_{n+1} - f_n.\]

Now, with

\[f_n = \frac{1}{N} \sum_m F_m \exp(i2\pi nm/N),\]

we have

\[g_n = \frac{1}{N} \sum_m F_m \exp(i2\pi nm/N)[\exp(i2\pi m/N) - 1]\]

which shows that the DFT filter for this operation is given by

\[\exp(i2\pi m/N) - 1 = \cos(2\pi m/N) - 1 + i \sin(2\pi m/N).\]

Similarly, the DFT filter that characterizes the centre differenced approximation to a second order derivative (i.e. \(f_{n+1} - 2f_n + f_{n-1}\)) is given by

\[\exp(i2\pi m/N) + \exp(-i\pi m/N) - 2 = 2[\cos(2\pi m/N) - 1].\]

Note that these filters differ from the application of Fourier filters in discrete form, i.e. \(i\omega_m\) and \(-\omega^2_m\). In particular, their response at high frequencies is significantly different.

Also, note that a process such as \(f_{n+1} - 2f_n + f_{n-1}\) for example can be considered in terms of a discrete convolution of the signal \(f_n\) with \((1, -2, 1)\) and in this sense we can write

\[(1, -2, 1) \otimes f_n \iff 2[\cos(2\pi m/N) - 1]F_m.\]
We conclude this appendix with a schematic diagram of the relationships between the Fourier transform, the complex Fourier series and the discrete Fourier transform as given below. Table A.1 illustrates qualitatively the principal properties associated with each of these approaches to Fourier analysis which should be borne in mind when their utilities are exercised. In particular, the relationship between the Fourier transform and the corresponding discrete Fourier transform (and hence between analogue and digital signal processing) should be understood in terms of the fact that the DFT does not produce identical results (due to numerical error associated with the discretization of arrays that are of finite duration) to those of the Fourier transform. Nevertheless, the use of Fourier theory can be used to investigate the analysis of (analogue) signals to design processing methods that can be implemented using the DFT.
Table 1: Schematic diagram of the relationship between the Fourier transform, the (complex) Fourier series and the discrete Fourier transform (DFT).

<table>
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<th>Fourier Transform</th>
<th>Complex Fourier Series</th>
<th>Discrete Fourier Transform</th>
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<td>Periodic Signals</td>
<td>Single Period Signals</td>
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Appendix B: Maxwell’s Equations and the Magnetic Vector Potential

This appendix is provided for completeness with regard to the analysis given in Chapter 7 and is based on an edited version of the material given in ‘Digital Image Processing: Mathematical and Computational Methods,’ (Chapter 4, Section 4.2) by Jonathan M Blackledge, Horwood Publishing, 2005; ISBN: 1-898563-49-7.

There are two forms of Maxwell’s equation, the ‘microscopic’ and the ‘macroscopic’ forms. We first consider these equations in their microscopic form (for individual charged particles) and go on to consider the macroscopic form of Maxwell’s equations (for the case when there are many particles per cubic wavelength), [44], [45] and [46].

The motions of electrons (and other charged particles) give rise to electric and magnetic fields. These fields are described by the following equations which are a complete mathematical descriptions for the physical laws.

**Coulomb’s law**

\[ \nabla \cdot \mathbf{E} = \rho \]  \hspace{1cm} (B.1)

**Faraday’s law of induction**

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]  \hspace{1cm} (B.2)
No free magnetic monopoles exist

$$\nabla \cdot \mathbf{B} = 0 \quad (B.3)$$

Modified (by Maxwell) Ampere’s law

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j} \quad (B.4)$$

where $\mathbf{E}$ is the electric field (Coulombs), $\mathbf{B}$ is the magnetic field (Wb/m$^2$), $\mathbf{j}$ is the current density (A/m$^2$), $\rho$ is the charge density (Coulombs/m$^3$) and $c \simeq 3 \times 10^8 \text{ ms}^{-1}$ is the speed of light (in a perfect vacuum). These microscopic Maxwell’s equations are used to predict the pointwise electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields given the charge and current densities ($\rho$ and $\mathbf{j}$ respectively). The differential operator $\nabla$ is defined as follows:

$$\nabla \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

for unit vectors ($\mathbf{i}, \mathbf{j}, \mathbf{k}$).

