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Minimal modules over valuation domains☆

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Abstract

Let R be a valuation domain. We say that a torsion-free R-module is *minimal* if it is isomorphic to all its submodules of finite index. Here, the usual concept of finite index for groups is replaced by the more appropriate (for module theory) definition: a submodule H of the module G is said to be of finite index in G if the quotient G/H is a finitely presented torsion module. We investigate minimality in various settings and show *inter alia* that over a maximal valuation domain, all torsion-free modules are minimal. Constructions of non-minimal modules are given by utilizing realization theorems of May and the authors. We also show that direct sums of minimal modules may fail to be minimal.

MSC: 13G05; 13A15; 13A17

0. Introduction

The motivation for the present investigation of modules over valuation domains with certain special properties comes, via Abelian group theory, from topology. Recall that a topological space Y is said to be H-connected if and only if every proper local homeomorphism ϕ from a space $X \to Y$ is a (global) homeomorphism. If the space Y is assumed to be a compact manifold, then Y is simply connected provided it has dimensions 1 and 2. For dimensions $\geqslant 4$, this is known not to be so. In dimension 3, however, one encounters the

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so-called Poincaré Conjecture since an H-connected three-manifold is a homology sphere. In an attempt to circumvent this difficulty, the notion of an *h*-connected space was considered: this is a connected first countable Hausdorff space *Y*, which has the property that given any connected covering $\phi : X \to Y$, *X* is homeomorphic to *Y* via *some* homeomorphism. If *Y* is an *h*-connected manifold then its fundamental group $\pi_1(Y)$ has the property: if $\varphi : H \to \pi_1(Y)$ is a monomorphism and $|G : H^{\varphi}| < \infty$, then $H^{\varphi} \cong \pi_1(Y)$ via *some* isomorphism. Thus, a (not necessarily Abelian) group, which has the property that is isomorphic to all its normal subgroups of finite index, is called an *hc-group*. (See [13] and references therein for further details.) Interestingly, in the category of Abelian groups, the same class of groups arose from considerations of a certain type of 'quasi-minimality' arising naturally in point-set topology. Recall that if $\mathscr{C}$ is a category with an equivalence relation on the objects of $\mathscr{C}$, and $\preceq$ is the quasi-order induced from a given property $\mathscr{P}$ by setting $A \preceq B$ if *A* is equivalent to some subobject of *B* having the property $\mathscr{P}$, then an object *M* is minimal with respect to $\preceq$ if $X \preceq M$ implies *X* is equivalent to *M*. When $\mathscr{C}$ is the category of Abelian groups with isomorphism of groups as the equivalence and $\mathscr{P}$ is the property of being of finite index, then *hc-groups* are precisely the minimal objects. These groups have been investigated by Óhógáin and Goldsmith in [7,11] where they have been termed *minimal*.

Since every proper quotient of the ring of integers is finite, the notion of an Abelian group *G* having a subgroup of *H* of finite index is both straightforward and natural: the quotient group G/H is finite. However, for modules over an arbitrary ring *R*, this concept is no longer so natural and one needs to find a meaningful extension of the notion if it is to play any significant role.

In the present paper, we focus on torsion-free modules over a valuation domain *R*. In the first section, we introduce the notion of a submodule *H* of *G* being of 'finite index' if the quotient G/H is *finitely presented and torsion*. With this new understanding of finite index, we say that a torsion-free *R*-module *G* is *minimal* if it is isomorphic to each of its submodules of finite index. Our principal result in that section is that every torsion-free module over a maximal valuation is minimal. We find classes of torsion-free *R*-modules, which are always minimal, for instance, free modules or modules with basic rank 1.

The second section is devoted to the construction of non-minimal, indecomposable torsion-free modules of finite rank over largely general valuation domains. We utilize the so-called realization theorems of May and the authors [6,9]. On the other hand, we show that over special valuation domains (first constructed by Nagata, [10]), every torsion-free module is minimal.

In the final section, we consider the problem of whether direct sums and summands of minimal modules remain minimal. Although we obtain some positive results, direct sums of minimal *R*-modules may fail to be minimal when *R* is not almost maximal.

1. Preliminaries and first results

Our notation is standard and any undefined terms may be found in the texts [4,5].

Throughout the sequel, *R* will denote a valuation domain and *Q* its field of quotients; the maximal ideal of *R* shall always be denoted by *P*. Since in the present paper there is no danger of confusion, when we say that R is a discrete valuation domain we automatically mean that R has rank 1, i.e. R is Noetherian.

Let M and H be torsion-free R-modules, with $H \leqslant M$. Inspired by the corresponding definition in the Abelian groups case, and with a little abuse of language, we say that *H has finite index in M* if M/H is a *finitely presented torsion* R-module, i.e. $M/H \cong \bigoplus_{i=1}^{n} R/r_i R$, for suitable $0 \neq r_i \in R$. An R-module M is said to be *minimal* if M is isomorphic to all of its submodules of finite index.

Throughout the paper we shall deal with torsion-free modules. Thus, an R-module is automatically assumed to be torsion-free and reduced, if not otherwise specified.

It is useful to recall that, over a valuation domain R, finitely generated submodules of finitely presented R-modules are finitely presented as well. Indeed, if $G/K < F/K$, where F is free of finite rank, and K and G/K are finitely generated, then G is a finitely generated submodule of the free module F, whence G is free, since R is a valuation domain. Therefore, G/K is finitely presented.

It is convenient to summarize some properties of this new notion of finite index; many standard ideas remain valid, one exception being that if A is of finite index in C and B is a submodule of C containing A, then B is not necessarily of finite index in C. For instance, take any valuation domain R with non-finitely generated maximal ideal P. Pick $0 \neq t \in P$. We have $tR < P < R$, and, by definition, tR has finite index in R, while P is not of finite index in R. In fact, R/P is not finitely presented since P is not finitely generated.

