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## Lattices in Finite Real reflection Groups

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## LATTICES IN FINITE REAL REFLECTION GROUPS

THOMAS BRADY AND COLUM WATT

**Abstract:** For a finite real reflection group  $W$  with Coxeter element  $\gamma$  we give a uniform proof that the closed interval,  $[I, \gamma]$  forms a lattice in the partial order on  $W$  induced by reflection length. The proof involves the construction of a simplicial complex which can be embedded in the type  $W$  simplicial generalised associahedron.

## 1. INTRODUCTION.

Let  $W$  be a finite real reflection group. Associated to  $W$  is a finite type Artin group or generalised braid group,  $A(W)$ . Much of the work on finite type Artin groups takes Garside's paper, [11], as its starting point, using the set of fundamental reflections as a generating set for  $W$  and a corresponding standard generating set for  $A(W)$ . Recently it has been shown that when the set of *all* reflections is used as a generating set for  $W$  and a corresponding generating set is used for  $A(W)$ , a parallel theory can be constructed. In particular, the new positive monoid embeds in  $A(W)$  and new  $K(\pi, 1)$ 's for  $A(W)$  have been constructed. The larger generating set gives  $A(W)$  a second structure as a Garside group. The larger generating set is proposed in [5], but Daan Krammer had used it independently in unpublished work. In the case of the braid group  $B_n$ , where  $W$  is the symmetric group  $\Sigma_n$ , the larger generating set coincides with the band generators from [4]. The second Garside structure for  $B_n$  is described in [1] and the structure for general finite  $W$  in [2]. The general construction of  $K(A(W), 1)$ 's is described in [9], using ideas from [3].

A central result needed in the development of this parallel theory is that the closed interval,  $[I, \gamma]$ , bounded by the identity  $I$  and a Coxeter element  $\gamma$ , forms a lattice in the partial order on  $W$  induced by reflection length. The lattice property is used to prove the embedding of the positive monoid and the asphericity of the new  $K(\pi, 1)$ 's. Existing proofs of the lattice property use the classification of finite real reflection groups with different methods applied to the different groups. The symmetric group is handled in [6], the  $C_n$  and  $D_n$  groups in [8]. Bessis treats all cases in [2].

In this paper we give a new proof of the lattice property that is independent of the classification of finite real reflection groups. For this purpose we introduce a new simplicial complex which is a geometric model for the partially ordered set  $[I, \gamma]$ . If  $n$  is the rank of  $W$ , this simplicial complex will lie in the  $(n - 1)$ -sphere,  $S^{n-1}$ , in  $\mathbf{R}^n$  and its vertex set will consist of a set of positive roots for  $W$ .

The layout of the paper is as follows. In section 2, we recall definitions and results about the partial order and spherical simplicial complexes. In section 3 we extend some of the material from [13] and prove some properties of the dot products of roots of  $W$  with vertices of a Petrie polygon. In section 4 we define the simplicial complex  $X(\gamma)$  and derive some of its properties. In section 5 we define the subcomplexes  $X(\sigma)$  for  $\sigma \leq \gamma$ . After some preparatory work in section 6, we characterise the geometric realisation of  $X(\sigma)$  and we prove the lattice property in section 7. We conclude in section 8 by explaining the connection between our construction and the generalised associahedra of [10]. In the process, we give a new proof that the latter is a spherical simplicial complex.

## 2. THE PARTIAL ORDER.

Let  $W$  be a finite, irreducible real reflection group with reflection set  $\mathcal{R}$ . We recall from [8] the partial order on  $W$  given by the reflection length function  $l$ . For  $u, w \in W$  we say

$$u \leq w \quad \Leftrightarrow \quad l(w) = l(u) + l(u^{-1}w),$$

where  $l(v)$  is the smallest positive integer  $k$  such that  $v$  can be written as a product of  $k$  reflections from  $\mathcal{R}$ . Thus  $u \leq w$  if and only if there is a shortest factorisation of  $u$  as a product of reflections which is a prefix of a shortest factorisation of  $w$ .

We note that the partial order on  $W$  is the restriction to  $W$  of a partial order on the orthogonal group  $O(n)$  which is introduced in [7]. Some implications of this relationship are investigated in [8]. Some of the notation and results from those papers will be used here. In particular, if  $A \in O(n)$ , we associate to  $A$  two subspaces of  $\mathbf{R}^n$ , namely

$$M(A) = \text{im}(A - I) \quad \text{and} \quad F(A) = \ker(A - I),$$

which we call the moved space of  $A$  and the fixed space of  $A$  respectively. We recall that  $M(A) = F(A)^\perp$  and note that  $F(A) = F(A^{-1})$ . The main result of [7] implies that if  $V$  is a subspace of  $M(A)$  then there is a unique  $B \in O(n)$  satisfying  $M(B) = V$  and  $B \leq A$ . It was

noted in [8] that a group element  $\alpha \leq \gamma$  is characterised by its reflection set  $S_\alpha$ , where

$$S_\alpha = \{R \in \mathcal{R} \mid R \leq \alpha\}.$$

It was also noted in [8] that whenever  $\alpha, \beta$  and  $\delta$  are elements of the interval  $[I, \gamma]$  in  $W$  with the properties that  $\delta \leq \alpha, \beta$  and  $S_\delta = S_\alpha \cap S_\beta$ , then  $\delta$  must be the greatest lower bound of  $\alpha$  and  $\beta$  in  $W$ ; that is, if  $\tau \in W$  satisfies  $\tau \leq \alpha, \beta$  then  $\tau \leq \delta$ .

We record for convenience here some results which either appear in [7] and [8] or are elementary consequences.

$$(1) \quad \text{If } \alpha \leq \beta \text{ then } M(\alpha) \subset M(\beta) \text{ and } F(\beta) \subset F(\alpha).$$

If  $R(\vec{v})$  denotes the reflection in  $\vec{v}^\perp$  then

$$(2) \quad \vec{v} \in M(\alpha) \Rightarrow R(\vec{v})\alpha = \alpha R[\alpha^{-1}(\vec{v})].$$

$$(3) \quad \text{If } \alpha \leq \beta \leq \delta \text{ then } \alpha^{-1}\beta \leq \alpha^{-1}\delta \text{ and } \beta\alpha^{-1} \leq \delta\alpha^{-1}.$$

$$(4) \quad \text{If } \alpha, \beta \leq \delta \text{ and } M(\alpha) \subset M(\beta) \text{ then } \alpha \leq \beta.$$

If  $l(\alpha) = k$  and  $R_i$  are reflections then

$$(5) \quad \alpha = R_1 \dots R_k \Rightarrow M(R_j) \subset M(R_1\alpha) \text{ for } j = 2, \dots, k.$$

$$(6) \quad \alpha \leq \beta \leq \delta \Rightarrow \beta^{-1}\delta \leq \alpha^{-1}\delta \text{ and } \delta\beta^{-1} \leq \delta\alpha^{-1}.$$

If  $R_1$  and  $R_2$  are distinct reflections then

$$(7) \quad R_1R_2 \leq \alpha \Leftrightarrow R_2 \leq R_1\alpha \Leftrightarrow R_1 \leq \alpha R_2.$$

The complexes that we define in sections 4 and 5 are spherical simplicial complexes. We will use (often without mention) the following facts about spherical simplices. Any linearly independent set of vectors in  $\mathbf{R}^n$  determines a spherical simplex provided the angle between each pair of vectors is less than  $\pi$ . For such a set the spherical simplex is obtained by intersecting the unit sphere with the positive cone on those vectors. The spherical simplex determined by the linearly independent set  $\{v_1, v_2, \dots, v_k\}$  will be denoted by  $\langle v_1, v_2, \dots, v_k \rangle$ . Since any fixed set of positive roots for  $W$  lies in a single open halfspace, each linearly independent subset of such a set of positive roots determines a spherical simplex.

3. ROOTS OF  $W$  AND VERTICES ON THE PETRIE POLYGON.

In [13] Steinberg gives new proofs of several results about irreducible, finite real reflection groups without using the classification of these groups. He uses the fact that, when the group is finite, the fundamental reflections can be partitioned into two mutually orthogonal subsets. We will use his numbering of the roots of  $W$  in the construction of our simplicial complex. Fix a fundamental chamber  $C$  of the standard tessellation with inward unit normals  $\alpha_1, \dots, \alpha_n$  and let  $R_1, \dots, R_n$  be the corresponding reflections. Since  $W$  is finite we can assume that it is possible to choose the ordering on the inward normals so that  $S_1 = \{\alpha_1, \dots, \alpha_s\}$  and  $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$  are orthonormal sets (from Lemma 2.2 of [13]). Let  $\gamma$  denote the Coxeter element given by  $\gamma = R_1 R_2 \dots R_n$  and let  $h$  denote the order of  $\gamma$  in  $W$ . We note that  $M(\gamma)$  is all of  $\mathbf{R}^n$ . As in [13], we set  $\rho_i = R_1 R_2 \dots R_{i-1} \alpha_i$ , where the  $\alpha$ 's and the  $R$ 's are indexed cyclically modulo  $n$ . We will use the following explicit formulae which are easily verified.

$$\rho_i = \begin{cases} \alpha_i & \text{for } i = 1, \dots, s \\ -\gamma(\alpha_i) & \text{for } i = s + 1, \dots, n \\ \gamma(\rho_{i-n}) & \text{for } i > n. \end{cases}$$

It is proved in [13] that the positive roots relative to  $C$  are  $\rho_1, \dots, \rho_{nh/2}$  while the negative roots are  $\rho_{(nh/2)+1}, \dots, \rho_{nh}$ . Furthermore the last  $n - s$  positive roots are a permutation (possibly trivial) of  $S_2$ . For each  $i$ , let  $R(\rho_i)$  denote the reflection in  $W$  with fixed subspace  $\rho_i^\perp$  which is given by

$$R(\rho_i)(x) = x - 2(\rho_i \cdot x)\rho_i.$$

**Note 3.1.** *If  $\rho_i, \rho_{i+1}, \dots, \rho_{i+n-1}$  are  $n$  consecutive roots in this ordering then  $R(\rho_i)R(\rho_{i+1}) \dots R(\rho_{i+n-1}) = \gamma^{-1}$ .*

Let  $\{\beta_1, \dots, \beta_n\}$  be the dual basis to  $\{\alpha_1, \dots, \alpha_n\}$ . Thus

$$\beta_i \cdot \alpha_j = \delta_{ij}, \text{ for } 1 \leq i, j \leq n, \text{ where } \delta_{ij} \text{ is the Kronecker delta.}$$

With a slight change of notation from [13] we define the vectors  $\mu_i$  for  $1 \leq i \leq nh$  by  $\mu_i = R_1 R_2 \dots R_{i-1} \beta_i$ , where the  $\beta$ 's and the  $R$ 's are again indexed cyclically modulo  $n$ . As with the  $\rho$ 's the following explicit formulae are readily verified.

$$\mu_i = \begin{cases} \beta_i & \text{for } i = 1, \dots, n \\ \gamma\mu_{i-n} & \text{for } i > n. \end{cases}$$

**Theorem 3.2.**  $\gamma\mu_i = \mu_i - 2\rho_i$  for  $1 \leq i \leq nh$ .

*Proof:* Because of the recursions satisfied by both  $\rho_i$  and  $\mu_i$  it is sufficient to establish this result for  $1 \leq i \leq n$ . In this case  $\mu_i = \beta_i$  and, since  $R_j(\beta_i) = \beta_i$  if  $j \neq i$ , we have

$$\begin{aligned} \gamma\beta_i &= (R_1 \dots R_n)\beta_i \\ &= (R_1 \dots R_i)\beta_i \\ &= (R_1 \dots R_{i-1})(\beta_i - 2\alpha_i) \\ &= \beta_i - 2(R_1 \dots R_{i-1})\alpha_i \\ &= \beta_i - 2\rho_i \quad \text{as required.} \end{aligned}$$

**Corollary 3.3.** *With  $\rho_i$  and  $\mu_i$  defined as above we have*

- (a):  $(\gamma - I)\mu_i = -2\rho_i$
- (b):  $\mu_i \cdot \rho_i = 1$
- (c):  $\mu_i \in F(R(\rho_i)\gamma)$

*Proof:* Part (a) is immediate from theorem 3.2. Part (b) follows from theorem 3.2 and the fact that  $\mu_i \cdot \mu_i = \gamma\mu_i \cdot \gamma\mu_i$ . From theorem 3.2 and part (b) we see that  $\gamma(\mu_i) = R(\rho_i)(\mu_i)$ , which is equivalent to statement (c).

