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Inverse Scattering Transform for the Degasperis-Procesi Equation

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Abstract

We develop the Inverse Scattering Transform (IST) method for the Degasperis-Procesi equation. The spectral problem is an $\mathfrak{sl}(3)$ Zakharov-Shabat problem with constant boundary conditions and finite reduction group. The basic aspects of the IST such as the construction of fundamental analytic solutions, the formulation of a Riemann-Hilbert problem, and the implementation of the dressing method are presented.

1 Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 3\kappa u_x + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \qquad \text{(DP)}$$

where $\kappa > 0$ is a constant, was first discovered in [11] in a search for asymptotically integrable PDEs. Equation (DP) is a bi-Hamiltonian system and admits interesting traveling wave solutions [12]. It arises as a model equation in the study of two-dimensional water waves propagating over a flat bed [39, 10, 36]. Given the intricate structure of the full governing equations for water waves, it is natural to seek, in various physical regimes, simpler approximate model equations. In the case of two-dimensional waves propagating mainly in one direction, two parameters appear in the non-dimensionalisation of the governing equations: the wave-amplitude parameter ε (indicating how close the waves are to a flat-surface flow) and the long-wave parameter δ (measuring the ratio of the approximate wavelength to the average water depth) [38]. The relative sizes of these two fundamental parameters determine the different physical regimes for water waves. The most studied regime is the shallow-water regime (also called the small amplitude or long-wave regime) for which $\delta \ll 1$ and $\varepsilon \sim \delta^2$. In this parameter range, due to a balance between nonlinearity and dispersion, various integrable systems like the Korteweg-de Vries (KdV) equation arise as approximations to the governing equations [2]. The integrability of these equations implies that the powerful method of inverse scattering can be used to obtain detailed qualitative (and even quantitative) information about the wave dynamics. In particular, since linear water wave theory cannot provide an approximation of solitary waves (see the discussion in [8]), an important outcome of studies of the KdV equation was a deeper understanding of the dynamics of solitary water waves [38].

The shallow amplitude regime is, however, not appropriate for the study of large amplitude waves, whose behavior is more nonlinear than dispersive. To model such waves, which are characterized by a relatively large value of ε, it is natural to investigate the parameter regime in which $\delta \ll 1$ and $\varepsilon \sim \delta$. Also in this parameter range is a reduction to a simple wave equation at leading order possible, but since the dimensionless parameter ε is larger, the nonlinear effects are stronger than in the shallow water regime. Of particular interest in this regard is that a stronger nonlinearity could allow for the occurence of wave-breaking—a fundamental phenomenon in the theory of water waves that is not captured by the KdV equation (see the discussion in [7]). In the shallow-water, moderate amplitude regime $[\delta \ll 1,\, \varepsilon \sim \delta]$ several model equations can be derived as approximations to the governing equations for water waves. However, among the various equations that arise in this way, there are only two which admit a bi-Hamiltonian structure (see [10, 35]): the Camassa-Holm (CH) equation [4] and the DP equation (DP). For these equations, $u(x,t)$ represents the horizontal velocity of the water at a certain depth at time t and position x; the depth ratios for CH and DP are different [39, 10] and are encoded in (DP) in the positive non-dimensional parameter $\kappa > 0$. In the limit $\kappa \to 0$ the solitary waves of (DP) are peakons and are given explicitly by

$$u_c(x,t) = c\, e^{-|x-ct|}, \qquad x,\, t \in \mathbb{R},$$

where $c > 0$ denotes the speed of the wave [12, 44]. Due to the presence of a peak at the wave crest, these waves have to be understood as weak solutions [42]. This feature is characteristic for waves of great height (i.e. waves of largest amplitude that are exact solutions of the governing equations for water waves, see the discussion in [6, 8, 51]). The dynamics of the peakon interactions for DP was elucidated in [45, 46, 13].

In this paper, we will develop an inverse scattering approach for smooth localized solutions to (DP). More precisely, we consider solutions $u(x,t)$ of class C^1 in time and of Schwartz class regularity with respect to the spartial x-variable (i.e. the solution is smooth and decays to zero faster than any polynomial as $|x| \to \infty$). Moreover, we will assume that the solution satisfies the following inequality initially (at time $t = 0$):

$$q = u - u_{xx} + \kappa > 0. \tag{1.1}$$

Note that well-posedness and global existence of solutions for (DP) holds within the class of Schwartz functions if the initial data satisfy (1.1), and in this case the validity of (1.1) is ensured at any later time $t > 0$ (see [14, 31, 43]).

The paper is organized as follows. In Section 2 we present the Lax pair formulation of (DP) and study the symmetry properties of the isospectral problem and of the scattering matrix. Section 3 is devoted to the associated Riemann-Hilbert problem, while the Zakharov-Shabat dressing method is implemented in Section 4.

2 Spectral problem

2.1 Lax pair

Equation (DP) admits the Lax pair formulation [12, 33]

$$\begin{cases} \varphi_{xxx} - \varphi_x - q\zeta^3\varphi = 0, \\ \varphi_t - \frac{1}{\zeta^3}\varphi_{xx} + u\varphi_x - u_x\varphi = 0, \end{cases} \tag{2.2}$$

where $\zeta \in \mathbb{C}$ is the spectral parameter and $\varphi(x,t)$ is a scalar function. Third-order spectral problems appear as eigenvalue problems for the Boussinesq, the Sawada-Kotera and the Kaup-Kuperschmidt equation [41] and they were first investigated by Kaup [40] who established the analyticity properties of the fundamental solutions, constructed scattering data and presented an inverse scattering approach in the cases when only the continuous spectrum is present and when only the bound state spectrum is present. The third order problem (2.2) however is a weighted spectral problem which requires additional care. For example, a weighted second order spectral problem arises in the inverse scattering for the CH equation, its analysis is described in details in [9].

