Conference papers   School of Mathematics

2022

New Bounds on the Real Polynomial Roots

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Abstract

The presented analysis determines several new bounds on the real roots of the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$ (with $a_n > 0$). All proposed new bounds are lower than the Cauchy bound $\max\left\{1, \sum_{j=0}^{n-1} |a_j/a_n|\right\}$. Firstly, the Cauchy bound formula is derived by presenting it in a new light – through a recursion. It is shown that this recursion could be exited at earlier stages and, the earlier the recursion is terminated, the lower the resulting root bound will be. Following a separate analysis, it is further demonstrated that a significantly lower root bound can be found if the summation in the Cauchy bound formula is made not over each one of the coefficients $a_0, a_1, \ldots, a_{n-1}$, but only over the negative ones. The sharpest root bound in this line of analysis is shown to be the larger of 1 and the sum of the absolute values of all negative coefficients of the equation divided by the largest positive coefficient. The following bounds are also found in this paper: $\max\left\{1, \left(\sum_{j=1}^{q} B_j/A_l\right)^{1/(l-k)}\right\}$, where $B_1, B_2, \ldots, B_q$ are the absolute values of all of the negative coefficients in the equation, k is the highest degree of a monomial with a negative coefficient, A_l is the positive coefficient of the term $A_l x^l$ for which $k < l \leq n$.

Key words: polynomial equation, root bounds, Cauchy polynomial, Cauchy theorem, Cauchy and Lagrange bounds, Descartes' rule of signs

2020 Mathematics Subject Classification: 12D10, 26C10

The real roots of the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$ are bound from above by the unique positive root of the associated Cauchy polynomial $|a_n| x^n - |a_{n-1}| x^{n-1} - \cdots - |a_0|$. The Cauchy formula yields that the upper bound on the real

roots of the equation is $\max\left\{1, \sum_{j=0}^{n-1} |a_j/a_n|\right\}$ – see Section 8.1 in [1]. In the first part of the analysis presented in this paper, this formula is derived from a different perspective – through a recursion – following the idea of splitting polynomials into two parts and studying the "interaction" between these parts [2] (i.e. studying the intersection points of their graphs) – a method successfully used also for the full classification of the roots of the cubic [2] and the quartic equation [3] in terms of the equation coefficients. This recursion involves bounding the unique positive root of a particular equation with the unique positive root of a subsidiary equation of degree one less. The recursion ends with the determination of the root of a linear equation and this root is exactly $\sum_{j=0}^{n-1} |a_j/a_n|$. If, instead, the recursion is terminated at an earlier stage – that of a quadratic, cubic, or quartic equation – the resulting root bound will be lower, as shown in this work. All of these analytically determinable new bounds are stricter than the Cauchy bound.

It is separately shown, following a different line of analysis, that one does not have to sum over all coefficients a_j in $\sum_{j=0}^{n-1} |a_j/a_n|$, but only over the negative ones. This results in a significantly lower root bound than the Cauchy bound. It is demonstrated further that this new bound can be made even lower by finding a denominator greater than $|a_n|$. The sharpest upper bound of the real roots of the general polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_n > 0$ that can be found following this analysis, is either 1 or the smallest of the unique positive roots of all Cauchy polynomials that can be extracted from this polynomial with preservation of all terms with negative coefficients – whichever is larger. The latter is the sum of the absolute values of all negative coefficients of the equation divided by the largest positive coefficient.

The following bounds are also found in this paper: $\max\left\{1, \left(\sum_{j=1}^{q} B_j/A_l\right)^{\frac{1}{l-k}}\right\}$, where $B_1, B_2, \ldots, B_q$ are the absolute values of all of the negative coefficients in the equation, k is the highest degree of a monomial with a negative coefficient, A_l is the positive coefficient of the term $A_l x^l$ with $k < l \leq n$.

The lower bound on the unique positive root of a Cauchy polynomial is also determined.

Definition 1 ([1]). A *Cauchy polynomial* of degree n is a polynomial in x such that the coefficient of x^n is positive and the coefficients of all of its remaining terms are negative.

That is, the Cauchy polynomial has the form $|a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_0|$. As there must be at least one term with negative coefficient, a monomial cannot be a Cauchy polynomial.