**Note 3.4.** *We observe that parts (b) and (c) of corollary 3.3 characterise  $\mu_i$  as the unique vector in the one-dimensional subspace  $F(R(\rho_i)\gamma)$  satisfying  $\mu_i \cdot \rho_i = 1$ .*

**Note 3.5.** *The vectors  $\mu_i$  are shown in [13] to be the vertices of a Petrie polygon. In particular the vectors  $\mu_1, \dots, \mu_n$  are the vertices of the fundamental chamber  $C$  while the other  $\mu$ 's are obtained by appropriate reflections using  $\gamma(\mu_i) = R(\rho_i)(\mu_i)$ . It follows from theorem 3.2 that the invertible linear transformation  $I - \gamma$  interchanges the vertices of a Petrie polygon with the set of all roots; in so doing, it takes the first  $nh/2$  vertices to the set of positive roots.*

In what follows we make extensive use the properties of the matrix of dot products  $[\mu_i \cdot \rho_j]$ . Before establishing the general result we will present an example.

**Example 3.6.** *Consider the symmetry group  $I_3$  of the icosahedron. A simple system for  $I_3$  is*

$$\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (1/2)(-1, -\tau, \tau - 1)$$

where  $\tau = 2 \cos(\pi/5) = (1 + \sqrt{5})/2$  is the positive root of  $x^2 - x - 1$ . Thus  $\tau$  satisfies  $\tau^2 = \tau + 1$  and the other root of  $x^2 - x - 1$  is  $-1/\tau = 1 - \tau$ . The dual basis to  $\{\alpha_1, \alpha_2, \alpha_3\}$  is

$$\beta_1 = (1, 0, \tau), \beta_2 = (0, 1, \tau + 1), \beta_3 = (0, 0, 2\tau).$$

From the formula for a reflection we compute

$$R_1(\alpha_1) = -\alpha_1, \quad R_1(\alpha_2) = \alpha_2, \quad R_1(\alpha_3) = \alpha_3 + \alpha_1.$$

$$R_2(\alpha_1) = \alpha_1, \quad R_2(\alpha_2) = -\alpha_2, \quad R_2(\alpha_3) = \alpha_3 + \tau\alpha_2.$$

$$R_3(\alpha_1) = \alpha_1 + \alpha_3, \quad R_3(\alpha_2) = \alpha_2 + \tau\alpha_3, \quad R_3(\alpha_3) = -\alpha_3.$$

It will also be useful to know that

$$\begin{aligned} R_1R_2R_3(\alpha_1) &= \tau\alpha_2 + \alpha_3 \\ R_1R_2R_3(\alpha_2) &= \tau\alpha_1 + \tau\alpha_2 + \tau\alpha_3 \\ R_1R_2R_3(\alpha_3) &= -\alpha_1 - \tau\alpha_2 - \alpha_3 \end{aligned}$$

These values can be used to generate the following table.

$i$	$\rho_i$	$\mu_i$
1	$(1, 0, 0)$	$(1, 0, \tau)$
2	$(0, 1, 0)$	$(0, 1, \tau + 1)$
3	$(1/2)(1, \tau, \tau - 1)$	$(0, 0, 2\tau)$
4	$(1/2)(-1, \tau, \tau - 1)$	$(-1, 0, \tau)$
5	$(1/2)(\tau, \tau - 1, 1)$	$(0, -1, \tau + 1)$
6	$(1/2)(\tau - 1, 1, \tau)$	$(-1, -\tau, \tau + 1)$
7	$(1/2)(\tau, -\tau + 1, 1)$	$(0, -\tau, 1)$
8	$(1/2)(-\tau + 1, 1, \tau)$	$(-\tau, -\tau, \tau)$
9	$(0, 0, 1)$	$(-\tau, -\tau - 1, 1)$
10	$(1/2)(-\tau, \tau - 1, 1)$	$(-\tau, -1, 0)$
11	$(1/2)(\tau - 1, -1, \tau)$	$(-1, -\tau - 1, 0)$
12	$(1/2)(-\tau + 1, -1, \tau)$	$(-\tau, -\tau - 1, -1)$
13	$(1/2)(1, -\tau, \tau - 1)$	$(0, -\tau, -1)$
14	$(1/2)(-\tau, -\tau + 1, 1)$	$(-\tau, -\tau, -\tau)$
15	$(1/2)(-1, -\tau, \tau - 1)$	$(-1, -\tau, -\tau - 1)$

The entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of the following table is  $\mu_i \cdot \rho_j$ .

1	0	1	0	$\tau$	$\tau$	$\tau$	1	$\tau$	0	$\tau$	1	1	0	0
0	1	$\tau$	$\tau$	$\tau$	$\tau^2$	1	$\tau^2$	$\tau^2$	$\tau$	$\tau$	$\tau$	0	1	0
0	0	1	1	$\tau$	$\tau^2$	$\tau$	$\tau^2$	$2\tau$	$\tau$	$\tau^2$	$\tau^2$	1	$\tau$	1
-1	0	0	1	0	1	0	$\tau$	$\tau$	$\tau$	1	$\tau$	0	$\tau$	1
0	-1	0	0	1	$\tau$	$\tau$	$\tau$	$\tau^2$	1	$\tau^2$	$\tau^2$	$\tau$	$\tau$	$\tau$
-1	$-\tau$	-1	0	0	1	1	$\tau$	$\tau^2$	$\tau$	$\tau^2$	$2\tau$	$\tau$	$\tau^2$	$\tau^2$
0	$-\tau$	-1	-1	0	0	1	0	1	0	$\tau$	$\tau$	$\tau$	1	$\tau$
$-\tau$	$-\tau$	$-\tau$	0	-1	0	0	1	$\tau$	$\tau$	$\tau$	$\tau^2$	1	$\tau^2$	$\tau^2$
$-\tau$	$-\tau^2$	$-\tau^2$	-1	$-\tau$	-1	0	0	1	1	$\tau$	$\tau^2$	$\tau$	$\tau^2$	$2\tau$
$-\tau$	-1	$-\tau$	0	$-\tau$	-1	-1	0	0	1	0	1	0	$\tau$	$\tau$
-1	$-\tau^2$	$-\tau^2$	$-\tau$	$-\tau$	$-\tau$	0	-1	0	0	1	$\tau$	$\tau$	$\tau$	$\tau^2$
$-\tau$	$-\tau^2$	$-2\tau$	$-\tau$	$-\tau^2$	$-\tau^2$	-1	$-\tau$	-1	0	0	1	1	$\tau$	$\tau^2$
0	$-\tau$	$-\tau$	$-\tau$	-1	$-\tau$	0	$-\tau$	-1	-1	0	0	1	0	1
$-\tau$	$-\tau$	$-\tau^2$	-1	$-\tau^2$	$-\tau^2$	$-\tau$	$-\tau$	$-\tau$	0	-1	0	0	1	$\tau$
-1	$-\tau$	$-\tau^2$	$-\tau$	$-\tau^2$	$-2\tau$	$-\tau$	$-\tau^2$	$-\tau^2$	-1	$-\tau$	-1	0	0	1

The second table in example 3.6 exhibits certain symmetry properties which are valid in the general case and which we now address.

**Theorem 3.7.** *The quantities  $\mu_i \cdot \rho_j$  have the following properties.*

- (a):  $\mu_i \cdot \rho_j = -\mu_{j+n} \cdot \rho_i$  for all  $i$  and  $j$ .
- (b):  $\mu_i \cdot \rho_j \geq 0$ , for  $1 \leq i \leq j \leq nh/2$ .
- (c):  $\mu_{i+t} \cdot \rho_i = 0$ , for  $1 \leq t \leq n-1$  and for all  $i$ .
- (d):  $\mu_j \cdot \rho_i \leq 0$  for  $1 \leq i < j \leq nh/2$ .

*Proof.* (a). Since  $\gamma\mu_i = \mu_i - 2\rho_i$  by Theorem 3.2, we compute

$$\begin{aligned}
-\mu_{j+n} \cdot \rho_i &= -\gamma\mu_j \cdot (-1/2)(\gamma\mu_i - \mu_i) \\
&= 1/2(\gamma\mu_j \cdot \gamma\mu_i - \gamma\mu_j \cdot \mu_i) \\
&= 1/2(\mu_j \cdot \mu_i - \gamma\mu_j \cdot \mu_i) \\
&= 1/2(\mu_j - \gamma\mu_j) \cdot \mu_i \\
&= \rho_j \cdot \mu_i.
\end{aligned}$$

(b) Since  $\rho_p \cdot \mu_q = \gamma\rho_p \cdot \gamma\mu_q$  we can assume that  $1 \leq i \leq n$  and  $i < j \leq nh/2$ . The result follows from the fact that  $\rho_j$  is a positive root and  $\mu_i$  is part of the dual basis to  $\{\alpha_1, \dots, \alpha_n\}$ .

(c) Since  $\gamma = R(\rho_{i+t})R(\rho_{i+t-1}) \dots R(\rho_i) \dots R(\rho_{i+t-n+1})$  (by note 3.1) equation (5) implies that  $\rho_i \cdot \mu_{i+t} = 0$ .

(d) If  $j < i+n$ , this follows from part (c). If  $j \geq i+n$ , this property follows from parts (a) and (b).

**Corollary 3.8.** *If  $1 \leq i_1 < i_2 < \dots < i_m < k \leq nh/2$ , then  $\rho_k$  does not lie in the positive cone on  $\{\rho_{i_1}, \dots, \rho_{i_m}\}$ .*



*Proof:* If the vector  $\vec{x}$  is expressible as a non-negative linear combination of  $\rho_{i_1}, \dots, \rho_{i_m}$  then  $\mu_k \cdot \vec{x} \leq 0$ , since  $\mu_k \cdot \rho_{i_j} \leq 0$ , for  $j = 1, \dots, m$ . As  $\mu_k \cdot \rho_k = 1$ , the vector  $\rho_k$  is not expressible in this manner.

For the construction of the simplicial complex of section 4 we will also use the following result.

**Lemma 3.9.** *Suppose  $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$  is a consistently ordered subset of  $\{\rho_1, \dots, \rho_{nh/2}\}$ , and for each  $i$  let  $\mu(\sigma_i) = -2(\gamma - I)^{-1}\sigma_i$ . Then the following are equivalent.*

- (a):  $l(R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma) = n - k$ .
- (b):  $\mu(\sigma_i) \cdot \sigma_j = 0$  whenever  $i > j$ .

*Proof:* (a)  $\Rightarrow$  (b): Let  $\delta = R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma$  and assume that (a) is true. If  $i > j$  then equation (2) gives

$$\begin{aligned} \gamma &= R(\sigma_k) \dots R(\sigma_i) \dots R(\sigma_j) \dots R(\sigma_1)\delta \\ &= R(\sigma_i) \dots R(\sigma_j) \dots R(\sigma_1)\delta R(\gamma^{-1}\sigma_k) \dots R(\gamma^{-1}\sigma_{i+1}). \end{aligned}$$

Since  $l(\delta) = n - k$ , it follows from equation (5) that  $R(\sigma_j) \leq R(\sigma_i)\gamma$ . Thus  $\sigma_j \in M(R(\sigma_i)\gamma)$  and hence  $\mu(\sigma_i) \cdot \sigma_j = 0$  (since  $\mu(\sigma_i) \in F(R(\sigma_i)\gamma)$ ).

(b)  $\Rightarrow$  (a): Assume that (b) is true. Then the  $k \times k$  matrix  $[\mu(\sigma_i) \cdot \sigma_j]$  is upper triangular with ones on the diagonal (corollary 3.3 and part (c) of theorem 3.7). Thus this matrix is nonsingular and it follows that each of  $\{\sigma_1, \dots, \sigma_k\}$  and  $\{\mu(\sigma_1), \dots, \mu(\sigma_k)\}$  is a linearly independent set.

Next we show that  $R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_i) \leq \gamma$  by reverse induction on  $i$ . The case  $i = k$  is immediate since  $M(\gamma) = \mathbf{R}^n$ . Now suppose that  $i < k$  and that  $R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_{i+1}) \leq \gamma$ . If  $i + 1 \leq j \leq k$  then using equation (6) and equation (1) we get

$$\begin{aligned} R(\sigma_j) &\leq R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_{i+1}) \leq \gamma \\ \Rightarrow R(\sigma_j)\gamma &\geq R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma \\ \Rightarrow F[R(\sigma_j)\gamma] &\subseteq F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma] \\ \Rightarrow \mu(\sigma_j) &\in F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma] \end{aligned}$$

Since the  $\mu(\sigma_j)$  are linearly independent and  $F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma]$  has dimension  $k - i$  by the inductive hypothesis, the set  $\{\mu(\sigma_{i+1}), \dots, \mu(\sigma_k)\}$  is a basis for the subspace  $F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma]$ . Since  $\sigma_i$  is orthogonal to each vector in this basis it follows that

$$\sigma_i \in M[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma]$$

and hence  $R(\sigma_k) \dots R(\sigma_{i+1})R(\sigma_i) \leq \gamma$ , which completes the inductive step.

