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Computation of the Stochastic Volatility and Lévy Index using the Kolmogorov-Feller Equation with Applications to Carbon Price Data Analysis

Jonathan Blackledge, Marc Lamphiere and Afshin Panahi

***Abstract*— We derive new algorithms for computing time variations in the Stochastic Volatility and the Lévy index using a standard financial price model and a Green's function solution to the Kolmogorov-Feller equation. A principal condition upon which the algorithms are based is the *Phase Only Condition* which allows the Power Spectral Density Function of a financial time series (specifically the log price differences) to be taken to be a constant. The paper is composed of four component parts: (i) the Stochastic Volatility is derived and studied numerically; (ii) the Kolmogorov-Feller equation is studied and solved to provide a model for the stochastic characteristics of a financial time series using the Lévy Characteristic Function; (iii) a method for computing the Lévy index is proposed given price data and the Stochastic Volatility of the data; (iv) numerical algorithms are designed and example results presented. Although the models proposed and the algorithms developed are applicable to financial time series in general, in this paper, we consider a study of the Stochastic Volatility and Lévy index for Carbon price data. This is because of the increasing importance of 'Carbon trading' with regard to climatic control and the emission of Carbon Dioxide and other green-house gases. The results presented therefore represent a study of a financial indicator (in particular the Lévy index) that may be of value for future energy commodities trading, and, in particular, Carbon price risk assessment modelling.**

***Index Terms*— Kolmogorov-Feller equation, Green's function solution, Lévy processes, financial signal analysis, Stochastic Volatility, Carbon price risk assessment modelling, energy commodities trading.**

I. Introduction

The diffusion-type and jump-type properties associated with the Kolmogorov-FokkerPlanck and Kolmogorov-Feller equations (e.g. [1]-[4] and references therein) make them suitable for modelling stochastic functions, and, in particular, the non-stationary dynamic behaviour of financial time series. In this paper, we consider a Green's function solution to the Kolmogorov-Feller Equation (KFE) which requires an iterative approach for which a sufficient condition for convergence is derived. The solution obtained is then investigated with regard to Lévy processes and a new algorithm considered for computing the Lévy index from the autocorrelation function of the input data. This is based on the application of a phase only condition, a condition that is similar to the 'white noise condition' used to design digital filters such as the matched filter, for example. A case study is considered using Carbon price data for which correlations between the price index, Stochastic Volatility and the Lévy index are considered.

The material presented in this paper represents an initial study of financial data relating to energy commodities trading in general. Energy commodities trading is becoming an increasingly important source of global trading activitiy given the rapid growth in the renewable energies industry, emission reduction schemes to reduce the Carbon footprint (e.g. the 1997 Kyoto protocol and the 2011 Durban Climate Change Conference agreements) the regulation of the commodities markets [5].

The structure of the paper is as follows: Section II revisits the standard price model from which a new (stochastic) volatility index is derived using the phase only condition. The time dependent behaviour of this index is then briefly studied for the purpose of latter applications using recent Carbon price data traded under the European Union Emissions Trading Scheme (EU ETS) which is a major pillar of EU climate policy. Section III revisits the Black-Scholes equation illustrating the relationship between this equation and Gaussian statistics. Section IV considers the relationship between Lévy processes and the fractional diffusion equation. In both cases, the Black-Scholes equation and the fractional diffusion equation are derived from the fundamental evolution equation for a random walk process which is introduced in Section III. Different versions of the Kolmogorov-Feller equation are introduced in Section V which is followed by Section VI where a general (Green's function based) solution to the (Classical) KFE is derived. A specific solution is then considered for random processes that conform to the (symmetric) Lévy Characteristic Function and a numerical algorithm developed for computing the Lévy index. The material presented in Section VI represents the principal original contribution to the field of study and, in Section VII, is applied to the computation of the Stochastic Volatility and Lévy index for two example sets of Carbon price data.

II. Computation of the Stochastic Volatility

Price models involve the derivation and solution of a variety of stochastic differential and partial differential equations. A standard model for the price of a stock as a function of time $s(t)$ is

$$\frac{d}{dt}s(t) = \mu s(t) + \sigma s(t)u(t) \qquad (1)$$

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where μ is the 'Drift', σ is the 'Volatility' and $u(t)$ is a stochastic function. This model is based on the idea that prices appear to be the previous price plus some random change and that these price changes are independent, i.e. asset price changes appear to be random and independent, prices being taken to follow some random walk-type behaviour. This is the basis for including a stochastic function $u(t)$. However the size of price movements also depends on the size of the price itself. The model is therefore revised to include this effect, the stochastic term $u(t)$ being replaced by $u(t)s(t)$ where σ determines the degree of randomness taken to influence a price changes. In general, μ and σ vary with time, and, in the context of equation (1), $\sigma(t)$ is referred to as the 'Stochastic Volatility', e.g. [6], [7] and [8]. The drift function $\mu(t)$ tends to vary over longer periods of time reflecting the long term trends associated with a price index.

In principle, $u(t)$ could be any stochastic function with statistical behaviour conforming to a range of Probability Density Functions. A conventional model is to assume that the log price changes are Gaussian distributed so that $u(t)$ is taken to be a zero-mean Gaussian distributed function. If this function is taken to have a fixed standard deviation of 1, then the volatility becomes a measure of the standard deviation, at least, for a (zero-mean) Gaussian model. The stock price model given by equation (1) then provides a method for estimating the volatility σ in terms of a lower bound as we shall now show.

A. Application of the Phase Only Condition

Let

$$f(t) = \mu + \sigma u(t)$$

where

$$f(t) = \frac{1}{s(t)}\frac{d}{dt}s(t) = \frac{d}{dt}\ln s(t)$$

and μ and σ are taken to be constant. We first obtain an estimate of the Drift which can be done by exploiting the fact that, if $u(t)$ is taken have a zero mean value over $t \in [0, T]$, then

$$\int_0^T f(t)dt = \int_0^T \mu dt + \sigma \int_0^T u(t)dt \sim \mu T$$

so that

$$\mu \sim \frac{1}{T}\int_0^T f(t)dt \tag{2}$$

To obtain an estimate for the Volatility, we now consider the case when the stochastic function $u(t)$ is a phase only function, i.e. given that

$$U(\omega) = \int_{-\infty}^{\infty} u(t)\exp(-i\omega t)dt$$

where ω is the (angular) frequency,

$$U(\omega) = A\exp[i\theta(\omega)] \tag{3}$$

the amplitude spectrum A being taken to be a constant for all values of ω. We also consider $u(t)$ to be a band-limited function $\omega \in [-\Omega/2, \Omega/2]$ with bandwidth Ω and a function of compact support $t \in [-T/2, T/2]$. Using Minkowski's identity for Euclidean norms,

$$\|f(t)\|_2 \leq \|\mu\|_2 + \|\sigma u(t)\|_2$$

where

$$\|x(t)\|_2 := \left(\int \mid x(t) \mid^2 dx\right)^{\frac{1}{2}}$$

so that we can write

$$\sigma\|u(t)\|_2 \geq \|f(t)\|_2 - \mu\sqrt{T}$$

where μ is given by equation (2). Using Parseval's Theorem (Rayleigh's Energy Theorem), the condition expressed by equation (3) allows us to write

$$\int_{-T/2}^{T/2} \mid u(t) \mid^2 dt = \frac{1}{2\pi}\int_{-\Omega/2}^{\Omega/2} \mid U(\omega) \mid^2 d\omega = \frac{\Omega A^2}{2\pi}$$

We can therefore write

$$\sigma \geq \frac{1}{A}\sqrt{\frac{2\pi}{\Omega}}(\|f(t)\|_2 - \mu\sqrt{T})$$

so that

$$\sigma_{\min} = \frac{1}{A}\sqrt{\frac{2\pi}{\Omega}}(\|f(t)\|_2 - \mu\sqrt{T}) \tag{4}$$

which yields an expression for the lower bound of the Volatility.

