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Meaning Making Of The Mathematics In Engineering: The Case Of Linear Models In Statistical Signal Processing

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POTENTIAL FOR MEANING MAKING OF THE MATHEMATICS IN ENGINEERING: THE CASE OF LINEAR MODELS IN STATISTICAL SIGNAL PROCESSING

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ABSTRACT

Meaning making of the mathematics involved in engineering problems can boost students' learning, in general. Zooming in to a particular engineering course in signal processing, called Estimation, Detection, and Classification, given to 3rd-year students at NTNU, the potential for meaning making has been investigated using a mix of directed and summative content analysis methods for the specific content Linear models. The findings show that an attempt is made to present the linear model-based estimators in reduced complexity, i.e., without detailed, rigorous proofs that demand solid prior knowledge and concept image from the learner. The 18-page chapter is dominated by advanced mathematical symbols from different mathematical concepts with higher cognitive demanding tasks and activities, which can increase complexity in meaning making. Four types of representations (context, verbal, symbols, and graphs) and multimodal approaches (writing and mathematical symbols) are used to create the potential for meaning making to the user. Symbolic representation dominates the pages creating a higher extraneous cognitive load on the learner. Whereas examples and contexts contribute to lowering the complexity in the potential for meaning making of the mathematics in the chapter. This preliminary study does not include the instructors' and students' active meaning making processes yet.

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1 INTRODUCTION

Mathematics is viewed, in general, as a service subject for engineering fields [1]. Understanding mathematical contents like algebra, calculus, partial differential equations, and other mathematical concepts could facilitate success in engineering studies [2, 3]. [2] investigated what engineering faculty members meant about “mathematical maturity” to get their desired outcomes from core mathematics courses.They found out that “the mathematically mature” student would have strong mathematical modeling skills supported by the ability to extract meaning from symbols and the ability to use computational tools as needed” p. 97. However, many engineering students perceive mathematics instrumentally and think of it as a subject of many rules and procedures [1,5] and struggle to make meaning of mathematical content in their various engineering studies [4]. Such belief and attitude could lead to perceiving the subject as an obstacle to engineering study [2, 4].

Guided by three research questions, [1] reviewed research journals, books, and proceedings to understand the recent state-of-the-art overview of the emergent field of mathematics in engineering. They aimed to develop a deeper understanding of the characteristics of “the current teaching and learning practices in mathematics that can inform the design and implementation of future innovative practices in engineering education” p.163. One of the research questions is about the ‘resources’ used and if they are well suited for innovative practices. The term ‘resource’ is defined as anything that can ‘re-source’ the learning activity of learners, in this case, engineering students [1, 6]. Hence, resources include textbooks, educational technologies, and others. This preliminary study focuses on a textbook used by engineering students.

[7] conceptualized a textbook as a learning tool embedded in a tertiary educational setting. For Randahl, “by a learning tool mean a cognitive tool that promotes cognitive processes related to meaningful learning of mathematics” p. 34. Meaning making of the mathematics involved in a textbook could be one aspect of boosting students’ learning and innovative skill [8, 9], for engineering students as well. We are especially interested in textbooks as learning tools for the student's potential meaning making of the given curriculum [7, 9,10]. As a cognitive tool, textbooks can facilitate or hinder students’ meaning making process, which can also be explained via the concepts of cognitive load and cognitive demand.

Using a particular engineering course in signal processing, and specific content in a textbook, Linear models, the potential for meaning-making has been investigated. Linear models is one of the most critical classes of models that represents a more complex phenomenon in a simplified abstraction. Several service courses in mathematics and statistics cover this content. Meaning making of this model is expected from the students in several engineering problems. Hence, the potential for the meaning of the textbook on the linear models of this textbook is investigated. The data is analyzed using directed and summative content analysis methods [11]. The aim is to investigate the possible meaning in the student textbook used for teaching the linear models-based estimator in the engineering course guided by the research questions: how is the linear model-based estimator presented in the given course material? And what are the potentials for meaning making of the mathematical contents in connection to the content linear model-based signal processing estimation problems?