Finally, since  $l(R(\sigma_k) \dots R(\sigma_2)R(\sigma_1)) = k$  (by linear independence of  $\sigma_1, \dots, \sigma_k$ ) and  $R(\sigma_k) \dots R(\sigma_2)R(\sigma_1) \leq \gamma$ , we can conclude that the length of  $R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma$  is  $n - k$ .

#### 4. DEFINITION OF THE COMPLEX $X(\gamma)$ .

Before describing the simplicial complex  $X$  which will be used to prove the lattice property, we give some motivation for its definition. By [8], the map  $M$ , which associates to an orthogonal transformation  $\alpha$  its moved space  $M(\alpha)$ , restricts to give a poset isomorphism of the interval  $[I, \gamma]$  in  $W$  onto its image. Here the set of subspaces of  $\mathbf{R}^n$  is partially ordered by inclusion and is, in fact, a lattice whose meet operation is given by subspace intersection. However, the set of subspaces of the form  $M(\sigma)$  for  $\sigma \leq \gamma$  is not closed under intersection. There are many examples of group elements  $\alpha, \beta \leq \gamma$  for which there is no element  $\delta \leq \gamma$  in  $W$  satisfying  $M(\delta) = M(\alpha) \cap M(\beta)$ . (For example, if  $W = A_3 = \Sigma_4$  then the reflections are transpositions and every 4-cycle is a Coxeter element. Consider  $\gamma = (1, 2, 3, 4)$  and the elements  $\alpha$  and  $\beta$  given by

$$\alpha = (1, 3)\gamma = (1, 2)(3, 4) \text{ and } \beta = (2, 4)\gamma = (1, 4)(2, 3).$$

Each of  $\alpha$  and  $\beta$  has a two-dimensional moved space, these moved spaces intersect inside the three-dimensional space  $M(\gamma)$  in a one-dimensional subspace, but no transposition precedes both  $\alpha$  and  $\beta$ .) To get around this problem we fix a set of simple roots and replace  $M(\alpha)$  by the positive cone on the positive roots associated to those reflections which precede  $\alpha$ . Two surprising things happen: (i) the intersection of the positive cones associated to two elements  $\alpha$  and  $\beta$  of  $[I, \gamma]$  is equal to the positive cone associated to some other element  $\delta$  of  $[I, \gamma]$  (see proof of theorem 7.8) and (ii) this collection of positive cones intersects the unit sphere in a spherical simplicial complex which we denote  $X(\gamma)$ . We begin this section by defining  $X(\gamma)$ . Much of our later work is devoted to showing that  $X(\gamma)$  is indeed a simplicial complex.

**Definition 4.1.** *We define a set,  $X = X(\gamma)$ , of simplices by declaring that*

- *the vertex set is  $\{\rho_1, \rho_2, \dots, \rho_{nh/2}\}$ ,*
- *that an edge joins  $\rho_i$  to  $\rho_j$  whenever  $i < j$  and  $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$  and*
- *$\langle \rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k} \rangle$  forms a  $(k - 1)$ -simplex if the vertices are distinct and pairwise joined by edges.*

**Note 4.2.** *It follows from lemma 3.9 that a set of  $k \leq n$  distinct vertices,  $\{\rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k}\}$ , with  $1 \leq i_1 < i_2 < \dots < i_k \leq nh/2$ , determines a  $k - 1$  simplex of  $X$ , provided*

$$l[R(\rho_{i_1})R(\rho_{i_2}) \dots R(\rho_{i_k})\gamma] = n - k.$$

*The largest possible dimension for a simplex of  $X$  is  $n - 1$  and at least  $nh/2 - n + 1$  of these occur by part (c) of theorem 3.7 and lemma 3.9.*

In the case of  $I_3$  the complex  $X$  is illustrated in Figures 1 and 2. The first figure shows the vertices  $\rho_i$  and the moved spaces of the length two elements in  $[I, \gamma]$ , while the second figure shows the simplicial complex  $X$ .

## 5. SUBCOMPLEXES

In this and the following sections we will consider a fixed element  $\sigma \leq \gamma$ . We let  $P = \{\tau_1, \tau_2, \dots, \tau_t\}$  be the set of positive roots whose reflections precede  $\sigma$ . We assume that  $P$  is ordered consistently with the total order from section 3 and we let the corresponding ordered set of vertices of the Petrie polygon be  $\{\mu(\tau_1), \mu(\tau_2), \dots, \mu(\tau_t)\}$ . In this section we will characterise the simple system associated to  $\sigma$  and we will analyse the case where  $l(\sigma) = 2$ . We let  $X(\sigma)$  be the collection of simplices of  $X(\gamma)$  whose vertices lie in  $P$ .

Let  $k = l(\sigma)$ . Then  $P$  is the set of positive roots for the, possibly reducible, rank  $k$  reflection group  $\{\alpha \in W \mid M(\alpha) \subseteq M(\sigma)\}$ . Thus  $P$  contains a simple system  $\Delta = \{\delta_1, \delta_2, \dots, \delta_k\}$ , which we assume is ordered consistently with the total order from section 3.

**Theorem 5.1.** *The ordered elements  $\delta_1, \delta_2, \dots, \delta_k$  of  $\Delta$  are determined recursively by the fact that  $\delta_i$  is the last positive root in the subspace  $M(\sigma R(\delta_k)R(\delta_{k-1}) \dots R(\delta_{i+1}))$ . In particular,  $\delta_k = \tau_t$ .*

*Proof:* Since  $\tau_t$  is the last root in  $P$  we can apply Corollary 3.8 to  $\tau_t$  and the set  $P - \{\tau_t\}$  to deduce that  $\tau_t$  cannot be a positive linear combination of the elements of  $P - \{\tau_t\}$ . Thus  $P - \{\tau_t\}$  cannot contain a simple system and  $\delta_k = \tau_t$ .

Next we show that  $U_k = M(\sigma R(\delta_k))$  contains  $\{\delta_1, \dots, \delta_{k-1}\}$ . To begin, we show that  $U_k = M(\sigma) \cap \mu^\perp$ , where  $\mu$  is the Petrie polygon vertex  $\gamma(\mu(\delta_k))$ . Note that  $U_k = M(\sigma R(\delta_k))$  has dimension  $k - 1$ . By parts (a) and (b) of Theorem 3.7, if  $1 \leq i \leq t$  then

$$\mu \cdot \tau_i = \gamma(\mu(\tau_t)) \cdot \tau_i = -\tau_t \cdot \mu(\tau_i) \leq 0.$$

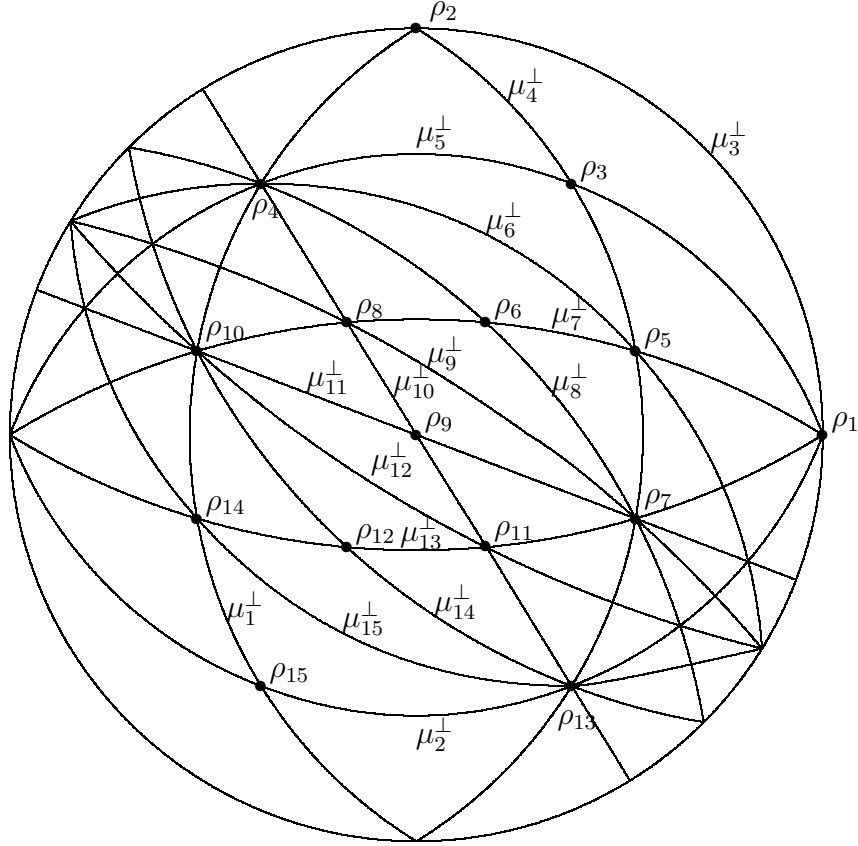


FIGURE 1.

For  $i = t$ , we obtain  $\mu \cdot \tau_t = -1$ . Thus  $\tau_t$  is not an element of  $M(\sigma) \cap \mu^\perp$ , and this subspace must also have dimension  $k - 1$ . Since  $M(\sigma R(\delta_k))$  is contained in  $M(\sigma)$  and

$$M(\sigma R(\delta_k)) \subseteq M(\gamma R(\delta_k)) = M(R[\gamma(\delta_k)]\gamma) = \mu(\gamma(\delta_k))^\perp = \gamma(\mu(\delta_k))^\perp$$

(by equation (3) and equation (2)) it follows that  $U_k \subseteq M(\sigma) \cap \mu^\perp$ , and hence these subspaces are equal.

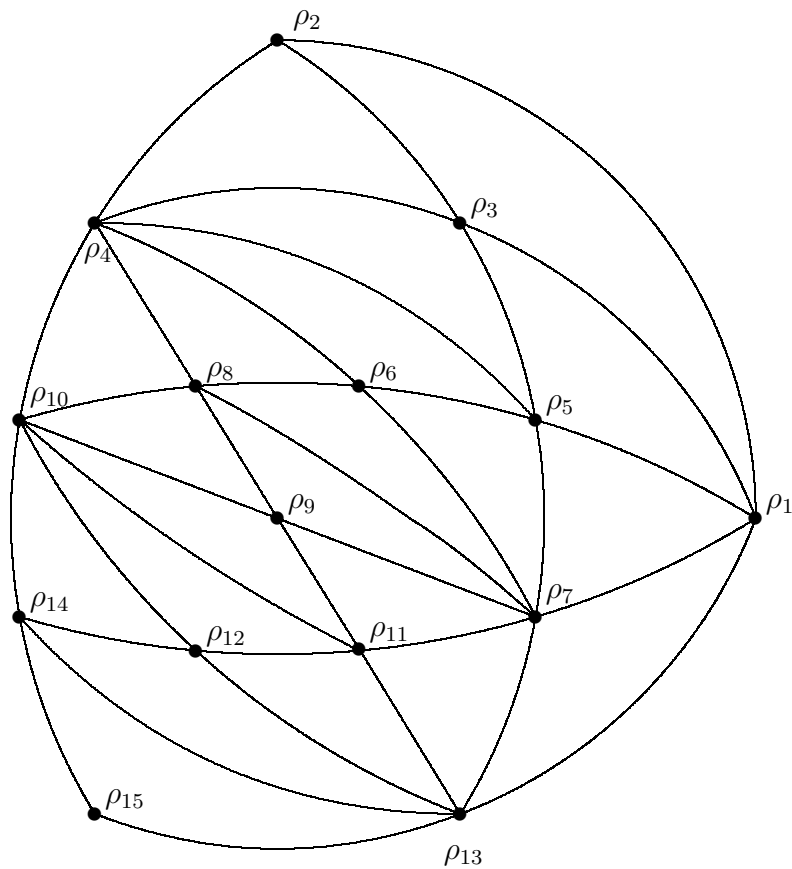


FIGURE 2.

Now suppose  $\tau$  is an arbitrary element of  $P \cap U_k$  and write

$$\tau = a_1\delta_1 + \cdots + a_k\delta_k, \text{ with each } a_i \geq 0.$$

Since  $\tau \in \mu^\perp$  and  $\delta_i \cdot \mu \leq 0$  for  $i = 1, \dots, k$ , we must have  $a_i = 0$  or  $\delta_i \cdot \mu = 0$  for each  $i$ . Thus  $\tau$  is a linear combination of simple roots in  $M(\sigma) \cap \mu^\perp = U_k$ . However, the span of  $P \cap U_k$  is all of  $U_k$  since

$l(\sigma R(\tau_t)) = k - 1$ . Hence the  $k - 1$  simple roots  $\delta_1, \dots, \delta_{k-1}$  must lie in  $U_k$ .