Our aim is to develop the inverse scattering approach for (DP). The Lax representation (2.2) however is inconvenient for this purpose. It turns out that the problem simplifies a lot if the Lax representation is written in the form of Zakharov-Shabat (ZS) type spectral problem [54, 50, 49, 48, 15, 25, 17, 18, 19]. This allows us to take advantage of the existing inbuilt symmetries of the considered equation. For this reason we write the above Lax pair in matrix form as

$$\begin{cases} \phi_x = \tilde{L}\phi, \\ \phi_t = \tilde{M}\phi, \end{cases} \tag{2.3}$$

where

$$\tilde{L} = \begin{pmatrix} -1 & \zeta & 0 \\ 0 & 0 & \zeta \\ \zeta q & 0 & 1 \end{pmatrix}, \qquad \tilde{M} = \begin{pmatrix} u + u_x + \frac{1}{3\zeta^3} & -\zeta u - \frac{1}{\zeta^2} & \frac{1}{\zeta} \\ \frac{q+u_x+u_{xx}}{\zeta} & -\frac{2}{3\zeta^3} & \frac{1}{\zeta^2} - \zeta u \\ \frac{-qu\zeta^3+q_x-u_x+u_{xxx}}{\zeta^2} & \frac{q+u_x+u_{xx}}{\zeta} & -u - u_x + \frac{1}{3\zeta^3} \end{pmatrix}$$

and $\phi(x,t)$ is a $SL(3)$ - matrix-valued function, whose columns, considered as vectors, represent the three linearly independent solutions of the matrix equation. Let $G(x,t)$ be a $SL(3)$ matrix and let us take a change of variables $\phi = G\psi$. It transforms (2.3) into

$$\begin{cases} \psi_x = L\psi, \\ \psi_t = M\psi, \end{cases}$$

where

$$L = G^{-1}\tilde{L}G - G^{-1}G_x, \qquad M = G^{-1}\tilde{M}G - G^{-1}G_t.$$

Letting $\omega = e^{2\pi i/3}$ and

$$G = \frac{1}{\sqrt{3}} \begin{pmatrix} q^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{1/3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix},$$

we find $L = \zeta q^{1/3} J - \tilde{Q}$ where $\tilde{Q} = Q^* \left(1 - \frac{q_x}{3q}\right)$

$$J = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q^* = \frac{1}{3}(1-\omega) \begin{pmatrix} 0 & \omega+1 & 1 \\ 1 & 0 & \omega+1 \\ \omega+1 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

Let us change the variables according to

$$y = x + \int_{-\infty}^{x} \Big[\Big(\frac{q(x')}{\kappa}\Big)^{1/3} - 1\Big]\mathrm{d}x', \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \Big(\frac{q(x)}{\kappa}\Big)^{1/3}. \tag{2.5}$$

The t-variable can be viewed as an additional parameter rather than a second independent variable. This will be clear in the following section and is due to the fact that the t-dependence of the scattering data can be explicitly computed in relatively simple form. For the sake of simplicity in what follows we usually omit the t-dependence of the variables, unless this dependence is necessary for the computations. The spectral problem

$$\psi_x + (\tilde{Q} - \zeta q^{1/3} J)\psi = 0 \tag{2.6}$$

can be written in the form

$$\psi_y + (Q - \lambda J)\psi = 0, \tag{2.7}$$

where

$$\lambda = \zeta\kappa^{1/3}, \quad Q = Q^* h, \quad h(x) = \Big(\frac{q(x)}{\kappa}\Big)^{-1/3} + \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{q(x)}{\kappa}\Big)^{-1/3}, \tag{2.8}$$

and $\lim_{x\to\pm\infty} h(x) = 1$.

Suppose that $x = X(y)$. It is possible to recover $q(x)$ from $h(X(y))$. First, we notice that asymptotically $y \to x$ when $x \to -\infty$. Since

$$\int_{-\infty}^{\infty} \Big[\Big(\frac{q(x)}{\kappa}\Big)^{1/3} - 1\Big]\mathrm{d}x$$

is an integral of motion [12], for $x \to \infty$ we have that x and y differ only by a constant. Let us introduce for convenience

$$f(y) = \Big(\frac{q(X(y))}{\kappa}\Big)^{1/3}. \tag{2.9}$$

Then f, and therefore q, can be recovered from $h(y) \equiv h(X(y))$ by solving the first order differential equation that follows from (2.8):

$$\frac{\mathrm{d}f}{\mathrm{d}y} + h(y)f = 1, \qquad f(\pm\infty) = 1. \tag{2.10}$$

We obtain the solution in parametric form

$$q(X(y)) = \kappa f^3(y), \qquad x \equiv X(y) = y + \int_{-\infty}^{y} \Big(\frac{1}{f(y')} - 1\Big)\mathrm{d}y', \tag{2.11}$$

or, with the convolution interpreted in the sense of distributions [34],

$$q(x) = \kappa \int_{-\infty}^{\infty} \delta(x - X(y)) f^2(y)\mathrm{d}y, \tag{2.12}$$

where the function $X(y)$ is defined in (2.11) and $f(y)$ is the solution of (2.10). Finally, $h(y)$, which is actually $h(X(y))$, can be obtained from the scattering data of the spectral problem (2.7) which is a ZS-type spectral problem, however with constant boundary conditions, since $Q(y) = Q^* h(y)$ and $\lim_{y\to\pm\infty} h(y) = 1$. For other ZS spectral problems for multicomponent systems with constant boundary conditions we refer to [3, 23, 15].

2.2 Automorphisms

The specific form of the potential Q in (2.7) is due to the symmetry of the problem under the action of three distinct automorphisms. In other words, the fact that Q is determined by a single real (scalar) function, rather than 6 complex functions (which is the case for an arbitrary $sl(3)$ potential) is a consequence of its invariance under one $\mathbb{Z}_3$ automorphism and two $\mathbb{Z}_2$ automorphisms. The automorphisms lead to the reduction of the independent components of the potential and their action extends to the spectrum and the eigenfunctions. They form a group, known as *a reduction group* [47, 24, 25, 22, 20, 21, 27, 28, 29, 30].

2.2.1 $\mathbb{Z}_3$ automorphism

The spectral problem (2.7) has a manifest $\mathbb{Z}_3$ symmetry:

$$CL(\omega\lambda)C^{-1} = L(\lambda), \tag{2.13}$$

where

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Indeed, one can check that

$$CJC^{-1} = \frac{1}{\omega}J, \qquad CQC^{-1} = Q$$

from where (2.13) follows immediately. Furthermore, one can verify that

$$CM(\omega\lambda)C^{-1} = M(\lambda), \tag{2.14}$$

so that the $\mathbb{Z}_3$ automorphism (2.13) is an automorphism of the graded Kac-Moody algebra where $L(\lambda)$ and $M(\lambda)$ take their values [16]. This holds for all automorphisms of the spectral problem.

2.2.2 $\mathbb{Z}_2$ automorphisms

The spectral problem possess two additional $\mathbb{Z}_2$ automorphisms, one of which reflects the reality of $h(y)$. The first one is

$$\Gamma\overline{L(\overline{\omega\lambda})}\Gamma^{-1} = L(\lambda), \tag{2.15}$$

where

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second $\mathbb{Z}_2$ automorphism is

$$ZL^{\dagger}(-\bar{\lambda})Z^{-1} = -L(\lambda), \tag{2.16}$$

where

$$Z = \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the dagger stands for a matrix Hermitian conjugation.