Analysing time variations in the Stochastic volatility provides an important measure on the dynamics of a financial signal. To this end, we consider computing the time dependent Stochastic Volatility $\sigma(x)$ which, in terms of equation (4), is given by

$$\sigma_{\min}(x) = \frac{1}{A}\sqrt{\frac{2\pi}{\Omega}}(\|f(x+t)\|_2 - \mu(x)\sqrt{T})$$

where

$$\|f(x+t)\|_2 = \left(\int_0^T \mid f(x+t) \mid^2 dt\right)^{\frac{1}{2}}$$

and

$$\mu(x) = \frac{1}{T}\int_0^T f(x+t)dt$$

B. Numerical Estimation

Consider a discrete signal denoted by the array $f_n, \quad n = 1, 2, 3, ..., N$ where a uniform sampling interval of Δt is assumed. In this case, the discrete version of equation (4) becomes

$$\sigma_{\min} = \frac{1}{A}\sqrt{\frac{2\pi}{\Omega}}(\sqrt{\Delta t}\|f_n\|_2 - \mu\sqrt{T})$$

where we invoke the usual definition for a vector (Euclidean) norm, i.e.

$$\|f_n\|_2 := \left(\sum_{n=1}^{N} \mid f_n \mid^2\right)^{\frac{1}{2}}$$

and

$$\mu = \frac{\Delta t}{T}\sum_{n=1}^{N} f_n$$

The sampling interval Δt of f_n is related to the sampling interval $\Delta\omega$ of the Discrete Fourier Transform of f_n by the equation

$$\Delta t \Delta\omega = \frac{2\pi}{N}$$

and since the bandwidth of the discrete spectrum of f_n is $N\Delta\omega$ is is clear that $\Delta t = 2\pi/\Omega$. Thus, given that the support of the signal is $T = N\Delta t$, we note that

$$T = \frac{2\pi N}{\Omega}$$

and therefore obtain

$$\sigma_{\min} = \frac{2\pi}{A\Omega}(\|f_n\|_2 - \sqrt{N}\mu)$$

where

$$\mu = \frac{1}{N}\sum_{n=1}^{N} f_n$$

The scaling constant $2\pi/(A\Omega)$ can then be used to define a re-scaled Stochastic Volatility given by

$$\hat{\sigma} := \sigma_{\min}\frac{A\Omega}{2\pi}$$

thereby yielding the expression

$$\hat{\sigma} = \|f_n\|_2 - \sqrt{N}\mu$$

Writing this result explicitly in terms of the price s_n we then have

$$\hat{\sigma} = \left(\sum_{n=1}^{N-1}\left|\ln\left(\frac{s_{n+1}}{s_n}\right)\right|^2\right)^{\frac{1}{2}} - \frac{1}{\sqrt{N-1}}\sum_{n=1}^{N-1}\ln\left(\frac{s_{n+1}}{s_n}\right) \tag{5}$$

To compute the 'Stochastic Volatility' σ_m, N is taken to determine the size of the data sampling window or 'look-back' window which is moved along the time series one element at a time so that we can write

$$\hat{\sigma}_m = \left(\sum_{n=1}^{N-1}\left|\ln\left(\frac{s_{m+n+1}}{s_{m+n}}\right)\right|^2\right)^{\frac{1}{2}} - \frac{1}{\sqrt{N-1}}\sum_{n=1}^{N-1}\ln\left(\frac{s_{m+n+1}}{s_{m+n}}\right) \tag{6}$$

Equation (5) may be compared with other estimates for the Stochastic Volatility (e.g. [9] - [12]) such as the Maximum Likelihood (ML) estimate given by,

$$\hat{\sigma}^2_{\mathrm{ML}} = \frac{1}{N-1}\sum_{n=1}^{N-1}(\ln s_{n+1} - \ln s_n)^2 - \frac{1}{(N-1)^2}(\ln s_N - \ln s_1)^2$$

The phase only condition used to derive equations (5) and (6) is equivalent to modelling the stochastic function $u(t)$ in terms of a random walk in the (complex) Fourier domain where the amplitude of each step is the same. The effect of this condition on the statistical characteristics of the function (apart from scaling) is not significant as illustrated in Figure 1. In this example, a uniformly sampled zero-mean Gaussian distributed function $u_1(t)$ consisting of 1000 elements with a variance and standard deviation of 1 is generated. A second stochastic function $u_2(t)$ is computed using the phase components $\theta(\omega)$ of the Fourier transform of $u_1(t)$, i.e. if

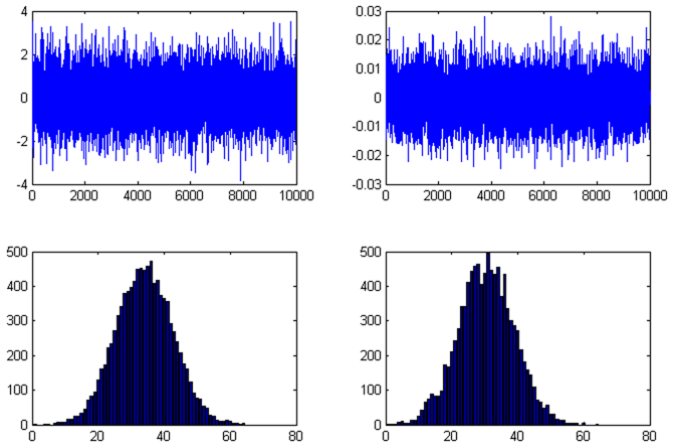

Fig. 1. A 1000 element zero-mean Gaussian distributed function (top-left) with unit variance and standard deviation, the phase only function (top-right) and the 64-bin histograms of each function (bottom-left and bottom-right respectively).

$$U_1(\omega) = A(\omega)\exp[i\theta(\omega)] = \int_{-\infty}^{\infty} u_1(t)\exp(-i\omega t)dt$$

then

$$u_2(t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}[\cos\theta(\omega) + i\sin\theta(\omega)]\exp(i\omega t)d\omega\right)$$

From Figure 1, where the computational tasks have been undertaken using a Discrete Fourier Transform, it is clear that the zero-mean Gaussian distributed characteristics of $u_1(t)$ are preserved in $u_2(t)$ apart from a reduction in scale (i.e. $\|u_2(t)\|_\infty < \|u_1(t)\|_\infty$ due to the Fourier amplitude of $u_2(t)$ being unity for all frequency components. A comparison of the 300 element random walk patterns in the (complex) Fourier domain associated with $u_1(t)$ and $u_2(t)$ (i.e. plots of $A(\omega)\sin\theta(\omega)$.v.$A(\omega)\cos\theta(\omega)$ and $\sin\theta(\omega)$.v.$\cos\theta(\omega)$, respectively) is given in Figure 2.

C. Carbon Price Data Analysis

The world wide concern over climate change and dwindling fossil fuel reserves, has resulted in a steady move towards more stringent Carbon emission and energy targets. Putting a price on Green-House Gas (GHG) emissions is now a cornerstone policy of climate change mitigation. It is generally accepted that without these price measures, it will be significantly more difficult (and expensive) to implement the economic transformation required to put the world on track to meet the Copenhagen Accord goal of limiting temperature rise to 2^oC.

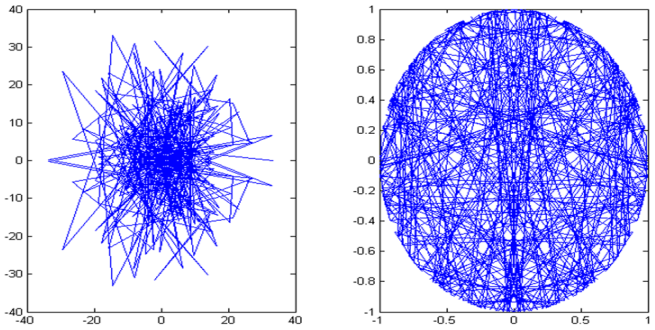

Fig. 2. Fourier domain 300 element random walk patterns for $u_1(t)$ (left) and $u_2(t)$ (right).

Delivering a rapid turn-around in energy investment patterns to meet this target necessitates clear, strong, and sustained policy measures. In response to this need, many countries have implemented, or are in the process of developing, emissions trading schemes. Currently, the largest of these schemes is that which operates in the EU, namely the European Union Emissions Trading Scheme, more commonly known as the EU ETS. The EU ETS currently covers more than 10,000 installations with a net heat excess of 20 MW in the energy and industrial sectors which are collectively responsible for close to half of the EU emissions of Carbon Dioxide and 40% of its total GHG emissions.

The expansion of the EU ETS, along with the predicted worldwide growth in Carbon trading indicates a global emissions trading market with a turnover in excess of $3 Trillion by 2020 (Reuters, 2011). This compares with 2010, where the market for Carbon credits was worth $136 billion (Point Carbon, 2011) indicating a growth of fourteen fold over the coming decade alone.

The significance of establishing a financial market place for Carbon can be observed in terms of the increase of emissions trading schemes, the growth in volumes traded and through the wider impacts of Carbon on energy and equity markets. The emergence of Carbon as a new commodity and asset class has introduced new interrelationships that reach across the spectrum of energy commodities such as oil, gas, coal and power. While early movers have been able to capitalize on new investment opportunities, the Carbon market has also introduced a new set of variabilitys to which a traditional approach to risk management may not necessarily lend itself. Moreover, it is increasingly challenging to understand the drivers of Carbon prices, their volatilities and hence risks. There is therefore, a clear need to develop a specific set of tools and techniques applicable to Carbon trading, analysis, and risk management.