The theorem follows since we can now apply the same arguments as above to the shorter element  $\sigma' = \sigma R(\delta_k)$ .

**Note 5.2.** *In an analogous way we can show that  $\delta_i$  is the first root in  $M(R(\delta_1)R(\delta_2)\dots R(\delta_{i-1})\sigma)$  and, in particular,  $\delta_1 = \tau_1$ .*

**Definition 5.3.** *We will refer to the intersection with  $S^{n-1}$  of the positive cone on  $\{\delta_1, \delta_2, \dots, \delta_k\}$  as the fat simplex associated to  $\sigma$ .*

**Theorem 5.4.** *If  $\sigma \leq \gamma$  has length 2 then  $X(\sigma)$  consists of the  $t$  roots from  $P$  and the  $(t - 1)$  1-cells given by  $\langle \tau_i, \tau_{i+1} \rangle$  for  $i = 1, 2, \dots, t - 1$ .*

*Proof:* Suppose  $\sigma \leq \gamma$  has length two. Since both  $R(\tau_i)\sigma$  and  $\sigma R(\tau_i)$  have length one for each  $i$ , each subspace  $M(\sigma) \cap \mu(\tau_i)^\perp$  contains precisely one positive root  $\tau_j$  and each  $\tau_j$  lies in precisely one subspace of the form  $M(\sigma) \cap \mu(\tau_i)^\perp$ . It follows that the  $t \times t$  matrix  $A = [\mu(\tau_i) \cdot \tau_j]$  has precisely one zero in each row and column. Note that the diagonal entries are all 1, the entries below the diagonal are non-positive, while the entries above the diagonal are non-negative (by theorem 3.7). By theorem 5.1 and note 5.2,  $\sigma = R(\tau_1)R(\tau_t)$  and hence  $\tau_t \cdot \mu(\tau_1) = 0$ . Let  $i_1 > 1$  be the value for which  $\sigma = R(\tau_{i_1})R(\tau_1)$ . Thus  $\mu(\tau_{i_1}) \cdot \tau_1 = 0$  and  $X(\sigma)$  contains the edge  $\langle \tau_1, \tau_{i_1} \rangle$ . As row  $i_1$  contains only one zero and  $i_1 \leq t$ , we must have  $\mu(\tau_{i_1}) \cdot \tau_t > 0$  (by theorem 3.7). Now, since  $\{\tau_1, \tau_t\}$  is a simple system, we can write each  $\tau_j$  as a non-negative linear combination of  $\tau_1$  and  $\tau_t$ . Thus we obtain  $\tau_j \cdot \mu(\tau_{i_1}) \geq 0$ . It follows that each entry in row  $i_1$  of  $A$  is non-negative. Since every row after the second has at least two entries below the diagonal and no more than one of these can be zero we get  $i_1 \leq 2$ . Since we know that  $i_1 \neq 1$  we deduce that  $i_1$  must be equal to 2. If  $t = 2$  the proof is complete.

The rest of the proof uses induction. We assume that  $r < t$  and we have

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = \dots = R(\tau_r)R(\tau_{r-1}).$$

We know that  $\sigma = R(\tau_{i_r})R(\tau_r)$  for some  $i_r > r$ . Thus

$$\mu(\tau_{i_r}) \cdot \tau_r = 0 \text{ while } \mu(\tau_{i_r}) \cdot \tau_t > 0.$$

If  $j \geq r$  then corollary 3.8 and the fact that  $M(\sigma)$  is two dimensional implies that  $\tau_j$  can be expressed as a non-negative linear combination of  $\tau_r$  and  $\tau_t$ . As in the case  $r = 1$ , we obtain  $\tau_j \cdot \mu(\tau_{i_r}) \geq 0$  for  $r \leq j \leq t$ . It follows that each of the last  $t - r + 1$  entries in row  $i_r$  of  $A$  is non-negative. Since row  $j$  has  $j - 1$  entries below the diagonal and no more

than one of these can be zero we get  $i_r \leq r + 1$ . Since we know that  $i_r > r$  we deduce that  $i_r$  must be equal to  $r + 1$ .

In the case  $l(\sigma) = 2$ , the action of the simple reflections on  $P$  can now be deduced.

**Lemma 5.5.** *If  $l(\sigma) = 2$  and  $P = \{\tau_1, \dots, \tau_t\}$  is the ordered set of positive roots for reflections preceding  $\sigma$  then*

$$R(\tau_1)(\tau_i) = \begin{cases} \tau_{t-i+2} & \text{for } 2 \leq i \leq t \\ -\tau_1 & \text{for } i = 1, \end{cases}$$

and  $R(\tau_t)(\tau_i) = \begin{cases} \tau_{t-i} & \text{for } 1 \leq i \leq t-1 \\ -\tau_t & \text{for } i = t. \end{cases}$

*Proof:* We will consider the case of  $\tau_1$ . The case of  $\tau_t$  is similar. From the proof of theorem 5.4 it follows that the only expressions for  $\sigma$  as a product of two reflections are

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = R(\tau_3)R(\tau_2) = \dots = R(\tau_t)R(\tau_{t-1}).$$

Since  $R(\tau_1)$  permutes  $P - \{\tau_1\}$  (lemma A in I.4.3 of [12]) and

$$\sigma = R(\tau_1)R(\tau_t) = (R(\tau_1)R(\tau_t)R(\tau_1))R(\tau_1) = R(R(\tau_1)\tau_t)R(\tau_1)$$

we deduce that  $R(\tau_1)\tau_t = \tau_2$ . In general, suppose that  $1 < i + 1 \leq t$  and that  $R(\tau_1)\tau_{i+1} = \tau_{t-(i+1)+2}$ . Conjugate  $R(\tau_t)R(\tau_1) = \sigma^{-1} = R(\tau_i)R(\tau_{i+1})$  by  $R(\tau_1)$  to obtain

$$R(\tau_1)R(\tau_t) = \sigma = R(R(\tau_1)\tau_i)R(R(\tau_1)\tau_{i+1}) = R(R(\tau_1)\tau_i)R(\tau_{t-i+1}).$$

Comparing with the list of factorisations for  $\sigma$ , we deduce that  $R(\tau_1)\tau_i = \tau_{(t-i+1)+1} = \tau_{t-i+2}$ . The result follows by reverse induction on  $i$  starting at  $i = t$ .

**Lemma 5.6.** *Let  $\rho_i$  and  $\rho_j$  be distinct positive roots for which  $R(\rho_i)R(\rho_j) \leq \gamma$ .*

(a) *If  $i < j$  then  $\rho_i \cdot \rho_j \leq 0$ .*

(b) *If  $i > j$  then  $\rho_i \cdot \rho_j \geq 0$ .*

*Proof:* Let  $P = \{\tau_1, \dots, \tau_t\}$  be the ordered set of positive roots whose reflections precede  $\sigma = R(\rho_i)R(\rho_j)$ . By theorem 5.4,  $\{\tau_1, \tau_t\}$  is the corresponding simple system and the only expressions for  $\sigma$  as a product of two reflections are

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = R(\tau_3)R(\tau_2) = \dots = R(\tau_t)R(\tau_{t-1}).$$

Since  $\sigma = R(\tau_1)R(\tau_t)$  is the only case in which the roots appear in increasing order, if  $i < j$  then we must have  $\rho_i = \tau_1$  and  $\rho_j = \tau_t$ . Part (a) now follows from  $\rho_i \cdot \rho_j = \tau_1 \cdot \tau_t \leq 0$  (since  $\{\tau_1, \tau_t\}$  is a simple system).

If  $i > j$  then  $\rho_i = \tau_{r+1}$  and  $\rho_j = \tau_r$  for some  $r \geq 1$ . Thus we need to verify that  $\tau_i \cdot \tau_{i+1} \geq 0$  for  $i = 1, \dots, t-1$ . Using lemma 5.5 and the facts that reflection is an isometry and  $\{\tau_1, \tau_t\}$  is a simple system, we obtain

$$\tau_1 \cdot \tau_2 = R(\tau_1)\tau_1 \cdot R(\tau_1)\tau_2 = -\tau_1 \cdot \tau_t \geq 0.$$

For the same reasons, if  $i \geq 2$  then

$$\tau_i \cdot \tau_{i+1} = R(\tau_t)R(\tau_1)\tau_i \cdot R(\tau_t)R(\tau_1)\tau_{i+1} = R(\tau_t)\tau_{t-i+2} \cdot R(\tau_t)\tau_{t-i+1} = \tau_{i-2} \cdot \tau_{i-1}$$

and the result follows by induction on  $i$ .

**Note 5.7.** *It is immediate from lemma 5.6 that the edges of  $X$  can now be characterised by the following geometric criterion. There is an edge joining the distinct vertices  $\rho_i$  and  $\rho_j$  if and only if the vectors subtend a non-obtuse angle and one of  $R(\rho_i)R(\rho_j)$  or  $R(\rho_j)R(\rho_i)$  precedes  $\gamma$ .*

## 6. WALLS OF FAT SIMPLICES

In this section we continue to investigate a fixed element  $\sigma$  of length  $k$  which precedes  $\gamma$  in  $W$ . Using the same notation as in the preceding section, we find a dual basis to  $\{\delta_1, \dots, \delta_k\}$  (corollary 6.8). We also determine the first top-dimensional simplex of  $X(\sigma)$  in the lexicographic order.

**Definition 6.1.** *For each  $i = 1, \dots, k$  we define  $\epsilon_i$  to be the root given by*

$$\epsilon_i = R(\delta_1) \dots R(\delta_{i-1})\delta_i.$$

**Proposition 6.2.** *For each  $i = 1, \dots, k$ , the vector  $\epsilon_i$  is a positive root. Moreover*

$$\sigma = R(\epsilon_k)R(\epsilon_{k-1}) \dots R(\epsilon_1)$$

and for each  $i = 1, \dots, k$

$$\sigma = R(\epsilon_i)R(\delta_1) \dots R(\delta_{i-1})R(\delta_{i+1}) \dots R(\delta_k).$$

*Proof:* The first statement is a special case of lemma D in section I.4.3 of [12]. The remaining statements follow from

$$\begin{aligned} R(\epsilon_i) &= R(R(\delta_1) \dots R(\delta_{i-1})\delta_i) \\ &= [R(\delta_1)R(\delta_2) \dots R(\delta_{i-1})]R(\delta_i)[R(\delta_{i-1}) \dots R(\delta_2)R(\delta_1)] \end{aligned}$$

since each reflection has order 2.

**Corollary 6.3.** *The walls of the spherical simplex on the vertices  $\{\delta_1, \dots, \delta_k\}$  in the subspace  $M(\sigma)$  are given by intersecting the planes  $\mu^\perp(\epsilon_i)$  with the unit sphere in  $M(\sigma)$ .*



*Proof:* The second identity in proposition 6.2 implies that the set  $\{\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_k\}$  is contained in the subspace  $\mu^\perp(\epsilon_i)$ , for  $i = 1, \dots, k$ .

**Note 6.4.** *We will see in the next section that this simplex is the geometric realisation of  $X(\sigma)$ .*

**Definition 6.5.** *For each  $i = 1, \dots, t$ , let  $\mu'(\tau_i)$  be the orthogonal projection of  $\mu(\tau_i)$  into  $M(\sigma)$ .*

**Proposition 6.6.** *For each  $i, j \in \{1, \dots, t\}$ , we have  $\mu'(\tau_i) \cdot \tau_j = \mu(\tau_i) \cdot \tau_j$  and  $\mu'(\tau_i) \in F(R(\tau_i)\sigma)$ . In particular,*

$$\sigma(\mu'(\tau_i)) = R(\tau_i)\mu'(\tau_i) = \mu'(\tau_i) - 2\tau_i.$$

*Proof:* Write  $\mu(\tau_i) = \mu'(\tau_i) + \vec{y}$ , where  $\mu'(\tau_i) \in M(\sigma)$  and  $\vec{y} \in [M(\sigma)]^\perp = F(\sigma)$ . Then  $\mu(\tau_i) \cdot \tau_j = \mu'(\tau_i) \cdot \tau_j$  for any  $1 \leq j \leq t$ . Now  $\mu(\tau_i) \in F(R(\tau_i)\gamma) \subset F(R(\tau_i)\sigma)$ , because  $R(\tau_i)\sigma \leq R(\tau_i)\gamma$  by equation (3). But  $\vec{y}$  is also an element of  $F(R(\tau_i)\sigma)$ , since  $F(R(\tau_i)\sigma)$  contains  $F(\sigma)$  by equation (1). It follows that  $\mu'(\tau_i)$  is an element of  $F(R(\tau_i)\sigma)$ . The final claim of the proposition follows from this.