One can easily check that all $\mathbb{Z}_2$ and $\mathbb{Z}_3$ automorphisms are compatible, in the sense that the order of their application to the potential does not matter.

3 Construction of Fundamental Analytic Solutions

3.1 Asymptotic behavior

Let us define the asymptotic values

$$L_\infty(\lambda) = \lim_{y\to\pm\infty} L(y,\lambda), \qquad M_\infty(\lambda) = \lim_{y\to\pm\infty} M(y,\lambda).$$

Then

$$L_\infty = \lambda J - Q^*$$

where Q^* is defined in (2.4). We also find that

$$M_\infty(\lambda) = \frac{\kappa}{3\lambda^3}\begin{pmatrix} 3\omega^2\lambda^2 & \omega^2(\omega-1)\lambda - \omega & \omega^2(1-\omega)\lambda - \omega^2 \\ (1-\omega)\lambda - \omega^2 & 3\omega\lambda^2 & (\omega-1)\lambda - \omega \\ \omega(\omega-1)\lambda - \omega & \omega(1-\omega)\lambda - \omega^2 & 3\lambda^2 \end{pmatrix} \tag{3.17}$$

and that

$$[L_\infty, M_\infty] = 0.$$

Since L_∞ and M_∞ commute, they can be simultaneously diagonalized. Let $U(\lambda)$ be a $SL(3)$ matrix such that

$$L_\infty(\lambda) = U(\lambda)\Lambda(\lambda)U^{-1}(\lambda), \qquad M_\infty(\lambda) = U(\lambda)A(\lambda)U^{-1}(\lambda),$$

with

$$\Lambda(\lambda) = \mathrm{diag}(\Lambda_1(\lambda), \Lambda_2(\lambda), \Lambda_3(\lambda)), \qquad A(\lambda) = \mathrm{diag}(A_1(\lambda), A_2(\lambda), A_3(\lambda)),$$

where $\Lambda_1, \Lambda_2, \Lambda_3$ and A_1, A_2, A_3 are the eigenvalues of L_∞ and M_∞ respectively.

The eigenvalues of $L_\infty(\lambda) = \lambda J - Q^*$ are the solutions $\Lambda(\lambda)$ of the characteristic equation

$$\Lambda^3 - \Lambda - \lambda^3 = 0.$$

Introducing a new spectral parameter k such that

$$\lambda(k) = 3^{-1/2}k\Big(1 + \frac{1}{k^6}\Big)^{1/3} = 3^{-1/2}k\left(1 + \frac{1}{3k^6} + \dots\right) \tag{3.18}$$

we have

$$\lambda^3 = 3^{-3/2}\Big(k^3 + \frac{1}{k^3}\Big) \tag{3.19}$$

and the following expression for the eigenvalues of L_∞:

$$\Lambda_j(k) = 3^{-1/2}\Big(\omega^j k + \frac{1}{\omega^j k}\Big). \tag{3.20}$$

Furthermore, $\lambda(k)$ has the property $\lambda(\omega k) = \omega\lambda(k)$ and also

$$\lambda(k) \to 3^{-1/2}k \qquad \text{when} \qquad |k| \to \infty. \tag{3.21}$$

The characteristic polynomial of the matrix $3\kappa^{-1}\lambda^3 M_\infty$, cf. (3.17), is

$$P(w) = w^3 - 3w - 27\lambda^6 + 2, \tag{3.22}$$

with roots

$$w_j(k) = \omega^j k^2 + \omega^{-j}k^{-2},$$

where $\lambda(k)$ is given in (3.18). Thus the eigenvalues of M_∞ are

$$A_j(k) = \frac{\kappa w_j(k)}{3\lambda^3(k)}. \tag{3.23}$$

It remains to determine the ordering of the eigenvalues A_j that is consistent with the ordering of Λ_j, the eigenvalues of L_∞. To this end we consider the asymptotic expressions when $k \to \infty$. Then

$$L_\infty \to \frac{kJ}{\sqrt{3}}, \qquad M_\infty \to \frac{\sqrt{3}\kappa}{k}J^2,$$

or

$$\Lambda_j \to \frac{k\omega^j}{\sqrt{3}}, \qquad A_j \to \frac{\sqrt{3}\kappa}{k}\omega^{2j}. \tag{3.24}$$

Thus, from (3.23) we have

$$A_j(k) \quad = \quad \sqrt{3}\kappa\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{k^3 + k^{-3}}; \tag{3.25}$$

3.2 Scattering matrix

Consider the modified Lax pair

$$\begin{cases} \psi_y = L\psi, \qquad L = \lambda J - Q(y) \\ \psi_t = M\psi - \psi M_\infty(\lambda). \end{cases} \tag{3.26}$$

The compatibility condition holds for any choice of matrix replacing $M_\infty(\lambda)$, i.e. the modified Lax pair gives rise to the same equation as the original Lax pair. For convenience we use the spectral parameter k. Let $\psi^\pm(y,t,k)$ be the solutions of (3.26) such that

$$\lim_{y\to\pm\infty} \psi^\pm(y,t,k) = U(k)e^{\Lambda(k)y}U^{-1}(k).$$

Note that due to the modified second equation in (3.26) the asymptotic values do not depend on t. The two solutions $\psi^+(y,t,k)$ and $\psi^-(y,t,k)$ are not linearly independent, i.e. they are related by a linear transformation when k is a point of the spectrum. Thus, the expression $\hat{\psi}^+(y,t,k)\psi^-(y,t,k)$ depends on t and k, but not on y. (From now on we write $\hat{B}$ for the inverse of a matrix B). We define the scattering matrix $T(t,k)$ by

$$T(t,k) = \hat{U}(k)\hat{\psi}^+(y,t,k)\psi^-(y,t,k)U(k), \tag{3.27}$$

The explicit form of the t-dependence of T is quite simple:

Proposition 3.1 *The time-evolution of the scattering matrix is given by*

$$T(t,k) = e^{A(k)t}T(0,k)e^{-A(k)t}. \tag{3.28}$$

Proof. From (3.27) it follows that

$$\psi^+ UT = \psi^- U. \tag{3.29}$$

Differentiating both sides with respect to t, we obtain

$$\psi_t^+ UT + \psi^+ UT_t = \psi_t^- U.$$

Replacing $\psi_t^\pm$ from the t-part of the Lax pair (3.26), we find

$$(M\psi^+ - \psi^+ M_\infty)UT + \psi^+ UT_t = (M\psi^- - \psi^- M_\infty)U.$$

In view of (3.29) this becomes

$$-\psi^+ M_\infty UT + \psi^+ UT_t = -\psi^+ UT\hat{U}M_\infty U.$$

We conclude that $T(t,k)$ evolves according to $T_t = -[T, \hat{U}M_\infty U]$, or

$$T_t = [A, T].$$

Therefore the time-evolution of the scattering matrix is given by (3.28). □

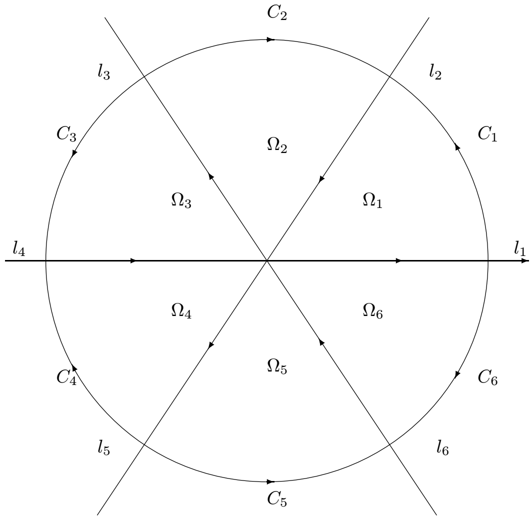

Figure 1: Domains of analyticity and integration contours.

3.3 Fundamental analytic solutions

An important role in the theory of inverse scattering play the so called *fundamental analytic solutions* (FAS) of the spectral problem. We will explain the construction of FAS for the system (2.7) or rather for the related spectral problem

$$\xi_y + \tilde{Q}\xi + [\xi, \Lambda(k)] = 0, \qquad \tilde{Q} = \hat{U}(Q - Q^*)U, \tag{3.30}$$

in the domains Ω_ν, $\nu = 1, \ldots, 6$ occupying the space in the complex k-plane separated by the rays $\{l_\nu\colon \arg(k) = (\nu - 1)\frac{\pi}{3}\}$ with $\nu = 1$, ..., 6 (see Figure 1). It is easy to notice that the components of $\tilde{Q}(y)$ are functions in the Schwartz class.

Let us introduce an ordering for each sector Ω_ν as follows. We will say that

$$r \underset{\nu}{<} s \qquad \text{iff} \qquad \Re[\Lambda_r(k)] < \Re[\Lambda_s(k)] \qquad \text{when} \qquad k \in \Omega_\nu. \tag{3.31}$$

One can verify that $1 \underset{1}{<} 2 \underset{1}{<} 3$ since for $k \in \Omega_1$, $\Re[\Lambda_1(k)] < \Re[\Lambda_2(k)] < \Re[\Lambda_3(k)]$. Similarly, $1 \underset{2}{<} 3 \underset{2}{<} 2$, etc. Then we can prove the following result:

Proposition 3.2 *The solution $\xi^\nu(y, k)$ of the following system of integral equations*

$$\begin{aligned} \xi^\nu_{rs}(y,k) &= \delta_{rs} - \int_{-\infty}^{y} e^{-[\Lambda_r(k)-\Lambda_s(k)](y-y')}[\tilde{Q}(y',k)\xi^\nu(y',k)]_{rs}\,dy', \quad r \underset{\nu}{\geq} s; & (3.32)\\ \xi^\nu_{rs}(y,k) &= -\int_{\infty}^{y} e^{-[\Lambda_r(k)-\Lambda_s(k)](y-y')}[\tilde{Q}(y',k)\xi^\nu(y',k)]_{rs}\,dy', \quad r \underset{\nu}{<} s. & (3.33)\end{aligned}$$

is also a solution of spectral problem (3.30) and is analytic for $k \in \Omega_\nu$.

Proof. The first part of the statement follows by direct computation. The analyticity follows from the fact that the real part of $\Lambda_r(k) - \Lambda_s(k)$ is nonnegative when $k \in \Omega_\nu$ and $r \underset{\nu}{\geq} s$ etc. □

By direct computation we prove the following useful result.

Proposition 3.3 *The solution $\xi(y,k)$ of (3.30) is related to a solution $\psi(y,k)$ of (2.7) via*

$$\xi(y,k) = \hat{U}(k)\psi(y,k)U(k)B(k)e^{-\Lambda(k)y}$$

where $B(k)$ is some nondegenerate matrix.

Using the two propositions we compute the asymptotics

$$\lim_{y\to\infty}[e^{-\Lambda(k)y}\xi^{\nu}(y,k)e^{\Lambda(k)y}]_{rs} = 0, \qquad r \underset{\nu}{<} s, \qquad k \in \Omega_\nu.$$

Therefore

$$\lim_{y\to\infty} e^{-\Lambda(k)y}\xi^{\nu}(y,k)e^{\Lambda(k)y} = T_\nu^-(k)D_\nu^+(k), \qquad k \in \Omega_\nu, \tag{3.34}$$

where $T_\nu^-(k)$ is a lower -triangular matrix with units on the diagonal with respect to the ν-ordering and $D_\nu^+(k)$ is a diagonal matrix. Similarly,

$$\lim_{y\to-\infty} e^{-\Lambda(k)y}\xi^{\nu}(y,k)e^{\Lambda(k)y} = S_\nu^+(k), \qquad k \in \Omega_\nu, \tag{3.35}$$

where $S_\nu^+(k)$ is an upper-triangular matrix with units on the diagonal with respect to the ν-ordering.

A more-subtle result, the proof of which follows exactly the lines of the proof of Theorem 3.3. from [26], gives the asymptotics for $k \in l_\nu$:

Proposition 3.4 *When $k \in l_\nu$, both $[S_\nu^+(k)]_{rs} = 0$ and $[T_\nu^-(k)]_{rs} = 0$ if $\Re(\Lambda_r(k)) \neq \Re(\Lambda_s(k))$.*

In other words, on the line $k \in l_\nu$ nonzero could be only the entries that have the numbers r_0, s_0 of the two eigenvalues that define l_ν as $\Re(\Lambda_{r_0}(k)) = \Re(\Lambda_{s_0}(k))$. For example, when $\nu = 2$, $r_0 = 2$, $s_0 = 3$. Notice that if in $\Omega_{\nu-1}$ $r_0 \underset{\nu-1}{<} s_0$, in the neighboring Ω_ν (l_ν separates $\Omega_{\nu-1}$ and Ω_ν) the ordering changes such that $r_0 \underset{\nu}{>} s_0$. Therefore, if the matrix $S_{\nu-1}^+(k)$ is upper triangular with respect to the $\nu-1$ ordering, it is lower-triangular with respect to the ν-ordering and we will denote it as $S_\nu^-(k)$, etc.

