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# On Adjoint Entropy of Abelian Groups\*

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## Abstract

The theory of endomorphism rings of algebraic structures allows, in a natural way, a systematic approach based on the notion of entropy borrowed from dynamical systems. In the present work we introduce a ‘dual’ notion based upon the replacement of the finite groups used in the definition of algebraic entropy, by subgroups of finite index. The basic properties of this new entropy are established and a connection to Hopfian groups is investigated.

## 1 Introduction

The notion of, but not the name, algebraic entropy for an endomorphism of an Abelian group, first appeared in a brief sketch at the end of a paper by Adler, Konheim and McAndrew [1] on entropy of continuous self-maps of compact topological spaces. In a follow-up paper in 1975, Michael Weiss, [19], elaborated on the ideas in [1] and formally introduced the name algebraic entropy. His work laid down the basic properties of this entropy and revealed the fundamental connection between the algebraic entropy of an endomorphism and the topological entropy of its adjoint under Pontryagin duality. In recent times these concepts have been re-examined and developed significantly – see for example, [7], [16] and [17]. The fundamental concept in algebraic entropy is the notion of the trajectory of a finite subgroup of a group  $G$  under an endomorphism  $\phi$  of  $G$ ; this essentially restricts the notion to torsion groups. However, Peter Vámos has pointed out the possibility of an analogy between algebraic entropy and multiplicity in the sense used in [18] and has suggested that a ‘dual’ concept of entropy may be of interest. This paper is focused on developing the basic properties of this notion and determining some elementary but fundamental results. The dualizing principle is to replace ‘finite subgroups’ by ‘subgroups of finite index’. One immediate consequence is that the new notion is then not restricted to torsion groups. It is an easy exercise to show that a non-divisible Abelian group  $G$  always has subgroups of finite index; indeed in ‘most situations’ the number of such subgroups is large, being equal to  $2^{|G|}$ . It is also worth remarking that one can approach this dual entropy by working with the algebraic entropy of the adjoint mapping arising from the theory of Pontryagin duality – such an approach has been exploited in [6] and there is, of course, some overlap between the present work and that paper; although one can often work directly with subgroups of finite index, we have found it convenient, particularly in the early part of Section 2, to quote without proof, results from that work.

The choice of name for this new concept is not without possible contention. As was pointed out in [6], the new notion does not behave as a perfect duality. Given the connections that we shall establish here with Hopfian groups and those known to exist between algebraic entropy and co-Hopfian groups, there is some argument in favour of calling the new entropy “algebraic co-entropy” or “co-algebraic entropy”. However for this to be perfectly logical, one would need to interchange

the well-established terminologies for Hopfian and co-Hopfian groups. Consequently we feel it better to use the terminology “adjoint entropy” introduced in [6], even though we do not need to explicitly use the connection to Pontryagin duality in this work.

In the first section of this work we provide the basic definition of this adjoint entropy, establish its existence and derive some elementary properties of it. A point of particular interest is the connection between groups of zero adjoint entropy and Hopfian groups. (Recall that a group, not necessarily Abelian, is said to be Hopfian if each epimorphism is an automorphism.) We also show that the adjoint entropy of an Abelian  $p$ -group is zero if, and only if, the group is the direct sum of a divisible group and a finite group; if the reduced part of an Abelian  $p$ -group is infinite, then the adjoint entropy is also infinite. This is in strong contrast to the situation for algebraic entropy where groups of power the continuum exist having zero algebraic entropy. In the second section we examine the situation for torsion-free groups showing, *inter alia*, that the classification of torsion-free Abelian groups with zero adjoint entropy is essentially impossible. In the final section we deal with mixed groups and exploit an idea in an unpublished result of the late A.L.S. Corner to establish that in the local situation, a mixed Abelian group  $G$  with countable torsion-free quotient, having zero adjoint entropy, necessarily splits.

Finally we remark that in the sequel all groups are additively written Abelian; the fundamental notions for such groups, which we use without comment, may be found in the standard texts [9, 10] and [15].

## 2 Basic Definitions and Elementary Properties

In this section we introduce the basic notions and derive important, but elementary, consequences. There is considerable overlap with the paper [6] and we shall refer freely to that work for proofs of many of the results in this section. Suppose that  $f$  is an endomorphism of  $G$  and  $N$  is a subgroup of finite index in  $G$ , then if  $f^{-1}(N)$  denotes the pre-image of  $N$  under  $f$  and  $f^{-k}(N)$  means the  $k^{\text{th}}$  pre-image of  $N$ , so that  $f^{-k}(N) = \underbrace{f^{-1}(f^{-1} \dots (f^{-1}(N)))}_n$ , then  $f^{-k}(N)$  is again of finite index in  $G$ .

**Definition 2.1** *If  $N$  is a finite index subgroup of  $G$  and  $f \in \text{End}(G)$ , then we define the  $n^{\text{th}}$ -cotrajectory of  $f$  with respect to  $N$ , by*

$$C_n(f, N) = N \cap f^{-1}(N) \cap \dots \cap f^{-(n-1)}(N).$$

*If there is no danger of confusion, we simply write  $C_n$  in place of  $C_n(f, N)$ .*

Since  $G/C_n(f, N)$  can be embedded into  $\prod_{i=0}^{n-1} G/f^{-i}(N)$  where  $f^0(N) = N$ , it follows that  $G/C_n(f, N)$  is finite. Set  $c_n(f, N) = |G/C_n(f, N)|$ , then  $c_n(f, N)$  is a natural number, which we shall frequently

abbreviate to  $c_n$ . In the following we always assume that  $N$  is a finite index subgroup of  $G$ . It is easy to see that

$$C_1(f, N) \geq C_2(f, N) \geq \cdots C_n(f, N) \geq \cdots$$

By the usual isomorphism theorem, we have

$$\frac{G/C_{n+1}}{C_n/C_{n+1}} \cong \frac{G}{C_n}.$$

Since  $G/C_{n+1}(f, N)$  is finite, then so is  $C_n(f, N)/C_{n+1}(f, N)$ . Thus we may write  $c_{n+1} = c_n \delta_n$  where  $\delta_n = |C_n(f, N)/C_{n+1}(f, N)|$ .

We have the following relationships; for a proof see Lemma 2.2 in [6].

**Proposition 2.2** (i) For each natural number  $n$ ,  $\delta_{n+1}$  divides  $\delta_n$ ;  
(ii) the sequence  $\{c_n\}$  is either stationary or  $c_{n+1} = c_n \delta$  for some integer  $\delta > 1$ , for all  $n$  large enough. In particular,  $|G/C_n(f, N)| = b_0 \delta^{n-k}$  for sufficiently large  $n$ , where  $b_0$  and  $k$  depend only on the finite index subgroup  $N$ .