Proposition 1.1. *Let $A < B < C$ be R-modules.*

(i) *If A has finite index in C, then A has finite index in B if and only if B/A is finitely generated.*
(ii) *If A has finite index in B and B has finite index in C, then A has finite index in C.*

Proof. If A has finite index in B, then it is immediate that B/A is finitely generated. Conversely suppose B/A is finitely generated. Then B/A is also finitely presented, being contained in the finitely presented module C/A. We conclude that A is of finite index in B.

To establish (ii) note that we have an exact sequence

$$0 \to B/A \to C/A \to C/B \to 0,$$

where B/A and C/B are finitely presented. We have to show that C/A is also finitely presented. The above sequence shows that C/A is finitely generated; write $C/A = F/K$, where F is free of finite rank. Then $B/A = H/K$ for a suitable submodule H of F, and $C/B \cong F/H$. Since C/B is finitely presented, we derive that H is finitely generated, in view of Proposition 2.1, page 152 of [5] (an immediate application of Schanuel's lemma). Since R is a valuation domain, finitely generated submodules of free modules are free; so H is free. Again by Proposition 2.1 of [5], from H/K finitely presented we get K finitely generated. We conclude that $C/A = F/K$ is finitely presented, as desired. □

Remark 1. One may wonder why we did not give a more obvious definition for a submodule H to be finite index in M, viz. requiring that M/H be a finite direct sum of cyclic torsion modules. We avoided this definition mainly because in such a case not even R would be a minimal module (unless R is a discrete valuation domain). In fact, whenever R contains a non-finitely generated ideal I, we have R/I cyclic and $R \ncong I$.

Proposition 1.2. *Let H be a submodule of finite index of a torsion-free R-module M. Let $x_1, \ldots, x_n$ be elements of M such that $M/H = \bigoplus_{i=1}^{n} R(x_i + H)$, where $Ann(x_i + H) = r_i R \neq R$, for suitable $r_i \in R$. Then $X = \langle x_1, \ldots, x_n \rangle$ is a free pure submodule of M, and $\{x_1, \ldots, x_n\}$ is a basis of X. Moreover, $Y = \bigoplus_{i=1}^{n} R(r_i x_i)$ coincides with $X \cap H$. In particular, Y is a pure submodule of H.*

Proof. Note that $\mathrm{Ann}(x_i + H) = r_i R \neq R$ implies that $x_i \notin H$ for all $i \leqslant n$; set $y_i = r_i x_i \in H$.

Firstly, we verify that $X = \langle x_1, \ldots, x_n \rangle = Rx_1 \oplus \cdots \oplus Rx_n$. In fact, assume for a contradiction that $\sum_{i=1}^{n} a_i x_i = 0$, for suitable $a_i \in R$ not all zero. One of the non-zero coefficients, say a_j, divides all the a_i, $1 \leqslant i \leqslant n$. Then, since M is torsion-free, possibly dividing the preceding equality by a_j, we may assume that $a_j = 1$. Reducing modulo H we get $x_j + H \in \langle x_i + H : i \neq j \rangle$, which is impossible. As a consequence, we also have $Y = \langle y_1, \ldots, y_n \rangle = Ry_1 \oplus \cdots \oplus Ry_n$.

Our next step is to prove that X is a pure submodule of M. Assume that $z \in M$ is such that $0 \neq tz = \sum_{i=1}^{n} b_i x_i$; we have to show that t divides all the coefficients b_i. Suppose not: let b_j be a coefficient which divides all the b_i, $1 \leqslant i \leqslant n$; then t is a proper multiple of b_j. Since M is torsion-free, dividing the preceding equality by b_j, we may reduce to the case when $b_j = 1$ and $t \in P$. Recall now that $z + H = \sum_{i=1}^{n} c_i(x_i + H)$, for suitable $c_i \in R$, whence $t(z + H) = \sum_{i=1}^{n} tc_i(x_i + H) = \sum_{i=1}^{n} b_i(x_i + H)$. It follows that $(1 - tc_j)(x_j + H) \in \langle x_i + H : i \neq j \rangle$, a contradiction, since $t \in P$, whence $1 - tc_j$ is a unit of R.

It remains to prove that $Y = X \cap H$, so that X pure in M implies Y pure in H, since all the modules are torsion-free. Pick an arbitrary $h \in X \cap H$; we have $h = \sum_{i=1}^{n} d_i x_i$, for suitable $d_i \in R$. Then $\sum_{i=1}^{n} d_i(x_i + H) = 0$, and therefore $d_i \in r_i R$, for all $i \leqslant n$. It follows that $h = \sum_{i=1}^{n} (d_i/r_i) y_i \in Y$. Since h was arbitrary, we conclude that $X \cap H \subseteq Y$. The opposite inclusion is trivial. □

The proof of the following lemma may be found in the book by Fuchs and Salce [5,Theorem XII.2.3].

Lemma 1.3. *Let F be a free R-module of finite rank, which is a pure submodule of a finite direct sum W of uniserial R-modules. Then F is a direct summand of W.*

The preceding Proposition 1.2 and Lemma 1.3 allow us to find some interesting classes of modules, which are automatically minimal.

Proposition 1.4. *Let M be a torsion-free R-module.*

(i) *If $M = PM$, then M is minimal. In particular, divisible modules are minimal.*
(ii) *If M is free, then M is minimal.*
(iii) *If I is a fractional ideal of R, then I is minimal.*

Proof. (i) We will show that in this case a submodule of finite index of M necessarily coincides with M. Therefore, M is trivially minimal. In the notation of Proposition 1.2, let us assume for a contradiction that $M = \langle x_1, \ldots, x_n \rangle + H$, where $M/H = \bigoplus_{i=1}^n R(x_i + H)$ and $0 \neq R(x_i + H) \cong R/r_i R$, for all $i \leqslant n$. In view of Proposition 1.2, the submodule $X = \bigoplus_{i=1}^n Rx_i$ is free and pure in M. In particular, $x_i \notin PM$ for all i. But this is contrary to the assumption $M = PM$.