The asymptotics on the ray $k \in l_\nu$ therefore can be written in the form:

$$\begin{aligned}
\lim_{y\to\infty} e^{-\Lambda(k)y}\xi^{\nu}(y,k)e^{\Lambda(k)y} &= T_\nu^-(k)D_\nu^+(k), && (3.36)\\
\lim_{y\to-\infty} e^{-\Lambda(k)y}\xi^{\nu}(y,k)e^{\Lambda(k)y} &= S_\nu^+(k), && (3.37)\\
\lim_{y\to\infty} e^{-\Lambda(k)y}\xi^{\nu-1}(y,k)e^{\Lambda(k)y} &= T_\nu^+(k)D_\nu^-(k), && (3.38)\\
\lim_{y\to-\infty} e^{-\Lambda(k)y}\xi^{\nu-1}(y,k)e^{\Lambda(k)y} &= S_\nu^-(k), && (3.39)
\end{aligned}$$

where the matrices S_ν^+, T_ν^+ (resp. S_ν^-, T_ν^-) are upper-triangular (resp. lower triangular) with units on the diagonal (with respect to the ν-ordering) and the matrices $D^\pm$ are diagonal. They provide the Gauss decomposition of the scattering matrix with respect to the ν-ordering [16, 26], i.e.

$$T_\nu(k) = T_\nu^-(k)D_\nu^+(k)\hat{S}_\nu^+(k) = T_\nu^+(k)D_\nu^-(k)\hat{S}_\nu^-(k), \qquad k \in l_\nu \tag{3.40}$$

where T^-, S^- are lower-triangular matrices with units along the diagonal, T^+, S^+ are upper-triangular matrices with units along the diagonal, and $D^\pm$ are diagonal matrices. From the definition (3.27) of T and (3.40) it follows that on the lines $k \in l_\nu$,

$$\begin{aligned}
\psi^+(y,k)U(k)T_\nu^-(k)D_\nu^+(k) &= \psi^-(y,k)U(k)S_\nu^+(k), && (3.41)\\
\psi^+(y,k)U(k)T_\nu^+(k)D_\nu^-(k) &= \psi^-(y,k)U(k)S_\nu^-(k). && (3.42)
\end{aligned}$$

Now, using the results from Proposition 3.3 and (3.36)– (3.42) we can relate the eigenfunctions ξ^ν with $\psi^\pm$ for $k \in l_\nu$:

$$\begin{aligned}\xi^\nu(y,k) &= \hat{U}(k)\psi^-(y,k)U(k)S_\nu^+(k)e^{-\Lambda(k)y} \\ &= \hat{U}(k)\psi^+(y,k)U(k)T_\nu^-(k)D_\nu^+(k)e^{-\Lambda(k)y}\end{aligned} \tag{3.43}$$

$$\begin{aligned}\xi^{\nu-1}(y,k) &= \hat{U}(k)\psi^-(y,k)U(k)S_\nu^-(k)e^{-\Lambda(k)y} \\ &= \hat{U}(k)\psi^+(y,k)U(k)T_\nu^+(k)D_\nu^-(k)e^{-\Lambda(k)y}.\end{aligned} \tag{3.44}$$

The symmetries (2.13), (2.15) and (2.16) impose constraints to the eigenfunctions, the scattering matrix and its factors e.g.

$$C\xi^{\nu+2}(y,\omega k)\hat{C} = \xi^\nu(y,k), \tag{3.45}$$

$$\Gamma\bar{\xi}^{5-\nu}\left(y,\bar{\omega}\bar{k}\right)\hat{\Gamma} = \xi^\nu(y,k), \tag{3.46}$$

$$Z[\xi^{4-\nu}\left(y,-\bar{k}\right)]^\dagger\hat{Z} = \hat{\xi}^\nu(y,k), \tag{3.47}$$

etc. where the ν-indices should be taken modulo 6.

Thus, independent is only the data on one of the lines, say l_0, and all the rest can be recovered from (3.45) – (3.47).

4 Riemann-Hilbert problem

Any two solutions $\xi^\nu(y,k;t)$ and $\xi^{\nu-1}(y,k;t)$ are related on $k \in l_\nu$ due to (3.43) – (3.44):

$$\xi^\nu(y,k;t) = \xi^{\nu-1}(y,k;t)G^\nu(y,k;t), \tag{4.1}$$

$$G^\nu(y,k;t) = e^{\Lambda(k)y+A(k)t}\hat{S}_\nu^-(k)S_\nu^+(k)e^{-\Lambda(k)y-A(k)t} \qquad k \in l_\nu, \tag{4.2}$$

$$\lim_{k\to\infty}\xi^\nu(y,k;t) = \mathbf{1}. \tag{4.3}$$

We can show that the relations (4.1) – (4.3) constitute a Riemann-Hilbert problem (RHP) for the matrix-valued functions $\xi^\nu(y,k;t)$, each being analytic for $k \in \Omega_\nu$. In other words, the solution of the original Inverse Scattering Problem reduces to a RHP

Proposition 4.1 *The RHP (4.1) – (4.3) has a unique solution $\xi^\nu(y,k;t)$, with an analytic continuation for $k \in \Omega_\nu$.*

For simplicity we consider the case where $\xi^\nu(y,k;t)$ do not have any singularities in the k-plane. Suppressing y and t-dependence for convenience where possible, we can write an analytic continuation for $\xi^\eta(y,k)$ ($k \in \Omega_\eta$) as follows:

$$\xi^\eta(y,k) = \sum_{\nu=1}^{6}\frac{(-1)^{\nu+1}}{2\pi i}\oint_{\partial\Omega_\nu}\frac{\xi^\nu(y,k')\mathrm{d}k'}{k'-k}, \tag{4.4}$$

where the orientations of the contours $\partial\Omega_\nu$ are shown on Figure 1 and C_ν belong to the infinite circle. Due to (4.3) we have

$$\begin{aligned}\xi^\eta(y,k) &= \frac{1}{2\pi i}\sum_{\nu=1}^{6}\left(\int_{l_\nu}\frac{\xi^\nu(y,k')\mathrm{d}k'}{k'-k} - \int_{l_{\nu+1}}\frac{\xi^\nu(y,k')\mathrm{d}k'}{k'-k} + (-1)^{\nu+1}\int_{C_\nu}\frac{\mathbf{1}\mathrm{d}k'}{k'-k}\right) \\ &= \mathbf{1} + \frac{1}{2\pi i}\sum_{\nu=1}^{6}\left(\int_{l_\nu}\frac{\xi^\nu(y,k')-\xi^{\nu-1}(y,k')}{k'-k}\mathrm{d}k'\right),\end{aligned} \tag{4.5}$$

where the orientation of l_ν is always from 0 to ∞. From (4.1) it follows that

$$\xi^\eta(y,k) \;=\; \mathbf{I} + \frac{1}{2\pi i}\left(\sum_{\nu=1}^{6}\int_{l_\nu}\frac{\xi^{\nu-1}(y,k')[G^\eta(y,k')-\mathbf{I}]\mathrm{d}k'}{k'-k}\right), \qquad k\in\Omega_\eta. \qquad (4.6)$$