Figure 3 shows the time variations in the volatility computed using equation (6) and a look-back window of 100 elements for Close-of-Day EUA (European Union Allowance) Carbon Price Value from 26-11-2008 to 30-03-2011. From this result it is clear that the Volatility has a downward trend over the period considered from the latter part of 2009 to March 2011. This result is further compounded in Figure 4 which shows the time variations in the volatility using a 100 element look-back window for Close-of-Day CER (Certified Emission Reductions) Carbon Price Value from 22-04-2009 to 30-03-2011. In both examples, the downward trend in the volatility indicates that Carbon prices appear to becoming increasingly stable. This suggests that, in the long term, and, in addition to other energy commodities, Carbon trading may provide a relatively stable environment in which to build an investment portfolio. The m-code for computing and plotting the data

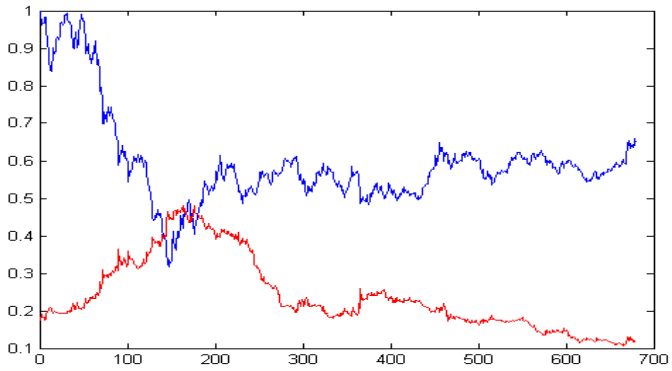

Fig. 3. Close-of-Day EUA (European Union Allowance) Carbon Price Value from 05-01-2009 to 30-03-2011 (Blue) and the Stochastic Volatility (Red) computed using equation (6) for a look-back window of 100 elements.

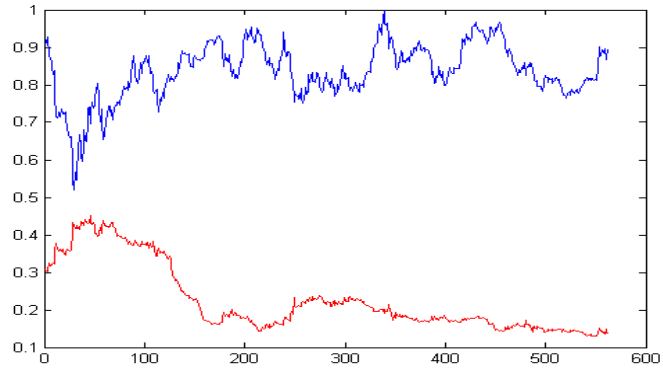

Fig. 4. Close-of-Day CER (Certified Emission Reductions) Carbon Price Value from 18-07-2009 to 30-03-2011 (Blue) and the Stochastic Volatility (Red) computed using equation (6) for a look-back window of 100 elements.

and the Stochastic Volatility given in Figure 3 and Figure 4 is given in Appendix A.

III. Gaussian Statistics and the Black-Scholes Equation

In Gaussian statistical models, the stochastic function $u(t)$ in equation (1) is chosen to have a Gaussian Probability Density Function (PDF). The rationale for this is that price movements are presumed to be an aggregation of smaller ones and sums of independent random contributions have a Gaussian PDF due to the Central Limit Theorem. This is equivalent to arguing that all financial time series used to construct an 'averaged signal are statistically independent. However, this argument is not fully justified because it assumes

that the reaction of investors to one particular stock market is independent of investors in other stock markets which, in general, will not be the case as each investor may have a common reaction to economic issues that transcend any particular stock. In other words, asset management throughout the markets relies on a high degree of connectivity, and, the arrival of new information sends 'shocks through the market as people react to it and then to each others reactions.

The use of Gaussian statistical models relates to the concept of the Efficient Market Hypothesis (EMH) which states that the current price of an asset fully reflects all available information relevant to it and that new information is immediately incorporated into the price. Thus, in an efficient market, models for asset pricing are concerned with the arrival of new information which is taken to be independent and random. This model is the basis for the Black-Scholes equation which assumes that the underlying statistics associated with an economic signal are Gaussian.

The Black-Scholes equation can be written in terms of the diffusion equation

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial t}u(x,t) = 0 \tag{7}$$

subject to an initial condition $u_0(x) \equiv u(x, t = 0)$. The parameter σ is the volatility and $u(x,t) = C(x,t)\exp(rt)$ where C is the Call Option and r is the 'risk free rate'. Black-Scholes analysis involves developing solution for $u(x,t)$ and thereby the call options which depend on obtaining good estimates for the volatility and risk.

The Black-Scholes equation can be derived from the evolution equation

$$u(x, t+\tau) = u(x,t) \otimes_x p(x) \tag{8}$$

where $p(x)$ is the PDF and $\otimes_x$ denotes the (non-causal) convolution integral over x, i.e.

$$u(x,t) \otimes_x p(x) \equiv \int_{-\infty}^{\infty} u(x-\lambda, t)p(\lambda)d\lambda$$

This equation describes the concentration of particles in an interval of time τ that move over a short distance with a probability $p(x)$ due to some random walk effect. The statistical evolution associated with this random walk effect is taken to be characterised by the function $p(x)$.

Using the convolution theorem, equation (8) becomes [13]

$$U(k, t+\tau) = U(k,t)P(k,\tau) \tag{9}$$

where

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)\exp(-ikx)dx,$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)\exp(-ikx)dk$$

and

$$P(k,\tau) = \int_{-\infty}^{\infty} p(x,t)\exp(-ikx)dx,$$

$$p(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(k,\tau)\exp(-ikx)dk$$

If we consider a Gaussian Characteristic Function given by

$$P(k) = \exp(-\sigma^2k^2\tau/2)$$

then, under the condition $\sigma^2k^2\tau/2 << 1$, we can approximate the Characteristic Function by

$$P(k) = \exp(-\sigma^2k^2\tau/2) = 1 - \sigma^2k^2\tau/2$$

Equation (9) can then be written as

$$\frac{U(k,t+\tau) - U(k,t)}{\tau} = -U(k,t)\sigma^2k^2/2$$

Thus

$$\lim_{\tau\to 0}\frac{U(k,t+\tau)-U(k,t)}{\tau} = \frac{\partial}{\partial t}U(k,t) = -U(k,t)\sigma^2k^2/2 \tag{10}$$

where we note that the limiting condition used to derive this equation (i.e. $\tau \to 0$) is consistent with the condition used to approximate the characteristic function $P(k)$ to the form $1 - \sigma^2k^2\tau/2$. Noting that

$$\frac{\partial^2}{\partial x^2}u(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k^2U(k,t)\exp(-ikx)dk$$

inverse Fourier transformation of equation (10) yields equation (7).

The general solution to equation (7) is well known and given by [14]

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}u_0(x) \otimes_x \exp[-x^2/(2\sigma^2t)], \quad t > 0$$

where u_0 is the initial condition, i.e. $u_0 \equiv u(x, t = 0)$. For a stochastic source term $n(x,t)$ where equation (7) is extended to the form

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial t}u(x,t) = n(x,t)$$

the general solution becomes[14]

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}u_0(x) \otimes_x \exp[-x^2/(2\sigma^2t)]$$

$$+n(x,t) \otimes_x \otimes_t \frac{1}{\sigma\sqrt{2\pi t}}\exp[-x^2/(2\sigma^2t)]$$

where $\otimes_t$ denotes the causal convolution integral, i.e. for two piecewise continuous functions $f(t)$ and $g(t)$

$$f(t) \otimes_t g(t) \equiv \int_0^t f(t-\tau)g(\tau)d\tau$$

The purpose of this section has been to show that the Black-Scholes equation is intimately connected with the assumption that the stochastic function $u(x,t)$ is characterised by a statistical model that is Gaussian (i.e. the PDF $p(x)$ is a Gaussian function). This yields a deterministic model that is based on the classical diffusion equation. The principal problem with such a model is that it fails to include rare but extreme events that can occur in real financial times series. In the following

section, we consider the case when $u(x,t)$ is characterised by a Lévy distribution which may be viewed as a generalization of a Gaussian process.