For a fixed finite index subgroup  $N$  of  $G$ , and an endomorphism  $f$  we define the real number

$$I_n(f, N) = \log |G/C_n(f, N)|.$$

Clearly we have

$$0 \leq I_1(f, N) \leq I_2(f, N) \leq I_3(f, N) \leq \cdots$$

Hence, we may formally define the cotrajectory of  $f$  with respect to  $N$

$$I(f, N) = \lim_{n \rightarrow \infty} \frac{I_n(f, N)}{n}.$$

The next result shows that the definition of a cotrajectory makes sense; for a proof see Proposition 2.3 in [6].

**Proposition 2.3** Given an endomorphism  $f$  of  $G$ , and a finite index subgroup  $N$  of  $G$ , then either

- (1)  $I(f, N) = 0$ , which happens exactly if the  $f$ -cotrajectory of  $N$  is stationary; or
- (2)  $I(f, N) = \log(\delta)$ , where  $\delta = |C_n(f, N)/C_{k+1}(f, N)|$  for all large enough  $k$ , which happens exactly if the sequence of  $f$ -cotrajectories of  $N$  is strictly decreasing.

It is easy to show that  $I(f, N)$  is actually equal to  $\inf(I_n(f, N)/n)$ ; either observe that the sequence  $I_n(f, N)$  is sub-additive and apply the well-known Fekete Lemma or check directly that the sequence  $I_n(f, N)/n$  is eventually monotonic decreasing.

**Definition 2.4** We define the adjoint entropy,  $\text{ent}^*$ , of an endomorphism  $f$  of a group  $G$  as

$$\text{ent}^*(f) = \sup_{N \in \mathcal{N}} I(f, N),$$

where  $\mathcal{N}$  is the set of all subgroups of finite index in  $G$ . The adjoint entropy of the group  $G$  is defined as

$$\text{ent}^*(G) = \sup_{f \in \text{End}(G)} \text{ent}^*(f).$$

It follows immediately from the definition that every endomorphism of a group  $G$  has zero adjoint entropy if  $G$  is either finite or divisible; the latter following from the fact that the only finite index subgroup of a group  $G$  is the whole group  $G$  itself.

Our next proposition is intended to give an overview of the basic properties of this new adjoint entropy. For proofs of the various parts we shall again refer to the appropriate sections of [6]; variants of some of these will appear in the forthcoming thesis of Gong [13].

**Proposition 2.5** (i) For any group  $G$ , if  $H$  is an  $f$ -invariant subgroup of  $G$ , then  $\text{ent}^*(f) \geq \text{ent}^*(\bar{f})$ , where  $\bar{f}: G/H \rightarrow G/H$  is the induced endomorphism;

(ii) if  $H$  is an  $f$ -invariant finite index subgroup of  $G$ , then  $\text{ent}^*(f) \geq \text{ent}^*(f|_H)$ ;

(iii) for every nonnegative integer  $t$ ,  $\text{ent}^*(f^t) = t \cdot \text{ent}^*(f)$ . If  $f$  is an automorphism,  $\text{ent}^*(f^t) = |t| \cdot \text{ent}^*(f)$  for every integer  $t$ . As a result, either  $\text{ent}^*(G) = 0$  or  $\text{ent}^*(G) = \infty$ ;

(iv) for an arbitrary group  $G$  and an endomorphism  $\phi$ , either  $\text{ent}^*(\phi) = 0$  or  $\text{ent}^*(\phi) = \infty$ ;

(v) if  $V$  is a vector space over the field of  $p$  elements, then an endomorphism  $\phi$  has zero adjoint entropy if, and only if,  $\phi$  is algebraic.

*Proof* For parts (i) and (ii) see Lemmas 4.7 and 4.9 in [6]; a proof of the first statement of (iii) is given in [6, Lemma 4.4], while the second statement is an immediate corollary. Parts (iv) and (v) are much deeper results and may be found in Theorems 7.5 and 7.6 of [6].  $\square$

At this stage it seems appropriate to include some explicit examples.

**Example 2.6** (i) Let  $B$  be the standard basic group,  $B = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$ , where the order of  $e_i$  is  $o(e_i) = p^i$ . If  $\sigma, \tau$  are the endomorphisms defined by  $\sigma(e_i) = pe_{i+1}$ ;  $\tau(e_{i+1}) = e_i$  for  $i \geq 1$  and  $\tau(e_1) = 0$  (i.e.  $\sigma, \tau$  are the forward and backward Bernoulli shifts respectively), then  $\text{ent}^*(\sigma) = 0$ , while  $\text{ent}^*(\tau) = \infty$ ;

(ii) if  $V$  is a countable dimensional vector space over the field of  $p$  elements and  $\phi$  is the forward Bernoulli shift on  $V$ , then  $\text{ent}^*(\phi) = \infty$ .

*Proof* Both results may be obtained by straightforward, if somewhat tedious, direct calculation. We prefer to base our proofs on [6]. From Proposition 2.5 (i) we have that  $\text{ent}^*(\tau) \geq \text{ent}^*(\bar{\tau})$  and it

follows from [6, Proposition 6.2] that the latter term is  $\infty$ . Note that the induced map  $\bar{\sigma}$  induced on  $B/pB$  by  $\sigma$  is the zero map and the result follows from Corollary 7.7 in [6].

Part (ii) follows immediately from Proposition 2.5 (iv) and (v) since  $\phi$  is not algebraic.  $\square$

Our first observation is a simple characterization of when an endomorphism has zero adjoint entropy.

**Lemma 2.7** *Let  $f$  be an endomorphism of  $G$ . Then  $f$  has zero adjoint entropy if, and only if, for any finite index subgroup  $N$  of  $G$ , there is a positive integer  $K$ , depending on  $N$ , such that  $C_n(f, N) \leq f^{-n}(N)$  for all  $n \geq K$ .*

*Proof* If  $\text{ent}^*(f) = 0$ , then for any given finite index subgroup  $N$ , there is some natural number  $K$  such that for all  $n \geq K$ , we have  $\delta_n = |C_n(f, N)/C_{n+1}(f, N)| = 1$ . This implies that  $C_n(f, N) = C_{n+1}(f, N)$  and so  $C_n(f, N) \subset f^{-n}(N)$ . The converse is clear.  $\square$

The result above may be reformulated in a way which makes it more convenient for some applications, as follows:

**Corollary 2.8** *The adjoint entropy of an endomorphism  $\phi$  is zero if, and only if, for any finite index subgroup  $N$  of  $G$ , there exists a finite index subgroup  $N_1$  of  $G$  with  $N_1 \leq N$  such that  $\phi(N_1) \leq N_1$ .*

*Proof* For the necessity it suffices to take  $N_1 = C_K(f, N)$ , where the integer  $K$  is chosen as in Proposition 2.7 above. For the sufficiency, note that if  $N_1 \leq N_2$ , then  $I(\phi, N_1) \geq I(\phi, N_2)$ . So it suffices to show that  $\phi(N_1) \leq N_1$  implies  $I(\phi, N_1) = 0$ . But this is obvious, since  $N_1 \leq \phi^{-1}(N_1)$ .  $\square$

Our next result is a weak form of the so-called Addition Theorem – see [7], Section 3.