By including a modification to Ampere’s law, i.e. the inclusion of the ‘displacement current’ term $\partial \mathbf{E}/\partial (ct)$, Maxwell provided a unification of electricity and magnetism compounded in the equations above.

### B.1 Linearity of Maxwell’s Equations

Maxwell’s equations are linear because if

$$\rho_1, \ J_1 \rightarrow \mathbf{E}_1, \ \mathbf{B}_1$$

and

$$\rho_2, \ J_2 \rightarrow \mathbf{E}_2, \ \mathbf{B}_2$$

then

$$\rho_1 + \rho_2, \ J_1 + J_2 \rightarrow \mathbf{E}_1 + \mathbf{E}_2, \ \mathbf{B}_1 + \mathbf{B}_2$$
where \( \rightarrow \) means ‘produces’. This is because the operators \( \nabla \cdot \), \( \nabla \times \) and the time derivatives are all linear operators.

### B.2 Solution to Maxwell’s Equations

The solution to these equations is based on exploiting the properties of vector calculus and, in particular, identities involving the curl.

Taking the curl of equation (B.2), we have

\[
\nabla \times \nabla \times E = -\frac{1}{c} \nabla \times \frac{\partial B}{\partial t}
\]

and using the identity

\[
\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E
\]

together with equations (B.1) and (B.4), we get

\[
\nabla \rho - \nabla^2 E = -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial E}{\partial t} + j \right)
\]

or, after rearranging,

\[
\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \nabla \rho + \frac{1}{c} \frac{\partial j}{\partial t}.
\]  \(\text{(B.5)}\)

Taking the curl of equation (B.4), using the identity above, equations (B.2) and (B.3) and rearranging the result gives

\[
\nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = -\nabla \times j.
\]  \(\text{(B.6)}\)

Equations (B.5) and (B.6) are inhomogeneous wave equations for \( E \) and \( B \). These equations are related or coupled to the vector field \( j \) (which is related to \( B \)). If we define a region of free space where \( \rho = 0 \) and \( j = 0 \), then both \( E \) and \( B \) satisfy the equation

\[
\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0.
\]
This is the homogeneous wave equation. One possible solution of this equation (in Cartesian coordinates) is

\[ f_x = p(z - ct); \quad f_y = 0, \quad f_z = 0 \]

which describes a wave or distribution \( p \) moving along \( z \) at velocity \( c \). Thus, we have shown that in free space when

\[
\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \\
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.
\]

Maxwell’s equations describe the propagation of an electric and magnetic (or electromagnetic field) in terms of a wave travelling at the speed of light. After developing the origins of the vector calculus, Maxwell derived the wave equations for an electromagnetic field in a paper entitled *A Dynamical Theory of the Electromagnetic Field*, first published in 1865 and arguably one of the greatest intellectual achievements in the history of physics.

### B.3 General Solution to Maxwell’s (Microscopic) Equations

The solution to Maxwell’s equation in free space is specific to the charge density and current density being zero. We now investigate a method of solution for the general case [47], [42]. The basic method of solving Maxwell’s equations (i.e. finding \( \mathbf{E} \) and \( \mathbf{B} \) given \( \rho \) and \( \mathbf{j} \)) involves the following:

(i) Expressing \( \mathbf{E} \) and \( \mathbf{B} \) in terms of two other fields \( \mathbf{U} \) and \( \mathbf{A} \).

(ii) Obtaining two separate equations for \( \mathbf{U} \) and \( \mathbf{A} \).

(iii) Solving these equations for \( \mathbf{U} \) and \( \mathbf{A} \) from which \( \mathbf{E} \) and \( \mathbf{B} \) can then be computed.

For any vector field \( \mathbf{A} \)

\[ \nabla \cdot \nabla \times \mathbf{A} = 0. \]
Hence, if we write

\[ \mathbf{B} = \nabla \times \mathbf{A} \]  

(B.7)

then equation (B.3) remains unchanged. Equation (B.2) can then be written as

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{A} \]

or

\[ \nabla \times \left( \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \]

The field \( \mathbf{A} \) is called the Magnetic Vector Potential. For any scalar field \( U \)

\[ \nabla \times \nabla U = 0 \]

and thus equation (B.2) is satisfied if we write

\[ \pm \nabla U = \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \]

or

\[ \mathbf{E} = -\nabla U - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \]  

(B.8)

where the minus sign is taken by convention. \( U \) is called the Electric Scalar Potential.