**Proposition 6.7.** *For  $1 \leq i \leq k$ , we have  $\mu(\epsilon_i) \cdot \delta_i = 1$ .*

*Proof:* Fix  $i$  and let  $\sigma' = R(\delta_1) \dots R(\delta_i)$  (which precedes  $\sigma$  and hence  $\gamma$ ). Write  $\mu(\epsilon_i) = \mu''(\epsilon_i) + \vec{z}$  where  $\mu''(\epsilon_i) \in M(\sigma')$  and  $\vec{z} \in M(\sigma')^\perp = F(\sigma')$ . Applying proposition 6.6 to  $\sigma'$  yields,

$$\begin{aligned} \mu(\epsilon_i) \cdot \delta_i &= \sigma'[\mu(\epsilon_i)] \cdot \sigma'[\delta_i] \\ &= \sigma'[\mu''(\epsilon_i) + \vec{z}] \cdot \sigma'[\delta_i] \\ &= [\mu''(\epsilon_i) - 2\epsilon_i + \vec{z}] \cdot R(\delta_1) \dots R(\delta_{i-1})R(\delta_i)[\delta_i] \\ &= [\mu(\epsilon_i) - 2\epsilon_i] \cdot R(\delta_1) \dots R(\delta_{i-1})[-\delta_i] \\ &= [\mu(\epsilon_i) - 2\epsilon_i] \cdot [-\epsilon_i] \\ &= 1 \quad \text{as required.} \end{aligned}$$

The following corollary is immediate from corollary 6.3 and proposition 6.7

**Corollary 6.8.** *The dual basis to  $\{\delta_1, \dots, \delta_k\}$  is  $\{\mu'(\epsilon_1), \dots, \mu'(\epsilon_k)\}$ .*

The order  $\epsilon_1, \dots, \epsilon_k$  need not be consistent with the global ordering. However the induced order still determines a factorisation of  $\sigma$  (proposition 6.11 below). First we make a definition.

**Definition 6.9.** *Let  $\theta_1, \dots, \theta_k$  be the reordering of  $\epsilon_1, \dots, \epsilon_k$  which is consistent with the global order.*

**Lemma 6.10.** *If  $i < j$  but  $\epsilon_i > \epsilon_j$  then  $R(\epsilon_j)R(\epsilon_i) = R(\epsilon_i)R(\epsilon_j)$ .*

*Proof:* Assume that  $i < j$  and that  $\epsilon_i > \epsilon_j$ . We need to show that  $\epsilon_i \cdot \epsilon_j = 0$ . Since  $\{\delta_1, \dots, \delta_k\}$  is a simple system, each of the dot products  $\delta_i \cdot \delta_j$  is non-positive for  $i \neq j$ . It follows that  $R(\delta_{j-1})\delta_j$  is a non-negative linear combination of  $\delta_{j-1}$  and  $\delta_j$  and, by induction, that  $R(\delta_{i+1}) \dots R(\delta_{j-1})\delta_j$  is a non-negative linear combination of  $\delta_{i+1}, \delta_{i+2}, \dots, \delta_j$ . Hence,  $\delta_i \cdot R(\delta_{i+1}) \dots R(\delta_{j-1})\delta_j \leq 0$ . Now, since  $i < j$ , we can compute that

$$\begin{aligned} \epsilon_i \cdot \epsilon_j &= R(\delta_1) \dots R(\delta_{i-1})\delta_i \cdot R(\delta_1) \dots R(\delta_{j-1})\delta_j \\ &= \delta_i \cdot R(\delta_i) \dots R(\delta_{j-1})\delta_j \\ &= R(\delta_i)\delta_i \cdot R(\delta_{i+1}) \dots R(\delta_{j-1})\delta_j \\ &= -\delta_i \cdot R(\delta_{i+1}) \dots R(\delta_{j-1})\delta_j \\ &\geq 0. \end{aligned}$$

However, since  $i < j$ , the identity  $\sigma = R(\epsilon_k) \dots R(\epsilon_j) \dots R(\epsilon_i) \dots R(\epsilon_1)$  implies that  $R(\epsilon_j)R(\epsilon_i)$  precedes  $\sigma$  and hence  $\gamma$ . Now part (a) of lemma 5.6 implies that  $\epsilon_i \cdot \epsilon_j \leq 0$  (because  $\epsilon_i > \epsilon_j$ ). We conclude that  $\epsilon_i \cdot \epsilon_j = 0$ , as required.

**Proposition 6.11.** *The elements  $\theta_1, \dots, \theta_k$  satisfy the identity*

$$\sigma = R(\theta_k) \dots R(\theta_1).$$

*Proof:* We know that  $\sigma = R(\epsilon_k) \dots R(\epsilon_1)$ . If  $\epsilon_i = \theta_i$  for each  $i$ , there is nothing to prove. Otherwise, repeated application of lemma 6.10 yields the required result.

**Corollary 6.12.** *The simplex  $\langle \theta_1, \dots, \theta_k \rangle$  is the first top dimensional simplex of  $X(\sigma)$  in the lexicographic order.*

*Proof:* Since  $\theta_1 < \theta_2 < \dots < \theta_k$  and  $\sigma = R(\theta_k) \dots R(\theta_1)$ , the  $(k-1)$ -simplex  $\langle \theta_1, \dots, \theta_k \rangle$  is in  $X(\sigma)$ . As  $\{\delta_1, \dots, \delta_k\}$  is the simple system corresponding to  $P$ ,  $\{\theta_1, \dots, \theta_k\}$  is a rearrangement of  $\{\epsilon_1, \dots, \epsilon_k\}$  and

$$\mu(\epsilon_a) \cdot \delta_b = \begin{cases} 0 & \text{for } a \neq b \\ 1 & \text{for } a = b, \end{cases}$$

it follows that  $\mu(\theta_i) \cdot \tau_j \geq 0$  for  $1 \leq i \leq k$  and  $1 \leq j \leq t$ . However, theorem 3.7 implies that whenever  $\tau_j < \theta_i$ , the dot product  $\mu(\theta_i) \cdot \tau_j$  is non-positive, and hence it must be zero. Therefore

$$\{\tau_j \mid \tau_j < \theta_i\} \subseteq M(\sigma) \cap \mu(\theta_i)^\perp \cap \mu(\theta_{i+1})^\perp \cap \dots \cap \mu(\theta_k)^\perp$$

which is an  $(i-1)$ -dimensional subspace of  $\mathbf{R}^n$  since the  $\theta$ 's are linearly independent. Now if  $\langle \tau_{i_1}, \dots, \tau_{i_k} \rangle$  is a  $(k-1)$ -simplex of  $X(\sigma)$  with

$\tau_{i_1} < \tau_{i_2} < \cdots < \tau_{i_k}$ , then  $\langle \tau_{i_1}, \dots, \tau_{i_j} \rangle$  is a  $(j - 1)$ -simplex for each  $j \leq k$  and this forces  $\tau_{i_j} \geq \theta_j$ .

We finish this section with two examples. The first shows that the first  $k$  roots of  $P$  may not span a top dimensional simplex. The second example illustrates that even in the case  $\sigma = \gamma$  the  $\epsilon$ 's may not be ordered consistently with the global order.

**Example 6.13.** *Consider the symmetry group of the 4-dimensional cube. One simple system of unit vectors for this group is*

$$\begin{aligned}\alpha_1 &= (1, 0, 0, 0), \alpha_2 = (\sqrt{2}/2)(0, 1, 0, -1), \\ \alpha_3 &= (\sqrt{2}/2)(-1, 0, 0, 1), \alpha_4 = (\sqrt{2}/2)(0, -1, 1, 0).\end{aligned}$$

The element  $\gamma = [1, 2, 3, 4]$  (in the notation of [8]) is one of the Coxeter elements determined by this simple system, where

$$[1, 2, 3, 4](x, y, z, w) = (-w, x, y, z).$$

The dual basis to  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  is

$$\begin{aligned}\beta_1 &= (1, 1, 1, 1), \beta_2 = (\sqrt{2})(0, 1, 1, 0), \\ \beta_3 &= (\sqrt{2})(0, 1, 1, 1), \beta_4 = (\sqrt{2})(0, 0, 1, 0).\end{aligned}$$

The 16 positive roots and the 16 corresponding Petrie polygon vertices can be computed according to section 3. The element  $\sigma = [1, 2, 3]$  precedes  $\gamma$ , has length three and its nine positive roots are tabulated below.

$i$	$\tau_i$	$\rho$ subscript	$\mu(\tau_i)$
1	$(1, 0, 0, 0)$	1	$(1, 1, 1, 1)$
2	$(\sqrt{2}/2)(1, 1, 0, 0)$	3	$(\sqrt{2})(0, 1, 1, 1)$
3	$(0, 1, 0, 0)$	5	$(-1, 1, 1, 1)$
4	$(\sqrt{2}/2)(1, 0, 1, 0)$	6	$(\sqrt{2})(0, 0, 1, 1)$
5	$(\sqrt{2}/2)(0, 1, 1, 0)$	7	$(\sqrt{2})(-1, 0, 1, 1)$
6	$(0, 0, 1, 0)$	9	$(-1, -1, 1, 1)$
7	$(\sqrt{2}/2)(-1, 1, 0, 0)$	12	$(\sqrt{2})(-1, 0, 0, 0)$
8	$(\sqrt{2}/2)(-1, 0, 1, 0)$	14	$(\sqrt{2})(-1, -1, 0, 0)$
9	$(\sqrt{2}/2)(0, -1, 1, 0)$	15	$(\sqrt{2})(0, -1, 0, 0)$

We use theorem 5.1 to determine the simple system  $\{\delta_1, \delta_2, \delta_3\}$ . First  $\delta_3 = \tau_9$  and hence  $\sigma R(\delta_3) = [1, 2]$ . It follows that  $\delta_2 = \tau_7$  and, since  $\sigma R(\delta_3)R(\delta_2) = [1]$ , that  $\delta_1 = \tau_1$ . Thus

$$\Delta = \{(1, 0, 0, 0), (\sqrt{2}/2)(-1, 1, 0, 0), (\sqrt{2}/2)(0, -1, 1, 0)\}.$$

Next we calculate

$$\epsilon_1 = \delta_1 = \tau_1, \quad \epsilon_2 = R(\delta_1)\delta_2 = \tau_2, \quad \text{and} \quad \epsilon_3 = R(\delta_1)R(\delta_2)\delta_3 = \tau_4.$$

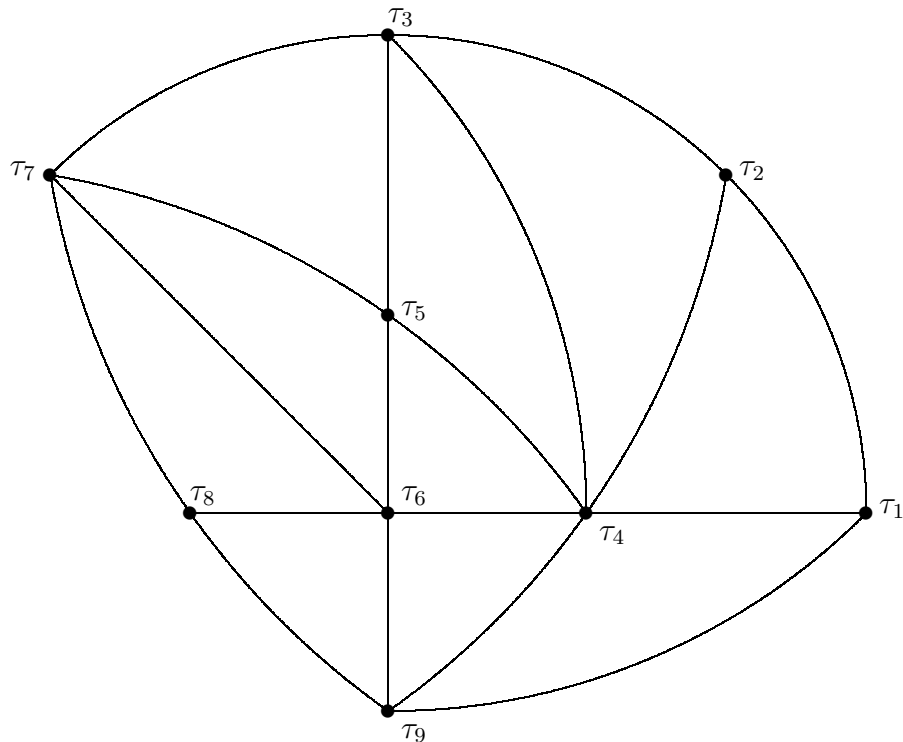


FIGURE 3.