**Proposition 2.9** *Let  $f$  be an endomorphism of  $G$ ,  $H$  an  $f$ -invariant subgroup of  $G$ , and  $\bar{f} : G/H \rightarrow G/H$  the induced endomorphism. If  $\text{ent}^*(f|_H) = 0$ , then  $\text{ent}^*(f) = \text{ent}^*(\bar{f})$ .*

*Proof* For any finite index subgroup  $N/H$  of  $G/H$ ,  $N$  is also a finite index subgroup of  $G$ , so we have

$$I_n(\bar{f}, N/H) = \frac{1}{n} \log \left| \frac{G/H}{C_n(\bar{f}, N/H)} \right| = \frac{1}{n} \log \left| \frac{G/H}{(C_n(f, N)/H)} \right| = I_n(f, N).$$

Hence, by definition,  $\text{ent}^*(f) \geq \text{ent}^*(\bar{f})$ .

Our next step is to prove the reverse inequality,  $\text{ent}^*(f) \leq \text{ent}^*(\bar{f})$ . For any finite index subgroup  $N$  of  $G$ , the intersection  $N \cap H$  is of finite index in  $H$ . Since  $\text{ent}^*(f|_H) = 0$ , by Lemma 2.7 we have some fixed natural number  $m$  such that  $(N \cap H) \cap (f|_H)^{-1}(N \cap H) \cap \dots \cap (f|_H)^{-(m-1)}(N \cap H) \leq (f|_H)^{-m}(N \cap H)$ . This is equivalent to  $H \cap C_m(f, N) \leq H \cap f^{-m}(N)$ , since  $H$  is  $f$ -invariant

in  $G$ . As  $N$  is of finite index in  $G$ , there exists a finite index subgroup  $M$  of  $G/H$  such that  $f^{-m}(N) \geq C_m(f, N) \cap M$ , e.g. let  $M = H + C_{m+1}(f, N)$ . This gives

$$f^{-1}(C_m(f, N)) \geq C_m(f, N) \cap M$$

and by induction on  $k > 0$ , we get

$$f^{-k}(C_m(f, N)) \geq C_m(f, N) \cap C_k(f, M).$$

Now let  $n > m$  say,  $n = m + k$  for some  $k > 0$ . Then  $C_n(f, N) \geq C_m(f, N) \cap C_k(f, M)$ , whence we have  $I_n(f, N) \leq I_m(f, N) + I_k(f, M)$ . But  $m$  is fixed, so, by letting  $n \rightarrow \infty$ ,  $(1/n)I_m(f, N) = 0$ . We thus can deduce that  $I(f, N) \leq I(f, M) = I(\bar{f}, M/H) \leq \text{ent}^*(\bar{f})$  as required.  $\square$

There is, however, no possibility of the Addition Theorem holding in general for adjoint entropy: as we noted above, the adjoint entropy of a divisible group is necessarily 0. As observed in [6], if  $B$  is a standard basic  $p$ -group and  $D$  is its divisible hull, then any endomorphism  $\phi$  of  $B$  extends to an endomorphism of  $D$ ; in particular the left shift on  $B$  extends to a mapping  $\psi$  say, on  $D$  and  $\text{ent}^*(\psi) = 0$  while  $\text{ent}^*(\psi|_B) = \text{ent}^*(\phi) = \infty$ . This same example also shows that even the weak version of Addition Theorem stating that  $\text{ent}^*(\psi) \geq \text{ent}^*(\psi|_B)$ , fails. However in the special case of pure subgroups, we do in fact have such a result. We need some preliminary lemmas before embarking on this proof.

**Lemma 2.10** *Let  $B$  be a pure subgroup of  $G$ , then for any positive integer  $n$ ,  $B/nB$  is a direct summand of  $G/nB$*

*Proof* First we show that  $B/nB$  is pure in  $G/nB$ . Suppose that  $b + nB = k(g + nB) = kg + nB$ , then  $b = kg + nb_1$  for some  $b_1 \in B$ . Thus,  $b - nb_1 = kg = kb_2$  for some  $b_2 \in B$  by the purity of  $B$  in  $G$ . So  $b + nB = kb_2 + nB$  and  $B/nB$  is pure in  $G/nB$  as claimed. On the other hand,  $B/nB$  is bounded by  $n$ , so  $B/nB$  is a direct summand of  $G/nB$ .  $\square$

We now fix some notation for the remainder of this discussion: let  $B$  be a pure subgroup of  $G$ ,  $M$  a finite index subgroup of  $B$  and  $n$  a natural number such that  $nB \leq M$ . Let  $X$  be a fixed but arbitrary subgroup of  $G$  such that  $\frac{G}{nB} = \frac{B}{nB} \oplus \frac{X}{nB}$ .

**Lemma 2.11** *There is an injection from the set of all finite index subgroups of  $B$  into the set of all finite index subgroups of  $G$ , given by  $M \mapsto M + X$ .*

*Proof* Firstly, we show that  $M + X$  is of finite index in  $G$ . We claim that for any  $g + (M + X)$ ,  $g \in G$ , there is a  $b \in B$  such that  $g + (M + X) = b + (M + B)$ : since  $\frac{G}{nB} = \frac{B}{nB} \oplus \frac{X}{nB}$ , then  $g + nB = b + nB + x + nB$ . In particular,  $g = b + x + nb_1$  for some  $b_1 \in B$ . Thus,  $g + (M + X) =$



$b + x + nb_1 + (M + X) = b + (M + X)$  since  $nb_1 \in nB \leq M$ . Now consider a mapping  $f$  from  $B$  to  $\frac{G}{M+X}$  by  $f(b) = b + (M + X)$ . Clearly it is a homomorphism and as we have just shown,  $f$  is surjective; the kernel of  $f$  is easily seen to be  $\{b \in B | b + (M + X) = 0\}$ . Thus,  $\ker f = (M + X) \cap B$ . Assuming that we have established that  $(M + X) \cap B = M$ , we have the isomorphism

$$\frac{B}{M} \cong \frac{G}{M + X}.$$

Hence  $M + X$  is finite index in  $G$ . Furthermore, the mapping  $M \mapsto M + X$  is then an injection, since the composite  $M \mapsto M + X \mapsto (M + X) \cap B$  is the identity.

Thus it remains to show that  $(M + X) \cap B = M$ . By the modular law,  $(M + X) \cap B = M + (X \cap B)$ , so it suffices to prove that  $X \cap B \leq M$ . To see this, pick any element  $b \in X \cap B$ , then  $b + nB \in B/nB$ . In the same way,  $b + nB \in X/nB$  for  $b \in X$ . Thus,  $b + nB \in \frac{B}{nB} \cap \frac{X}{nB} = 0$ . Hence  $b \in nB \leq M$ . Thus,  $(M + X) \cap B = M$ .  $\square$

Hence, we have the following for pure  $\phi$ -invariant subgroups of a group.