(ii) In the same notation, let $M = X + H$, where M is free. Then X is a free pure submodule of M. Since X has finite rank, it is contained in a finite rank direct summand N of M. Therefore, by Lemma 1.3, X is a direct summand of N, whence X is also a direct summand of M, say $M = X \oplus M_1$. Let $Y = X \cap H$. By Proposition 1.2, $Y = \bigoplus_{i=1}^n R(r_i x_i)$ is free. We have

$$M_1 \cong M/X = (X + H)/X \cong H/(X \cap H) = H/Y.$$

Since M_1 is free, we get $H \cong Y \oplus M_1$. We conclude that $H \cong M$ since $X \cong Y$.

(iii) Since R is a valuation domain, then either I is principal, and so is free, or $I = PI$. In either case the desired result follows from the previous sections of the proposition. □

Recall that a torsion-free R-module M is said to be *separable* if every finite subset of M is contained in a direct summand W of M, where W is a finite direct sum of uniserial R-modules (see [4]).

Theorem 1.5. *Let H be a submodule of finite index of a torsion-free R-module M. If H is separable, then H is isomorphic to M. In particular, M is separable, too.*

Proof. We follow the notation of Proposition 1.2. Write $M = X + H$ and $Y = X \cap H$. Since H is separable, and $Y \cong X$ is free of finite rank, then Y is contained in a finite direct sum of uniserial modules W, which is a direct summand of H. By the preceding lemma, Y is a direct summand of W, and therefore of H. Write $H = Y \oplus H_1$ and observe that $X \cap H_1 = 0$, since $M = X + H$. It follows readily that $M = X \oplus H_1$. We conclude that M is isomorphic to H, as desired. □

Corollary 1.6. *Let R be a maximal valuation domain. Then every torsion-free R-module is minimal.*

Proof. Recall that uniserial modules over maximal valuation domains are pure-injective (see Theorem XIII.4.6 of [5]). Moreover, a torsion-free module over a valuation domain is separable if and only if every rank-one pure submodule is a direct summand—see e.g. Property (C), p. 551 of [5]. It follows immediately that every torsion-free R-module is separable. □

Corollary 1.7. *If M is a torsion-free separable minimal R-module, then a direct summand of M is again minimal.*

Proof. Suppose that $M = A \oplus B$ and that H is a submodule of A of finite index. Then $H \oplus B$ is clearly of finite index in M and hence is isomorphic to M. In particular, $H \oplus B$

is separable, which in turn implies H is separable since over valuation domains, summands of separable modules are again separable (see Property (D), p. 551 of [5]). It now follows directly from the theorem above that H is isomorphic to A and so the latter is again minimal. □

For the general notion of *basic submodule* of an R-module M we refer to [4,5]. We recall that, when M is torsion-free, a basic submodule B of M is any submodule which is maximal with respect to the properties:

(1) B is a direct sum of uniserial modules;
(2) B is a pure submodule of M.

In the torsion-free setting, basic submodules always exist and are unique, up to isomorphism. The *basic rank* of M is defined to be the rank of its basic submodules; it plays a role somewhat analogous to that played by the p-rank of a torsion-free Abelian group.

In this context, we have a result analogous to Óhógáin's result (see [11]) that a torsion-free Abelian group with p-rank at most 1 for all primes p is minimal.

We shall need a preliminary lemma, apparently not yet openly stated in the literature.

Lemma 1.8. *If B is a basic submodule of the torsion-free R-module module M, then $M = B + PM$.*

Proof. Pick an arbitrary $x \in M$, $x \notin B$. Since B is pure in M, we have $Rx \cap B = 0$. Then $Rx \oplus B$ is not pure in M, by the defining properties of basic submodules. Now $Rx \oplus B$ not pure amounts to the existence of $y \in M$ and $t \in P$ such that $ty \in Rx \oplus B \setminus P(Rx \oplus B)$. Let $ty = rx + b$, where $r \in R$ and $b \in B$. Then $r \notin P$; otherwise $b \in PB$, since B is pure, whence $ty \in P(Rx \oplus B)$, contrary to our assumption. It follows that r is a unit, so that $x = r^{-1}(-b + ty) \in B + PM$. Since x was arbitrary, we get the desired conclusion $M = B + PM$. □

Let us now examine the case when R is a discrete valuation domain of rank 1 (equivalently, a local principal ideal domain), with maximal ideal $P = \pi R$, and M is a reduced torsion-free R-module. In that case any basic submodule B of M is free, since M is reduced and the only uniserial R-modules are either R or Q. Moreover, from Lemma 1.8 we get $M/B = \pi(M/B)$, which implies that M/B is divisible. It then follows immediately that the basic rank of M coincides with the $R/\pi R$-dimension of $M/\pi M$ (which is equal to $\dim(B/\pi B)$). Therefore, when R is a discrete valuation domain, we have properties mirroring those satisfied by basic subgroups of Abelian groups.

Theorem 1.9. *Let M be a torsion-free R-module with basic rank* 1. *Then M is minimal*.

Proof. Suppose that there exists a proper submodule H of M such that M/H is finitely presented. Then, by Proposition 1.2, we may write $M = X + H$, where X is free and pure in M. Since M has basic rank 1, it follows that necessarily X is a basic submodule of M, whence X is cyclic, say $X = Rx$. Let $\mathrm{Ann}(x + H) = tR$. We want to show that, necessarily, $H = tM$, whence $M \cong H$ follows at once.

We readily see that $tM \subseteq H$, since $tM = t(Rx + H)$ and $tx \in H$.