The ordering  $\epsilon_1, \epsilon_2, \epsilon_3$  is already subordinate to the global ordering. Thus  $\theta_i = \epsilon_i$  in this case. Figure 3 shows the subcomplex  $X(\sigma)$  for this example.

**Example 6.14.** Consider the group  $A_3 = \Sigma_4$ . We have seen that the four-cycle  $\gamma = (1, 2, 3, 4)$  is a Coxeter element. Choose the simple system

$$\alpha_1 = (\sqrt{2}/2)(1, 0, -1, 0), \alpha_2 = (\sqrt{2}/2)(-1, 1, 0, 0), \alpha_3 = (\sqrt{2}/2)(0, 0, 1, -1),$$

corresponding to the factorisation  $\gamma = (1, 3)(1, 2)(3, 4)$ . For notational convenience we will identify roots with the transpositions they determine. The global order is the following.

$$(1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 2).$$

We use theorem 5.1 to find the simple system. First  $\delta_3 = (1, 2)$  since this is the last root. Since  $(1, 2, 3, 4)(1, 2) = (1, 3, 4)$ ,  $\delta_2 = (3, 4)$ . Finally,  $\delta_1 = (1, 3)$ . Thus  $\epsilon_1 = (1, 3) = \tau_1$ ,  $\epsilon_2 = (1, 4) = \tau_3$  and  $\epsilon_3 = (2, 3) = \tau_2$ , making the ordering on the  $\epsilon$ 's inconsistent with the global order.

## 7. CHARACTERISATION OF $|X(\sigma)|$ AND PROOF OF THE LATTICE PROPERTY.

In this section we show that if  $\sigma \leq \gamma$  then  $X(\sigma)$  is a simplicial complex. We characterise the geometric realisation  $|X(\sigma)|$  of the subcomplex  $X(\sigma)$  and use this characterisation to prove that the interval  $[I, \gamma]$  in  $W$  is a lattice. Throughout this section we continue to use the notation of the earlier sections.

We begin with a technical result (proposition 7.2) about the separation properties of the hyperplanes  $\{\mu(\tau_i)^\perp\}$  which is used in the proof of theorem 7.6. It depends on proposition 7.1 which is stated for convenient reference and which concerns the action of  $\sigma^{-1}$  on  $P$ . Using the factorisation  $\sigma^{-1} = R(\delta_k) \dots R(\delta_1)$ , this is a special case of theorem B in section I.4.3 of [12].

**Proposition 7.1.** *If  $\tau_s \in P$  then  $\sigma^{-1}(\tau_s) \in -P$  if and only if  $\tau_s = \epsilon_i$  for some  $i$  with  $1 \leq i \leq k$ .*

**Proposition 7.2.** *Let  $\tau_a$  and  $\tau_s$  be elements of  $P$  such that  $\tau_a \geq \theta_k$ ,  $\tau_s < \tau_a$ ,  $\tau_s \notin \{\theta_1, \theta_2, \dots, \theta_{k-1}, \theta_k\}$  and  $\tau_a \in \mu(\tau_s)^\perp$ . Then we can find two roots  $\tau_b$  and  $\tau_c$  in  $P$  with  $\tau_b, \tau_c < \tau_a$ , which both lie on the hyperplane  $\mu(\tau_a)^\perp$ , but which are separated by the hyperplane  $\mu(\tau_s)^\perp$ .*

*Proof:* Since  $\tau_a \cdot \mu(\tau_s) = 0$ ,  $\tau_a \in M(R(\tau_s)\gamma)$  and the element  $R(\tau_s)R(\tau_a)$  precedes  $\gamma$  by equation (7).

**Construction of  $\tau_b$ :** Let  $Q = \{\tau_i \in P \mid R(\tau_i) \leq R(\tau_s)R(\tau_a)\}$ . Since  $\tau_s < \tau_a$ , the proof of lemma 5.6 implies that (i) the set  $\{\tau_s, \tau_a\}$  is a simple system which spans  $Q$  and (ii)  $\tau_s$  is the first root and  $\tau_a$  the last root in the induced ordering of  $Q$ . By lemma 5.5, the element  $\tau_b$  given by  $\tau_b = R(\tau_a)\tau_s$  is in  $Q$  and  $s < b < a$ . Since

$$R(\tau_a)R(\tau_b) = R(\tau_a)(R(\tau_a)R(\tau_s)R(\tau_a)) = R(\tau_s)R(\tau_a) \leq \gamma,$$

we deduce that  $\tau_b \cdot \mu(\tau_a) = 0$ . Finally  $\tau_b \cdot \mu(\tau_s) > 0$  because

$$\mu(\tau_s) \cdot \tau_b = \mu(\tau_s) \cdot R(\tau_a)\tau_s = \mu(\tau_s) \cdot \{\tau_s - 2(\tau_a \cdot \tau_s)\tau_a\} = 1$$

since  $\mu(\tau_s) \cdot \tau_s = 1$  and  $\mu(\tau_s) \cdot \tau_a = 0$ .

**Construction of  $\tau_c$ :** As  $\tau_s$  is not an element of  $\{\theta_1, \dots, \theta_k\} = \{\epsilon_1, \dots, \epsilon_k\}$ , the root  $\tau_c$  defined by  $\tau_c = \sigma^{-1}\tau_s$  is an element of  $P$  (by

proposition 7.1). Since  $R(\tau_s)R(\tau_a) \leq \gamma$  and  $M[R(\tau_s)R(\tau_a)] \subset M(\sigma)$  we deduce by equation (4) that  $R(\tau_s)R(\tau_a) \leq \sigma$ . This gives  $R(\tau_a) \leq R(\tau_s)\sigma = \sigma R(\tau_c)$  by equation (2) and hence  $R(\tau_a)R(\tau_c) \leq \sigma \leq \gamma$  by equation (7). It follows that  $\tau_c \cdot \mu(\tau_a) = 0$ . Now  $\tau_c \cdot \mu(\tau_s) < 0$  since

$$\begin{aligned} \mu(\tau_s) \cdot \tau_c &= \mu'(\tau_s) \cdot \tau_c \\ &= \mu'(\tau_s) \cdot \sigma^{-1}(\tau_s) \\ &= \sigma(\mu'(\tau_s)) \cdot \tau_s \\ &= \{\mu'(\tau_s) - 2\tau_s\} \cdot \tau_s \text{ by proposition 6.6} \\ &= 1 - 2 = -1. \end{aligned}$$

Since  $\mu(\tau_s) \cdot \tau_c < 0$ , theorem 3.7 implies that  $\tau_c < \tau_s$  and hence  $\tau_c < \tau_a$ , as required.

**Definition 7.3.** *If  $\tau_i$  is a root in  $P$  and  $\rho$  is any positive root, we define*

- $\mu(\tau_i)^+ = \{x \in \mathbf{R}^n \mid x \cdot \mu(\tau_i) \geq 0\}$  (a positive halfspace),
- $\mu(\tau_i)^- = \{x \in \mathbf{R}^n \mid x \cdot \mu(\tau_i) \leq 0\}$  (a negative halfspace),
- $X(\sigma, \rho) =$  the set of simplices of  $X$  whose vertices both lie in  $M(\sigma)$  and precede  $\rho$  in the total order,
- $c[X(\sigma, \rho)] =$  the positive cone on the set  $|X(\sigma, \rho)|$ .
- $Y(\sigma, \rho) =$  the positive cone on those roots which both lie in  $M(\sigma)$  and precede  $\rho$  in the total order,
- $Z(\sigma, \tau_i) = M(\sigma) \cap \mu(\theta_1)^+ \cap \cdots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i+1})^- \cap \cdots \cap \mu(\tau_t)^-$ , for  $\tau_i \geq \theta_k$ .

**Theorem 7.4.** *The set  $X(\sigma, \tau_i)$  is a simplicial complex for each  $i = 1, \dots, t$ .*

*Proof:* We use induction on  $i$ . First note that  $X(\sigma, \tau_1) = \{\langle \tau_1 \rangle\}$ , a zero-dimensional simplicial complex.

Assume now that  $i \geq 1$  and that  $X(\sigma, \tau_i)$  is a simplicial complex. By definition, if  $\tau_j \in X(\sigma, \tau_i)$  then  $\langle \tau_j, \tau_{i+1} \rangle \in X(\sigma, \tau_{i+1})$  if and only if  $\tau_j \cdot \mu(\tau_{i+1}) = 0$ . It follows from theorem 3.7 that the only vertices of  $X(\sigma, \tau_i)$  that are not joined to  $\tau_{i+1}$  by an edge in  $X(\sigma, \tau_{i+1})$ , lie in the interior of the halfspace  $\mu(\tau_{i+1})^+$ . Hence,  $\mu(\tau_{i+1})^\perp \cap |X(\sigma, \tau_i)|$  is a simplicial complex. Now each simplex in  $X(\sigma, \tau_{i+1}) \setminus X(\sigma, \tau_i)$  is of the form  $\langle \tau_{a_1}, \dots, \tau_{a_b}, \tau_{i+1} \rangle$  where  $\tau_{a_1} < \cdots < \tau_{a_b} < \tau_{i+1}$  and where  $\tau_{a_c} \in \mu(\tau_{i+1})^\perp$  for  $c = 1, \dots, b$ . Thus the simplex  $\langle \tau_{a_1}, \dots, \tau_{a_b} \rangle$  of  $X(\sigma, \tau_i)$  is contained in  $\mu(\tau_{i+1})^\perp$ . Using this, it is straightforward to verify that the intersection of any two simplices in  $X(\sigma, \tau_{i+1})$  is itself a simplex in  $X(\sigma, \tau_{i+1})$ .

**Corollary 7.5.** *For each  $\sigma \leq \gamma$ ,  $X(\sigma)$  is a simplicial complex of dimension  $l(\sigma) - 1$ . In particular,  $X(\gamma)$  is a simplicial complex of dimension  $n - 1$ .*

**Theorem 7.6.** *For  $\theta_k \leq \tau_i \leq \tau_t$ , we have  $c[X(\sigma, \rho)] = Y(\sigma, \tau_i) = Z(\sigma, \tau_i)$ .*

*Proof:* It suffices to show that  $Z(\sigma, \tau_i)$  is contained in  $c[X(\sigma, \tau_i)]$  because  $c[X(\sigma, \rho)]$  is contained in  $Y(\sigma, \rho)$  (by definition) and  $Y(\sigma, \tau_i)$  is contained in  $Z(\sigma, \tau_i)$  when  $\tau_i \geq \theta_k$  (by theorem 3.7). The proof is by induction on  $i$ , starting at the value  $i_0$  for which  $\tau_{i_0} = \theta_k$ .

**Base step:** Let  $F_0 = X(\sigma, \tau_{i_0-1})$  be the subcomplex of  $X(\gamma)$  whose vertex set is  $\{\tau_1, \tau_2, \dots, \tau_{i_0-1}\}$ . Since  $\langle \theta_1, \theta_2, \dots, \theta_k \rangle \in X(\sigma)$  (by corollary 6.12) and  $\theta_1, \theta_2, \dots, \theta_{k-1} \in F_0$ , it follows that  $\langle \theta_1, \theta_2, \dots, \theta_{k-1} \rangle \in F_0$ . Furthermore, if  $\sigma_0 = R(\theta_k)\sigma$ , then  $F_0 = X(\sigma_0, \tau_{i_0-1})$ . By induction on  $k = l(\sigma)$ , we can assume that the assertion of the theorem is valid if  $\sigma$  is replaced by the length  $k - 1$  element  $\sigma_0$  and hence  $c[X(\sigma_0, \tau_{i_0-1})] = Z(\sigma_0, \tau_{i_0-1})$ . (The base case of this inner induction is trivial since  $k = 1$  corresponds to a rank 1 group.) Thus  $|F_0|$  is convex and  $(k - 2)$ -dimensional.

Now let  $V_0 = X(\sigma, \tau_{i_0})$  which has vertex set  $\{\tau_1, \tau_2, \dots, \tau_{i_0}\}$ . Since  $\mu(\theta_k) \cdot \tau_j = 0$  whenever  $\tau_j < \theta_k$  (as in the proof of corollary 6.12), it follows that  $|V_0|$  is the cone with base  $|F_0|$  and apex  $\tau_{i_0}$ . Thus  $|V_0|$  is convex and  $(k - 1)$ -dimensional.