**Proposition 2.12** *If  $B$  is a pure  $\phi$ -invariant subgroup of  $G$ , then  $\text{ent}^*(\phi) \geq \text{ent}^*(\phi|_B)$ .*

*Proof* Let  $M$  be an arbitrary finite index subgroup of  $B$ . Then it follows from Lemma 2.11 that there is a finite index subgroup  $N$  of  $G$  such that  $N \cap B = M$ . Assume for the moment that we have shown that  $B \cap C_k(\phi, N) = C_k(\theta, M)$ , where we have written  $\theta$  for the restriction  $\phi|_B$ . Since  $I(\theta, M) = \inf(I_n(\theta, M)/n)$ ,  $I(\theta, M) \leq I_n(\phi, N)/n$  for all  $n$ . Thus  $I(\theta, M) \leq \inf I_n(\phi, N)/n = I(\phi, N)$ . Hence

$$\text{ent}^*(\theta) = \sup_M I(\theta, M) \leq \sup_N I(\phi, N) \leq \sup_{N^*} I(\phi, N^*) = \text{ent}^*(\phi),$$

where the first supremum ranges over all finite index subgroups  $M$  of  $B$ , the second ranges over the corresponding finite index subgroups  $N$  which have the form  $N = M + X$  as given by Lemma 2.11 and the third supremum ranges over all finite index subgroups  $N^*$  of  $G$ . The proof is completed by the following lemma.  $\square$

**Lemma 2.13** *Let  $N$  be a finite index subgroup of  $G$ ,  $B$  a  $\phi$ -invariant subgroup of  $G$  and set  $M = N \cap B$ . Then if  $\theta = \phi|_B$ , we have  $C_k(\theta, M) = B \cap C_k(\phi, N)$ .*

*Proof* The proof is based on a straightforward computation of the cotrajectories. First we note that  $\phi^{-i}(N) \cap B = \theta^{-i}(N \cap B)$ : if  $x \in \phi^{-i}(N) \cap B$ , then  $\phi^i(x) \in N, x \in B$ , thus,  $\phi^i(x) = \theta^i(x) \in B$  as  $\theta B \leq B$ , hence  $\theta^i(x) \in N \cap B$ . On the other hand, if  $x \in \theta^{-i}(N \cap B)$ , then  $\theta^i(x) \in N \cap B$ .

Thus,  $x \in B$  and  $\phi^i(x) = \theta^i(x) \in N$ . This implies  $x \in \phi^{-i}(N) \cap B$  as claimed. Hence we have

$$\begin{aligned}
B \cap C_k(\phi, N) &= (N \cap \phi^{-1}(N) \cap \dots \cap \phi^{-(k-1)}(N)) \cap B \\
&= (N \cap B) \cap (\phi^{-1}(N) \cap B) \cap \dots \cap (\phi^{-(k-1)}(N) \cap B) \\
&= (N \cap B) \cap (\theta^{-1}(N \cap B) \cap \dots \cap (\theta^{-(k-1)}(N \cap B))) \\
&= C_k(\theta, M)
\end{aligned}$$

□

An immediate consequence of Proposition 2.12 is the rather obvious, but useful:

**Corollary 2.14** *If  $B$  is a pure subgroup of  $A$  and every endomorphism of  $B$  lifts to an endomorphism of  $A$ , then  $\text{ent}^*(B) \leq \text{ent}^*(A)$ ; in particular if  $B$  is a direct summand of  $A$  or if  $A$  is the completion of  $B$  in the natural or  $p$ -adic topologies,  $\text{ent}^*(B) \leq \text{ent}^*(A)$ .*

The injection in Lemma 2.11 above, need not necessarily be a bijection, but it is if we make the additional assumption that the subgroup  $B$  is dense in  $G$  i.e.  $G/B$  is divisible.

**Proposition 2.15** *Let  $B$  be a pure dense subgroup of  $G$ , then there is a bijection between the set of all finite index subgroups of  $B$  and those of  $G$ . In particular, if  $B$  is basic in  $G$ , there is such a bijection.*

*Proof* Suppose that  $N$  is an arbitrary finite index subgroup of  $G$ . Then there is an integer  $n$  such that  $nG \leq N$ . Set  $M = N \cap B$  and observe that  $N$  is of finite index in  $B$  and that  $nB \leq N \cap B = M$ . Since  $B$  is dense in  $G$ , we have that  $G = B + kG$  for all natural numbers  $k$ ; in particular  $G = B + nG$ . It follows easily that  $\frac{G}{nB} = \frac{B}{nB} \oplus \frac{nG}{nB}$ . Thus the mapping  $M \mapsto M + nG$  is an injection by Lemma 2.11 above. To show that this mapping is a bijection, we note that by modularity

$$M + nG = (N \cap B) + nG = N \cap (B + nG) = N \cap G = N.$$

The statement about basic subgroups is then, of course, immediate. □

In the above situation we can improve on Proposition 2.12

**Proposition 2.16** *If  $B$  is a pure dense subgroup of the group  $G$  and  $\phi$  is an endomorphism of  $G$  leaving  $B$  invariant, then  $\text{ent}^*\phi = \text{ent}^*\phi|_B$ .*

*Proof* The proof is a simple modification of the proof of Proposition 2.12 above. When  $B$  is dense in  $G$ , we have just seen that the correspondence between finite index subgroups of  $B$  and those of  $G$  is given by  $M \mapsto N + kG$  where  $kG \leq N$ . It follows that  $kG \leq C_n(\phi, N)$  for all  $n$  and so  $I_n(\phi|_B, M) = I_n(\phi, N)$ . Moreover, as the correspondence is a bijection, the range of the supremum

used in the argument in Proposition 2.12, is the whole of the set of finite index subgroups of  $G$ ; this ensures the desired equality.  $\square$

There is another situation where we can relate the adjoint entropy of a group and a non-pure subgroup:

**Proposition 2.17** *For any group  $G$  and any endomorphism  $\phi$  of  $G$ ,  $\text{ent}^*(\phi) \geq \text{ent}^*(\phi|_{p^n G})$  for all positive integers  $n$ . Moreover  $\text{ent}^*(p^n G) \leq \text{ent}^*(G)$ .*

*Proof* Let  $n$  be a fixed but arbitrary integer. For any finite index subgroup  $M$  of  $p^n G$ , consider the subgroup  $N = p^{-n}M = \{g \in G | p^n g \in M\}$ . Claim  $N$  is a finite index subgroup of  $G$ : define a homomorphism  $f$  from  $G$  to  $p^n G/M$  by  $f(g) = p^n g + M$ . Clearly  $f$  is onto and the kernel of  $f$  is exactly  $N$ . Thus  $G/N \cong p^n G/M$  and so is finite. For convenience of notation, let  $\psi = \phi|_{p^n G}$ .