We want to verify the opposite inclusion $H \subseteq tM$. First we show that $H \subseteq PM$. Indeed, since Rx is basic in M, by the preceding lemma we have $M = Rx + PM$. Now assume, for a contradiction, that there exists $h \in H \setminus PM$. We may write

$$h = ax + qm$$

for suitable $a \in R$, $q \in P$, $m \in M$. Then a is a unit, since $h \notin PM$. Moreover $m = bx + h'$, where $b \in R$, $h' \in H$. It follows that $(a + qb)x \in H$, which is impossible since $a + qb$ is a unit and $x \notin H$. We may now verify that $H \subseteq tM$. Take any $h \in H$. Since the basic submodule of M is free of rank 1, the purification $\langle h \rangle_*$ of h in M is also free. Accordingly, there exists $q \in R$ such that $h \in qM \setminus qPM$. Therefore, we may write

$$h = q(cx + h''), \quad c \in R, h'' \in H.$$

Since $h'' \in PM$ and $h \notin qPM$, we get that c is a unit. Then $qcx \in H$ implies $qx \in H$, whence $q \in tR$. It follows that $h \in tM$, as desired. $\square$

2. Non-minimal modules

The purpose of the present section is to show how to construct non-minimal indecomposable modules of finite rank over largely general valuation domains. However, we will see that the important class of Nagata valuation domains constitutes a noticeable exception.

We begin with a pair of lemmas valid for commutative domains.

Lemma 2.1. *Let A be a commutative ring, $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ pairwise distinct maximal ideals of A, and M an A-module such that $\mathfrak{M}_i M \neq M$ for all $i \leqslant k$. Then there exists $z \in M$ such that $z \notin \bigcup_{i \leqslant k} \mathfrak{M}_i M$.*

Proof. Since $\mathfrak{M}_j \not\supseteq \bigcap_{i \neq j} \mathfrak{M}_i$ for any $j \leqslant k$, we get $M = \mathfrak{M}_j M + (\bigcap_{i \neq j} \mathfrak{M}_i)M$, whence, in particular, $\mathfrak{M}_j M \not\supseteq (\bigcap_{i \neq j} \mathfrak{M}_i)M$. It follows that for every $j \leqslant k$ there exists $z_j \in (\bigcap_{i \neq j} \mathfrak{M}_i) M \setminus \mathfrak{M}_j M$. It is now clear that the element $z = z_1 + \cdots + z_k$ fulfills our requirements. $\square$

The following lemma and its proof are inspired by Arnold [1, Theorem 5.9(a), p. 52].

Lemma 2.2. *Let R be a commutative domain and let A be a commutative torsion-free R-algebra with $\mathfrak{M}$ a maximal ideal of A. If M is a torsion-free R-module such that*: (1) $End_R(M) = A$ *and* (2) M/tM *is a finitely generated R-module for some $t \in R \cap \mathfrak{M}$, then $\mathfrak{M}M \neq M$.*

Proof. Assume for a contradiction that $\mathfrak{M}M = M$. If $N = M/tM$, we have $\mathfrak{M}N = N$, as well. Since N is a finitely generated A-module, a classical result shows that $(1 + y)N = 0$, for some $y \in \mathfrak{M}$ (see e.g. Theorem 76 of [8]). It follows that $(1 + y)M \subseteq tM$. Since M is a torsion-free R-module, we infer that $(1 + y)/t$ is a well-defined endomorphism of M, so that $1 + y \in tA \subseteq \mathfrak{M}$ and hence we get the desired contradiction $1 \in \mathfrak{M}$. $\square$

Theorem 2.3. *Let R be a valuation domain and A a free R-algebra of rank* $n \geqslant 2$ *containing only finitely many maximal ideals* $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$, $k \geqslant 1$. *Suppose that there exists a torsion-free finite rank R-module M satisfying the following conditions*: (1) $End_R(M) = A$ *and* (2) *there exists* $t \in P$ *such that* M/tM *and* A/tA *are isomorphic as R-modules*. *Then, if* $z \in M$ *is such that* $z \notin \bigcup_{i \leqslant k} \mathfrak{M}_i M$, *the module* $N = Rz + tM$ *is non-minimal*. *Moreover, if A is an integral domain, then both M and N are indecomposable*.

Proof. It is clear that $P \subset \mathfrak{M}_i$, for every $i \leqslant k$. Also, observe that the existence of *z* is ensured by Lemmas 2.1 and 2.2; indeed $\mathfrak{M}_i M < M$, for every $i \leqslant k$, since $M/tM \cong A/tA \cong (R/tR)^n$ is a finitely generated *R*-module.

Let us first show that $\mathrm{Ann}_R(z + tM) = tR$, so that *tM* has finite index in *N*. Suppose not and assume that $sz \in tM$ with $t/s \in P$. It follows that $z \in (t/s)M \subset PM \subset \mathfrak{M}_i M$—contradiction.

We now verify that *N* is non-minimal by proving that $N \ncong M \cong tM$. Assume, for a contradiction, that there exists $f \in \mathrm{End}_R(M) = A$ such that $f(M) = N$. Now $f \notin \mathfrak{M}_i$ for all $i \leqslant k$, since $z \in f(M) \backslash \bigcup_{i \leqslant k} \mathfrak{M}_i M$. Therefore, *f* is a unit of *A* and $M = N$. It follows that

$$M/tM = N/tM \cong Rz/(tM \cap Rz)$$

is a cyclic *R*-module. We have reached the desired contradiction, since $M/tM \cong A/tA \cong (R/tR)^n$ is a direct sum of $n \geqslant 2$ cyclic modules.