IV. Lévy Statistics and the Fractional Diffusion Equation

Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions. Lévy's original approach to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function (i.e. the Fourier transform) $P(k)$ of such a distribution $p(x)$) was first shown by Lévy to be given by (for symmetric distributions only)

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2$$

where a is a constant and γ is the Lévy index. For $\gamma \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed. For example, if the result were a random walk with a step length distribution governed by $p(x), \quad \gamma > 2$, then the result would be normal (Gaussian) diffusion, i.e. a Brownian process. For $\gamma < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. For values of γ between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form (as shown in Appendix B)

$$p(x) \sim \frac{1}{\mid x \mid^{1+\gamma}}, \quad x \to \infty.$$

This type of random walk is called a Lévy flight and is an example of a fractal walk. Given equation (8), Lévy processes are consistent with a fractional diffusion equation for the stochastic function $u(x,t)$ as shown below following a similar approach to that taken in Section III.

Consider the Characteristics Function

$$P(k) = \exp(-a\tau \mid k \mid^{\gamma})$$

which, under the condition $\tau \to 0$, can be approximated by

$$P(k) = 1 - a\tau \mid k \mid^{\gamma}$$

We can then write the evolution equation - equation (9) - as

$$\frac{U(k,t+\tau) - U(k,t)}{\tau} = -a \mid k \mid^{\gamma} U(k,t)$$

which for $\tau \to 0$ yields the fractional diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = a\frac{\partial^{\gamma}}{\partial \mid x \mid^{\gamma}}u(x,t), \quad \gamma \in (0,2]$$

where we have used the Fourier transform based definition of a fractional derivative,

$$\frac{\partial^{\gamma}}{\partial \mid x \mid^{\gamma}}u(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mid k \mid^{\gamma} U(k,t)\exp(ikx)dk$$

A solution to this equation with the singular initial condition $u(x,0) = \delta(x)$ is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx - at \mid k \mid^{\gamma})dk$$

which is itself Lévy distributed. This derivation of the fractional diffusion equation reveals its physical origin in terms of Lévy processes characterising the evolution of the stochastic function $u(x,t)$, i.e. Lévy's characteristic function.

For a stochastic source function of the form $n(x,t) = \delta(x)n(t)$, the solution, for $x \to 0$, to the equation

$$a\frac{\partial^{\gamma}}{\partial \mid x \mid^{\gamma}}u(x,t) - \frac{\partial}{\partial t}u(x,t) = n(x,t)$$

can be shown to be given by (ignoring scaling constants) [15]

$$u(t) = \frac{1}{t^{1-1/\gamma}} \otimes_t n(t), \quad t > 0$$

For a white noise source $n(t)$ this equation provides describes a random scaling fractal signal $u(t)$.

V. The Kolmogorov-Feller Equation

The approach used to deriving the Black-Scholes equation and the fractional diffusion equation given in Section III and Section IV respectively, is based on applying a condition to approximate an exponential Characteristic Function. In this context, the KFE provides a model for the stochastic function $u(x,t)$ that is independent of a specific PDF or an approximated form. There is surprisingly relatively little published material on the Kolmogorov-Feller equation and its applications, particularly in the area of financial time series modelling. Thus, in the following sub-sections, we consider the derivations of the KFE and its various forms [16]-[21].

A. The Classical KFE

For an arbitrary Characteristic Function $P(k)$ with PDF $p(x)$, the evolution equation is

$$u(x,t+\tau) = u(x,t) \otimes_x p(x)$$

Cosider a Taylor series for the function $u(x,t+\tau)$, i.e.

$$u(x,t+\tau) = u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) + \frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(x,t) + ...$$

For $\tau << 1$

$$u(x,t+\tau) = u(x,t) + \tau\frac{\partial}{\partial t}u(x,t)$$

we obtain the 'Classical KFE,

$$\tau\frac{\partial}{\partial t}u(x,t) = -u(x,t) + u(x,t) \otimes_x p(x) \qquad (11)$$

B. The Generalised KFE

Equation (11) is based on a critical assumption which is that the time evolution of the field $u(x,t)$ is influenced only by short term events and that longer term (historical) events have no influence of the behaviour of the field, i.e. the 'system' described by equation (11) has no 'memory'. This statement is the physical basis upon which we introduce the condition $\tau << 1$ thereby allowing the Taylor series expansion of the $u(x,t+\tau)$ to be made to first order. The question then arises as to how longer term temporal influences can be modelled, other than by taking an increasingly larger number of terms in the Taylor expansion of $u(x,t+tau)$ which is not of practical analytical value. For arbitrary values of τ,

$$\tau\frac{\partial}{\partial t}u(x,t)+\frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(x,t)+...=-u(x,t)+u(x,t)\otimes_x p(x)$$

We can model the effect on a solution for $u(x,t)$ of the series on the left hand side of this equation in terms of a 'memory function' $m(t)$ and write

$$\tau m(t)\otimes_t\frac{\partial}{\partial t}u(x,t)=-u(x,t)+u(x,t)\otimes_x p(x) \qquad (12)$$

This is the Generalised KFE which reduces to the Classical KFE when

$$m(t)=\delta(t)$$

Note that for any memory function for which there exists a function or class of functions of the type $\hat{m}(t)$, say, such that

$$\hat{m}(t)\otimes_t m(t)=\delta(t)$$

then we can write equation (12) in the form

$$\tau\frac{\partial}{\partial t}u(x,t)=-\hat{m}(t)\otimes_t u(x,t)+\hat{m}(t)\otimes_t u(x,t)\otimes_x p(x) \quad (13)$$

where the Classical KFE is recovered when $\hat{m}(t)=\delta(t)$. Also note that the memory function can be 'cast' as follows: Let

$$u_1(x,t)=\tau\frac{\partial}{\partial t}u(x,t) \quad \text{and} \quad u_2(x,t)=\frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(x,t)+...$$

so that

$$m(t)\otimes_t u_1(x,t)=u_1(x,t)+u_2(x,t)$$

Taking Laplace transforms of this equation and using the convolution theorem, we can then write

$$M(p)U_1(x,p)=U_1(x.p)+U_2(x,p)$$

and

$$m(t)=\frac{1}{2\pi i}\int\limits_{\epsilon-i\infty}^{\epsilon+i\infty}\left[1+\frac{U_2(x,p)}{U_1(x,p)}\right]\exp(pt)dp$$

where U_1 and U_2 are the Laplace transforms of u_1 and u_2 respectively.

C. The Time Fractional KFE

Any solution obtained to the Generalised KFE will be dependent upon the choice of memory function $m(t)$ used. There are a number of choices that can be considered each or which is taken to be a 'best characteristic' of the stochastic system in terms of the influence of its time history. However, it may be expected that the time history of physically significant random systems is relatively localised in time. Thus, memory functions of the type $\exp(-t)$, for example, may be expected to apply. However, there is a class of memory function that gives rise to the Fractional KFE and, in this sense, is a relative of the (time) Fractional Diffusion Equation discussed in Section IV. The memory function in question is the Mittag-Leffler function

$$m(t)=\frac{1}{\Gamma(1-\beta)t^{\beta}}, \quad 0<\beta<1$$

where

$$\hat{m}(t)=\frac{1}{\Gamma(\beta-1)t^{2-\beta}}$$

given that

$$\int\limits_0^{\infty}\frac{\exp(-st)}{\Gamma(\beta)t^{1-\beta}}dt=\frac{1}{s^{\beta}} \quad \text{and} \quad \int\limits_0^{\infty}\delta(t)\exp(-st)dt=1$$

D. The Stochastic KFE

Suppose we write the evolution equation - i.e. equation (8) after applying a Taylor series expansion of the function $u(x,t+\tau)$ - in the form

$$\tau\frac{\partial}{\partial t}u(x,t)=-u(x,t)+u(x,t)\otimes_x p(x)+\frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(x,t)+...$$

The second and higher order terms (which are ignored in the derivation of the Classical KFE) represent a source of error with regard to a solution of the Classical KFE. Thus we consider the equation

$$\tau\frac{\partial}{\partial t}u(x,t)=-u(x,t)+u(x,t)\otimes_x p(x)+n(x,t)$$

where $n(x,t)$ is a random variable which is taken to be a result of the combination of errors associated with the neglect of higher order terms. This idea has a synergy with the application of the Born approximation used to solve forward and inverse scattering problems where higher order terms in the Born series (which describes double, triple and higher order scattering processes) are taken to introduce 'system noise' into an equation describing single scattering events [22]. The problem is then reduced to modelling an appropriate stochastic function which, because of the Central Limit Theorem, is often chosen to be Gaussian distributed.