For each positive integer  $k$  consider the cotrajectory  $C_k(\phi, N)$ : if  $x \in C_k(\phi, N)$ , then certainly  $p^n x \in M$  by definition of  $N$  and for each  $r$  ( $1 \leq r \leq k-1$ ) we have that  $x \in \phi^{-r}(N)$ , so that  $\phi^r(p^n x) = p^n \phi^r(x) \in M$ . Since  $\phi^r(p^n x) = \psi^r(p^n x)$ , we deduce that  $p^n x \in \psi^{-r}(M)$ . Thus we have that  $p^n C_k(\phi, N) \leq C_k(\psi, M)$ . Since the kernel of the map  $\chi : G \rightarrow p^n G/C_k(\psi, M)$  contains  $C_k(\phi, N)$ , we conclude that  $p^n G/C_k(\psi, M)$  is an epimorphic image of  $G/C_k(\phi, N)$  and so  $(\log |p^n G/C_k(\psi, M)|/k) \leq (\log |G/C_k(\phi, N)|/k)$  for all  $k$ . Hence we have that  $I(\psi, M) \leq I(\phi, N)$  and so

$$\text{ent}^*(\psi) = \sup_M I(\psi, M) \leq \sup_{N=p^{-n}M} I(\phi, N) \leq \sup_{\text{all } N} I(\phi, N) = \text{ent}^*(\phi).$$

To establish the final assertion, let  $\psi$  be an arbitrary endomorphism of  $p^n G$ . Since  $G/p^n G$  is a direct sum of cyclic groups, it follows and is well known – see for example [14] – that there is an endomorphism  $\phi$  of  $G$  such that  $\phi|_{p^n G} = \psi$ . By the first part of the proposition  $\text{ent}^*(\psi) \leq \text{ent}^*(\phi)$  and so the result follows.  $\square$

We now return to the investigation of groups with zero adjoint entropy. We begin with a simple result:

**Proposition 2.18** *If the adjoint entropies  $\text{ent}^*(G)$ ,  $\text{ent}^*(H)$  are both zero, and either  $\text{Hom}(G, H) = 0$  or  $\text{Hom}(H, G) = 0$ , then  $\text{ent}^*(G \oplus H) = 0$ .*

*Proof* For any endomorphism  $\phi$  of  $G \oplus H$ , one can write  $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$  where  $\alpha \in \text{End}(G)$ ,  $\beta \in \text{End}(H)$ ,  $\delta \in \text{Hom}(H, G)$ ,  $\gamma \in \text{Hom}(G, H)$ . We show that if  $\text{ent}^*(G) = \text{ent}^*(H) = 0$  and  $\delta = 0$  or  $\gamma = 0$ , then  $\text{ent}^*(\phi) = 0$ . Without loss in generality, suppose  $\delta = 0$ . Then  $H$  is a  $\phi$ -invariant subgroup of  $G \oplus H$  and by Proposition 2.9, since  $\phi|_H = \beta$  one has that  $\text{ent}^*(\phi|_H) = \text{ent}^*(\beta) = 0$ . Hence  $\text{ent}^*(\phi) = \text{ent}^*(\bar{\phi})$ , where  $\bar{\phi} \in \text{End}((G \oplus H)/H)$ . On the other hand,  $(G \oplus H)/H \cong G$ . This implies that  $\text{ent}^*(\bar{\phi}) = 0$  as  $\text{ent}^*(G) = 0$ .  $\square$

The next lemma is the key observation concerning zero adjoint entropy.

**Lemma 2.19** *Suppose that an epimorphism  $f$  of  $G$  has zero adjoint entropy, then for any finite index subgroup  $N$  of  $G$ , the kernel of  $f$  is contained in  $N$ .*

*Proof* We first note that if  $f$  has zero adjoint entropy, then, by Lemma 2.7, there exists some integer  $k \geq 1$ , depending on  $N, f$ , such that for all  $i \geq 0$ ,  $C_k = C_k(f, N) = C_{k+i}$ . Hence  $C_k = C_{k+1} = C_k \cap f^{-k}(N)$ , so  $C_k \subset f^{-k}(N)$  and similarly  $C_k = C_{k+1} = C_{k+2} = C_{k+1} \cap f^{-(k+1)}(N)$  and an easy induction shows that  $C_k \subset f^{-(k+i)}(N)$  for all  $i \geq 0$ . Moreover,  $f^{-k}(C_k) = f^{-k}(N \cap f^{-1}(N) \cap \dots \cap f^{-(k-1)}(N)) = f^{-k}(N) \cap f^{-(k+1)}(N) \cap \dots \cap f^{-(2k-1)}(N)$ . Hence we have

$$C_k \subset f^{-k}(C_k) \tag{1}$$

Now assume that  $f$  is surjective, we first observe that

$$\frac{G}{C_k} \cong \frac{G}{f^{-k}(C_k)} \tag{2}$$

To see this, we define a mapping  $\phi : G \rightarrow G/C_k$  by  $g \mapsto f^k(g) + C_k$  and note that  $\phi$  is onto since  $f, f^k$  are onto. The isomorphism follows since  $\text{Ker} \phi = f^{-k}(C_k)$ . By Equation 1,  $\frac{G/C_k}{f^{-k}(C_k)/C_k} \cong \frac{G}{f^{-k}(C_k)}$  is meaningful. It then follows from Equation 2 that  $\frac{G/C_k}{f^{-k}(C_k)/C_k} \cong \frac{G}{C_k}$  and so

$$|G/C_k| = |G/C_k| |f^{-k}(C_k)/C_k| \tag{3}$$

However, the subgroups  $C_k$  are of finite index in  $G$  and so all the cardinalities in Equation 3 are finite. Hence  $|f^{-k}(C_k)/C_k| = 1$ , so that  $f^{-k}(C_k) = C_k$ . Clearly  $\text{Ker} f^k \subset f^{-k}(C_k) = C_k$  and so  $\text{Ker} f \subset C_k \subset N$ . □

We will use the notation  $U(G)$  for the first Ulm subgroup of a group  $G$ , so that  $U(G) = \bigcap_{n \geq 1} nG$ ; recall that  $U$  is then a radical.

The following result is well known and its proof is omitted:

**Lemma 2.20** *For any group  $G$ , the intersection of all finite index subgroup is precisely  $U(G) = \bigcap_{n \geq 1} nG$ .*

Recall the notion of Hopficity: a group is said to be Hopfian if every epimorphism is an automorphism. Such groups have been the subject of investigation for a long time but many problems relating to them are still unresolved. Our next result shows a connection with adjoint entropy and should be compared to a corresponding observation in [7, Proposition 2.9].

**Theorem 2.21** *If  $G$  is a reduced group with zero adjoint entropy and  $U(G)$  is Hopfian, then  $G$  is Hopfian.*

*Proof* Suppose that  $f$  is an epimorphism of  $G$ . Then  $f$  has zero adjoint entropy and so, by Lemma 2.19, we have  $\text{Ker } f \subset N$  for any finite index subgroup of  $G$ . Thus,  $\text{ker } f \subset \bigcap N = \bigcap nG = U(G)$ . Since  $f$  is epic,  $G \cong G/\text{ker } f$  and so  $U(G) \cong U(G/\text{ker } f)$ . However,  $\text{ker } f \subset U(G)$  and as  $U$  is a radical, we have  $U(G/\text{ker } f) = U(G)/\text{ker } f$ . Thus,  $U(G) \cong U(G)/\text{ker } f$ . If  $\text{ker } f \neq 0$ , then  $U(G)$  would have a proper isomorphic quotient contrary to  $U(G)$  being Hopfian. So we conclude  $\text{ker } f = 0$  and  $G$  is Hopfian as required.  $\square$

Since a reduced torsion-free group has trivial first Ulm subgroup, we have:

**Corollary 2.22** *If  $G$  is a torsion-free, reduced group with zero adjoint entropy, then it is Hopfian.*

At first sight Theorem 2.21 might seem to be a promising source of Hopfian  $p$ -groups, but this hope is dashed by the final result of this section.