The containment of  $Z(\sigma, \tau_{i_0})$  in  $c[X(\sigma, \tau_{i_0})]$  is demonstrated by examining the supports of the facets of the positive cone,  $c[V_0]$ , on  $|V_0|$ . Each support is of the form  $M(\sigma) \cap \mu(\tau_j)^\perp$  for some  $\tau_j$ .

One of the facets of  $c[V_0]$  contains  $|F_0|$  and hence it has support  $M(\sigma) \cap \mu(\tau_{i_0})^\perp$ . Each of the other facets of  $c[V_0]$  contains the vertex  $\tau_{i_0} = \theta_k$  and hence its support is of the form  $M(\sigma) \cap \mu(\tau_j)^\perp$  for some  $j \neq i_0$ . In fact,  $\tau_j$  must belong to  $\{\theta_1, \dots, \theta_{k-1}\} \cup \{\tau_j \mid \tau_j > \theta_k\}$  in this case. For,  $\mu(\tau_j)^\perp$  cannot separate  $|F_0|$  (because  $|F_0| \subset |V_0|$  which is contained in one of the half-spaces bounded by  $\mu(\tau_j)^\perp$ ) and hence if  $\tau_j \notin \{\theta_1, \dots, \theta_{k-1}\}$  then proposition 7.2 implies that we must have  $\tau_j > \theta_k$ . Thus the set of facets of  $c[V_0]$  is of the form

$$\{M(\sigma) \cap \mu(\tau_j)^\perp \mid \tau_j \in \{\theta_{i_1}, \dots, \theta_{i_a}, \theta_k\} \cup \{\tau_{j_1}, \dots, \tau_{j_b}\}\}$$

where  $j_l > i_0$  for  $l = 1, 2, \dots, b$ . It follows that

$$\begin{aligned} Z(\sigma, \tau_{i_0}) &= M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i_0+1})^- \cap \dots \cap \mu(\tau_t)^- \\ &\subseteq M(\sigma) \cap \mu(\theta_{i_1})^+ \cap \dots \cap \mu(\theta_{i_a})^+ \cap \mu(\theta_k)^+ \cap \\ &\quad \mu(\tau_{j_1})^- \cap \dots \cap \mu(\tau_{j_b})^- \\ &= c[X(\sigma, \tau_{i_0})]. \end{aligned}$$

**Inductive step:** Assume now that  $i \geq i_0$  and that  $c[X(\sigma, \tau_i)] = Z(\sigma, \tau_i)$ . Let  $F$  and  $V$  be the subcomplexes of  $X(\gamma)$  whose sets of vertices are

$$\{\tau_j \mid 1 \leq j \leq i \text{ and } \mu(\tau_{i+1}) \cdot \tau_j = 0\}$$

and

$$\{\tau_{i+1}\} \cup \{\tau_j \mid 1 \leq j \leq i \text{ and } \mu(\tau_{i+1}) \cdot \tau_j = 0\}$$

respectively. Then  $|V|$  is a cone with base  $|F|$  and apex  $\tau_{i+1}$ . We prove that the closure,  $Z$ , of  $Z(\sigma, \tau_{i+1}) \setminus Z(\sigma, \tau_i)$  is contained in the positive cone,  $c[V]$ , on  $|V|$ . Since  $|V|$  is contained in  $|X(\sigma, \tau_{i+1})|$ , it will then follow that  $Z(\sigma, \tau_{i+1}) = Z \cup Z(\sigma, \tau_i)$  is contained in  $c[X(\sigma, \tau_{i+1})]$ , as required.

First we show that  $F$  is  $(k-2)$ -dimensional. Denote the length  $(k-1)$  element  $R(\tau_{i+1})\sigma$  by  $\sigma'$ . Note that  $\sigma' \leq R(\tau_{i+1})\gamma$  so that  $M(\sigma') \subset \mu(\tau_{i+1})^\perp$ . Apply the procedures of sections 5 and 6 to  $\sigma'$  (i) to obtain a simple system  $\{\delta'_1, \delta'_2, \dots, \delta'_{k-1}\}$  for the set  $P'$  of positive roots in  $M(\sigma')$  and (ii) to calculate the reordering  $\{\theta'_1, \theta'_2, \dots, \theta'_{k-1}\}$  of the set  $\{\epsilon'_1, \epsilon'_2, \dots, \epsilon'_{k-1}\}$  for which

$$\sigma' = R(\theta'_{k-1}) \dots R(\theta'_2) R(\theta'_1)$$

where  $\theta'_1 < \theta'_2 < \dots < \theta'_{k-1}$  and where  $\epsilon'_j$  is given by

$$\epsilon'_j = R(\delta'_1) R(\delta'_2) \dots R(\delta'_{j-1}) \delta'_j \text{ for } j = 1, \dots, k-1.$$

Since  $\{\theta'_1, \theta'_2, \dots, \theta'_{k-1}\} \subset \mu(\tau_{i+1})^\perp$ , if we show that  $\tau_{i+1} > \theta'_j$  for  $j = 1, \dots, k-1$ , then it will follow that  $\langle \theta'_1, \theta'_2, \dots, \theta'_{k-1} \rangle \in F$  and hence  $F$  is  $(k-2)$ -dimensional.

Fix  $j \in \{1, \dots, k-1\}$ . From  $\epsilon'_j \cdot \mu(\tau_{i+1}) = 0$  we deduce that  $\epsilon'_j \neq \tau_{i+1}$  and that  $R(\epsilon'_j) \leq R(\tau_{i+1})\gamma$ . Therefore the length two element  $\sigma'' = R(\tau_{i+1})R(\epsilon'_j)$  precedes  $\gamma$  and hence  $\sigma$  by equation (4).

Assume now that  $\epsilon'_j > \tau_{i+1}$  and let  $Q = P \cap M(\sigma'')$ . Then, as in the proof of lemma 5.5,  $\{\tau_{i+1}, \epsilon'_j\}$  is the simple system for  $Q$ , with  $\tau_{i+1}$  the first root and  $\epsilon'_j$  the last root. Thus  $R(\tau_{i+1})\epsilon'_j$  is a positive root and  $R(\tau_{i+1})\epsilon'_j > \tau_{i+1}$  (by lemma 5.5).



Consider now  $(\sigma')^{-1}(\epsilon'_j)$  which is negative, by proposition 7.1. However

$$(\sigma')^{-1}\epsilon'_j = (R(\tau_{i+1})\sigma)^{-1}\epsilon'_j = \sigma^{-1}(R(\tau_{i+1})\epsilon'_j).$$

Since  $R(\tau_{i+1})\epsilon'_j$  is positive, proposition 7.1 implies that  $R(\tau_{i+1})\epsilon'_j = \epsilon_a$  for some  $a$ . Thus  $R(\tau_{i+1})\epsilon'_j$  precedes  $\tau_{i+1}$  (since each  $\epsilon_b$  does), which contradicts the earlier conclusion that  $R(\tau_{i+1})\epsilon'_j > \tau_{i+1}$ . Thus the assumption that  $\epsilon'_j > \tau_{i+1}$  must have been false.

Next we show that  $|F|$  is convex. Since  $c[X(\sigma, \tau_i)] = Z(\sigma, \tau_i)$  by the inductive hypothesis, we obtain

$$\begin{aligned} |F| &= M(R(\tau_{i+1})\sigma) \cap c[X(\sigma, \tau_i)] \cap S^{n-1} \\ &= M(R(\tau_{i+1})\sigma) \cap Z(\sigma, \tau_i) \cap S^{n-1} \end{aligned}$$

which is convex.

Now  $|V|$ , being a cone with a convex  $(k-2)$ -dimensional base, must itself be convex and  $(k-1)$ -dimensional. The proof that  $Z$  is contained in  $c[V]$  involves a close examination of the facets of  $c[V]$ . First one argues that each facet of  $c[V]$  is of the form  $M(\sigma) \cap \mu(\tau_j)^\perp$  for some

$$\tau_j \in \{\theta_{i_1}, \dots, \theta_{i_a}\} \cup \{\tau_{i+1}, \tau_{j_1}, \dots, \tau_{j_b}\},$$

where  $j_l > i+1$  for  $l = 1, \dots, b$ . (This step is similar to the corresponding argument for  $c[V_0]$ , with  $\tau_{i+1}$  taking the place of  $\theta_k$ .) It then follows that

$$\begin{aligned} Z &= M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i+1})^+ \cap \mu(\tau_{i+2})^- \cap \dots \cap \mu(\tau_t)^- \\ &\subseteq M(\sigma) \cap \mu(\theta_{i_1})^+ \cap \dots \cap \mu(\theta_{i_a})^+ \cap \mu(\tau_{i+1})^+ \cap \mu(\tau_{j_i})^- \cap \dots \cap \mu(\tau_{j_b})^- \\ &= c[V] \quad \text{as required.} \end{aligned}$$

**Corollary 7.7.** *For each  $\sigma \leq \gamma$  the set  $|X(\sigma)|$  is convex. Furthermore,  $|X(\sigma)|$  is the intersection with  $S^{n-1}$  of the positive cone on the set  $P$  of positive roots whose reflections precede  $\sigma$  and*

$$|X(\sigma)| = S^{n-1} \cap M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+.$$

*Proof:* Apply theorem 7.6 with  $i = t$ .

We are now in a position to prove that  $[I, \gamma]$  is a lattice.

**Theorem 7.8.** *If  $W$  is a finite real reflection group equipped with the partial order  $\leq$  defined by reflection length and  $\gamma$  is a Coxeter element for  $W$ , then the interval  $[I, \gamma]$  is a lattice.*

*Proof:* Choose a simple system and Coxeter element  $\gamma$  for  $W$  as in section 3. For  $\sigma \leq \gamma$  and  $\sigma \neq I$ , construct the simplicial complex  $X(\sigma)$

and define  $X(I)$  to be the empty set  $\emptyset$ . We have seen that  $X(\sigma)$  has dimension  $l(\sigma) - 1$  and its vertex set is

$$P_\sigma = \{\rho_i \mid 1 \leq i \leq nh/2 \text{ and } R(\rho_i) \leq \sigma\}.$$

Furthermore,  $X(\sigma)$  is a subcomplex of  $X(\gamma)$  and by corollary 7.7,  $|X(\sigma)|$  is convex.

Suppose now that  $\alpha$  and  $\beta$  both precede  $\gamma$  and consider the sub-complex  $X(\alpha) \cap X(\beta)$  of  $X(\gamma)$ . Since  $|X(\alpha) \cap X(\beta)| = |X(\alpha)| \cap |X(\beta)|$ , the set  $|X(\alpha) \cap X(\beta)|$  is convex by corollary 7.7. If  $d$  denotes the dimension of  $X(\alpha) \cap X(\beta)$  then  $|X(\alpha) \cap X(\beta)|$  is a union of  $d$ -dimensional simplices and, for each  $d$ -simplex  $\langle v_0, \dots, v_d \rangle$  in  $X(\alpha) \cap X(\beta)$ , the linear subspaces  $\text{span}(\{v_0, \dots, v_d\})$  and  $\text{span}(X)$  coincide. We associate an element  $\sigma \leq \gamma$  to  $X(\alpha) \cap X(\beta)$  as follows. Choose a  $d$ -simplex  $\langle v_0, \dots, v_d \rangle \in X$  and assume (without loss of generality) that  $v_0 \leq v_1 \leq \dots \leq v_d$ . Then

$$R(v_0)R(v_1)\dots R(v_d) \leq \gamma^{-1} \text{ by definition of } X(\gamma)$$

and we define  $\sigma$  to be  $R(v_d)\dots R(v_1)R(v_0)$ . Since  $\text{span}(v_0, \dots, v_d) = M(\sigma) = \text{span}(X(\alpha) \cap X(\beta))$ , it follows that  $M(\sigma)$  contains and is spanned by the vertex set of  $X(\alpha) \cap X(\beta)$ . However this vertex set is  $P_\alpha \cap P_\beta$  and the theorem now follows by our remarks in section 2.

## 8. RELATIONSHIP WITH GENERALISED ASSOCIAHEDRA.

In this section we embed the complex  $X(\gamma)$  in a larger simplicial complex  $EX(\gamma)$  whose vertex set consists of all positive roots and the negatives of the simple roots, and we show that the geometric realisation of  $EX(\gamma)$  is a sphere. If  $W$  is crystallographic, then we show that  $EX(\gamma)$  is simplicially isomorphic to the simplicial generalised associahedron for  $W$  (defined in [10]).