Finally, assume that *A* is an integral domain. Then *M* is trivially indecomposable. However, *N* is also indecomposable. Suppose $\pi : N \to N$ is an idempotent in $\mathrm{End}_R(N)$. Then the composition $t\pi : M \to N$ can be regarded as an endomorphism of *M*, say *f*. Thus, $f^2 = tf$ in $\mathrm{End}_R(M) = A$, whence either $f = 0$ or $f = t1_A$. In the first case we get $f(N) = 0$, whence $\pi = 0$, since *N* is torsion-free. If $f = t1_A$, then for any $u \in N$ we get $tu = f(u) = t\pi(u)$. By torsion-freeness, we infer that π is the identity on *N*. Thus, $\mathrm{End}_R(N)$ contains only trivial idempotents and *N* is indecomposable. □

Before proving our next theorem, let us make some remarks. Assume that the field of quotients *Q* of *R* is not algebraically closed. Then there exists a non-trivial finite field extension $L = Q[x]$ of *Q*, where we may clearly assume that *x* is integral over *R*. Consider the integral closure *D* of *R* in *L*; by classical results on integral closures (for instance, see [3]), we know that *D* has only finitely many maximal ideals. Since *x* is integral over *R*, we have $D \supseteq R[x]$; equality does not hold in general. However, $R[x]$ has only finitely many maximal ideals, since *D* is integral over $R[x]$ and the 'lying over' property holds (see Theorem 44 of [8]).

In the remainder of this section we shall denote by $\hat{Q}$, $\hat{R}$ and $\hat{A}$ the completions as *R*-modules of *Q*, *R* and *A*, respectively.

Theorem 2.4. *Let R be a valuation domain such that Q is not algebraically closed and* $[\hat{Q} : Q] \geqslant 6$. *Then there exist non-minimal indecomposable torsion-free R-modules of finite rank*.

Proof. Since *Q* is not algebraically closed there exists, as noted above, a non-trivial finite-dimensional extension $L = Q[x]$ of *Q* such that $R[x]$ contains only finitely many maximal ideals. Set $A = R[x]$, and let $n = [L : Q] \geqslant 2$. Then $A = R \oplus Rx \oplus \ldots Rx^{n-1}$ is a free R-module.

Our goal is to construct an R-module M satisfying conditions (1) and (2) of Theorem 2.3.

In this situation we are in a position to apply the techniques of [6] and [9]. Note that in [9] it was observed that it is enough to assume $[\hat{Q} : Q] \geqslant 6$. We follow the notation of those papers. By Theorem 1 of [6] (see also Lemma 1 of [9]), for a suitable choice of $\alpha, \delta \in \hat{R} \backslash R$, the torsion-free R-module

$$M = \langle A1_A + A\alpha + A\delta \rangle_* \subset \hat{A}$$

is such that $\mathrm{End}_R(M) = A$ (here the symbol $\langle\ \rangle_*$ denotes the purification in $\hat{A}$).

Fix $t \in P$; to complete the proof we have to show that $M/tM \cong A/tA$. Consider firstly any $z = b_1 1_A + b_2\alpha + b_3\delta \in A1_A + A\alpha + A\delta$, where $b_i \in A$. For every $r \in P$, there exists $a_r, d_r \in R$ such that $\alpha - a_r \in r\hat{R}$ and $\delta - d_r \in r\hat{R}$. If we set $\lambda_r = b_1 + b_2 a_r + b_3 d_r \in A$, it follows that

$$z - \lambda_r 1_A \in r\hat{A} \cap M = rM.$$

Now an arbitrary element of M may be written in the form z/r, with $z = b_1 1_A + b_2\alpha + b_3\delta \in A1_A + A\alpha + A\delta$, where $b_i \in A$ and $r \in P$ are suitably chosen. In a similar way as above, we write $\lambda_{rt} = b_1 + b_2 a_{rt} + b_3 d_{rt} \in A$. Then $z - \lambda_{rt} 1_A \in rt\hat{A}$, and from this relation we first get $\lambda_{rt}/r 1_A \in \hat{A}$, whence $\mu_{rt} = \lambda_{rt}/r \in A$. Therefore, we also see that

$$z/r \equiv \mu_{rt} 1_A \bmod t\hat{A} \cap M = tM.$$

Since $z/r \in M$ was arbitrary, we conclude that the map $\phi : A \to M/tM$ defined by $\phi(\mu) = \mu(1_A + tM)$, for $\mu \in A$, is an epimorphism. Recall that L is the field of fractions of A. Then if $\mu 1_A \in tM \subset t\hat{A}$, we get $\mu/t \in \hat{A} \cap L = A$, whence $\mu \in tA$. Thus, we see that tA is the kernel of ϕ, so that M/tM and A/tA are isomorphic as A-modules, and therefore isomorphic as R-modules, as well. □

The fact that the non-minimal modules furnished by the above theorem are indecomposable is particularly interesting in the light of the forthcoming Theorem 3.5. In fact, in that result we will produce easy examples of non-minimal decomposable modules of rank 2 over any not almost maximal valuation domain.

It is worth noting that Q is not algebraically closed when R is not Henselian; see e.g. [12]. Consequently, Theorem 2.4 applies to any non-Henselian valuation domain R satisfying the degree condition.

Our next aim is to show that the preceding results do not hold, in general, if we deal with Henselian valuation domains. We will provide examples of non-maximal valuation domains, which are even discrete valuation domains, such that all their torsion-free modules of finite rank are minimal.

We consider the important class of discrete valuation domains called “Nagata valuation domains” in [17]. These are discrete valuation rings R of rank one such that $[\hat{Q} : Q] = p^k$, where $p > 0$ is the characteristic of Q, k is a positive integer, and, as above, $\hat{Q}$ denotes the completion of Q in the topology of the valuation. These types of discrete valuation domains were first constructed in Nagata’s book [10, Example E33, p. 207]. They are Henselian, since, by construction, $\hat{Q}$ is a purely inseparable extension of Q. Since they are not complete, they are, of course, not maximal.

However, note that Theorem 2.4 applies to Nagata valuations domains, whenever, in the above notation, $p^k \geqslant 6$.

Let R be a Nagata valuation domain such that $[\hat{Q} : Q] = 2$. Then it was shown in [17] (see also [2] for generalizations) that the finite-rank, torsion-free indecomposable modules, all have rank $\leqslant 2$ and moreover those of rank 2 are all isomorphic [17, Theorem 8]. We are now in a position to establish the following result.