**Proposition 2.23** *If  $G$  is an infinite reduced  $p$ -group, then there is an endomorphism  $\phi \in \text{End}(G)$  such that  $\text{ent}^*(\phi) = \infty$ .*

*Proof* Our proof is a simple modification of an old argument due to Szele used to show that a basic subgroup of a  $p$ -group is always an endomorphic image. Let  $B$  be any basic subgroup of  $G$ . If  $B$  is bounded then so also is  $G$ , and hence  $G$  would be an infinite direct sum of cyclic groups. As the left shift on such a group has infinite adjoint entropy, we are finished.

From the proof of Proposition 2.9 we see that  $\text{ent}^*(G) \geq \text{ent}^*(G/p^\omega G)$  since  $p^\omega G$  is a fully-invariant subgroup of  $G$ . Thus it suffices to prove the claim in the case that  $p^\omega G = 0$ , i.e.  $G$  is an infinite separable  $p$ -group and is embedded as a pure subgroup of the torsion-completion of any of its basic subgroups.

Suppose then that  $B$  is unbounded. Then  $B$  has a canonical direct summand  $H$  such that the exponents of the successive cyclic generators of  $H$  are at least doubling, i.e. if the generators are  $h_i$ , then  $e(h_{i+1}) \geq 2e(h_i)$ . The left shift on  $H$  extends to an endomorphism  $\phi$  of  $\bar{B}$  whose restriction to  $G$  maps  $G$  into  $B$  – see the introductory paragraph of [8, Theorem 32.1]. However, it follows from Propositions 2.16 and 2.12 that  $\text{ent}^*(\phi) = \text{ent}^*(\phi|_B) \geq \text{ent}^*(\phi|_H)$ , and since the latter is easily seen to be infinite, we have the desired result.  $\square$

We immediately deduce

**Corollary 2.24** *The adjoint entropy of a reduced  $p$ -group is zero if and only if the group is finite; the adjoint entropy is infinite if and only if the group is infinite.*

### 3 Adjoint entropy of torsion free groups

In this section we consider torsion-free groups only. As we have seen in Theorem 2.21 above, a torsion-free group with zero adjoint entropy is necessarily Hopfian. Our first example shows that the class of torsion-free Hopfian groups is much wider than the class of groups with zero adjoint entropy. We have chosen to use an example of Corner [3] which displays, in some sense, rather extreme behaviour.

**Example 3.1** *There are Hopfian groups  $A, B$  such that  $A \oplus B$  is not Hopfian and each of  $\text{ent}^*A, \text{ent}^*B$  is infinite.*

*Proof* Let  $a_k, b_k, x_k, y_k (k \in \mathbb{Z})$  be a basis of a vector space  $V$  over the rationals and let  $p, q_k, r, s (k \in \mathbb{Z})$  be distinct prime numbers. Define  $A, B$  as subgroups of  $V$  by

$$A = \sum_k \{p^{-\infty} a_k, q_k^{-\infty} x_k, \frac{1}{r}(a_k + x_k)\}$$

$$B = \sum_k \{p^{-\infty} b_k, q_k^{-\infty} y_k, \frac{1}{s}(a_k + x_k)\}$$

where  $p^{-\infty} a_k$  is an abbreviation for the set of elements  $p^{-m} a_k (m = 0, 1, 2, \dots)$ . It was shown in [3, Example 2] that  $A, B$  are torsion free Hopfian groups, but that  $A \oplus B$  is not Hopfian. Here we compute their adjoint entropies. It is clearly enough to consider  $\text{ent}^*(A)$ .

We first construct an endomorphism of  $A$ . Since  $\{p^{-\infty} a_k (k \in \mathbb{Z})\}, \{q_k^{-\infty} x_k\} (k \in \mathbb{Z})$  are fully invariant subgroups of  $A$ , we can construct an endomorphism  $\phi$  of  $A$  as follows:

$$\begin{aligned} \phi : \mu a_1 &\rightarrow 0, \\ \phi : a_{i+1} &\rightarrow r a_i, \alpha a_{i+1} \rightarrow \alpha r a_i, i \geq 1 \\ \phi : x_i &\rightarrow r x_i, \beta_i x_i \rightarrow \beta_i r x_i, i \geq 1 \\ \phi : \gamma \frac{1}{r}(a_1 + x_1) &\rightarrow \gamma x_1 \\ \phi : \gamma \frac{1}{r}(a_{i+1} + x_{i+1}) &\rightarrow \gamma a_i + \gamma x_{i+1}, i \geq 1, \end{aligned}$$

where  $\mu, \alpha$  are rationals with denominators power of  $p$ ,  $\beta_i, i = 1, 2, \dots$  are rationals with denominators power of  $q_i$  respectively, and  $\gamma$  is an integer.

Suppose that  $t$  is a prime number which is different from  $p, q_k, r, s (k \in \mathbb{Z})$ , we claim that the subgroup

$$N = t\{p^{-\infty} a_1\} + r\{\frac{1}{r}(a_1 + x_1)\} + \sum_{k=1}^{\infty} \{p^{-\infty} a_{k+1}, q_k^{-\infty} x_k, \frac{1}{r}(a_{k+1} + x_{k+1})\}$$

is of finite index in  $A$ .

**Lemma 3.2** *The quotient group  $A/N$  is finite.*

*Proof* The elements in  $A/N$  are of the form  $\frac{\mu}{p^l}a_1 + \frac{\nu}{r}(a_1+x_1)+N$  where  $0 < |\mu| < p$ . Since  $(p, t) = 1$ , when  $l \geq 1$ , there exist two integers  $\alpha, \beta$  such that  $\alpha p^l + \beta t = u$ , then  $\frac{\mu}{p^l}a_1 + N = \frac{\alpha p^l + \beta t}{p^l}a_1 + N = \alpha a_1 + N$ . When  $l \leq 0$ ,  $\frac{\mu}{p^l}a_1 + N = p^{-l}\mu a_1 + N$ . On the other hand,  $r(\frac{1}{r}(a_1 + x_1) + N) = a_1 + x_1 + N = a_1 + N$ . In any case,  $A/N$  is a cyclic group with the generator  $\frac{1}{r}(a_1 + x_1) + N$ , and clearly, the order of  $\frac{1}{r}(a_1 + x_1) + N$  is  $tr$ . So  $|A/N| = tr$  is finite.  $\square$

**Proposition 3.3** *The cotrajectory  $I(\phi, N) > 0$  and thus  $\text{ent}^*(A) = \infty$*

*Proof* First we note that  $a_1 \notin N$ . Furthermore,  $r^n a_1 \notin N$  for any natural  $n$ ; for if were, then since  $(t, r^n) = 1$ , there are two integers  $\alpha, \beta$  such that  $\alpha t + \beta r^n = 1$ , giving  $\alpha t a_1 + \beta r^n a_1 = a_1$ , but the left hand side is in  $N$ —a contradiction! Now, since  $r a_1 \notin N$ , and  $\phi(a_2) = r a_1$ , we have  $a_2$  is not in  $\phi^{-1}(N)$ . On the other hand  $\phi(r a_2) = r \phi(a_2) = r^2 a_1 \notin N$ , thus,  $r a_2 \notin \phi^{-1}(N)$ . By induction,  $a_{k+1} \notin \phi^{-k}(N)$ .