2 THEORETICAL FRAMEWORK

2.1 Meaning making in Mathematics

By defining meaning making as “ the process by which people interpret situations, events, objects, or discourses, in the light of their previous knowledge and experience” [12:1809], asserts learning as a meaning making process in light of the different educational, psychological and philosophical perspectives which includes: cultural-historical psychology, pragmatism, constructivism, and social constructionism. From the standpoint of these perspectives, “to learn something means to establish a meaningful relation to the subject matter so that it makes sense to the learner” [12:1809]. [13] claimed, from a social-cultural perspective, “Meanings of concepts are not necessarily conceived of as referring to something “objective” in the world but as something embedded in the social and cultural practices in which they evolve” p. 150. Hence, meaning making is a dynamic process, and mediating artifacts, such as textbooks, can facilitate learning [8, 9].

For [5], understanding mathematical concepts can provide two different meanings: instrumental and relational. The learners make instrumental meaning, i.e., learning an increasing number of fixed plans, by which they can find their way from particular starting points to required finishing points. While relational meaning, according to [12], consists of building up a conceptual structure (schema) from which its possessor can produce unlimited plans for getting from any starting point within his schema to any finishing point making meaning making is a complex process. [8] expressed the difficulty of discerning students’ conceptual understanding and preferred to investigate students’ mathematical meaning making. To characterize the process, [8] connected the study of students meaning making with the SEFI/Niss competence framework, which has eight subcategories [14].

2.2 Potential Meaning Making of textbooks

In a doctoral study, [15] investigated learners’ meaning making as a combination of i) their prior knowledge, ii) the information they access as they progress with the content, iii) the resource available to support their learning, and iv) the constraints imposed on that content by the wider environment. Textbooks are one of the resources that provide opportunities to facilitate the meaning making process. In another doctoral study titled, ‘Engineering students approaching the mathematics textbook as a potential learning tool – opportunities and constraints’, [7] conceptualized the textbook as a learning tool embedded in a tertiary educational setting. For [7], there are three perspectives on the process of approaching the textbook as a learning tool, that is, as potentially as a meaning making tool: the epistemological perspective referring to the nature of mathematical knowledge; the cognitive perspective focusing on the individual student ability to engage in the making process which can be related to the prior knowledge of the student as well as the concept definition and concept image; and didactic perspective focusing on the way the textbook is embedded in the institution. The assumption is that the learners are expected to use the textbook as a cognitive tool that promotes cognitive processes related to the meaningful learning of the mathematical content.

In general, textbooks facilitate the student’s meaning making process at different stages of learning. Intending to explore the role of the textbook in a Swedish classroom as the teacher-student interaction, [16] used three theoretical perspectives: the choice of educational content and contextualization, interaction to negotiate meaning making, and the use of the textbook as a potentially implemented curriculum. According to [17], meaning is a difficult concept, but it can come to presence through signs or semiotics. [9] used a multimodal approach to learning, where meaning making is central to textbook research. Assuming that modes (Writing, images, mathematical symbols, speech, moving images, etc.) carry the potential for meaning making, [9] investigated the potential for enabling communication between a Year 1 child Swedish and textbooks. The study showed a great complexity in the potential for meaning making in children's work with mathematics textbooks. Another study by [7], focusing on teachers' and students' interaction as influenced by textbooks in a grade eight classroom in Sweden, claims that textbooks are designed in a certain way with a guiding view of learning, stated explicitly or not. [7] questions if textbooks may be a source for meaning making by themselves without a teacher or facilitator/tutor, i.e., the textbook might not have a potential for meaning making, and the student can ignore it. However, one can argue that meaning making is directly related to prior knowledge of the learner, and much is expected from a tertiary-level student to make meaning by interacting with the book's author.

At a tertiary level, as noted above, [7] situated the study in the context of the basic mathematics course taken by first-year engineering students to identify and explore the factors that might influence the role of the textbook proposed to first-year engineering students. In other words, the study focused on the potential of meaning making of the textbook embedded in the educational setting offering the basic mathematics course. Since engineering students are more mature than the learners at primary or secondary school, they can engage in the meaning making process individually or in a group with and without a facilitator or teacher. The problem, at this level, could be that different mathematical contents might show up in a single mathematical task, like in the linear model, a case considered in this study.