Substituting equation (B.8) into Maxwell’s equation (B.1) gives

\[ \nabla \cdot \left( \nabla U + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = -\rho \]

or

\[ \nabla^2 U + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\rho. \]  

(B.9)

Substituting equations (B.7) and (B.8) into Maxwell’s equation (B.4) gives

\[ \nabla \times \nabla \times \mathbf{A} + \frac{1}{c} \frac{\partial}{\partial t} \left( \nabla U + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{j} \]

Finally, using the identity

\[ \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]
we can write
\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial U}{\partial t} \right) = -j \]  \hspace{1cm} (B.10)

If we could solve equations (B.9) and (B.10) above for \( U \) and \( A \) then \( E \) and \( B \) could be computed. The problem here, is that equations (B.9) and (B.10) are coupled. They can be decoupled by applying a technique known as a ‘gauge transformation’ called the Lorentz gauge transformation, after Lorentz who was among the first to consider it as an approach to solving these equations. The idea is based on noting that equations (B.7) and (B.8) are unchanged if we let

\[ A \rightarrow A + \nabla X \]

and

\[ U \rightarrow U - \frac{1}{c} \frac{\partial X}{\partial t} \]

since \( \nabla \times \nabla X = 0 \). If this gauge function \( X \) is taken to satisfy the homogeneous wave equation

\[ \nabla^2 X - \frac{1}{c^2} \frac{\partial^2 X}{\partial t^2} = 0 \]

then

\[ \nabla \cdot A + \frac{1}{c} \frac{\partial U}{\partial t} = 0 \]  \hspace{1cm} (B.11)

which is called the Lorentz condition. With equation (B.11), equations (B.9) and (B.10) become

\[ \nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = -\rho \]  \hspace{1cm} (B.12)

and

\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -j \]  \hspace{1cm} (B.13)

respectively. These equations are non-coupled inhomogeneous wave equations.
References


[38] Voxel Sculpturing http://3d-coat.com/voxel-sculpting/


[40] Pendix, http://home.arcor.de/sercan-san/homepage/sangames/sites/project\pendix.html

[41] Pendix Sketch Based 3D modeling http://www.youtube.com/watch?v=LTU9PbEQ-14


Bibliography

The following references were accessed and used throughout the duration of the research programme leading to the submission of this PhD thesis.


BS-EN, *BS EN 61000-3-2, Ed.2:2001: Electromagnetic compatibility (EMC), Limits for harmonic current emissions (equipment input current up to and including 16 A per phase)*, 2001.


BS-IEC, *BS IEC 61000-3-4:1998 Electromagnetic compatibility (EMC), Limitation of emission of harmonic currents in low-voltage power supply systems for equipment with rated current greater than 16 A*, 1998.


BS-IEC, BS-IEC61000-3-8-1997 *Electromagnetic compatibility (EMC)* - Incorporating Amendment No 1, Limits Guide to Signalling on Low-Voltage Electrical Installations - Emission levels, frequency bands and electromagnetic disturbance levels., 1997


Voxologic, [http://www.voxologic.com](http://www.voxologic.com)
Voxel Sculpturing http://3d-coat.com/voxel-sculpting/


Pendix, http://home.arcor.de/sercan-san/homepage/sangames/sites/project\pendix.html

Pendix Sketch Based 3D modeling http://www.youtube.com/watch?v=LTU9PbEQ-14


S. M. Halpin, Comparison of IEEE and IEC harmonic standards, Power Engineering


B. H. Campbell, *Failed motors: rewind or replace?*, Industry Applications Magazine,


D. Lin, T. Batan, E. F. Fuchs, and W. M. Grady, *Harmonic losses of single-phase induction motors under non-sinusoidal voltages*, Energy Conversion, IEEE Transactions on,


IEC, *IEC 61000-3-12 ed1.0 Electromagnetic compatibility (EMC), Part 3-12: Limits, Limits for harmonic currents produced by equipment connected to public low-voltage systems*
with input current greater than 16 A and less than 75 A per phase, 2004.


J. G. Boudrias, *Power factor correction and energy saving with proper transformer, phase


B. Ilya and H. Greg, Uncertainties in the Measurement of Power Harmonics and Flicker,


Bleaney B I and Bleaney B, *Electricity and Magnetism*, Oxford University Press,