VI. Solution to the KFE

We now turn our attention to developing a solution to the Classical KFE, i.e. given equation (11), we seek a solution for $u(x,t)$ given $p(x)$. We require a general solution that allows us to investigate the effect of different PDFs $p(x)$ on the function $u(x,t)$ such that the associated inverse problem is

practically viable, i.e. where statistical parameter(s) associated with $p(x)$ can be computed from $u(x,t)$. This condition rules out an approach based on, for example, the transformation of equation (11) into Fourier-Laplace space to obtain a solution of the form

$$u(x,t) = \frac{1}{4\pi^2 i}\int_{c-i\infty}^{c+i\infty}\int_{\infty}^{\infty}\frac{\tau U(k,t=0)\exp(ikx)\exp(st)}{1+s\tau-P(k)}dkds$$

where

$$P(k) = \int_{-\infty}^{\infty} p(x)\exp(-ikx)dx$$

and

$$U(k,t=0) = \int_{-\infty}^{\infty} u(x,t=0)\exp(-ikx)dx$$

which requires an initial condition $u(x,t=0)$ to be realised. Thus, we consider a Green's function solution to equation (11) which we write in the form

$$\tau\frac{\partial}{\partial t}u(x,t) + u(x,t) = h(x,t) \tag{12}$$

where

$$h(x,t) = u(x,t)\otimes_x p(x)$$

The Green's function $g(\tau)$ is then given by the solution of

$$\tau\frac{\partial}{\partial t}g(\xi) + g(\xi) = \delta(\xi) \tag{13}$$

where $\xi = t - t_0$. Taking the Laplace transform of equations (12) and (13), we have

$$\tau(s\bar{u} - u_0) + \bar{u} = \bar{f} \tag{14}$$

and

$$\tau(s\bar{g} - g_0) + \bar{g} = 1 \tag{15}$$

respectively, where

$$\bar{u}(x,s) = \int_0^{\infty} u(x,t)\exp(-st)dt,$$

$$\bar{g}(s) = \int_0^{\infty} g(\xi)\exp(-s\xi)d\xi,$$

$u_0 = u(x,t=0)$ and $g_0 = g(\tau=0)$. Form equation (14) and (15), it is clear that

$$\bar{u}(x,s) = \bar{g}(s)\bar{h}(x,s) \tag{16}$$

under the condition that $u_0 = 0$ and $g_0 = 0$. Using the convolution theorem for the Laplace transform, equation (16) becomes

$$u(x,t) = g(t)\otimes_t h(x,t) \tag{17}$$

The Green's function, which is the solution to equation (13), is given by

$$g(t) = \frac{1}{\tau}\exp(-t/\tau)H(t)$$

where

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

From equation (17), it is clear that an iterative solution is required to solve for the function $u(x,t)$ and we therefore consider the iteration, for $m = 1, 2, 3, ...$

$$u_{m+1}(x,t) = g(t)\otimes_t u_m(x,t)\otimes_x p(x) \tag{18}$$

By transforming equation (18) into Fourier space and using the convolution theorem for the Fourier transform, we can write

$$U_{m+1}(k,t) = g(t)\otimes_t U_m(k,t)P(k)$$

where

$$U_m(k,t) = \int_{-\infty}^{\infty} u_m(x,t)\exp(-ikx)dx$$

and

$$P(k) = \int_{-\infty}^{\infty} p(x)\exp(-ikx)dx$$

$$U_2(k,t) = g(t)\otimes_t U_1(k,t)P(k)$$

$$U_3(k,t) = g(t)\otimes_t U_2(k,t)P(k)$$

$$= g(t)\otimes_t g(t)\otimes_t U_1(k,t)[P(k)]^2$$

and so on for $m = 4, 5, ..., M$, so that, by induction,

$$U_M(k,t) = \prod_{m=1}^{M}{}^{\otimes} g_m(t)\otimes_t U_1(k,t)[P(k)]^M \tag{19}$$

where

$$\prod_{m=1}^{M}{}^{\otimes} g_m(t) \equiv g(t)\otimes_t g(t)\otimes_t g(t)\otimes_t ...$$

the condition for convergence being given by $\|P(k)\| < \sqrt{2\tau}$ as shown in Appendix C. We note that the Laplace transform of g is given by

$$\int_0^{\infty}\frac{1}{\tau}\exp(-t/\tau)\exp(-st)dt = \frac{1}{1+\tau s}$$

and that, using the convolution theorem for Laplace transforms,

$$\int_0^{\infty}\prod_{m=1}^{M}{}^{\otimes} g_m(t)\exp(-st)dt = \frac{1}{(1+\tau s)^M}$$

Thus, since

$$\int_0^{\infty}\frac{t^M}{\tau}\exp(-t/\tau)\exp(-st)dt = \frac{M!}{(1+\tau s)^{1+M}}$$

it follows that

$$\prod_{m=1}^{M}{}^{\otimes} g_m(t) = \frac{\exp(-t/\tau)t^{M-1}}{\tau(M-1)!}$$

Finally inverse Fourier transforming equation (19) and using the convolution for Fourier transforms we can write

$$u_M(x,t) = \frac{\exp(-t/\tau)t^{M-1}}{\tau(M-1)!} \otimes_t u_1(x,t) \otimes_x \prod_{m=1}^{M} \otimes p_m(x) \tag{20}$$

where it is clear that

$$\lim_{t\to 0} u_M(x,t) = 0$$

and

$$\lim_{t\to \infty} u_M(x,t) = 0$$

If we now define the function

$$\hat{u}(x) = \int_0^\infty u_M(x,t)dt$$

then equation (20) can be reduced to the form

$$\hat{u}(x) = n(x) \otimes_x \prod_{m=1}^{M} \otimes p_m(x) \tag{21}$$

where

$$n(x) = \int_0^\infty \frac{\exp(-t/\tau)t^{M-1}}{\tau(M-1)!} \otimes_t u_1(x,t)dt$$

For a PDF $p(x)$ with Characteristic Function $P(k)$, using the convolution theorem for Fourier transforms, equation (21) transforms to

$$\hat{U}(k) = N(k)[P(k)]^M$$

If we now take $n(x)$ to be a phase only function, i.e. $N(k) = \exp[i\theta(k)]$ then the power spectrum becomes

$$\mid \hat{U}(k) \mid^2 = \mid [P(k)]^M \mid^2 = [P(k)]^{2M}$$

for real $P(k)$. Thus, using the correlation theorem, we obtain the result

$$c(x) = \prod_{m=1}^{2M} \otimes p_m(x) \tag{22}$$

where $c(x)$ is the correlation function given by, for real $\hat{u}(x)$,

$$c(x) = \int_{-\infty}^{\infty} \hat{u}(y)\hat{u}(x+y)dy \tag{23}$$

A. Application of Lévy's Characteristic Function

We consider the case when the Characteristic Function is given by $\exp(-a \mid k \mid^\gamma)$ and note that, in this case,

$$\mid \hat{U}(k) \mid^2 = \exp(-b \mid k \mid^\gamma)$$

where $b = 2aM$. Using the result derived in Appendix B, the correlation function is then given by

$$c(x) \sim \frac{1}{\mid x \mid^{1+\gamma}}, \quad x \to \infty \tag{24}$$

Thus, we derived a logarithmic scaling relationship between the Lévy index and the correlation function in terms of the equation

$$\ln c(x) \sim -(1+\gamma)\ln \mid x \mid \tag{25}$$

and a double logarithmic scaling relationship between the Lévy index and the Power Spectrum

$$2\ln \mid \ln \mid \hat{U}(k) \mid\mid = \ln b + \gamma \ln \mid k \mid \tag{26}$$

Application of equation (26) for characterising a non-stationary stochastic signal in terms of the Lévy index requires repeated application of a Discrete Fourier Transform on a moving window basis where as application of equation (25) only requires a least squares estimate of γ to be generated on a similar moving window basis.