2.3 Cognitive load and Cognitive demand

[18] reconceptualized mathematical cognition as a process of ascribing meaning to the mathematical objects of one's thinking and claimed that "mathematics cognition does not merely involve the attempt to recognize a previously unnoticed meaning of a concept but the attempt to ascribe meaning to the objects of one's thinking," [18:1234]. Instead, mathematics cognition evolves due to contextualization, complementizing, and complexifying. Such a complex process is a cognitively demanding activity. [19] characterized the cognitive demand of mathematical tasks (activities) into four levels: memorization, procedures without connection, procedures with connection, and doing mathematics. In light of these, an advanced meaning making process demands higher cognitive levels. John Sweller developed a cognitive load theory (CLT) in 1988. Cognitive load refers to a user's total amount of information the working memory can hold at any given time [20]. Hence working memory has a limited capacity. There are three types of cognitive load: Intrinsic, Extraneous, and Germane. Intrinsic load refers to the inherent difficulty level associated with a specific instructional topic that can vary from the learner's prior knowledge and experience [21]. At the same time, extraneous cognitive load refers to how information is presented to the learners to engage in working memories [20, 21]. Germane cognitive load is the learners' processing, construction, and automation to comprehend the content (material). Only these Germane cognitive

load is seen as favorable for learning [20]. Textbooks as cognition tool can create different cognitive loads.

2.4 Linear Models

This research focuses on the case of linear models. A large number of signal-processing estimation problems can be represented by a linear model [22]. This data model allows us to easily determine the estimator and its performance for both the classical and Bayesian approaches.The classical general linear model assumes that the data to be described as given in *Eq. (1).*

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w} \quad (1).$$

where $\mathbf{x}$ is an $N \times 1$ vector of observations, $\mathbf{H}$ is a known $N \times p$ observation matrix $N \times p$ of rank p, $\boldsymbol{\theta}$ is a vector of $p \times 1$ vector of parameters to be estimated, and $\mathbf{w}$ is a $N \times 1$ noise vector with a Gaussian Probability Distribution Function (PDF) with mean zero vector and covariance matrix $\mathbf{C}$, $\boldsymbol{\mathcal{N}}(\mathbf{0}, \mathbf{C})$. The PDF of $\mathbf{x}$ is

$$p(\mathbf{x}; \boldsymbol{\theta}) = \frac{\mathbf{1}}{(2\boldsymbol{\pi})^{N/2}\, \boldsymbol{det}^{1/2}\,(\mathbf{C})}\ \mathbf{exp}\left[-\frac{\mathbf{1}}{\mathbf{2}}\,(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^{T}\mathbf{C}^{-1}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right] \quad (2).$$

The Bayesian linear model further assumes that $\boldsymbol{\theta}$ is a $p \times 1$ random vector with Gaussian PDF, $\boldsymbol{\mathcal{N}}(\boldsymbol{\mu}_{\theta}, \mathbf{C}_{\theta})$, independent of $\mathbf{w}$. When assuming the linear model, it is possible to determine the optimal estimator $\hat{\boldsymbol{\theta}}$. The performance of any estimator obtained is critically dependent on the PDF assumptions.

3 METHODOLOGY

3.1 Context and Selected Textbook

This study is conducted in the Norwegian University of Science and Technology (NTNU) context. The author, a former doctoral candidate in the signal processing group, took the initiative to investigate the textbook used in a course in Statistical Signal Processing. The book Fundamentals of Statistical signal processing, Volume I, by [22], is one of the main course materials used in the mentioned course. It is selected to study the potential meaning making of the mathematics in engineering for convenience (convenience sampling): the researcher has taken the course and assisted it for two years during the doctoral study a decade ago. The textbook is heavily influenced by different mathematical contents, a natural candidate to start studying about meaning making of the mathematics in engineering. It consists of 15 chapters. According to the author, the book is intended as a graduate one-semester course with several student tasks (including the explanation, worked examples, and problems). For the present study, only chapter four, Linear Models, is considered as a key model used for several signal-processing problems in areas like estimation, detection, system identification, pattern recognition, machine learning, etc.