Our next step is to show that  $a_{k+1} \in N \cap \phi^{-k}(N) \cap \dots \cap \phi^{-(k-1)}(N)$ . Clearly,  $a_2 \in N$ , so is  $r a_2$ . But  $\phi(a_3) = r a_2 \in N$ , thus  $a_3 \in \phi^{-1}(N)$ . This means  $a_3 \in N \cap \phi^{-1}(N)$ . By induction,  $a_{k+1} \in N \cap \phi^{-k}(N) \cap \dots \cap \phi^{-(k-1)}(N)$ . Thus, the cotrajectory never stabilizes and  $I(\phi, N) > 0$ , as required.  $\square$

The classification of torsion-free groups with zero adjoint entropy is essentially an impossible task since the groups exist in such abundance. We justify this statement by looking at so-called realization theorems. The first of these was the famous theorem of Corner [4] that every reduced countable torsion-free ring is the endomorphism ring of a reduced countable torsion-free group. Corner's approach was to realize a ring  $\mathbf{A}$  as the endomorphism ring of a group  $G$  where  $G$  lies between the additive group of  $\mathbf{A}$  and that of  $\hat{\mathbf{A}}$ , where the completion is in either the natural or the  $p$ -adic topology. Moreover,  $G$  is a pure dense subgroup of  $\hat{\mathbf{A}}$  and the endomorphisms of  $G$  act on  $\mathbf{A}$  as scalar multiplication; for convenience, we shall say that a ring  $\mathbf{A}$  is *C-realizable by  $G$*  if there exists a group  $G$  with properties as described having endomorphism ring equal to  $\mathbf{A}$ . This result has been extended to much wider classes of groups than the countable ones and there is an extensive literature on the problem; a good survey of modern developments may be found in [11, Chapter 12]. These recent approaches share a fundamental approach that was already present in Corner's original work but have a significant difference: the ring  $\mathbf{A}$  is now realized as the endomorphism ring of a group  $G$  where  $G$  lies between the additive group  $B$  of a large direct sum of copies of  $\mathbf{A}$  and that of  $\hat{B}$ . Moreover,  $G$  is a pure dense subgroup of  $\hat{B}$  and the endomorphisms of  $G$  act on each summand  $\mathbf{A}$  as scalar multiplication; for convenience, we shall say that a ring  $\mathbf{A}$  is *realizable by  $G$*  if there exists a group  $G$  with properties as described having endomorphism ring equal to  $\mathbf{A}$ .

The next result is an immediate consequence of Propositions 2.16 and 2.12.

**Proposition 3.4** *Let  $\mathbf{A}$  be a ring,  $C$ -realizable on  $G$ , then  $\text{ent}^*(a) = \text{ent}^*(a|_{\mathbf{A}})$  for each endomorphism  $a$  of  $G$ ; if  $\mathbf{A}$  is realizable on  $G$ , then  $\text{ent}^*(a) \geq \text{ent}^*(a|_{\mathbf{A}})$  for each endomorphism  $a$  of  $G$ .*

With this result one may calculate the adjoint entropy of a wide variety of torsion-free groups. We content ourselves with:

**Example 3.5** *There exists arbitrary large torsion-free groups  $G$  which are indecomposable and have zero adjoint entropy and arbitrary large torsion-free groups  $H$  which are indecomposable and have infinite adjoint entropy.*

*Proof* Groups with the properties ascribed to  $G$  are obtained by realizing the ring of integers,  $\mathbb{Z}$  as endomorphism ring; it is well known that this is possible. To obtain groups of a type like  $H$ , observe that the ring  $\mathbb{Z}[X]$  has free additive group and hence may be realized on arbitrarily large groups – see [11, Chapter 12]. Multiplication by  $X$  corresponds to the forward (right) shift on  $\mathbb{Z}[X]$  and it is an easy consequence of Proposition 2.5 that this mapping has infinite adjoint entropy. Consequently  $H$  has infinite adjoint entropy.  $\square$

## 4 Adjoint entropy on mixed groups

First, we introduce an unpublished theorem due to A.L.S.Corner [5]– this is the result referred to as [U14] in [12] – concerning the existence of an epimorphism from a mixed group onto a basic subgroup of its torsion subgroup. Since this result (and Corner’s other unpublished works) will shortly be freely available on the Arrow website of the Dublin Institute of Technology (for details see [5] in the References section), we have not included the full details of that work here.

**Theorem 4.1** *Let  $G$  be an extension of a  $p$ -group  $T$  by a countable torsion-free group and let  $B$  be a basic subgroup of  $T$ . Then there exists an epimorphism from  $G$  onto  $B$ .*

In fact, we shall not exploit this result directly, rather we shall make use of a technical observation used by Corner in the proof.

It is clear from our discussion in the previous section that there is no possibility of classifying all mixed groups with zero adjoint entropy. We can, however, provide such a classification “modulo torsion-freeness” in a special but nonetheless reasonably general case.



**Theorem 4.2** *Let  $G$  be a mixed group of countable torsion-free rank with torsion subgroup a  $p$ -group  $T$  having an unbounded basic subgroup  $B$ . Then there is an endomorphism of  $B$  with infinite adjoint entropy which extends to a mapping  $G \rightarrow B$ . In particular  $\text{ent}^*(G) = \infty$ .*

*Note:* It suffices to prove the theorem under the additional hypothesis that  $G/T$  is divisible. For if  $G/T$  is not divisible, we may choose a group,  $G'$  say, such that  $G'/T$  is a divisible hull of  $G/T$ . Clearly an endomorphism of  $B$  extending to a map  $G' \rightarrow B$  will restrict to a mapping  $G \rightarrow B$ .