B. Numerical Algorithms

Given equations (24) and (26) we consider numerical algorithms for estimating the Lévy index using a least squares approach. On the basis of equation (24), we consider a model for the correlation function (positive half space) of the form (for $n = 1, 2, ..., N$)

$$\hat{c}(x_n) = a x_n^\alpha, \quad x_n > 0$$

where a is a constant of proportionality and $\alpha = -(1+\gamma)$. Given the (discrete) correlation function $c(x_n)$, where it is assumed that $c_n \equiv c(x_n) > 0 \forall x_n$, we consider estimating the constants a and α which minimise the error function

$$e(a,\alpha) = \| \ln \hat{c}_n - \ln c_n \|_2^2 \equiv \sum_{n=1}^{N} (\ln \hat{c}_n - \ln c_n)^2$$

Differentiating with respect to $A = \ln a$ and α it is trivial to show that

$$\alpha = \frac{\sum_{n=1}^{N} \ln c_n \sum_{n=1}^{N} \ln x_n - N \sum_{n=1}^{N} \ln c_n \ln x_n}{\left(\sum_{n=1}^{N} \ln x_n\right)^2 - N \sum_{n=1}^{N} (\ln x_n)^2} \tag{27}$$

and

$$a = \exp\left(\frac{\sum_{n=1}^{N} \ln c_n - \alpha \sum_{n=1}^{N} \ln x_n}{N}\right)$$

Similarly, using equation (26), it is easy to show that, for a discrete spectrum $U_n \equiv U(k_n)$, where $\mid U_n \mid > 0 \forall k_n$ and, a (discrete) spectral model of the form (for the positive half space)

$$2\ln \mid \ln \mid \hat{U}_n \mid\mid = \ln b + \gamma \ln k_n, \quad k_n > 0,$$

where $\hat{U}_n \equiv \hat{U}(k_n)$ we obtain,

$$\gamma = \frac{2N \sum_{n=1}^{N} \ln \mid \ln \mid U_n \mid\mid \ln k_n - 2 \sum_{n=1}^{N} \ln \mid \ln \mid U_n \mid\mid \sum_{n=1}^{N} \ln k_n}{\left(\sum_{n=1}^{N} \ln k_n\right)^2 - N \sum_{n=1}^{N} (\ln k_n)^2} \tag{28}$$

and

$$b = \exp\left(\frac{2 \sum_{n=1}^{N} \ln \mid \ln \mid U_n \mid\mid - \gamma \sum_{n=1}^{N} \ln k_n}{N}\right)$$

A study of the Lévy index computed via the amplitude spectrum $\mid U_n \mid$ as given by equation (28) will be published elsewhere. In this paper we focus on the computation of the Lévy index using the correlation function via equation (27). However, it should be noted that this method is based on an asymptotic approximation (given that equation (24) is strictly valid only for $\mid x \mid \rightarrow \infty$). Equation (27) will therefore not necessarily output values for the Lévy index that are within the bound $\gamma \in (0, 2]$ even for data that is entirely compatible with the model for $\hat{u}(x)$ given by equation (21). However, this approximation does not require the application of a Discrete Fourier Transform to compute the amplitude spectrum and is therefore less computationally intensive by the order of $N \log_2 N$. Moreover, irrespective of the numerical range out by equation (27), our focus is on the dynamic performance of γ (i.e. variations of γ with time that may reflect changes from diffusive $\gamma > 2$ to trending $\gamma < 2$ behaviour) and the potential of this parameter as a financial indicator with regard to Carbon and other price movements. In this respect, the output generated by equation (27) is normalized to unity.

VII. Computation of the Lévy Index for Carbon Price Data

The KFE discussed in Section V and the solution considered has been undertaken in an attempt to model the stochastic function given in equation (1). We therefore consider the stochastic function $\hat{u}(x)$ given by equation (21) to be related to the price $s(x)$, the Drift $\mu(x)$ and the Stochastic Volatility $\sigma(x)$ at any point in time x by

$$\hat{u}(x) = \left| \frac{1}{\sigma(x)} \left(\frac{d}{dx} \ln s(x) - \mu(x) \right) \right| \tag{29}$$

with the discrete form (using a backward differencing scheme)

$$\hat{u}_n = \left| \frac{1}{\hat{\sigma}_n} \left[\ln \left(\frac{s_n}{s_{n-1}} \right) - \mu_n \right] \right| \tag{30}$$

where $\hat{\sigma}_m$ is given by equation (6). In this context, we consider x to represent any value of time that provides an amplitude $\hat{u}(x)$ of a financial time series that is the 'sum' of inter-sample trades defined by $n(x)$ in equation (21). The absolute value given in equation (29) is undertaken to ensure compatibility with the model for $c(x)$ in equation (25) where it is required that $c(x) \geq 0 \forall x$ which, in turn, requires that $\hat{u}(x) \geq 0 \forall x$.

Figure 5 shows a plot of $\hat{u}_n$ for the price and Stochastic Volatility data used to generate Figure 3. Also shown in Figure 5 is the associated 100-bin histogram that is illustrative of the statistical characteristics of $\hat{u}_n$. Another important issue associated with defining $\hat{u}(x)$ in terms of equation (29) is that the autocorrelation of the log price differences is relatively featureless whereas the autocorrelation function of the absolute log price changes contains correlated features including a number of short range correlations [23].

Having obtained $\hat{u}_n, \quad n = 1, 2, ..., N$, the discrete correlation function is generated using the correlation sum

$$c_n = \sum_{m=1}^{N} \hat{u}_m \hat{u}_{m+n}$$

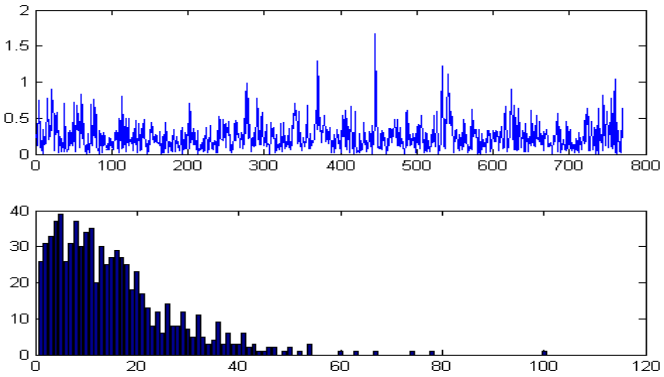

Fig. 5. Example plots of the stochastic function $\hat{u}_n$ (above) and the 100-bin histogram (below).

from which the Lévy index is then computed using equation (27). The process is then repeated on a movig window basis. This requires two look-back windows to be specified. The first window is required to compute the Stochastic Volatility, the Drift and thus $\hat{u}_n$ and the second window is required to compute the Lévy index. The size of these windows with regard to the length of the data stream and their size relative to each other effects the output in terms of the localised behaviour of the Stochastic Volatility relative to the price value and the Lévy index with regard to both price value and volatility. The windows used to compute the Drift and Stochastic Volatility should be relatively small compared to that used to compute the Lévy index. The reasons for this are twofold: (i) we require the values of the Drift and Stochastic Volatility to be local to the log price differences used in the computation of equation (30); (ii) the computation of the Lévy index requires a least squares fit to be undertaken that yields a statistically significant output thereby requiring relative large samples of data.

Given the scaling law expressed in equation (25), it is important to assess the compatibility of Carbon price data with this law. Figure 6 shows an example log-log plot of the correlation function for the price data given in Figure 3 and clearly illustrates that the expected scaling relationship is not uniform over the entire length of the data. For this reason, only the first half of the data is considered in the computation of the Lévy index where, with reference to Figure 6, it is clear that there is a linear scaling relationship between $\ln c_i$ and $\ln x_i$ which is compatible with that given in equation (25).

On the basis of the issues raised above with regard the computation of the Lévy index using equations (30) and (27), Appendix D provide the m-code used to generate the example results given in Figure 7 and Figure 8 where the data, the Stochastic Volatility (computed using a look-back window of 10) and the Lévy (computed using a look-back window of 40) have been normalised to unity.

VIII. Discussion

Compared to the diffusion and fractional diffusion equations discussed in Sections III and IV, respectively, the Kolmogorov-Feller equation, derived in Section V, represents a more accurate model for a stochastic field describing random processes

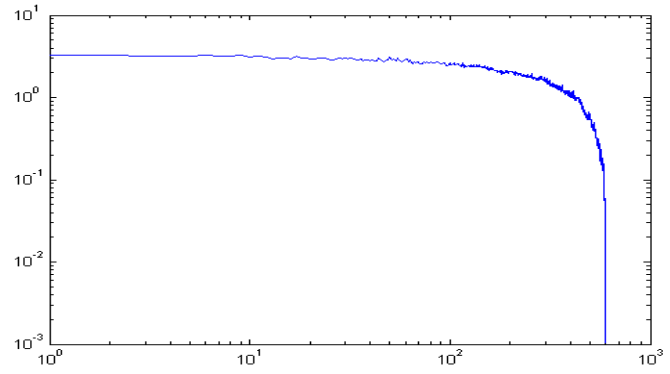

Fig. 6. A log-log plot of the correlation function obtained using the data provided in Figure 5.