3.2 Framework for analysis

Qualitative content analysis is "a strict and systematic set of procedures for the rigorous analysis, examination, replication, inference, and verification of the contents of written data" [23]. In this case, it can be used to study the potential for meaning making a document or textbook. According to [11], there are three kinds of content analysis: conventional, directed, and summative. Conventional content analysis is used when researchers try to avoid using preconceived categories; directed content analysis is guided by existing theory or prior research by identifying key concepts or variables as initial coding categories; and in summative content analysis, keywords are selected based on previous research or the researchers' interests [11, 20]. This study follows a mix of the second and the third approaches since Keywords derived from the researcher's interest based on the literature review are used for analysis. In this preliminary study, neither the engineering student's engagement in the meaning making process nor the teachers' work to facilitate the meaning making process is not included. Rather a mere look at the textbook used for an engineering course and the potential for meaning making is investigated. Table 1 summarizes the researcher's choice to analyze the contents of the chapter selected from the mentioned textbooks.

Table 1. The framework for textbook analysis.

Literature	**Meaning Making in mathematics**	**Keywords**
[15]	As a combination of i) their *prior knowledge*, ii) the information they access as they *progress with the content*, iii) the resource available to support their learning, and iv) the constraints imposed on that content by the wider environment.	Prior knowledge, progress with the content, resource
[8]	In light of the eight mathematics competencies: thinking mathematically, Reasoning mathematically, Posing and solving mathematical problems, *Modelling mathematically*, *Representing* mathematical entities, Handling mathematical symbols and *formalism*, Communicating in, with, and about mathematics, and Making use of aids and tools.	Representing, Modelling, mathematical symbols and formalism
[9]	A textbook with a *multimodal approach*, broadening mathematics representation, provides an individual potential for meaning making.	A multimodal approach
[7]	The epistemological perspective, including *conceptual and procedural knowledge*. The cognitive perspective, including the notions of previous knowledge, concept image, concept definition, and the didactical perspective, characterizes the educational setting that creates teaching-learning environments.	Conceptual and procedural knowledge
[5]	Three types of *understanding*: Instrumental understanding, and Relational understanding	Understanding
[19]	The *cognitive demand* of mathematical tasks (activities) into four levels: memorization, procedures without connection, procedures with connection, and doing mathematics	Cognitive demand
[20, 21]	There are three types of *cognitive load*: Intrinsic, Extraneous, and Germane.	Cognitive load

4 ANALYSIS AND DISCUSSION

4.1 Minimum Variance Unbiased (MVU) estimator for the Linear Model

Linear models allow us to model several signal-processing problems like estimation, detection, pattern recognition, machine learning, etc. In Chapter 2, [22] provided the minimum variance unbiased (MVU) estimator, which produces values close to the truth most of the time. In Chapter 3, for its easiness, the Cramer-Rao Lower Bound (CRLB) is presented as a bound on the variance of any unbiased estimator. In Chapter 4, [22] assumed a linear model, defined in equation (1). Steven M. Kay argues that a significant number of signal processing estimation problems can be represented by a data model that allows us to determine the MVU estimator quickly.

The chapter first provides an introduction and summary section and then provides the development of the MVU estimator for the Linear Model. Given the linear model in *Eq. (1),* a key model with PDF of $\mathbf{x}$ given in *Eq. (2)*, then $\hat{\boldsymbol{\theta}} = \mathbf{g}(x)$ will be the MVU estimator if:

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}(\boldsymbol{\theta})(\mathbf{g}(\mathbf{x}) - \boldsymbol{\theta}) \tag{3}$$

for some function $\mathbf{g}$ and $\mathbf{I}(\boldsymbol{\theta})$**,** which is the Fisher information matrix that determines the characteristics of statistical estimation. The linear model *Eq. (3)*

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial}{\partial \boldsymbol{\theta}} \left[-\boldsymbol{ln}\,(2\pi\sigma^2)^{\frac{N}{2}} - \frac{1}{2\sigma^2}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^T(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right] \quad (4).$$

$$= -\frac{1}{2\sigma^2}\frac{\partial}{\partial \boldsymbol{\theta}}\left[\mathbf{x}^T\mathbf{x} - 2\mathbf{x}^T\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{H}^T\mathbf{H}\boldsymbol{\theta}\right] \quad (5).$$