Proposition 2.5. *Let R be a Nagata valuation domain such that $[\hat{Q} : Q] = 2$. Then all torsion-free R-modules of finite rank are minimal.*

Proof. Recall that R is a discrete valuation domain and denote by πR its maximal ideal. Let M be a torsion-free R-module of finite rank and H a submodule of M of finite index. Recall that the maximal divisible submodule D of a torsion-free R-module M is a direct summand, and that the definition readily implies that a submodule H of finite index in M has to contain D. Now if $M = M_1 \oplus D$, we can write $H = H_1 \oplus D$, where $H_1 = M_1 \cap H \subseteq M_1$, so that $M/H \cong M_1/H_1$ and $M \cong H$ if and only if $M_1 \cong H_1$. In conclusion, M is minimal if and only if M_1 is minimal, and therefore in the remainder of the proof we assume that M is reduced.

Since H is of finite index in M, there exists an integer k such that $\pi^k M \leqslant H \leqslant M$, and so it follows that M and H have the same rank. We claim that the basic ranks of M and H coincide also. As observed after Lemma 1.8, since M and H are reduced, these two basic ranks equal the dimensions of the $R/\pi R$ spaces $M/\pi M$ and $H/\pi H$, respectively. Let $T = M/H$. Since H has finite index in M, we have $T = \bigoplus_{i=1}^{m} Rz_i$, where $Rz_i \cong R/\pi^{n_i} R$, for suitable positive integers n_i. Now consider the submodule $T[\pi] = \{t \in T : \pi t = 0\}$. It is clear that the $R/\pi R$-dimensions of $T/\pi T$ and $T[\pi]$ coincide (namely, they are both equal to m). We may now reproduce verbatim the proof of Theorem 0.2, p. 3 in [1] to obtain the following equality of $R/\pi R$-dimensions:

$$\dim(M/\pi M) + \dim(T[\pi]) = \dim(H/\pi H) + \dim(T/\pi T)$$

which, in our case, yields $\dim(M/\pi M) = \dim(H/\pi H)$.

Now, as recalled above, Theorem 8 of [17] shows that M and H are direct sums of indecomposable submodules of rank $\leqslant 2$. Note that none of these summands is divisible, since M and H are reduced and so we may write $M = F \oplus M_1$ and $H = G \oplus H_1$, where F and G are free modules, and M_1 and H_1 are direct sums of rank-two indecomposable modules. Let $\mathrm{rk}\ F = f$, $\mathrm{rk}\ G = g$, $\mathrm{rk}\ M_1 = 2h$, $\mathrm{rk}\ H_1 = 2k$. Then the basic ranks of M_1 and H_1 are h and k, respectively. The equalities of the ranks and basic ranks yield the equations $f + 2h = g + 2k$ and $f + h = g + k$, from which it follows that $f = g$ and $h = k$. Therefore, we at once get $F \cong G$. Moreover M_1 and H_1 have the same number of indecomposable rank-two direct summands. Since indecomposable rank-two modules are all isomorphic [17, Theorem 8], we have $M_1 \cong H_1$ as well. We conclude that $M \cong H$, as desired. □

Our final results in this section show that over certain discrete valuation domains, the so-called Baer–Specker module $\mathscr{P} = \prod_{\aleph_0} R$ is not minimal. Note that $\mathscr{P}$ has infinite rank. We are presently assuming that R is a discrete valuation domain. So, if R is complete the product $\mathscr{P}$ will, of course, be minimal by Corollary 1.6. Clearly then it is necessary to assume that R is not complete but there are technical reasons relating to the combinatorial nature of our proof, which necessitate some cardinality restrictions. So, suppose that R is a discrete valuation domain of cardinality λ with maximal ideal pR and that the residue class field R/pR has cardinality μ. We shall call the following cardinal inequality, the residue class field cardinality condition:

$$\lambda^{\aleph_0} < \mu^{\mu^{\aleph_0}}.$$

Notice that $\mu^{\mu^{\aleph_0}} = 2^{\mu^{\aleph_0}}$ and that if $\lambda \leqslant 2^{\aleph_0}$ then this condition is always satisfied.

Theorem 2.6. *If R is a non-complete discrete valuation domain satisfying the residue class field cardinality condition, then $\mathscr{P} = \prod_{\aleph_0} R$ is not minimal.*

Proof. Since R is not complete, it is a slender module—see Example XVI 6.10 in [5]—and so the algebra of endomorphisms of $\mathscr{P}$ is, as an R-module, isomorphic to $\prod_{\aleph_0} \bigoplus_{\aleph_0} R$; in particular it has cardinality $\lambda^{\aleph_0}$. Consequently, there are at most $\lambda^{\aleph_0}$ submodules of $\mathscr{P}$, which are isomorphic to $\mathscr{P}$.

Now consider the module $\mathscr{P}/p\mathscr{P} \cong \prod_{\aleph_0} R/pR$. This is a vector space over the residue class field R/pR and as such is isomorphic to $\bigoplus_{\kappa} R/pR$ for some cardinal κ. Since $\prod_{\aleph_0} R/pR$ is the dual space of $\bigoplus_{\aleph_0} R/pR$, its dimension κ is well known to be $\mu^{\aleph_0}$. Fix a one-dimensional summand, say, $\mathscr{P}/p\mathscr{P} = \langle e \rangle \oplus V$, where V is a R/pR-space of dimension κ. It is well known that the number of direct complements for $\langle e \rangle$ is equal to the dimension of the dual space of V, i.e., there are $\mu^{\kappa} = \mu^{\mu^{\aleph_0}}$ subspaces $V_\alpha (\alpha < \mu^{\kappa})$ with $\mathscr{P}/p\mathscr{P} = \langle e \rangle \oplus V_\alpha$. Let H_α be the pre-image of V_α so that $H_\alpha / p\mathscr{P} = V_\alpha$. Thus, the family $\{H_\alpha : \alpha < \mu^{\kappa}\}$ is a family of $\mu^{\mu^{\aleph_0}}$ submodules of $\mathscr{P}$ each of which is of finite index in $\mathscr{P}$. By assumption $\mu^{\mu^{\aleph_0}} > \lambda^{\aleph_0}$, so not all of these submodules H_α can be isomorphic to $\mathscr{P}$. Hence $\mathscr{P}$ is not minimal as required. $\square$