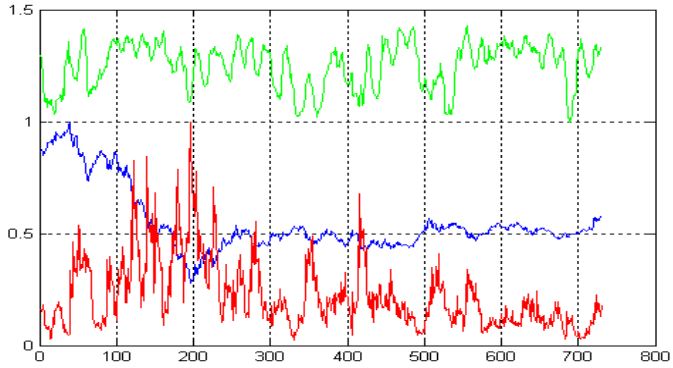

Fig. 7. Close-of-Day EUA (European Union Allowance) Carbon Price Value from 24-04-2008 to 30-03-2011 (Blue), the (normalised) Stochastic Volatility (Red) computed using equation (6) for a look-back window of 10 elements and the normalised Lévy index (Green) using a look-back window of 40.

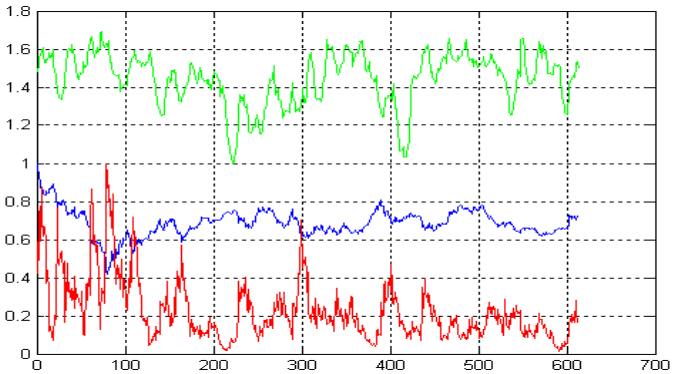

Fig. 8. Close-of-Day CER (Certified Emission Reductions) Carbon Price Value from 08-10-2008 to 30-03-2011 (Blue), the (normalised) Stochastic Volatility (Red) computed using equation (6) for a look-back window of 10 elements and the normalised Lévy index (Green) using a look-back window of 40.

compounded in the evolution equation - equation (8). In this sense, the computation of the Lévy index considered in this paper, is based on a more fundamental model than the approaches considered in [23] and [14], for example. However, the asymptotic approximation used to compute the index from the autocorrelation function given by equation (24) provides an estimate that is inevitably limited in accuracy compared to a spectral estimation method based on equation (26) albeit with improved computational performance. Thus, a future computational study should involve application of equation (28) using equation (30) to compute the discrete amplitude spectrum $\mid U_n \mid$. Further, equation (21) provides the basis for developing numerical methods that simulate stochastic fields for different Characteristic Functions $P(k)$ using the result

$$\hat{U}(k) = N(k)[P(k)]^M$$

where $N(k)$ is a white noise spectrum, obtained by taking the Fourier transform of equation (21).

In terms of the theoretical framework developed in this paper, a further area of investigation is to consider solutions to the Generalised and Fractional KFE as discussed in Sections V(B) and V(C), respectively. A similar Green's function approach may be considered in this respect using the methods of fractional calculus considered in [14], for example. Such a solution will introduce another parameter with potential value in the analysis of financial time series data, namely, the parameter β used to define the Mittag-Leffler function. Within the context of the Fractional KFE discussed in Section V(C), this parameter is a measure of the memory associated with a stochastic process defined by equation (8) and may therefore, in addition to the Lévy index, be of value in assessing trending behaviour in financial signals.

Given the algorithms developed in this paper, there are a number of different signals that can be analysed using the Lévy index including the price value itself and the Stochastic Volatility, for example. Further, given the model compounded in equation (29), there are a range of additional indicators that may be considered including various combinations involving the Stochastic Volatility and the Lévy index. For example, Figure 9 shows the result of cross-correlating (positive half-space) the Stochastic Volatility with the Lévy index and computing the absolute value of the gradient (using a forward differencing scheme) for the Close-of-Day EUA Carbon Price Value given in Figure 7. A further example is given in Figure 10 for FTSE Close-of-Data data over a period of 27 years. Examination of this correlation-based measure (as given in Figure 9 and Figure 10) reveals a prominent relationship between the amplitude value and those areas of the time series with significant and predominantly downward trends. It is left to the interested reader to investigate further this apparent correlation using the m-code provided in Appendix D and other financial time series.

Appendix A: M-Code for Computing the Stochastic Volatility

M-code for implementing the moving window process:

```
clear;%Clear memory.
```

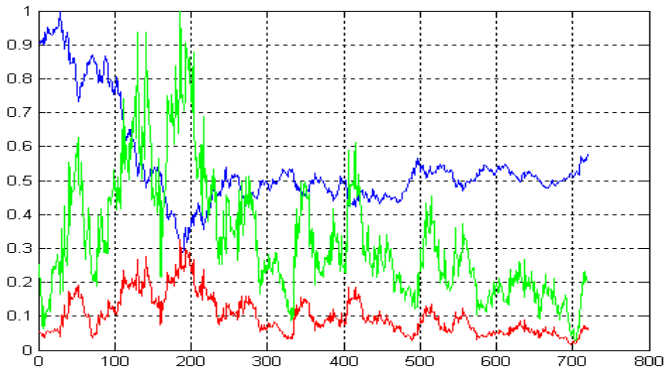

Fig. 9. Close-of-Day EUA (European Union Allowance) Carbon Price Value from 24-04-2008 to 30-03-2011 (Blue), the Stochastic Volatility (red - using a look-back window of 20) and the normalised absolute difference of the correlation function for the Stochastic Volatility and the Lévy index (Green - using a look=back window of 20.

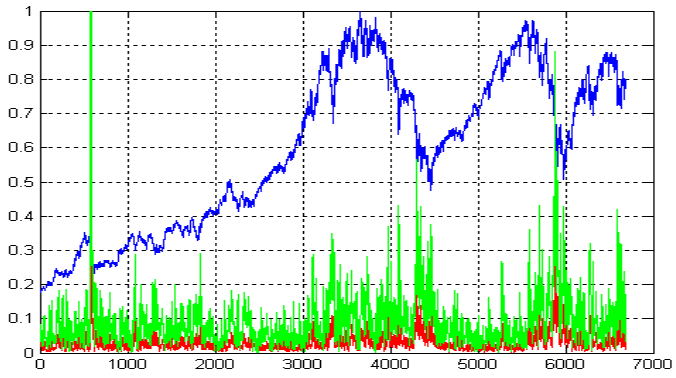

Fig. 10. FTSE values (close-of-day) from 27-06-1985 to 03-01-2012 (blue), the Stochastic Volatility (red - using a look-back window of 10) and the normalised absolute difference of the correlation function for the Stochastic Volatility and the Lévy index (Green - using a look=back window of 300.

```
%Read .txt file into data array
%consisting of n elements
fid=fopen('C:\PATH\Filename.txt','r');
[data n]=fscanf(fid,'%g',[inf]);
fclose(fid);
%Set length of look-back window
w=100;%where w<n
%Compute working array of length
m=n-w;
%Start process
for i=1:m
%Extract windowed data
%into signal - array s.
for j=1:w
    s(j)=data(j+i-1);
end
%Compute the Stochastic Volatility
sigma(i)=volatility(s,w);
end %end process.
%Compute the data for comparative plots.
i=1;
for j=1:m
    signal(i)=data(j+w-1);
    x(i)=i; i=i+1;
end
%Normalise the data.
signal=signal./max(signal);
%Plot the results
figure(1); plot(x,signal,'b-',x,sigma,'r-');
```

M-code for computing the Stochastic Volatility using function 'volatility':

```
function sigma=volatility(s,n)
%Function to compute the
%Stochastic Volatility.
%
%Compute the log price differences.
for i=1:n-1
    ds(i)=log(s(i+1)/s(i));
end
ds(n)=ds(n-1);%Set end point value.
%Compute first and second terms.
term1=sqrt(sum(abs(ds.*ds)));
term2=sum(ds)/sqrt(n);
%Return the volatility.
sigma=term1-term2;
```

Appendix B: Evaluation of the Lévy Distribution

We wish to show that the Characteristic Function

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2$$

is equivalent to a Probability Density Function given by

$$p(x) \sim x^{-(1+\gamma)}, \quad x \to \infty$$

i.e. we wish to prove the following:

Theorem

$$\frac{1}{\mid x \mid^{1+\gamma}} \leftrightarrow \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2, \quad x \to \infty$$

where $\leftrightarrow$ denotes transformation from real to Fourier space[1].