The Cauchy polynomial has, by Descartes' rule of signs, a unique positive root, say μ (as there is only one sign change in the sequence of its coefficients).

Theorem 1 (**Cauchy**). *The unique positive root μ of the Cauchy polynomial* $|a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_0|$ *provides a bound on the real roots of the general*

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polynomial equation of degree n, see [1]:

$$(1) \qquad a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0.$$

In this equation $a_n \neq 0$ and, without losing generality, it will be assumed further that $a_n > 0$.

If all of the coefficients of this equation have the same sign, then it will not have positive roots and therefore 0 will be the upper bound of the real roots (if real roots exist). This special case will not be considered further.

Before addressing the general equation (1), re-write the Cauchy polynomial as $\alpha_n x^n - \alpha_{n-1} x^{n-1} - \cdots - \alpha_0$, where all coefficients α_j are positive (the number of terms $\alpha_j x^j$ for which $j < n$ may be between 1 and n), and consider the associated equation

$$(2) \qquad \alpha_n x^n - \alpha_{n-1} x^{n-1} - \cdots - \alpha_0 = 0.$$

As the unique positive root of this equation is sought, one can assume, without loss of generality, that $\alpha_0 \neq 0$. (If α_0 happens to be zero, one identifies 0 as a root, reduces everywhere the powers of x and the indices of the coefficients by one unit and arrives at an equation of the same type, but of degree $n - 1$. If the "new" α_0 also happens to be zero, this procedure should be repeated until a non-zero coefficient is encountered – it is guaranteed to exist by the definition.)

Theorem 2 (Cauchy bound). *The real roots of equation* (1) *are bound by*

$$(3) \qquad \rho = \max \left\{ 1, \sum_{j=0}^{n-1} \frac{\alpha_j}{\alpha_n} \right\}.$$

Proof. This will be done by recursion. Equation (2) can be viewed as

$$(4) \qquad f_n(x) = \alpha_0,$$

where $f_n(x) \equiv \alpha_n x^n - \alpha_{n-1} x^{n-1} - \cdots - \alpha_1 x$, and also as

$$(5) \qquad x f_{n-1}(x) = \alpha_0,$$

where $f_{n-1}(x) \equiv \alpha_n x^{n-1} - \alpha_{n-1} x^{n-2} - \cdots - \alpha_1$ (since $x \neq 0$).

Due to $f_n(x) = x f_{n-1}(x)$, the two polynomials $f_n(x)$ and $f_{n-1}(x)$ have the same unique positive root r, that is, the graphs of the two functions $f_n(x)$ and $f_{n-1}(x)$ cross each other at the point r on the abscissa.

There is another intersection point between $f_n(x)$ and $f_{n-1}(x)$ for positive x and this intersection always happens at $x = 1$. This can be easily seen from

$$(6) \qquad f_n(1) = \alpha_n - \alpha_{n-1} - \cdots - \alpha_1 = f_{n-1}(1).$$
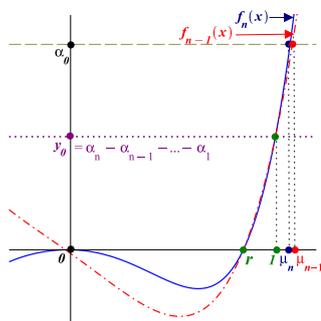
Fig. 1. $0 \le y_0 \le \alpha_0$

Fig. 2. $y_0 \le 0 \le \alpha_0$

Fig. 3. $0 < \alpha_0 < y_0$

When $\alpha_0 \ge \alpha_n - \alpha_{n-1} - \cdots - \alpha_1 \ge 0$, the upper bound of μ_n is μ_{n-1} and the lower bound of μ_n is 1
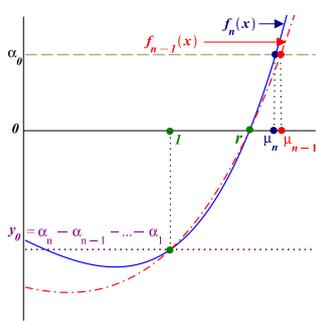
When $\alpha_0 \ge 0 \ge \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$, the upper bound of μ_n is μ_{n-1} and the lower bound of μ_n is 1
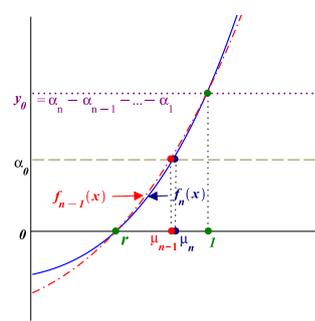
When $0 < \alpha_0 < \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$, the upper bound of μ_n is 1 and the lower bound of μ_n is μ_{n-1}