Further using identities,

$$\frac{\partial\, \mathbf{b}^T\boldsymbol{\theta}}{\partial \boldsymbol{\theta}} = \mathbf{b} \quad (6).$$

$$\frac{\partial\, \boldsymbol{\theta}^T\mathbf{A}\boldsymbol{\theta}}{\partial \boldsymbol{\theta}} = \mathbf{2A}\boldsymbol{\theta} \quad (7).$$

for $\mathbf{A}$ a symmetric matrix,

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{1}{\sigma^2}\left[\mathbf{H}^T\mathbf{x} - \mathbf{H}^T\mathbf{H}\boldsymbol{\theta}\right] \quad (8).$$

Assuming that $\mathbf{H}^T\mathbf{H}$ is invertible

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\mathbf{H}^T\mathbf{H}}{\sigma^2}\left[(\mathbf{H}^T\mathbf{H})^{-1}\,\mathbf{H}^T\mathbf{x} - \boldsymbol{\theta}\right] \quad (9).$$

Which is exactly in the form of *Eq. (3)*. Hence the MVU estimator of $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\,\mathbf{H}^T \quad (10).$$

and its covariance matrix is:

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \mathbf{I}^{-1}(\boldsymbol{\theta}) = \sigma^2(\mathbf{H}^T\mathbf{H})^{-1}. \quad (11).$$

The details of reflections on each step is attached in Appendix. Further, [22] provides three examples based on the problem contexts: curve fitting, Fourier analysis, and system identification, before presenting another subsection that extends (1) to a general linear model where the nose is not white. Therefore, it first deals with the whitening approach, and then the above procedure is repeated to develop a new estimator with the same form. In addition, the chapter provides two other practical examples, i.e., on Direct Current (DC) level in colored noise and DC level and exponential in white noise, to assimilate the material on parameter estimation effectively. In the end, there are 14 problems, either signal processing estimation problems or pure mathematics tasks, as [22] calls homework related to basic concepts. To reduce complexity for the learners, the two estimators, one for the white and another for the colored noise, are presented as theorems without rigorous, detailed proofs.

4.2 Impact of Prior knowledge and Understanding of Meaning Making

[24] extended instrumental and relational meaning making of mathematical concepts of [5] to advanced mathematical concepts. Instrumental understanding of a concept refers to the ability to state the definition of the concept, is aware of the important theorems associated with that concept, and can apply those theorems in specific

instances. While a relational understanding includes to understand the informal notion this concept was created to exhibit, why the definition is a rigorous demonstration of this intuitive notion, and why the theorems associated with this concept are true. Looking at the steps from *Eq. (3)* to *(11)*, the textbook provides opportunities for the two different meaning making in light of [24] definitions of understanding. Those who have an instrumental understanding meaning making may remember the rules and procedures applied at each step while those with a relational understanding meaning making can connect the mathematics why those mathematical concepts are working as such: For example, understanding why *Eq. (3)* and *(9)* are connected demands intuitive understanding of the different concept as a basis for constructing a formal argument. In turn, it demands a sold prior knowledge (Huthali, 2014) in the basic mathematical concepts like partial differentiation, which could be a difficult concept for many. In fact, [24] has extended a relational understanding of a concept as somewhat akin to Tall and Vinner's concept image [25]. In this case, the concept image of the partial derivative procedure is a total cognitive structure that is associated with this concept, which includes all the mental pictures and associated properties and processes. As an example, why the partial derivative is employed can be seen as an extension of the experience that when we derivate a function and set up the result to be zero, it is possible to get the extremum values of the function. As we are looking for the minimum variance estimator, it gives a sense of meaning to the learner. Concept image is "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" [25:152], which in turn has huge impact on the meaning making process.

4.3 Impact of Representation and Multimodal on Meaning making

Most parts of the pages of Chapter 4 in [22] are highly dominated by advanced mathematical symbols, very brief texts and few diagrams, no real-life related images. According to [9], different modes like writing, images, mathematical symbols, speech and moving images carry potential for meaning making. The textbook is highly populated with advanced mathematical symbols, which can contribute to the complexity in the meaning making process for learners. In [8] and [26], one finds mathematical competency model connected to mathematical meaning making. One of the elements mathematical competences is dealing with different representations of mathematical entities [26, 27]. In Chapter 4, one finds four types of representations: verbal, symbolic (most dominating ones), diagram (three figures) and context (like system identification, curve fitting, a fading noise, and so on). These four can assist the ability to interpret as well as translate and move between the different representations [27], boosting the meaning making process. However, the dominating advanced mathematical symbols used can be problematic.