Recall that we have partitioned the simple roots into two commuting sets,  $S_1 = \{\alpha_1, \dots, \alpha_s\}$  and  $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ . We will use the notation

$$-S_1 = \{-\alpha_1, \dots, -\alpha_s\} \quad \text{and} \quad -S_2 = \{-\alpha_{s+1}, \dots, -\alpha_n\}.$$

We note that the subscripting on the  $\rho$ 's can be applied to negative indices with the convention that  $\rho_{-k} = \rho_{nh-k}$ .

**Definition 8.1.** *We define a set,  $EX = EX(\gamma)$ , of simplices by declaring that*

- *the vertex set is the ordered set*

$$\{\rho_{-n+s+1}, \dots, \rho_{-1}, \rho_0\} \cup \{\rho_1, \rho_2, \dots, \rho_{nh/2}\} \cup \{\rho_{nh/2+1}, \dots, \rho_{nh/2+s}\},$$

- *that an edge joins  $\rho_i$  to  $\rho_j$  whenever  $i < j$ ,  $\rho_i \neq -\rho_j$  and  $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$  and*

- $\langle \rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k} \rangle$  forms a  $(k-1)$ -simplex if the vertices are distinct and pairwise joined by edges.

We note that it follows from the definitions that the simplices of  $X(\gamma)$  are simplices of  $EX(\gamma)$ . The extra vertices are precisely the negatives of the simple roots  $\{\alpha_1, \dots, \alpha_n\}$ . Firstly, for  $k = 1, \dots, n-s$   $\rho_{s+k} = -\gamma(\alpha_{s+k})$  implies that  $\rho_{-n+s+k} = -\alpha_{s+k}$ . Secondly the set  $\{\rho_{nh/2+1}, \dots, \rho_{nh/2+s}\}$  is a permutation of  $\{-\alpha_1, \dots, -\alpha_s\}$ .

**Theorem 8.2.**  *$EX(\gamma)$  is a simplicial complex and  $|EX(\gamma)|$  is a sphere of dimension  $n-1$ .*

*Proof:* Let  $C$  be the spherical cross-polytope whose set of vertices is  $\{\pm\alpha_1, \dots, \pm\alpha_n\}$ . Then  $C$  is a simplicial complex whose geometric realisation  $|C|$  is the unit sphere in  $\mathbf{R}^n$ . We will show that  $EX(\gamma)$  is a simplicial subdivision of  $C$ .

Consider the simplicial subdivision  $C'$  of  $C$  which is defined as follows. Let  $K$  be the sub-complex of  $C$  which consists of the simplex  $A = \langle \alpha_1, \dots, \alpha_n \rangle$  and all its faces. We proved earlier that  $|X(\gamma)| = |K|$ . Thus  $X(\gamma)$  is a simplicial subdivision of  $K$ .

Extend this subdivision to the rest of  $C$  as follows. Any simplex of  $C$  whose vertices are contained in  $\{-\alpha_1, \dots, -\alpha_n\}$  is not subdivided. If  $B$  is a simplex of  $C$  which contains both positive and negative roots then we can write

$$B = \langle \alpha_{i_1}, \dots, \alpha_{i_p} \rangle * \langle -\alpha_{j_1}, \dots, -\alpha_{j_q} \rangle$$

where  $i_1 < \dots < i_p$ ,  $j_1 < \dots < j_q$ ,  $*$  denotes the spherical join, and  $\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} = \emptyset$ . We extend the subdivision of the simplex  $\langle \alpha_{i_1}, \dots, \alpha_{i_p} \rangle$  (in  $X(\gamma)$ ) by taking the join of this subdivision with the simplex  $\langle -\alpha_{j_1}, \dots, -\alpha_{j_q} \rangle$ .

Suppose  $J \subset \{1, \dots, s\}$ ,  $K \subset \{s+1, \dots, n\}$  and that  $\{\rho_{i_1}, \dots, \rho_{i_a}\}$  is an ordered set of positive roots. Then  $\{-\alpha_j \mid j \in J \cup K\} \cup \{\rho_{i_1}, \dots, \rho_{i_a}\}$  is the vertex set for a simplex in  $C'$  or in  $EX(\gamma)$  if and only if

$$R(\rho_{i_1}) \dots R(\rho_{i_a}) \leq \left( \prod_{l \in K} R(\rho_l) \right) \gamma^{-1} \left( \prod_{m \in J} R(\rho_m) \right).$$

It now follows that  $C' = EX(\gamma)$  as required.

For convenience we recall some facts from [10] and express them in a manner consistent with our earlier notation. If  $W$  is a crystallographic finite reflection group then the simplicial generalised associahedron,  $GA(W)$ , for the simple system  $S_1 \cup S_2$  is a simplicial complex whose

set of vertices, denoted  $\Omega_{\geq -1}$  consists of all the positive roots and the negative simple roots. Two piecewise-linear involutions  $\tau_+$  and  $\tau_-$  are introduced in [10]. It can be shown that they are determined by

$$\tau_+(\beta) = \begin{cases} R_1 R_2 \dots R_s(\beta) & \text{if } \beta \notin -S_2 \\ \beta & \text{if } \beta \in -S_2, \end{cases}$$

$$\tau_-(\beta) = \begin{cases} R_{s+1} R_{s+2} \dots R_n(\beta) & \text{if } \beta \notin -S_1 \\ \beta & \text{if } \beta \in -S_1. \end{cases}$$

From this we deduce that the action of  $\tau_+ \tau_-$  on  $\Omega_{\geq -1}$  is given by

$$\tau_+ \tau_-(\beta) = \begin{cases} \gamma(\beta) & \text{if } \beta \notin (-S_1) \cup S_2 \\ -\beta & \text{if } \beta \in (-S_1) \cup S_2, \end{cases}$$

and the action of the inverse  $\tau_- \tau_+$  is given by

$$\tau_- \tau_+(\beta) = \begin{cases} \gamma^{-1}(\beta) & \text{if } \beta \notin (-S_2) \cup S_1 \\ -\beta & \text{if } \beta \in (-S_2) \cup S_1. \end{cases}$$

The compatibility degree  $(\alpha \parallel \beta)$  of any two elements  $\alpha, \beta \in \Omega_{\geq -1}$  is characterised in [10] by the conditions

- (i):  $(-\alpha_i \parallel \alpha) = \max\{[\alpha : \alpha_i], 0\}$ , where  $[\alpha : \alpha_i] = \alpha \cdot \mu_i$  for  $i = 1, \dots, n$ , and
- (ii):  $(\alpha \parallel \beta) = (\tau_{\pm} \alpha \parallel \tau_{\pm} \beta)$  for all  $\alpha$  and  $\beta$ .

We recall that two vertices  $\alpha$  and  $\beta$  in  $GA(W)$  are connected by an edge if and only if  $(\alpha \parallel \beta) = 0$ . Finally, we note that  $GA(W)$  is completely determined by its one-skeleton.

**Theorem 8.3.** *If  $W$  is crystallographic, then  $EX(\gamma)$  is simplicially isomorphic to the simplicial generalised associahedron  $GA(W)$ .*

*Proof:* We continue to use the earlier notation. First note that the vertex set  $\Omega_{\geq -1}$  of the associahedron is the same as that of  $EX(\gamma)$ , and we will refer to the vertices in both complexes using the ordering from  $EX(\gamma)$ . Since each of  $EX(\gamma)$  and  $GA(W)$  is determined by its one-skeleton, it suffices to show that two vertices  $\alpha$  and  $\beta$  are joined by an edge in  $EX(\gamma)$  if and only if they are joined by an edge in  $GA(W)$ .

To simplify notation we will write  $\rho_i \xrightarrow{GA} \rho_j$  or  $\rho_i \xrightarrow{EX} \rho_j$  if  $\rho_i$  and  $\rho_j$  are connected by an edge in the simplicial complex  $GA(W)$  or  $EX(\gamma)$  respectively. Note that the ordering on the vertices of  $EX(\gamma)$  is such that they are arranged into the following sets in the given order

$$-S_2, S_1, \gamma(-S_2), \gamma(S_1), \gamma^2(-S_2), \dots, \gamma^{-1}(-S_1), S_2, -S_1.$$

Suppose  $\rho_i < \rho_j$ . Then  $\rho_i \xrightarrow{EX} \rho_j$  is equivalent to  $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$ . First we can assume  $\rho_j \neq -\rho_i$  since neither complex contains an edge from  $\rho_i$  to  $-\rho_i$ .

**Case 1.** If  $\rho_i \in -S_2$  then  $\rho_i = -\alpha_p$  for some  $p$  satisfying  $s+1 \leq p \leq n$ . Then  $\rho_i \xrightarrow{GA} \rho_j$  if and only if  $(-\alpha_p \parallel \rho_j) = 0$ . However this is equivalent to  $R(\rho_j) \leq R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n)$  by equation (4). But  $R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n) = \gamma R(\alpha_p)$  since  $p > s$ . Thus  $\rho_i \xrightarrow{GA} \rho_j$  is equivalent by equation (7) to  $R(\rho_j)R(\rho_i) \leq \gamma$  which in turn is the criterion for  $\rho_i \xrightarrow{EX} \rho_j$ .

**Case 2.** If  $\rho_i \in S_1$ , then  $\rho_i = \alpha_p$ , for some  $p$  satisfying  $1 \leq p \leq s$  and  $\tau_- \tau_+(\rho_i) = -\rho_i = -\alpha_p \in -S_1$ .

**Case 2(a).** It is possible that  $\rho_j \in S_1$  also. If this is the case then  $\tau_- \tau_+(\rho_j) = -\rho_j \in -S_1$  also and applying  $\tau_- \tau_+$  to both roots gives

$$(\rho_i \parallel \rho_j) = (-\rho_i \parallel -\rho_j) = 0$$

since  $\rho_j$  and  $\rho_i$  are distinct simple roots. By definition of edges in the generalised associahedron we must have  $\rho_i \xrightarrow{GA} \rho_j$ . However, in this subcase,  $\rho_j = \alpha_q$  with  $1 \leq p < q \leq s$ , so that

$$R(\rho_j)R(\rho_i) = R(\rho_i)R(\rho_j) \leq \gamma$$

which is again equivalent to the criterion for  $\rho_i \xrightarrow{EX} \rho_j$ .

**Case 2(b).** Now we assume  $j > s$ . In this subcase, we know that  $\tau_- \tau_+(\rho_j) = \gamma^{-1}(\rho_j)$ . We have  $\rho_i \xrightarrow{GA} \rho_j$  if and only if  $(-\alpha_p \parallel \gamma^{-1}(\rho_j)) = 0$ . However this is equivalent by equation (4) to

$$R[\gamma^{-1}(\rho_j)] \leq R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n).$$

Since  $1 \leq p \leq s$ , we have  $R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n) = R(\alpha_p)\gamma$ . Thus  $\rho_i \xrightarrow{GA} \rho_j$  is equivalent to  $R(\rho_i)R[\gamma^{-1}(\rho_j)] \leq \gamma$  by equation (7). But

$$R(\rho_i)R[\gamma^{-1}(\rho_j)] \leq \gamma \Leftrightarrow R(\rho_i) \leq \gamma R[\gamma^{-1}(\rho_j)] = R(\rho_j)\gamma,$$

using equations (7) and (2). The last condition in turn is the criterion for  $\rho_i \xrightarrow{EX} \rho_j$ .

**Case 3.** If  $i > s$  let  $m$  be the smallest positive integer with the property that

$$(\tau_- \tau_+)^m(\rho_i) \in (-S_2) \cup S_1.$$

Now  $\rho_i \xrightarrow{GA} \rho_j$  if and only if  $(\tau_- \tau_+)^m(\rho_i) \xrightarrow{GA} (\tau_- \tau_+)^m(\rho_j)$ . However, Case 1 or Case 2 now applies to this new pair of roots and  $(\tau_- \tau_+)^m = \gamma^{-m}$  when applied to  $\rho_i$  and  $\rho_j$ . Thus  $\rho_i \xrightarrow{GA} \rho_j$  if and only if

$$R(\gamma^{-m}(\rho_j))R(\gamma^{-m}(\rho_i)) \leq \gamma,$$

by the proof in cases 1 and 2. But this last condition is equivalent to  $R(\gamma^{-m}(\rho_i)) \leq R(\gamma^{-m}(\rho_j))\gamma$ , or  $\rho_{i-mn} \cdot \mu_{j-mn} = 0$ . Since  $\gamma$  is an isometry this is equivalent to  $\rho_i \cdot \mu_j = 0$  and hence to  $\rho_i \xrightarrow{EX} \rho_j$ .

**Note 8.4.** *We observe that theorem 8.3 provides a new proof that the simplicial generalised associahedron is a simplicial complex whose geometric realisation is a sphere. This proof is independent of the classification of finite real reflection groups and extends the work of Fomin and Zelevinsky [10] to include the non-crystallographic finite reflection groups.*

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