If k approaches l_η from the left and right domains correspondingly, with the Sohotski-Plemelj-type formulae we obtain that

$$\begin{aligned}
\xi^\eta(k) &= \mathbf{I} + \frac{1}{2}\xi^{\eta-1}(k)[G^\eta(k)-\mathbf{I}] + \frac{1}{2\pi i}\left(\sum_{\nu=1}^{6} P.V.\int_{l_\nu}\frac{\xi^{\nu-1}(k')[G^\nu(k')-\mathbf{I}]\mathrm{d}k'}{k'-k}\right),\\
\xi^{\eta-1}(k) &= \mathbf{I} - \frac{1}{2}\xi^{\eta-1}(k)[G^\eta(k)-\mathbf{I}] + \frac{1}{2\pi i}\left(\sum_{\eta=1}^{6} P.V.\int_{l_\nu}\frac{\xi^{\nu-1}(k')[G^\nu(k')-\mathbf{I}]\mathrm{d}k'}{k'-k}\right),\\
k &\in l_\eta, \qquad \eta=1,2\ldots,6 \qquad (4.7)
\end{aligned}$$

From (4.7) and (4.1) we finally obtain a system of integral equations for $\xi^{\eta-1}(y,k)$:

$$\begin{aligned}
\frac{1}{2}\xi^{\eta-1}(y,k)[G^\nu(y,k)+\mathbf{I}] &= \mathbf{I} + \frac{1}{2\pi i}\left(\sum_{\nu=1}^{6} P.V.\int_{l_\nu}\frac{\xi^{\nu-1}(k')[G^\nu(y,k')-\mathbf{I}]\mathrm{d}k'}{k'-k}\right),\\
k &\in l_\eta, \qquad \eta=1,2\ldots,6. \qquad (4.8)
\end{aligned}$$

The solutions of (4.8) provide $\xi^{\eta-1}(y,k)$ when $k\in l_\eta$ and then the analytic continuation is given by (4.6). Apparently the scattering data are provided by $G^\nu(y,k;t)$ given in (4.2), i.e. by $S_\nu^-(k)$ and $S_\nu^+(k)$, $k\in l_\nu$. Using the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ automorphisms (2.13) – (2.16) i.e. (3.45) – (3.47) we can restrict the scattering data to $S_1^-(k)$ and $S_1^+(k)$, $k\in l_1$. However, according to Proposition 3.4, $S_1^+(k)$ has only one nontrivial component $[S_1^+(k)]_{12}$, which is exactly the only nontrivial component of $S_1^-(k)$. Again, the automorphisms (3.45) – (3.47) relate $S_2^-(k)$ to $S_1^-(k)$ which restricts the minimal set of scattering data to only one function, say $[S_1^+(k)]_{12}$. This function is the analogue of the *reflection coefficient* used for example in the Inverse Scattering for the KdV equation [48].

The potential $Q(y,t)$ of the scattering problem can be recovered from the following result:

Proposition 4.2 *If $\xi^\nu(y,k;t)$ are the solutions of the RHP (4.1) – (4.3) then*

$$\chi^\nu(y,k;t) = U(k)\xi^\nu(y,k;t)e^{\Lambda(k)y} \qquad (4.9)$$

satisfy (2.7) with

$$Q(y,t) = -\lim_{k\to\infty}\lambda(k)\left(\chi^\nu(y,k;t)J\hat{\chi}^\nu(y,k;t)-J\right). \qquad (4.10)$$

Proof. With arguments similar to those given in [49, 50, 16] one can prove that

$$\tilde{Q}(y,t) = -\lim_{k\to\infty}\lambda(k)\left(\xi^\nu(y,k;t)J\hat{\xi}^\nu(y,k;t)-J\right). \qquad (4.11)$$

From (4.11) and the definition of $\tilde{Q}$ in (3.30) it follows that

$$Q(y,t) = Q^* - \lim_{k\to\infty}\lambda(k)\left(\chi^\nu(y,k;t)J\hat{\chi}^\nu(y,k;t)-U(k)J\hat{U}(k)\right). \qquad (4.12)$$

Next, we notice that

$$\lim_{k\to\infty}\left(\Lambda(k)-\lambda(k)J\right)=0,$$

giving

$$\lim_{k\to\infty}\left(U(k)\Lambda(k)\hat{U}(k)-\lambda(k)U(k)J\hat{U}(k)\right)=0,$$

$$\lim_{k\to\infty}\left(L_\infty-\lambda(k)U(k)J\hat{U}(k)\right)=0,$$

or

$$\lim_{k\to\infty}\left(\lambda(k)J-Q^*-\lambda(k)U(k)J\hat{U}(k)\right)=0,$$

i.e.

$$Q^*=\lim_{k\to\infty}\lambda(k)\left(J-U(k)J\hat{U}(k)\right). \tag{4.13}$$

Clearly (4.10) follows from (4.12) and (4.13). □

Corollary *From (4.10) and the fact that* $tr[(Q^*)^2]=2$*, one can find*

$$h(y,t)=\frac{1}{2}tr(QQ^*) \tag{4.14}$$

and then $f(y,t)$ *can be computed from (2.10) and* $q(x,t)$ *from (1.1). Finally,* $u(x,t)$ *can be obtained from (2.11) and (2.12).*

5 Zakharov-Shabat dressing method

The Zakharov-Shabat dressing method [49, 50, 53, 54] allows the explicit construction of a solution with singularities $\chi^\nu(y,k;t)$, starting from a given regular solution of the RHP, say $\chi_0^\nu(y,k;t)$:

$$\chi^\nu(y,k;t)=g(y,k;t)\chi_0^\nu(y,k;t). \tag{5.1}$$

Note that in our case the analyticity regions for χ^ν do not coincide with those for ξ^ν due to the nontrivial $U(k)$-factors in (4.9).