Corollary 2.7. *If R is a discrete valuation domain of non-measurable cardinality λ and R is not complete, then the product $\prod_{\lambda} R$ is not minimal.*

Proof. Since λ is not measurable, the product $\prod_{\lambda} R$ is again reflexive [5, Corollary XVI,6.14] and so we can conclude that the cardinality of the endomorphism algebra of the product $\prod_{\lambda} R$ is $\lambda^{\lambda} = 2^{\lambda}$. But now the number of submodules of finite index in $\prod_{\lambda} R$ will be $\mu^{\mu^{\lambda}} \geqslant 2^{2^{\lambda}} > \lambda^{\lambda}$ and the result follows as above. $\square$

3. Direct summands and direct sums

We have already noted in Corollary 1.7 that a direct summand of a separable minimal module is again minimal.

It is worth noting and easily seen that a direct summand of a non-minimal module may be minimal. Just take any valuation domain R admitting a non-minimal R-module M_1. If F

is a free R-module of finite rank, then $M = M_1 \oplus F$ is not minimal (see the proof of the next proposition). Of course, the direct summand F of M is a minimal module.

Proposition 3.1. *Let the R-module M be of the form $M = G \oplus F$, where F is free of finite rank. Then M minimal implies that G is minimal.*

Proof. Suppose for a contradiction that M is minimal and G is not minimal. Then there is a submodule H of finite index in G which is not isomorphic to G. However, the submodule $H \oplus F$ is of finite index in M. Then one has $H \oplus F \cong M \cong G \oplus F$ and this implies that $H \cong G$ since F is finitely generated, and finitely generated modules over a valuation domain have the substitution, and hence the cancellation, property (see [5, Corollary V.8.3]). □

We will soon see that the converse of the preceding proposition fails whenever R is not almost maximal.

In the case of a Henselian domain we can say somewhat more.

Proposition 3.2. *Let R be a Henselian valuation domain and let M be a minimal R-module of finite rank. Then the indecomposable direct summands of M are minimal.*

Proof. Recall that Vámos [16] proved that indecomposable torsion-free modules of finite rank over a Henselian valuation domain have local endomorphism rings. Let us write $M = M_1 \oplus \cdots \oplus M_k$, where the M_i are indecomposable. This decomposition is unique, up to isomorphism, since $\mathrm{End}_R(M_i)$ is local for all $i \leqslant k$, and we may apply Azumaya's Theorem. Let us now assume, by contradiction, that M_1 (say) is non-minimal. Let H_1 be a submodule of finite index of M_1 such that $H_1 \ncong M_1$. It is then clear that $H = H_1 \oplus M_2 \oplus \cdots \oplus M_k$ is a submodule of M of finite index. Then $M \cong H$. Since $M_2, \ldots, M_k$ have local endomorphism rings, by iterated use of the substitution property (see [5] or [1] Chapter 8) we get $M_1 \cong H_1$. We have thus reached the desired contradiction. □

In the general situation, one can derive a closure property under the operation of direct sums for the class of minimal modules provided one is willing to impose some restrictions.

Theorem 3.3. *If $M = F \oplus X$, where F is a finite rank free R-module and X is a minimal R-module with $Ext^1_R(X, F)$ torsion-free, then M is minimal.*

Proof. Suppose that H is of finite index in M. Then since F is free of finite rank, the quotient $(H + F)/H$ is a finitely generated submodule of M/H and so by Proposition 1.1, H is of finite index in $H + F$. It follows from the usual Noether isomorphism that $H \cap F$ is of finite index in F, and hence we have $F \cong (H \cap F)$.

Since $(H+F)/H$ is finitely generated and M/H is finitely presented torsion, $M/(H+F)$ is finitely presented and so $(H + F)/F$ is of finite index in M/F. But $M/F \cong X$ and so $(H + F)/F \cong X$ since X is minimal. Thus, $H/(H \cap F) \cong X$. Moreover, M/H is finitely presented, so there is an element $r \in R$ such that $rM \leqslant H$. Thus, $rM + (H \cap F) = r(F \oplus X) + (H \cap F) = rX \oplus (H \cap F)$. But now $(H \cap F) \leqslant rH + (H \cap F) \leqslant rX \oplus (H \cap F)$ and so $rH + (H \cap F)$ splits over $(H \cap F)$. It follows from Lemma I.5.6 in [5] that this is equivalent to the extension $0 \to (H \cap F) \to H \to H/(H \cap F) \to 0$ being in the kernel of the mapping induced on $\mathrm{Ext}^1_R(H/(H \cap F), H \cap F)$ by multiplication by the scalar r, i.e. it is a torsion element of the module of extensions and this latter is just $Ext^1_R(X, F)$ since $H/(H \cap F) \cong X$ and $F \cong (H \cap F)$. However, by assumption, $\mathrm{Ext}^1_R(X, F)$ is torsion-free and so the extension $0 \to (H \cap F) \to H \to H/(H \cap F) \to 0$ must be the null element, i.e. H splits as $H = (H \cap F) \oplus Y$, where $Y \cong H/(H \cap F) \cong X$. It follows that $H \cong M$ and thus M is minimal as required. $\square$

Since $\mathrm{Ext}^1_R(X, F) \cong \prod \mathrm{Ext}^1_R(X, R)$ when F is free of *finite* rank, we can easily deduce:

Corollary 3.4. *If $M = F \oplus X$, where F is free of finite rank and X is minimal with $Ext^1_R(X, R) = 0$, then M is again minimal.*

Remark 2. In the previous theorem and corollary, the critical steps were to show that $H \cap F$ and $(H + F)/F$ were of finite index in the minimal modules F and M/F, respectively. For an arbitrary valuation domain, both of these facts follow if F is free of finite rank. We shall see shortly that for discrete valuation domains, the restriction that F be free of finite rank can be relaxed to obtain a significant generalization.