*Proof* Let  $B = \bigoplus_{n=1}^{\infty} B_n$  be an unbounded basic subgroup of  $T$ , where, to simplify notation, we assume that each  $B_n$  is a non-zero direct sum of cyclic groups of order  $p^n$ . For each  $n$ , fix a canonical summand  $C_n$  of  $B_n$  which is cyclic of order  $p^n$  generated by  $c_n$ . Clearly, for each  $n$ ,  $B_1 \oplus B_2 \oplus \cdots \oplus B_n$  is a maximal  $p^n$ -bounded summand of  $G$ :  $G = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus G'_n$ . Hence for an arbitrary element  $g \in G$ , we have  $g = g_1 + g_2 + \cdots + g_n + g'_n$ , where  $g_i \in B_i$  and  $g'_n \in G'_n$ . In this way one can associate with each element  $g \in G$ , a unique “vector”  $\langle g_1, g_2, \dots, g_n, \dots \rangle \in \prod_{n=1}^{\infty} B_n$ . Now it follows from Corner’s proof in [5] of Theorem 4.1 above, that there is a sequence of integers  $s_n$  with the properties (i)  $s_n \rightarrow \infty$  and (ii)  $H(g_n) - s_n \rightarrow \infty$  for each  $g \in G$ . (Here we are writing  $H(x)$  for the height in  $G$  of the element  $x \in G$ .) Since  $s_n \rightarrow \infty$ , for each integer  $k$  there is an integer  $N_k$  such that  $s_n \geq k$  for all  $n \geq N_k$ . Now define a sequence of integers  $M_r$  as follows: let  $M_1 = \max\{1, N_1\}$  and assuming that  $M_k$  has been defined, set  $M_{k+1} = \max\{N_{M_k}, M_k + 1\}$ . Note that the sequence  $\{M_k\}$  is, by construction, strictly monotonic increasing and if  $n \geq M_{k+1}$ , then  $s_n \geq M_k$ .

Now define a map  $\omega : B \rightarrow B$  by setting  $\omega(c_{M_{r+1}}) = c_{M_r}$  and mapping all other basis elements to 0. Claim that  $\omega$  extends to an endomorphism  $\hat{\omega} : G \rightarrow B$  where  $\hat{\omega}(g) = \sum_{n=1}^{\infty} \omega(g_n)$ . Clearly it will suffice to show that  $\omega(g_n)$  vanishes for all but a finite number of  $n$  and this certainly holds true when  $g_n$  has no component in  $C_n$  ( $n \in \{M_1, M_2, \dots\}$ ). In the remaining cases, if  $r$  is sufficiently large, we have

$$H(\omega(g_{M_{r+1}})) \geq H(g_{M_{r+1}}) \geq s_{M_{r+1}} \geq M_r,$$

where  $H(x)$  denotes the height of the element  $x$  in  $G$ . However,  $\omega(g_{M_{r+1}}) \in C_{M_r}$ , a cyclic summand of  $G$  of order  $p^{M_r}$ , so that the only element of  $C_{M_r}$  of height  $\geq M_r$  is precisely 0.

Thus  $\hat{\omega}$  is an endomorphism of  $G$  and its restriction to  $B$  acts as the backward shift on  $\bigoplus_{r=1}^{\infty} C_{M_r}$ ; it follows from Example 2.6(i) that  $\text{ent}^*(\hat{\omega}) = \infty$ .  $\square$

**Theorem 4.3** *Let  $G$  be a reduced mixed group of countable torsion-free rank having torsion subgroup a  $p$ -group  $T$ . Then  $\text{ent}^*(G) = 0$  if, and only if,  $G = T \oplus X$  where  $T$  is finite and  $X$  is a (countable) torsion-free group with  $\text{ent}^*(X) = 0$ .*

*Proof* Sufficiency follows from Proposition 2.18 and the fact that finite groups have zero adjoint entropy. Conversely suppose  $\text{ent}^*(G) = 0$ . It follows from Proposition 2.12 that  $\text{ent}^*(T) = 0$  and

so by Proposition 2.23,  $T$  is finite. Clearly then  $G$  splits as  $G = T \oplus X$  for some torsion-free countable group  $X$ . Since every endomorphism of  $X$  lifts trivially to an endomorphism of  $G$ ,  $\text{ent}^*(X) \leq \text{ent}^*(G) = 0$  and thus  $\text{ent}^*(X) = 0$  as required.  $\square$

It seems to be difficult to give an explicit description of mixed groups of countable torsion-free rank in the non-local situation. There is, however, one further situation which is easily described and which lies, in a certain sense, at the opposite end of the spectrum. Recall that a group  $G$  is said to be *cotorsion* if  $\text{Ext}(\mathbb{Q}, G) = 0$ . It is well known this is incompatible with  $G$  having countable torsion-free rank and that every cotorsion group  $G$  may be expressed as  $G = A \oplus T^\bullet$ , where  $A$  is torsion-free algebraically compact and  $T^\bullet = \text{Ext}(\mathbb{Q}/\mathbb{Z}, T)$  is the so-called cotorsion completion of the torsion subgroup  $T$  of  $G$  – see [9, Sections 54-58].

Suppose now that  $G$  is a reduced cotorsion group with  $\text{ent}^*(G) = 0$ . It follows immediately from Corollary 2.14 that  $\text{ent}^*(A) = 0 = \text{ent}^*(T^\bullet)$ . Now consider  $A$ : it is a reduced algebraically compact group and so, by Kaplansky's result (Proposition 40.1 in [9]),  $A = \prod_p A_p$  where each  $A_p$  is the completion of a free  $p$ -adic module. Since the  $A_p$  are fully invariant, it follows that  $\text{ent}^*(A_p) = 0$  for all primes  $p$ . It follows easily from Corollary 2.14 that each free  $p$ -adic module involved is of finite rank. Thus  $A$  is the direct product (over primes  $p$ ) of finite rank  $p$ -adic modules. The description of  $T^\bullet$  is equally straightforward: applying Corollary 2.14, we have that  $\text{ent}^*(T) = 0$  and so each primary component  $T_p$  of  $T$  must be finite. Moreover, we have that  $T^\bullet = \text{Ext}(\mathbb{Q}/\mathbb{Z}, T) = \prod_p \text{Ext}(\mathbb{Z}(p^\infty), T_p)$  since the  $T_p$  are  $q$ -divisible for all primes  $q \neq p$ . As  $T_p$  is finite, it follows that  $\text{Ext}(\mathbb{Z}(p^\infty), T_p) \cong T_p$  and hence  $T^\bullet \cong \prod_p T_p$  with each  $T_p$  being a finite  $p$ -group.

Summarizing we can deduce

**Proposition 4.4** *A reduced cotorsion group  $G$  has zero adjoint entropy if, and only if, it has the form  $G = \prod_p T_p \oplus \prod_p A_p$  where  $T_p$  is a finite  $p$ -group and  $A_p$  is the direct sum of a finite number of copies of the group of  $p$ -adic integers,  $J_p$ ; equivalently  $G = \prod_p F_p$  where  $F_p$  is a finitely generated  $p$ -adic module.*

*Proof* The necessity has been established above and the sufficiency follows easily since each  $T_p \oplus A_p$  is fully invariant. The equivalent statement follows by taking  $F_p = T_p \oplus A_p$ .  $\square$

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