The dressing factor g is analytic in the entire complex plane, with the exception of the points of the discrete spectrum. We make the following assumptions in our construction of a dressing factor. First, we allow only simple poles of g and $\hat{g}$. For simplicity, instead of the k-dependence we are going to revert to the λ - dependence, having in mind the previously defined relation $k(\lambda)$. The automorphisms act on $g(\lambda)$ as on a group element:

$$Cg(y,\omega\lambda)\hat{C}=g(y,\lambda), \tag{5.2}$$

$$\Gamma\bar{g}\left(y,\bar{\omega}\bar{\lambda}\right)\hat{\Gamma}=g(y,\lambda), \tag{5.3}$$

$$Zg^\dagger(y,-\bar{\lambda})\hat{Z}=\hat{g}(y,\lambda), \tag{5.4}$$

From these symmetries it follows that if g or $\hat{g}$ have a pole at, say, λ_0, then they have also poles at $-\lambda_0$, $\pm\omega\lambda_0$, $\pm\omega^2\lambda_0$, $\pm\bar{\lambda}_0$, $\pm\omega\bar{\lambda}_0$, $\pm\omega^2\bar{\lambda}_0$. It also follows that $\det(g)=1$. Our next assumption will be that λ_0 can be chosen real, so that the following choice of g is possible

$$g(y,\lambda;t)=\mathbf{I}+\sum_{j=1}^{3}\frac{\alpha_j(y,t)}{\lambda-\lambda_j}, \tag{5.5}$$

$$\hat{g}(y,\lambda;t)=\mathbf{I}-\sum_{j=1}^{3}\frac{Z\alpha_j^\dagger(y,t)\hat{Z}}{\lambda-\mu_j}, \tag{5.6}$$

for some residues $\alpha_j(y,t)$ where

$$\lambda_j = \omega^{j+1}\lambda_0, \qquad \mu_j = -\bar{\lambda}_j. \tag{5.7}$$

The property (4.3) of the dressed solution is preserved. Following the ZS construction from [54] (see also [16, 5]), we represent the residues in the form

$$\alpha_p = \sum_{j=1}^{3} |n_j\rangle \hat{R}_{jp} \langle m_p| \tag{5.8}$$

$$R_{jp} = \frac{\langle m_j|n_p\rangle}{\lambda_j - \mu_p} \tag{5.9}$$

where $|n_j\rangle$ is a vector-column, and $\langle m_j|$ is a vector-row. The last is defined as

$$\langle m_j| \quad = \quad \langle m_{0j}|\hat{\chi}^{(j)}(y,\lambda_j). \tag{5.10}$$

where $\langle m_{0j}|$ is a constant vector and $\hat{\chi}^{(j)}$ is the solution of the adjoint problem, analytic at λ_j. We define

$$|n_j\rangle \equiv Z\overline{|m_j\rangle} = \chi^{(\bar{j})}(\mu_j)|n_{0j}\rangle, \qquad \text{where} \qquad |m_j\rangle = (\langle m_j|)^T \tag{5.11}$$

and $\chi^{(\bar{j})}(\mu_j) = Z[\hat{\chi}^{(j)}(\lambda_j)]^\dagger \hat{Z}$, i.e. the constant vectors are related as follows:

$$|n_{0j}\rangle = Z\overline{|m_{0j}\rangle}. \tag{5.12}$$

$\chi^{(\bar{j})}$ is an eigenfunction, analytic at μ_j, since it is obtained by the automorphism (2.16) from $\hat{\chi}^{(j)}$, analytic at λ_j (and λ_j and μ_j are related by the same automorphism). With these definitions one can easily check that the matrix R (5.9) is Hermitian: $R = R^\dagger$.

The residues of $\hat{g}$ in (5.6) can be computed and represented in the form

$$Z\alpha_p^\dagger(y,t)\hat{Z} = \sum_{j=1}^{3} |n_p\rangle \hat{R}_{pj} \langle m_j| \tag{5.13}$$

One can now verify that $g\hat{g} = \mathbf{I}$. Indeed, this is satisfied iff at any singular point, say $\lambda = \mu_p$ the corresponding residues satisfy

$$\begin{aligned} \left(\mathbf{I} + \sum_{j=1}^{3} \frac{\alpha_j}{\mu_p - \lambda_j}\right)(Z\alpha_p^\dagger \hat{Z}) \quad &= \quad 0, \\ \alpha_p \left(\mathbf{I} - \sum_{j=1}^{3} \frac{(Z\alpha_j^\dagger \hat{Z})}{\lambda_p - \mu_j}\right) \quad &= \quad 0, \end{aligned} \tag{5.14}$$

etc. The identities (5.14) can be verified by (5.8) – (5.13). Our construction for g (5.5) – (5.6) satisfies also the automorphism (5.4). The automorphism (5.2) gives an additional relation between the constant vectors $|n_{0j}\rangle$:

$$|n_{0,j+1}\rangle = C|n_{0j}\rangle. \tag{5.15}$$

From (5.3) we have the further restriction

$$\Gamma\overline{|n_{01}\rangle} = |n_{02}\rangle, \qquad \Gamma\overline{|n_{03}\rangle} = |n_{03}\rangle. \tag{5.16}$$

The relations (5.15) show that only one of the vectors, say $|n_{01}\rangle$, determines the others (and also $\langle m_{0j}|$ due to (5.12)). The components $n_{01;j}$ ($j = 1, 2, 3$) of the vector $|n_{0,1}\rangle$ are not independent: as a consequence of (5.16) and (5.15) they satisfy $n_{01;1} = \bar{n}_{01;2}$ and $n_{01;3} = \bar{n}_{01;3}$. Thus, if $n_{01;3} \neq 0$ we can take $n_{01;3} = 1$ and then only one complex number, $n_{01;1} \equiv \rho_0$ determines

$$|n_{0,1}\rangle = (\rho_0, \bar{\rho}_0, 1)^T \tag{5.17}$$

and therefore $|n_{0,j}\rangle$ and $|m_{0,j}\rangle$.