Proof of Theorem

For $0 < \gamma < 1$, and since the characteristic function is symmetric, we have

$$p(x) = \mathrm{Re}[f(x)]$$

where

$$f(x) = \frac{1}{\pi} \int_0^{\infty} e^{ikx} e^{-|k|^{\gamma}} dk$$

$$= \frac{1}{\pi} \left(\left[\frac{1}{ix} e^{ikx} e^{-k^{\gamma}} \right]_{k=0}^{\infty} - \frac{1}{ix} \int_0^{\infty} e^{ikx} (-\gamma k^{\gamma-1} e^{-k^{\gamma}}) dk \right)$$

[1]The author acknowledges Dr K I Hopcraft, School of Mathematical Sciences, Nottingham University, England, for his advice in respect of this result.

$$= \frac{\gamma}{2\pi i x} \int_{-\infty}^{\infty} dk H(k) k^{\gamma-1} e^{-k^\gamma} e^{ikx}, \quad x \to \infty$$

where

$$H(k) = \begin{cases} 1, & k > 0 \\ 0, & k < 0 \end{cases}$$

For $0 < \gamma < 1$, $f(x)$ is singular at $k = 0$ and the greatest contribution to this integral is the inverse Fourier transform of $H(k)k^{\gamma-1}$. Noting that

$$\mathcal{F}^{-1}\left[\frac{1}{(ik)^\gamma}\right] \sim \frac{1}{x^{1-\gamma}}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and that

$$H(k) \leftrightarrow \delta(x) + \frac{i}{\pi x} \sim \delta(x), \quad x \to \infty$$

then, using the convolution theorem, we have

$$f(x) \sim \frac{\gamma}{i\pi x} \frac{i^{1-\gamma}}{x^\gamma}$$

and thus

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

For $1 < \gamma < 2$, we can integrate by parts twice to obtain

$$f(x) = \frac{\gamma}{i\pi x} \int_0^{\infty} dk k^{\gamma-1} e^{-k^\gamma} e^{ikx}$$

$$= \frac{\gamma}{i\pi x} \left[\frac{1}{ix} k^{\gamma-1} e^{-k^\gamma} e^{ikx}\right]_{k=0}^{\infty}$$

$$+ \frac{\gamma}{\pi x^2} \int_0^{\infty} dk e^{ikx} [(\gamma-1)k^{\gamma-2} e^{-k^\gamma} - \gamma(k^{\gamma-1})^2 e^{-k^\gamma}]$$

$$= \frac{\gamma}{\pi x^2} \int_0^{\infty} dk e^{ikx} [(\gamma-1)k^{\gamma-2} e^{-k^\gamma} - \gamma(k^{\gamma-1})^2 e^{-k^\gamma}], \quad x \to \infty$$

The first term of this result is singular and therefore provides the greatest contribution and thus we can write,

$$f(x) \simeq \frac{\gamma(\gamma-1)}{2\pi x^2} \int_{-\infty}^{\infty} H(k) e^{ikx} (k^{\gamma-2} e^{-k^\gamma}) dk$$

In this case, for $1 < \gamma < 2$, the greatest contribution to this integral is the inverse Fourier transform of $k^{\gamma-2}$ and hence,

$$f(x) \sim \frac{\gamma(\gamma-1)}{\pi x^2} \frac{i^{2-\gamma}}{x^{\gamma-1}}$$

so that

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

which maps onto the previous asymptotic as $\gamma \to 1$ from the above.

APPENDIX C: CONVERGENCE CRITERION

Consider the iteration

$$U_{m+1}(k,t) = g(t) \otimes_t U_m(k,t) P(k) \qquad (C1)$$

and let

$$U_{m+1} = U + \epsilon_{m+1}$$

and

$$U_m = U + \epsilon_m$$

where ϵ_m is the error at iteration m. Substituting these equations into equation (C1) it is then clear that

$$\epsilon_{m+1}(k,t) = g(t) \otimes_t \epsilon_m(k,t) P(k)$$

since

$$U(k,t) = g(t) \otimes_t U(k,t) P(k)$$

Consider the operator $\hat{G} = P(k) g(t) \otimes_t$ so that we can write

$$\epsilon_{m+1}(k,t) = \hat{G} \epsilon_m(x,t)$$

Then

$$\epsilon_1 = \hat{G} \epsilon_0$$

$$\epsilon_2 = \hat{G} \epsilon_1 = \hat{G} \hat{G} \epsilon_0 = \hat{G}^2 \epsilon_0$$

$$\epsilon_3 = \hat{G} \epsilon_2 = \hat{G} \hat{G} \epsilon_1 = \hat{G}^3 \epsilon_0$$

and so on. Thus we can write

$$\epsilon_m = \hat{G}^m \epsilon_0$$

and for global convergence we require that

$$\epsilon_m \to 0 \ \text{ as } \ m \to \infty$$

or

$$\lim_{m \to 0} \hat{G}^m \epsilon_0 = 0 \quad \forall \epsilon_0$$

This will occur if

$$\|\hat{G}^m\| < 1$$

and since

$$\|\hat{G}^m\| \leq \|\hat{G}\|^m$$

the condition for convergence becomes

$$\|\hat{G}\| \leq \|P(k)\| \times \|g(t)\| < 1$$

Taking Eucliden norms,

$$\|P(k)\| = \left[\int_{-\infty}^{\infty} |\ P(k)\ |^2\, dk\right]^{\frac{1}{2}},$$

$$\|g(t)\| = \frac{1}{\tau}\left[\int_0^{\infty} \exp(-2t/\tau) dt\right]^{\frac{1}{2}} = \frac{1}{\sqrt{2\tau}}$$

and the condition for global convergence becomes

$$\|P(k)\| < \sqrt{2\tau}$$

which completes the proof.

APPENDIX D: M-CODE FOR COMPUTING THE LÉVY INDEX

M-code for implementing the moving window process required to compute the Stochastic Volatility (using function Volatility given in Appendix A), the Drift and the Lévy index.

```
clear;%Clear memory.
%Read file from txt file into data array.
fid=fopen('C:\PATH\Filename.txt','r');
[data n]=fscanf(fid,'%g',[inf]);
fclose(fid);
%Set length of look-back window for
%computing the stochastic function to
w1=10;
%and set length of look-back window for
%computing the Levy Index to
w2=round(w1+30);
%Begin moving window process
%required to compute the
%Stochastic Volatility using
%a working array length of
m=n-w1;
for i=1:m
%Window the data.
for j=1:w1
   s(j)=data(i+j-1);
end
%Compute the Stochastic Volatility.
sigma(i)=volatility(s,w1);
mu(i)=drift(s,w1);
u(i)=abs((log(s(w1)/s(w1-1))...
    -mu(i))/sigma(i));
end
%Begin the moving window process
%required to compute the Levy
%Index using a working array
%length of
n=m-w2;
for i=1:n
%Window the data.
for j=1:w2
   uu(j)=u(i+j);
end
%Compute the autocorrelation function.
c=xcorr(uu,uu);%MATLAB function xcorr.
%Extract the first half of the data
%in the positive half of the
%autocorrelation function.
k=1;
for j=w2+1:2*w2-1
    cc(k)=c(j);
    k=k+1;
end
%Compute the Levy index.
gamma(i)=Levyindex(cc,w2);
%End the moving window process.
end
%Prepare the original signal and
%the Stochastic Volatility for a
%a comparative plot.
i=1;
for j=1:n
    signal(i)=data(j+w1+w2-1);
    stochvol(i)=sigma(j+w2);
    x(i)=i; i=i+1;%time element
end
%Normalise the data
signal=signal./max(signal);
stochvol=stochvol./max(stochvol);
gamma=gamma./max(gamma);
%and plot the results.
figure(1);
plot(x,signal,'b-',x,stochvol,'r-',...
     x,gamma,'g-');
grid on;
```

M-code for computing the Drift using function 'Drift':

```
function mu=drift(s,n)
%Function to compute the Drift.
%
%Compute the log price differences.
for i=1:n-1
    ds(i)=log(s(i+1)/s(i));
end
ds(n)=ds(n-1);%Set end point value
%Compute and return the Drift.
mu=sum(ds)/n;
```

M-code for computing the Lévy index using function 'Levyindex':

```
function gamma=Levyindex(data,N)
%Computation of the Levy index
%using the least squares algorithm.
%
%Compute the logarithm of the
%data for the first half segment.
N=N/2;
for i=1:N
ydata(i)=log(data(i));
xdata(i)=log(i);
end
%Compute each term of the
%least squares formula.
term1=sum(ydata).*sum(xdata);
term2=sum(ydata.*xdata);
term3=sum(xdata)^2;
term4=sum(xdata.^2);
%Compute and return the Levy index
gamma=(term1-(N*term2))/(term3-(N*term4));
gamma=-gamma-1;
```

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