Thus, the coordinates of the two points of intersection between $f_n(x)$ and $f_{n-1}(x)$ are $(r, 0)$ and $(1, y_0)$, where $y_0 = \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$. Next, one has to determine where the point $(1, y_0)$ is with respect to the "level" $y = \alpha_0 > 0$, prescribed by the free term of the Cauchy polynomial.

Clearly, when $y_0 = \alpha_n - \alpha_{n-1} - \cdots - \alpha_1 < 0$, this intersection point is in the fourth quadrant; when $y_0 = \alpha_n - \alpha_{n-1} - \cdots - \alpha_1 > 0$, it is in the first quadrant; and when $y_0 = \alpha_n - \alpha_{n-1} - \cdots - \alpha_1 = 0$, then the functions $f_n(x)$ and $f_{n-1}(x)$ cross only once at the abscissa at $r = 1$, i.e. $r = 1$ is a double root of the polynomial $f_n(x) - f_{n-1}(x)$.

As $\alpha_0 > 0$, there are three possible situations: either $0 \le y_0 \le \alpha_0$ (Fig. 1), or $y_0 \le 0 \le \alpha_0$ (Fig. 2), or $0 < \alpha_0 < y_0$ (Fig. 3).

Let μ_n denote the unique positive root of $f_n(x) = \alpha_0$ (note that $\mu = \mu_n$) and μ_{n-1} is the unique positive root of $f_{n-1}(x) = \alpha_0$. It is quite clear that in the first two cases (Fig. 1 and 2) one has $1 < \mu_n < \mu_{n-1}$, while in the third case (Fig. 3), $\mu_{n-1} < \mu_n < 1$ holds. Thus, when $\alpha_0 \ge \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$, one has that μ_{n-1} is an upper bound of μ_n (or μ) and 1 is a lower bound μ_n (or μ). On the contrary, when $\alpha_0 < \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$, one has that μ_{n-1} is the lower bound of μ_n (or μ) and 1 is the upper bound μ_n (or μ). In the latter case, one does not need to proceed further and should just take 1 as the upper bound of the unique positive root μ of the Cauchy polynomial and, hence, as the upper bound of the roots of the general equation (1).

If, however, the free term α_0 of the Cauchy polynomial is such that $\alpha_0 \ge \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$, one needs to find an upper bound of μ_{n-1} and this, in turn, will serve as upper bound of $\mu_n = \mu$ and, hence, on the roots of (1). This means to continue recursively by considering the equation $f_{n-1}(x) = \alpha_0$ and re-writing it as

$$(7) \qquad\qquad g_{n-1}(x) = \alpha_0',$$

where $g_{n-1}(x) = \alpha_n x^{n-1} - \alpha_{n-1} x^{n-2} - \cdots - \alpha_2 x = f_{n-1}(x) + \alpha_1$ and $\alpha_0' = \alpha_0 + \alpha_1$, on the one hand, and as

$$(8) \qquad\qquad x g_{n-2}(x) = \alpha_0',$$

with $g_{n-2}(x) = \alpha_n x^{n-2} - \alpha_{n-1} x^{n-3} - \cdots - \alpha_2$, on the other.