Let us denote $Q_0 \equiv Q^* h_0$, where h_0 is the 'undressed' potential. Then we have the following equation for g:

$$g_y + Qg - gQ_0 + \lambda[g, J] = 0, \tag{5.18}$$

satisfied identically for any z, i.e. any λ. This equation is satisfied for the construction (5.5) due to the fact that

$$\begin{aligned} (\langle m_j|)_y - \langle m_j|(Q_0 - \lambda_j J) &= 0, & (5.19)\\ (|n_j\rangle)_y + (Q_0 - \mu_j J)|n_j\rangle &= 0. & (5.20) \end{aligned}$$

The identity (5.18) for $\lambda \to \infty$ gives

$$Q = Q_0 + [J, \alpha_1 + \alpha_2 + \alpha_3]. \tag{5.21}$$

Multiplication of (5.21) by Q^* followed by taking of a trace gives

$$h(y,t) = h_0(y,t) + \frac{1}{2}\sum_{p,j=1}^{3} \hat{R}_{pj}\langle m_j|[Q^*, J]|n_p\rangle. \tag{5.22}$$

The reality of (5.22) can be checked once again by using the introduced properties of R, $|n_j\rangle$ and $|m_j\rangle$.

Thus, to each discrete eigenvalue, which represents an 'action' variable and is determined by only one real value, λ_0 (5.7) one can put into correspondence a conjugated 'angle' variable that is given by the independent component of the associated constant vector, ρ_0 (5.17). This accounts for the scattering data related to the discrete spectrum. Since λ_0 is real, in the k - plane the corresponding discrete spectrum value k_0 is also on the real line. It seems that k_0 is on the continuous spectrum, since l_1 and l_4 are on the real line. However, the continuous spectrum for the actual spectral problem (2.7) does not consist of the rays l_ν shown in Figure 1, but rather of the lines $\tilde{l}_\nu$ through the origin, where $\arg(k) = \pi/6 + \nu\pi/3$. This is due to the relation between the corresponding eigenfunctions (4.9) involving nontrivial k-dependence in the factor $e^{\Lambda(k)y}$.

To repeat the dressing procedure N times we will need N copies of the same type scattering data, $\{\rho_{0,j}, \lambda_{0,j}\}$ $j = 1, \ldots, N$. The result will be the N-soliton solution.

For example, the one-soliton solution can be obtained as follows. One can start the dressing from the 'trivial' solution $u \equiv 0$, i.e. $h_0 = 1$. Then there is a global analytic solution of the spectral problem, symbolically $e^{L_\infty(k)y + M_\infty(k)t}$, i.e.

$$\Psi(y,k;t) = U(k)e^{\Lambda(k)y + A(k)t}\hat{U}(k).$$

Since the last factor does not depend on y and t we can take simply (for all sectors)

$$\chi_0(y,k;t) = U(k)e^{\Lambda(k)y + A(k)t},$$

The entries of $U(k)$ are

$$\begin{aligned}
U_{1p}(k) &= (\omega^2-1)\lambda + (\omega-1)\Lambda_p + \omega^2,\\
U_{2p}(k) &= (\omega-1)\lambda + (\omega^2-1)\Lambda_p + \omega,\\
U_{3p}(k) &= 3(\Lambda_p^2 + \lambda\Lambda_p + \lambda^2) - 1.
\end{aligned}$$

The potential for the one-soliton solution is given by (5.22), then $f(y,t)$ (t is viewed as an additional parameter rather than a second independent variable) can be computed as a solution of linear first order ODE (2.10). Then $q(x,t)$ from (1.1) and therefore $u(x,t)$ can be obtained from (2.11) and (2.12).

6 Discussion

In the presented analysis we formulated the inverse scattering problem for the DP equation. We defined a set of scattering data for the problem:

(i) on the continuous spectrum the coefficient $[S_1^+(k)]_{12}$, $k \in l_1$;

(ii) on the discrete spectrum the set $\{\rho_{0,j}, \lambda_{0,j}\}$, $j = 1, \ldots, N$, where N is the number of the discrete eigenvalues.

The scattering data uniquely define the potential $h(y,t)$ and therefore a general N-soliton Schwartz-class solution $u(x,t)$ of DP equation. As it is the case for all systems, integrable by the Inverse Scattering Method, the mapping between the solution and the scattering data allows the interpretation of a generalised Fourier transform [1, 25, 26, 52].

The peakon solutions (peaked solitons) appear in the limit $\kappa \to 0$ [44], although such limit in the space of the scattering data will require further considerations.

The N-soliton solution of the Degasperis-Procesi equation is obtain in [45] by Hirota's method. E.g. the 1-soliton solution is

$$\begin{aligned}
u(y,t) &= \frac{\frac{8\kappa}{a_1}(a_1^2-1)(a_1^2-\frac{1}{2})}{\cosh\xi_1 + 2a_1 - \frac{1}{a_1}} && (6.23)\\
x &= y + \ln\left(\frac{\gamma_1 + 1 + (\gamma_1-1)e^{\xi_1}}{\gamma_1 - 1 + (\gamma_1+1)e^{\xi_1}}\right), && (6.24)
\end{aligned}$$

where the quantities ξ_i are

$$\xi_i = \nu_i\left(y - \frac{3\kappa}{1-\nu_i^2}t - y_{i0}\right), \qquad i = 1, 2, \ldots, N, \tag{6.25}$$

where ν_i and y_{i0} are constants, representing the scattering data,

$$a_i = \sqrt{\frac{1-\frac{1}{4}\nu_i^2}{1-\nu_i^2}}$$

as well as $\gamma_1 = \sqrt{\frac{(2a_1-1)(a_1+1)}{(2a_1+1)(a_1-1)}}$ are also constants depending on the scattering data.

The quantities, related to the 1-soliton solution (6.23) can also be computed:

$$\begin{aligned}
f(y,t) &= 1 + \frac{3\nu_1^2 e^{\xi_1}}{a_1(1-\nu_1^2)\left(1 + \frac{2}{a_1}e^{\xi_1} + e^{2\xi_1}\right)}\\
h(y,t) &= \frac{\cosh 2\xi_1 + \frac{4}{a_1}\cosh\xi_1 - \frac{3\nu_1^3}{a_1(1-\nu_1^2)}\sinh\xi_1}{\left(\cosh\xi_1 + \frac{2+\nu_1^2}{a_1(1-\nu_1^2)}\right)\left(\cosh\xi_1 + \frac{2}{a_1}\right)}.
\end{aligned}$$

One can establish that ν_i that determines the dispersion relation (6.25) is related to the spectral parameters,

$$\nu_i = \frac{1}{\sqrt{3}}\left(k_i + \frac{1}{k_i}\right),$$

where k_i is a the real discrete eigenvalue, corresponding to the real $\lambda_{0,i} = \lambda(k_i)$. The computation of the soliton solution by dressing method requires development of additional techniques addressing the technical difficulties arising in the computation.

There are interesting multidimensional versions of the DP equation. They are in general nonintegrable, but admit singular (peakon-type) solutions [32].

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