5 SUMMARY AND ACKNOWLEDGMENTS

Linear models allow us to model several signal-processing problems in estimation, detection, system identification, pattern recognition, machine learning, etc. It is undoubtedly one of the most critical classes of models that represents a more complex phenomenon in a simplified abstraction. Several service courses in mathematics and statistics cover this content. Meaning making of this model is expected from the students in several engineering problems. The findings show that Chapter 4 of the course material [22] provides different aspects of meaning making in mathematics for engineering students. The linear model-based estimator is presented in a reduced complexity, i.e., without rigorous, detailed proofs that demand a solid prior knowledge of many of the mathematical concepts that are involved. The 18-page chapter is highly dominated by advanced mathematical symbols from different mathematical concepts with higher cognitive demanding tasks and activities, which can increase complexity in the meaning making process by the learner, further reducing the opportunity for learning. Four types of representations (context, verbal, symbol, and graphs), as well as multimodal approaches (writing and mathematical symbols), are used, creating potential for meaning making by the user. However, symbolic representation dominates the pages creating a higher extraneous cognitive load on the learner. Where as examples and contexts contribute for lowering the complexity in the potential for meaning making in engineering students with the mentioned textbook.

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Appendix. Meaning making of the Mathematics related to the MVU estimator for the Linear Model

Equation Number	Explanation of the equation	Mathematical concepts and procedures	Key Words From theory
(1)	$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ represents a system of linear equations in a matrix equation of the form. x denote a finite snapshot of a time or space series with dimension N. $\mathbf{H}$ is a matrix of modes with $N \times p$ entries. $\boldsymbol{\theta}$ is a vector of $p \times \mathbf{1}$ vector of parameters to be estimated. $\mathbf{w}$ is a random variable representing noise. It is a $N \times 1$ noise vector with entries represented by Gaussian probability distribution function (PDF).	Linear equations, vectors, matrix, random variable, PDF Estimation	Prior knowledge, mathematical symbols and formalism, modeling
(2)	A probability density function, for the vector $\mathbf{x}$. **It is a** Gaussian function of a normally distributed random variable with expected value **0** vector and covariance **C**.	Exponential function, matrix, inverse and transpose of matrix, product of matrix and vector, Covariance	Prior knowledge, mathematical symbols and formalism, Cognitive load
(3)	The minimum variance unbiased (MVU) estimator , $\hat{\boldsymbol{\theta}} = \mathbf{g}(\boldsymbol{x})$, that fulfils the Cramer-Rao Lower Bound (CRLB) exists if (3) is satisfied. The proof is given in Chapter 3 with 3 pages long involving advanced mathematical concepts like Expectation , integral, partial differential equation(PDE)	Logarithm function, PDF, partial differential equation of a vector, Expectation , integral	Prior knowledge, progress with the content, Understanding
(4)	Applying the PDE on the logarithmic of (2)	PDE and PDF of vector and matrices	Conceptual and procedural knowledge
(5)	Procedure applied to simplify (4). First, the term disappeared since derivative of a constant is zero. The rest is expanded.	Derivation and simplification	mathematical symbols and formalism, procedural knowledge
(6) & (7)	Identities when applying the partial derivatives. It is like the extension of finding the derivative of $(bx)' = b$ and $(ax^2)' = 2ax$.	Partial derivative of a vector and a matrix over vector.	procedural knowledge
(8)	Using the identities in (6) and (7) and applying the PDE in (5). The first term disappears since it is independent of $\boldsymbol{\theta}$.	Partial derivation and simplification	Conceptual and procedural knowledge
(9)	Assuming the matrix $\mathrm{H}^T\mathrm{H}$ as invertible, it is factored out to the front from the expression in (8).	Factorization of a matrix	procedural knowledge
(10)	The MVU linear model-based estimator derived from comparing the expression in (3) and (9).	Comparing expressions	Conceptual and procedural knowledge
(11)	The performance of the estimator in (10) is expressed using the covariance matrix (11), derived from comparing the expression in (3) and (9).	Covariance, Comparing expressions	Understanding, Cognitive demand