As before, since $g_{n-1}(x) = x g_{n-2}(x)$, the polynomials $g_{n-1}(x)$ and $g_{n-2}(x)$ have the same unique positive root r', that is, the graphs of the two functions $g_{n-1}(x)$ and $g_{n-2}(x)$ cross each other at the point r' on the abscissa. Also as before, there is another intersection point between $g_{n-1}(x)$ and $g_{n-2}(x)$ for positive x and this intersection point is again $x = 1$:

$$(9) \qquad\qquad g_{n-1}(1) = \alpha_n - \alpha_{n-1} - \cdots - \alpha_2 = g_{n-2}(1).$$

Equations (7) and (8) have the same unique positive root μ_{n-1}. Let μ_{n-2} denote the unique positive root of $g_{n-2}(x) = \alpha_0'$. This positive root is an upper bound for μ_{n-2} provided that $\alpha_0' \geq \alpha_n - \alpha_{n-1} - \cdots - \alpha_2$. The latter is simply $a_0 \geq \alpha_n - \alpha_{n-1} - \cdots - \alpha_1$ and this is indeed the case, as it was presumed to hold when the recursion started.

The equation $g_{n-2}(x) = \alpha_0'$ is, in fact,

$$(10) \qquad\qquad \alpha_n x^{n-2} - \alpha_{n-1} x^{n-3} - \cdots - \alpha_3 x - \alpha_2 - \alpha_1 = \alpha_0.$$

Continuing in the vein of recursively bounding the root of each of these equations with the root of an equation of degree reduced by one unit, the linear equation

$$(11) \qquad\qquad \alpha_n x - \alpha_{n-1} - \alpha_{n-2} - \cdots - \alpha_0 = 0,$$

which terminates the recursion, immediately follows. The exact unique positive root of this equation is

$$(12) \qquad\qquad \mu_1 = \frac{\alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_0}{\alpha_n}.$$

Hence, one gets the Cauchy bound

$$(13) \qquad\qquad \rho = \max\{1, \mu_1\} = \max\left\{ 1, \sum_{j=0}^{n-1} \frac{\alpha_j}{\alpha_n} \right\}$$

for the roots of the general equation (1) – see (8.1.10) in [1] (there are no absolute values in the above formula as the α's are taken as positive). $\qquad\square$

Root bounds, sharper than the Cauchy bound (13), can be found by earlier termination of the recursion which led to (12). This is summarised through the following corollaries.

Exit the recursion at the stage of the quadratic equation which precedes the linear equation (11) with root μ_1. The unique positive root μ_2 of this quadratic equation will be smaller than μ_1 and hence, it will provide a sharper bound.

Corollary 2.1. *A root bound on the real roots of* (1), *sharper than the Cauchy bound* (13), *is provided by the unique positive root of*

$$(14) \qquad \alpha_n x^2 - \alpha_{n-1} x - \alpha_{n-2} - \cdots - \alpha_0 = 0,$$

namely

$$(15) \qquad \mu_2 = \frac{1}{2\alpha_n} \left[\alpha_{n-1} + \sqrt{\alpha_{n-1}^2 + 4\alpha_n(\alpha_{n-2} + \alpha_{n-3} + \cdots + \alpha_0)} \right].$$

One can exit the recursion even earlier.

Corollary 2.2. *The unique positive root μ_3 of the cubic equation*

$$(16) \qquad \alpha_n x^3 - \alpha_{n-1} x^2 - \alpha_{n-2} x - \alpha_{n-3} - \cdots - \alpha_0 = 0$$

provides a bound on the real roots of (1) *which is sharper than the bound* (15) *and sharper than the bound* (12): $\mu_3 < \mu_2 < \mu_1$.

Corollary 2.3. *The unique positive root μ_4 of the quartic equation*

$$(17) \qquad \alpha_n x^4 - \alpha_{n-1} x^3 - \alpha_{n-2} x^2 - \alpha_{n-3} x - \alpha_{n-4} - \cdots - \alpha_0 = 0$$

provides the sharpest bound that can be found analytically by the recursion: $\mu_4 < \mu_3 < \mu_2 < \mu_1$.

The Abel–Ruffini theorem prevents from terminating the recursion earlier, as it is neither possible to get the roots of an equation of degree five or higher in terms of radicals, nor is it possible to find a closed-form solution of such equations.

The Cauchy bound (13) can be made significantly sharper by following a different line of analysis.