Our next result shows a crucial fact that in general the direct sum of minimal modules may fail to be minimal. As a by-product, we will see that one cannot drop the condition relating to the vanishing of Ext in the above results. Even in the simplest possible case where F is free of rank 1, the direct sum of a minimal module and R need not be minimal; in fact, this phenomenon always occurs when R is not an almost maximal valuation domain.

For the convenience of the reader, we recall some notions and results that are explained in full detail in [15,14], or, more briefly, in [5].

Let J be an ideal of the valuation domain R. Let us assume that R/J is not complete in the topology of its ideals. Then, necessarily, J is a v-ideal, that is $J = \bigcap_{r \notin J} rR$. Thus the set of ideals $\mathscr{B} = \{rR/J : r \notin J\}$ forms a basis of neighborhoods of 0 for the ideal topology of R/J. Since R/J is not complete, there exists a Cauchy net, with respect to $\mathscr{B}$, with no limit in R/J. We may assume it to have the form $\{u_r + J : r \notin J\}$, where all the u_r are units of R.

For J an ideal of R, we denote by J^{-1} its inverse, namely $J^{-1} = (R : J) = \{a \in Q : aJ \leqslant R\}$. We have $J^{-1} = \langle r^{-1} : r \notin J\rangle$. Note that, if J is a non-zero v-ideal, then J^{-1} is a non-principal fractional ideal, which is, by Proposition 1.4 (iii), a minimal R-module.

We also recall that R is not almost exactly maximal if there exists a non-zero ideal J such that R/J is not complete.

Theorem 3.5. *Let R be a valuation domain, which is not almost maximal. Then the direct sums of minimal R-modules may not be minimal. Specifically, if the non-zero ideal J is such that R/J is not complete, then the R-module $R \oplus J^{-1}$ is not minimal.*

Proof. Let us first note that, since J is a non-zero v-ideal, then J^{-1} is a fractional ideal, whence, in particular, J^{-1} is minimal. Since R/J is not complete, we may choose a family of units $\{u_r : r \notin J\}$ of R in the way described above. Consider the vector space $Qx \oplus Qy$, where x, y are indeterminates. By the results in [15], the following submodule of $Qx \oplus Qy$ is indecomposable of rank 2:

$$H = \langle x, z_r = r^{-1}(x + u_r y) : r \notin J \rangle;$$

moreover, Rx is a basic submodule of H. Now choose $t \in J$ and note that the R-module $M = Rt^{-1}x \oplus J^{-1}y$ clearly contains H and is isomorphic to $R \oplus J^{-1}$. Since H is indecomposable, it is not isomorphic to M. To complete the proof, it remains to show that H has finite index in M. We will show that $M = Rt^{-1}x + H$ and $M/H \cong R/tR$. Take $r^{-1}y \in J^{-1}y$, where r is any element of R not in J. By direct computation we see that $r^{-1}y = u_r^{-1}(-(t/r)t^{-1}x + z_r) \in Rt^{-1}x + H$ (note that $t/r \in P$). This suffices to show that $M \subseteq Rt^{-1}x + H$. Since $x \in H$, it is clear that $\mathrm{Ann}(t^{-1}x + H) \supseteq tR$. Assume now that $at^{-1}x \in H$ for a suitable $a \in R$. Since Rx is pure in H, we have $x \notin PH$, and therefore $x \in ta^{-1}H$ implies $ta^{-1} \notin P$, so that $at^{-1} \in R$ and $a \in tR$. Thus it follows that $\mathrm{Ann}(t^{-1}x + H) \supseteq tR$, whence equality holds, and we get the desired conclusion. □

The requirement in the preceding theorem that the valuation domain R is not almost maximal is necessary. If R is a discrete valuation domain and thus is automatically almost maximal, we can derive a far-reaching generalization of Theorem 3.3 and Corollary 3.4 above.

Theorem 3.6. *Let R be a discrete valuation domain and let G and X be minimal R-modules such that* $Ext_R^1(X, G)$ *is torsion-free. Then* $M = G \oplus X$ *is minimal.*

Proof. Since R is Noetherian, the notions of finitely presented and finitely generated coincide: a submodule of a finitely generated module over a Noetherian domain is again finitely generated. Now suppose that H is of finite index in $M = G \oplus X$. Then M/H is finitely presented and $(H+G)/H \leqslant M/H$ is also finitely presented. Since $G/H \cap G \cong (H+G)/H$, we conclude that $H \cap G$ is of finite index in G. Furthermore, $M/(H + G)$, as the quotient of two finitely presented modules is again finitely presented, and so $(H + G)/G$ is of finite index in M/G. Thus, we have shown that $H \cap G$ and $(H + G)/G$ are of finite index in the minimal modules G and M/G, respectively. As noted in the remark after Corollary 3.4, this is sufficient to obtain the desired result by simply repeating the proof of Theorem 3.3, replacing F by the minimal module G. □

Corollary 3.7. *Let R be a discrete valuation domain and G a minimal R-module. If F is a free R-module of arbitrary rank, then* $M = G \oplus F$ *is minimal.*

Proof. Since F is free, $\mathrm{Ext}_R^1(F, G) = 0$, and hence is trivially torsion-free. □

In special circumstances, the preceding results enable one to characterize summands of minimal groups.

Corollary 3.8. *Let R be a discrete valuation domain and let M be an R-module of the form* $M = G \oplus F$*, where F is free of finite rank. Then M is minimal if and only if G is minimal.*

Proof. One implication follows from the corollary above. The converse follows from Proposition 3.1. □

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