Theorem 3. *The unique positive root μ of any Cauchy polynomial, extracted from the given polynomial, is a bound on the roots of the polynomial.*

Proof. Suppose that the number of terms with positive coefficients in the general equation (1) is p and that the number of terms with negative coefficients is q. Clearly, $p + q \leq n + 1$ (equality is achieved if none of the coefficients of the general equation (1) is equal to zero). For the coefficients $a_j > 0$, write A_j instead, and for the coefficients $a_j < 0$, write $-B_j$ instead. Clearly, $A_n \equiv a_n > 0$ and all the rest of the A's are non-negative. At least one of the B's is positive, the rest – non-negative (equation in which all coefficients have the same sign is no longer of interest).

Following the ideas, presented in [2], of splitting a polynomial into two parts and analysing the "interaction" between these parts in order to study the roots of the polynomial, one can re-write the general equation (1) as

$$(18) \quad A_l x^l - B_{n-m_1} x^{n-m_1} - B_{n-m_2} x^{n-m_2} - \cdots - B_{n-m_q} x^{n-m_q}$$
$$= -A_n x^n - A_{n-k_1} x^{n-k_1} - \cdots - \hat{A}_l x^l - \cdots - A_{n-k_{p-1}} x^{n-k_{p-1}},$$

where $\{k_1 < k_2 < \cdots < k_{p-1}, m_1 < m_2 < \cdots < m_q\}$ is a permutation of $\{1, 2, \ldots, n\}$, l is such that $n - m_1 < l \leq n$, and the hat on A_l indicates that the term $A_l x^l$ is missing from the right-hand side. At least one monomial $A_l x^l$ with $n - m_1 < l \leq n$ exists: $A_n x^n$. Note that one can select *any* one of the monomials with positive A's to be the only positive term on the left-hand side of (18), that is, one can form different Cauchy polynomials on the left-hand side of (18) with different positive terms. Thus, the choice of Cauchy polynomial on the left-hand side of (18) is not unique (unless the given polynomial is itself a Cauchy polynomial). Each of these Cauchy polynomials will have a unique positive root μ_l. Note that, if the free term of the equation happens to be positive, then the resulting Cauchy polynomial will have zero as a root. The Cauchy polynomial, due to having a positive coefficient in its leading term, is strictly positive for all $x > \mu_l$. The polynomial on the right-hand side of the equation is strictly negative for all $x > 0$ (should the free term of the original equation happen to be negative, the polynomial on the right-hand side will have 0 as a root). For all $x > \mu_l$, the two sides have opposite sign and therefore, there can be no roots of the equation for $x > \mu_l$, i.e. the root bound for the general equation (1) is μ_l – the unique positive root of any Cauchy polynomial extracted from the given polynomial. $\square$

One has to reiterate that different choices of l in $A_l x^l$ lead to different Cauchy polynomials (having different unique positive roots) on the left-hand side of (18). It will be the unique positive root of the particular Cauchy polynomial, appearing on the left-hand side of (18), that will provide a bound for the roots of the general equation (1).

Theorem 4. *An upper bound on the roots of the general polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with $a_n > 0$ is the smallest of the unique positive roots of all Cauchy polynomials that can be extracted from this polynomial.*

Proof. Suppose now that on the left-hand side of (18) the "ghost" term $-B_{l-1} x^{l-1} - B_{l-2} x^{l-2} - \cdots - B_{m-n_1+1} x^{n-m_1+1}$ with $B_{l-1} = B_{l-2} = \cdots = B_{m-n_1+1} = 0$ has been added to the Cauchy polynomial. Following the analysis that led to the derivation of (12) and (13), one sees that, instead of the bound given by the Cauchy formula (13), a significantly lower bound is obtained:

$$(19) \qquad \rho_l' = \max\left\{1, \sum_{j=1}^{q} \frac{B_{n-m_j}}{A_l}\right\}.$$

The summation is no longer over all coefficients of the equation (as in the Cauchy formula), but only over the absolute values of the negative ones (in units of A_l). There will be different bounds ρ_l' for different choices of l.

Clearly, the sharpest of these bounds will be the one with the biggest denominator: $A_{\max} = \max\{A_l \mid n \geq l > n - m_1\}$:

$$(20) \qquad \rho' = \max\left\{1, \sum_{j=1}^{q} \frac{B_{n-m_j}}{A_{\max}}\right\}$$

– the smallest of the unique positive roots of all Cauchy polynomials that can be extracted from the given polynomial. □

Theorem 5. *The quantity*

$$(21) \qquad \rho_l'' = \max\left\{1, \left(\sum_{j=1}^{q} \frac{B_{n-m_j}}{A_l}\right)^{\frac{1}{l-n-m_1}}\right\}, \qquad n - m_1 < l \leq n,$$

is a bound on the real roots of equation (1), *sharper than* (20).

Proof. Instead of introducing the "ghost" term $B_{l-1}x^{l-1} - B_{l-2}x^{l-2} - \cdots - B_{m-n_1+1}$ with $B_{l-1} = B_{l-2} = \cdots = B_{m-n_1+1} = 0$, this sharper new bound can be obtained if one terminates the recursion when equation

$$(22) \qquad A_l x^{l-n-m_1} - B_{n-m_1} - B_{n-m_2} - \cdots - B_{n-m_q} = 0$$

is reached. The obtained in such a way bound, for different values of l, is exactly (21). □

One can find the sharpest possible bound, ρ'', in this line of analysis as the minimum of all ρ_l'' in (21).

To determine the lower bound of the unique positive root of the Cauchy polynomial, re-write equation (2) for positive x as

$$(23) \qquad x^n\left(\alpha_n - \alpha_{n-1}\frac{1}{x} - \cdots - \alpha_0\frac{1}{x^n}\right) = 0.$$

In the variable $y = 1/x$, this equation is:

$$(24) \qquad \alpha_0 y^n + \alpha_1 y^{n-1} + \cdots + \alpha_{n-1}y - \alpha_n = 0.$$

Theorem 6. *The numbers in the sequence*

$$(25) \qquad \left\{\frac{\alpha_n}{\alpha_{n-1}}, \left(\frac{\alpha_n}{\alpha_{n-2}}\right)^{\frac{1}{2}}, \ldots, \left(\frac{\alpha_n}{\alpha_0}\right)^{\frac{1}{n}}\right\}$$

are all bounds on the real roots of (24) *and the smallest of these provides the sharpest root bound.*

Proof. As there are n Cauchy polynomials $\alpha_{n-t}y^t - \alpha_n$ $(t = 1, 2, \ldots, n)$ that can be extracted from $\alpha_0 y^n + \alpha_1 y^{n-1} + \cdots + \alpha_{n-1}y - \alpha_n$, equation (24) can be written in n equivalent ways:

$$(26) \qquad -\alpha_n + \alpha_{n-t}y^t = -\alpha_0 y^n - \alpha_1 y^{n-1} - \cdots - \hat{\alpha}_{n-t}y^t - \cdots - \alpha_{n-1}y.$$

For each of these, the unique positive root $(\alpha_n/\alpha_{n-t})^{1/t}$ of the Cauchy polynomial $\alpha_{n-t}y^t - \alpha_n$ provides an upper bound on the real roots of equation (24). The sequence (25) represents their full set. $\qquad\square$

This a Lagrange type of bound (the Lagrange bound is the sum of the two largest values in the sequence (25)).

The sharpest lower bound of the unique positive root of the Cauchy equation (2) will thus be the largest of the reciprocals of the numbers in the sequence (25).

It is a great honour to have this paper published in Comptes rendus de l'Académie bulgare des Sciences. This work is dedicated to the memory of my mother, Nadejda Vassileva Manova-Prodanova (14 Jan 1926 – 11 Apr 2016), who worked in the Bulgarian Academy of Sciences for many years as head of the Bibliography and Information Department and who treasured with love and pride every day of her work there.

REFERENCES

[1] Rahman Q. I., G. Schmeisser (2002) Analytic Theory of Polynomials, Oxford University Press.

[2] Prodanov E. M. (2020) On the determination of the number of positive and negative polynomial zeros and their isolation, Open Mathematics (De Gruyter), **18**, 1387–1412, doi: 10.1515/math-2020-0079, arXiv: 1901.05960.

[3] Prodanov E. M. (2021) Classification of the roots of the quartic equation and their Pythagorean tunes, International Journal of Applied and Computational Mathematics (Springer), **7**, Art. 218, doi: 10.1007/s40819-021-01152-w, arXiv: 2008.07529.

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